On the geography of symplectic 6-manifolds

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§0 Introduction

In this article we consider the geography of the six dimensional, 1-connected, symplectic manifolds. The main result is:

Theorem *Let* $\mathbf{X}$ *be a symplectic manifold. Then*

$$c_1^3(\mathbf{X}) \equiv c_3(\mathbf{X}) \equiv 0 \bmod 2, \;\; c_1c_2(\mathbf{X}) \equiv 0 \bmod 24$$

Conversely, any triple (a, b, c) *with* $a \equiv c \equiv 0 \bmod 2$ *and* $b \equiv 0 \bmod 24$ *occurs as a triple* (c_1^3, c_1c_2, c_3) *of Chern numbers of some 1-connected, compact symplectic 6-manifold.*

Remarks

- Let us recall that a symplectic manifold admits always almost complex structures. The arithmetical restrictions on the Chern numbers are proved in [**OV**] ; it is proved there the same result for almost complex 6-manifolds.
- We must say from the very beginning that, even if the notion of minimality in the symplectic case is unclear, the examples given in this paper are non-minimal bacause they are obtained blowing-up some basic building blocks.
- A similar study for complex 3-folds can be found in [**H**].

§1 Basic results

I shall recall some results used in the article in order to produce new examples of symplectic manifolds from given ones. The following proposition gives us such a method ; it is the symplectic connected sum construction, explained in detail in [**G**]. I give here only the case where I am interested:

Proposition 1.1 *Let* $\mathbf{X}_j,\ j = 1, 2$ *denote two symplectic varieties of dimension* $dim(\mathbf{X}_j) = n$ *and let* $\mathbf{Y}$ *be another symplectic variety of dimension* $dim(\mathbf{Y}) = n - 2$ *such that there are two symplectic embeddings* $i_j : \mathbf{Y} \longrightarrow i_j(\mathbf{Y}) = \mathbf{Y}_j \hookrightarrow \mathbf{X}_j$ *with the property:*

$$c_1(\mathcal{N}_{\mathbf{Y}_1|\mathbf{X}_1}) + c_1(\mathcal{N}_{\mathbf{Y}_2|\mathbf{X}_2}) = 0$$

Then one can make the symplectic connected sum of $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *along* $\mathbf{Y}$ *; the result will be a new symplectic manifold.*

Suppose now that there are two 2-spheres $\mathbf{L}_j \hookrightarrow \mathbf{X}_j$ *which meet* $\mathbf{Y}_j$ *transversally in one point. Then the symplectic sum can be done such that* $\mathbf{X}$ *contains an embedded 2-sphere also ; this is the (usual) connected sum of the* $\mathbf{L}_j$*'s.*

Proof See [**G**], page 538. □

Now we are interested to decide if such a symplectic sum is or not simply connected. This criterion is also known and used by Gompf in his article.

Proposition 1.2 *Let $\mathbf{X}$ be a symplectic variety and suppose that there is a symplectic codimension two variety $\mathbf{Y}$ which has the properties:*

- $\mathcal{N}_{\mathbf{Y}|\mathbf{X}} = \mathcal{O}_{\mathbf{Y}}$;
- *there is an embedded projective line $\mathbf{L} \hookrightarrow \mathbf{X}$ which cuts transversally $\mathbf{Y}$ in exactly one point.*

Then the homomorphism $i_* : \pi_1(\mathbf{X}-\mathbf{Y}) \longrightarrow \pi_1(\mathbf{X})$ induced by inclusion is in fact an isomorphism. In particular, if $\mathbf{X}$ if 1-connected, $\mathbf{X}-\mathbf{Y}$ will be also.

Proof The proof uses Weinstein's theorem which tels us that in our condition, there is a symplectically embedded neighborhood $\mathbf{D}_\epsilon \times \mathbf{Y} \hookrightarrow \mathbf{X}$ of $\mathbf{Y}$ (see for instance [**McDS**], page 98). Because $codim_{\mathbf{X}}\mathbf{Y} = 2$, any loop in $\mathbf{X}$ can be moved away from $\mathbf{Y}$ and so i_* is surjective. Let us now consider a loop $\gamma \hookrightarrow \mathbf{X}-\mathbf{Y}$ such that the image $i_*[\gamma] = 0$; this means that there is a homotopy $\Gamma : I^2 \longrightarrow \mathbf{X}$ of γ to a constant map. This homotopy can be taken in such a way that it meets $S(\epsilon) \times \mathbf{Y}$ transversally in a finite number of circles γ_k. We deduce that $[\gamma] = \prod_k [\gamma_k]$ in $\mathbf{X}-\mathbf{Y}$. In order to prove that $[\gamma] = 0$ in $\mathbf{X}-\mathbf{Y}$ it suffices to prove that each class $[\gamma_k]$ is so. We may move each circle γ_k until we reach the intersection circle $c := \mathbf{L} \cap (S(\epsilon) \times \mathbf{Y})$; but now c is contractible in $\mathbf{L} - \{point\} = \mathbf{C}$. □

Proposition 1.3 (i) *Suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are symplectic 4-manifolds which contain isomorphic symplectically embedded curves $\mathbf{Y}_1 \cong \mathbf{Y}_2 \cong \mathbf{Y}$ with trivial normal bundle. Then the Chern numbers of the symplectic connected sum $\mathbf{X} := \mathbf{X}_1 \sharp_{\mathbf{Y}} \mathbf{X}_2$ are the following:*

$$\begin{aligned} c_1^2(\mathbf{X}) &= c_1^2(\mathbf{X}_1) + c_1^2(\mathbf{X}_2) - 4c_1(\mathbf{Y}) \\ c_2(\mathbf{X}) &= c_2(\mathbf{X}_1) + c_2(\mathbf{X}_2) - 2c_1(\mathbf{Y}) \end{aligned}$$

(ii) *Suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are symplectic 6-manifolds which contain isomorphic symplectically embedded 4-folds $\mathbf{Y}_1 \cong \mathbf{Y}_2 \cong \mathbf{Y}$ with trivial normal bundle. Then the Chern numbers of the symplectic connected sum $\mathbf{X} := \mathbf{X}_1 \sharp_{\mathbf{Y}} \mathbf{X}_2$ are the following:*

$$\begin{aligned} c_1^3(\mathbf{X}) &= c_1^3(\mathbf{X}_1) + c_1^3(\mathbf{X}_2) - 6c_1^2(\mathbf{Y}) \\ c_1c_2(\mathbf{X}) &= c_1c_2(\mathbf{X}_1) + c_1c_2(\mathbf{X}_2) - 2(c_1^2(\mathbf{Y}) + c_2(\mathbf{Y})) \\ c_3(\mathbf{X}) &= c_3(\mathbf{X}_1) + c_3(\mathbf{X}_2) - 2c_2(\mathbf{Y}) \end{aligned}$$

Proof Regardless the dimension of the data, we may apply again Weinstein's theorem and to deduce that there are the symplectic embeddings $\varphi_j : \mathbf{Y} \times \mathbf{D}_\epsilon \longrightarrow \mathbf{X}_j$ such that the restrictions $\varphi_{j|\mathbf{Y}\times 0} = i_j$; here $\mathbf{D}_\epsilon$ denotes the disc of radius ϵ in the complex plane with the symplectic structure inherited from $\mathbf{R}^2$. Take the involution ψ of $\mathbf{D}_\epsilon - 0$ given in polar coordinates by: $\psi(r,\theta) := (\sqrt{\epsilon^2 - r^2}, -\theta)$. This is a symplectomorphism of the punctured disc. We can use it to identify (symplectically by $id_{\mathbf{Y}} \times \psi$) the two tubular neighborhoods $\mathbf{Y}_j \hookrightarrow \mathbf{X}_j$ in order to obtain the symplectic sum $\mathbf{X}$. Topologically, $\mathbf{X}$ is obtained identifying the cylinders $\mathbf{Y}_j \times S(r_0)$, where $S(r_0)$ denotes the cercle of radius

$r_0 := \epsilon/\sqrt{2}$. Let me consider now a form η on $\mathbf{X}$ of maximal rank on $\mathbf{X}$. The restrictions $\eta_{|\mathbf{X}_1-(\mathbf{D}_{r_0}\times\mathbf{Y}_1)}$ and $\eta_{|\mathbf{X}_2-(\mathbf{D}_{r_0}\times\mathbf{Y}_2)}$ can always be extended (not uniquely, of course) to $\mathbf{X}_1$ and $\mathbf{X}_2$; let us chose such extensions and denote them by η_1 and η_2. Remark now that the map $id_{\mathbf{Y}}\times\psi$ gives us a form θ on $\mathbf{Y}\times S^2$ obtained from η_1 and η_2 and identifying them on $\mathbf{Y}\times S(r_0)$ where they agree with η. With this choices, let us integrate η on $\mathbf{X}$:

$$\int_{\mathbf{X}}\eta=\int_{\mathbf{X}_1-(\mathbf{D}_{r_0}\times\mathbf{Y}_1)}\eta+\int_{\mathbf{X}_2-(\mathbf{D}_{r_0}\times\mathbf{Y}_2)}\eta=$$
$$\int_{\mathbf{X}_1}\eta_1+\int_{\mathbf{X}_2}\eta_2-\int_{\varphi_1(\mathbf{D}_{r_0}\times\mathbf{Y})\cup_{\psi}\varphi_2(\mathbf{D}_{r_0}\times\mathbf{Y})}\eta=\int_{\mathbf{X}_1}\eta_1+\int_{\mathbf{X}_2}\eta_2-\int_{\mathbf{Y}\times S^2}\theta$$

In order to compute the Chern numbers of the connected sum we may take connections on $\mathbf{X}_1$ and $\mathbf{X}_2$ having the property that they agree on $\mathbf{Y}_1\times\mathbf{D}_\epsilon$ and $\mathbf{Y}_2\times\mathbf{D}_\epsilon$ with a product connection on $\mathbf{Y}\times\mathbf{D}_\epsilon$. Let η_1 and η_2 be the forms representing the (same) Chern classes on $\mathbf{X}_1$ and $\mathbf{X}_2$ respectively and with respect to these connections. Because our choice we may glue together η_1 and η_2 obtaining a form η on $\mathbf{X}$ which will represent the Chern class of $\mathbf{X}$. Now, when we want to integrate η on $\mathbf{X}$, we have already at our disposal the forms η_1 and η_2, by the very construction. The form θ which appears in the formula above will represent some Chern class on $\mathbf{Y}\times S^2$. □

§2 Behaviour of the Chern numbers

In this paragraph I shall study the behaviour of the Chern numbers under blowing-ups. Let us note by pass that the blow-up procedure preserves the fundamental group as it can be seen using the van Campen theorem. In [**McD**] is presented the blowing-up procedure of a symplectic submanifold of a symplectic manifold ; I shall recall it, for the sake of completeness. Let $\mathbf{X}\hookrightarrow\mathbf{Y}$ be a symplectic submanifold. Let $N_{\mathbf{X}|\mathbf{Y}}$ be the normal bundle and take the projectivized bundle $\tilde{\mathbf{X}}:=\mathbf{P}(N_{\mathbf{X}|\mathbf{Y}})\longrightarrow\mathbf{X}$ (using a complex stucture which is compatible with the symplectic form ω). There is a tautological line bundle $\tilde{E}$ over $\tilde{\mathbf{X}}$ whose fiber over a point $(x,l_v)\in\tilde{\mathbf{X}}$ is $\{(x,\lambda v),\lambda\in\mathbf{C}\}$, the complex line correspondind to l_v ; in the complex case it corresponds to $N_{\tilde{\mathbf{X}}|\tilde{\mathbf{Y}}}=\mathcal{O}_{\tilde{\mathbf{X}}}(-1)$. There is the commuting diagram:

$$\begin{array}{ccc}\tilde{E} & \longrightarrow & \tilde{\mathbf{X}}\\ \varphi\downarrow & & \downarrow f\\ N_{\mathbf{X}|\mathbf{Y}} & \longrightarrow & \mathbf{X}\end{array}$$

Moreover, φ is a diffeomorphism if restricted to non-zero vectors of $\tilde{E}$. Therefore one can take a small disc sub-bundle of E around the zero section which is identified with a tubular neighborhood V of $\mathbf{X}\hookrightarrow\mathbf{Y}$. Then one take:

$$\tilde{\mathbf{Y}}:=(\mathbf{Y}-V)\cup_{\partial V}\varphi^{-1}(V)$$

to be the symplectic blow-up of $\mathbf{Y}$ along $\mathbf{X}$. What is still to be shown is that $\tilde{\mathbf{Y}}$ is really a symplectic manifold. This is done in §3 of McDuff's article and (if $h : \tilde{\mathbf{Y}} \longrightarrow \mathbf{Y}$ is the projection) the new symplectic form is essentially $\tilde{\omega} = h^*\omega + \epsilon a$ with $a \in H^2(\tilde{\mathbf{Y}})$ a form which is non-degenerate along the "exceptional divisor".

We may note that the Kähler form on the blow-up of a Kähler manifold is constructed in a similar way.

In the following, Σ will always denote a symplectic variety of dimension six.

• **Blow-up of a point** If Σ' is the blow-up of Σ, then the Chern numbers transform as follows:

$$\begin{aligned} c_1^3(\Sigma') &= c_1^3(\Sigma) - 8 \\ c_1c_2(\Sigma') &= c_1c_2(\Sigma) \\ c_3(\Sigma') &= c_3(\Sigma) + 2 \end{aligned}$$

• **Blow-up of a curve** Let $C \hookrightarrow \Sigma$ be an algebraic curve of genus g and consider the blow-up Σ' of Σ along C. Then the relations:

$$\begin{aligned} c_1^3(\Sigma') &= c_1^3(\Sigma) + 6(g-1) - 2\langle c_1(\mathcal{N}_{C|\Sigma}), [C]\rangle \\ c_1c_2(\Sigma') &= c_1c_2(\Sigma) \\ c_3(\Sigma') &= c_3(\Sigma) - 2(g-1) \end{aligned}$$

Proof In [**F**] the Chern classes of the blown-up manifold are computed in the complex case. There is the following commuting blow-up diagram:

$$\begin{array}{ccc} \tilde{\mathbf{X}} & \stackrel{j}{\hookrightarrow} & \tilde{\mathbf{Y}} \\ f\downarrow & & \downarrow h \\ \mathbf{X} & \stackrel{i}{\hookrightarrow} & \mathbf{Y} \end{array}$$

The key for the computations are four exact sequences:

[**F**] **Lemma 15.4**(page 299) *There are exact sequences:*

$$0 \longrightarrow N_{\tilde{\mathbf{X}}|\tilde{\mathbf{Y}}} \longrightarrow f^*N_{\mathbf{X}|\mathbf{Y}} \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O} \longrightarrow f^*N_{\mathbf{X}|\mathbf{Y}} \otimes N^*_{\tilde{\mathbf{X}}|\tilde{\mathbf{Y}}} \longrightarrow T_{\tilde{\mathbf{X}}} \longrightarrow f^*T_X \longrightarrow 0$$

$$0 \longrightarrow T_{\tilde{\mathbf{X}}} \longrightarrow j^*T_{\tilde{\mathbf{Y}}} \longrightarrow N^*_{\tilde{\mathbf{X}}|\tilde{\mathbf{Y}}} \longrightarrow 0$$

$$0 \longrightarrow T_{\tilde{\mathbf{Y}}} \longrightarrow h^*T_{\mathbf{Y}} \longrightarrow j_*F \longrightarrow 0$$

The first three are exact sequences of vector bundles on $\tilde{\mathbf{X}}$, the fourth an exact sequence of sheaves on $\tilde{\mathbf{Y}}$. F is the universal quotient bundle on $\tilde{\mathbf{X}} = \mathbf{P}(N_{\mathbf{X}|\mathbf{Y}})$.

The remark to be done is that these sequences are still exact in our case because the proof uses nowhere the fact that the underlying manifold of those fibrations is complex.

We can now turn back to our problem of computing the Chern numbers of $\tilde{\Sigma}$. Using the fact that c_3 represents the Euler number, one can see immediatly the transformation formulas for them ; c_1c_2 is invariant under blow-ups. Therefore the only one thing to be computed here is the transformation law for c_1^3. In the case where I blow-up a point, $c_1' = c_1 - 2\eta$, where η represents the class of the exceptional divisor $\mathcal{E}$. Then $(c_1')^3 = c_1^3 - 8\eta^3$; but $\eta^3 = \int_{\mathcal{E}} \eta^2 = 1$. Let us look now the case of blowing-up a curve: we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{E} & \stackrel{j}{\hookrightarrow} & \Sigma' \\ f\downarrow & & \downarrow h \\ C & \stackrel{i}{\hookrightarrow} & \Sigma \end{array}$$

The first Chern class is: $c_1' = h^*c_1 - \eta$

$$(c_1')^3 = c_1^3 - 3h^*(c_1^2)\eta + 3h^*(c_1)\eta^2 - \eta^3$$

here we use that:

$$h^*(c_1^2)\eta = \int_{\mathcal{E}} j^*h^*(c_1^2) = \int_{\mathcal{E}} f^*i^*(c_1^2) = 0$$

because $i^*(c_1^2) \in H^4(C)$ is zero.
Let me denote by $\zeta \in H^2(\mathcal{E})$ the class of the fiber and by $\alpha \in H^2(C)$ the fundamental class of C. In this case, one can see that $i^*c_1 = 2(1-g)\alpha + c_1(\mathcal{N})$. Using these notations we can compute:

$$h^*(c_1)\eta^2 = -\int_{\mathcal{E}} j^*h^*(c_1)\zeta = -\int_{\mathcal{E}} f^*i^*(2(1-g)\alpha + c_1(\mathcal{N}))\zeta = 2(g-1) - \langle c_1(\mathcal{N}), [C]\rangle$$

Computing the last term: $\eta^3 = \int_{\mathcal{E}} \zeta^2 = -\int_{\mathcal{E}} c_1(\mathcal{N})\zeta = -\langle c_1(\mathcal{N}), [C]\rangle$ □

§3 The building blocks of the construction

The folowing examples can be found in [**G**]. I shall recall the construction of those which will be used here.

A Consider $W_1 := \mathbf{P}^2 \sharp 9\overline{\mathbf{P}}^2$. Supposing that the nine points are in general position, we can think of them as the intersection points of two smooth cubics $\mathcal{C}$ and $\mathcal{C}'$ in $\mathbf{P}^2$. We may consider the linear system $\lambda\mathcal{C} + \mu\mathcal{C}'$, $[\lambda : \mu] \in \mathbf{P}^1$, which has base points exactly the nine given points. Blowing-up $\mathbf{P}^2$ in these points we separate the directions and obtain an elliptic fibration structure on W_1 over $\mathbf{P}^1$. Any of the exceptional divisors provide a symplectically embedded projective line which cuts in only one point the generic fiber which is,topologically, a torus of self-intersection 0. We are in the case of propositions 1.1 and 1.2 and we can do the symplectic sum of m pairs $(W_1, fiber, E)$, where E denotes an exceptional divisor. The resulting (elliptic) fibration W_m contains a projective line which cut transversally in one point the generic elliptic fiber. Let now W_m^* denote the blow-up of W_m in one point and let e_m be the corresponding exceptional divisor. Obviously W_m^* is 1-connected ; the Chern numbers are:

$$c_1^2(W_m^*) = -1$$
$$c_2(W_m^*) = 12m + 1$$

B This building block is the 4-manifold $S_{1,1}$ in [**G**], page 566-568. I won't give the construction but I shall mention its properties:

- it is symplectic and 1-connected ;
- it contains symplectically embedded curves F_1, F_2 of genus 1 and 2 respectively, with trivial normal bundle ; in addition, each of these curves are cut by (disjoint) spheres transversally, in one point ;
- its Chern numbers are:

$$c_1^2(S_{1,1}) = 1$$
$$c_2(S_{1,1}) = 23$$

C Consider a quartic curve in $\mathbf{P}^2$ having a single node. Blow it up to obtain a non-singular projective curve of degree 4 and genus 2 in $\mathbf{P}^2\sharp\overline{\mathbf{P}}^2$ having self-intersection 2. Blowing-up twelve more times one has a genus two curve F_2 of self-intersection 0 in $P_1 := \mathbf{P}^2\sharp 13\overline{\mathbf{P}}^2$. The 13^{th} exceptional divisor will be a projective line which intersects our curve F_2 transversally in only one point. The Chern numbers are: $c_1^2(P_1) = 9 - 13 = -4$, $c_2(P_1) = 3 + 13 = 16$. We can make the symplectic sum of l pairs (P_1, F_2, E), the result being the 1-connected symplectic manifold P_l containing a curve F_2 of self-intersection 0 and a projective line which cut it transversally in one point. We may compute the Chern numbers for this manifold using proposition 1.3:

$$c_1^2(P_l) = 4l - 8$$
$$c_2(P_l) = 20l - 4$$

§4 The construction of the examples

The basic tool in constructing of the examples is the symplectic connected sum (of pairs). I shall use the notation $\mathbf{X}_1\sharp_{\mathbf{Y}}\mathbf{X}_2$ to denote the symplectic connected sum of $\mathbf{X}_1$ and $\mathbf{X}_2$ along $\mathbf{Y}$; F_g will denote always a projective curve of genus g. Consider now first the symplectic four manifold:

$$\mathbf{X} := W_m^*\sharp_{F_1} S_{1,1}$$

where the connected sum is done away from the exceptional divisor e_m. This manifold is symplectic but also 1-connected. Indeed, as mentioned, W_m^* contains a 2-sphere which meets the generic fiber F_1 transversally in one point ; the curve $F_1 \hookrightarrow S_{1,1}$ is cut also transversally by such a 2-sphere. Now we can apply proposition 1.2 and deduce that $\mathbf{X}$ is 1-connected. It must be mentioned at this point that we have done again a connected sum of pairs: because in both W_m^* and $S_{1,1}$ there are 2-spheres cuting transversally F_1, we may do the identifications in such that $F_1 \hookrightarrow \mathbf{X}$ to be cutted again by a 2-sphere transversally in one point, according to proposition 1.2. We will use this property to show the simply connectedness of some of our examples. The Chern numbers are:

$$c_1^2(\mathbf{X}) = 0$$
$$c_2(\mathbf{X}) = 12m + 24$$

The constuction will be broken in several steps, according to the values of c_1c_2. Let me begin with a lemma which gives us a simple method to vary c_1^3 and c_3 independently.

Lemma *Suppose that we have a six-dimensional, 1-connected symplectic manifold Σ having the Chern numbers $(2a, 24b, 2c)$. Suppose that we have the additional properties:*

- *Σ contains a symplectically embedded product $U \times \mathbf{D}$, where U is an open set of a projective surface, E is a projective line in U such that $-\alpha = \langle c_1(\mathcal{N}_{E|\Sigma}), [E]\rangle \leq -1$ and $\mathbf{D} \hookrightarrow \mathbf{C}$ denotes a disc. This happens in the case where Σ is either a product $\mathbf{S} \times F$ ($\mathbf{S}$ is a surface containing the projective line E and F is an algebraic curve) or is obtained by a symplectic connected sum of such a product, away from E. In this case one can move E in the direction given by F ;*
- *there is a projective curve F_2 of genus two with trivial normal bundle disjointly from E .*

Then, just blowing up x points, distinct copies of E (r times) and of F_2 (z times), one can obtain all triples of Chern numbers of the form $(2a', 24b, 2c')$, where a', c' are arbitrary integers.

Proof In fact, the formulas given in §2 show that c_1c_2 is invariant and the other two Chern numbers are:

$$c_1^3 = 2(a - (3-\alpha)\cdot r - 4x + 3z)$$
$$c_3 = 2(c + r + x - z).$$

Imposing the condition that $c_1^3 = 2a', c_3 = 2c'$, we obtain:

$$x = \alpha \cdot r - (a' - a + 3(c' - c))$$
$$z = (1+\alpha)r - (a' - a + 4(c' - c))$$

and one can see that we can chose r big enough to insure the positivity of x and z. □

Now we can finally construct the examples. We will distinguish several cases, accordind to the value of c_1c_2. Let us begin with:

$\underline{c_1c_2/24 = 0}$ Let us consider $W_1^* \times F_1$; it is a symplectic manifold, but there are two problems: it is not 1-connected and there is no F_2 curve on it. We will see that considering the following

$$\Sigma := W_1^* \times F_1 \sharp_{F_1 \times F_1} S_{1,1} \times F_1$$

where we identify in cross the two F_1's, these problems disappear. Taking the connected sum with $S_{1,1} \times F_1$ has the effect of introducing the needed curve F_2. We have to show that this manifold is 1-connected.

In order to make the proof clear, let us denote by F_1' and F_1'' our curves. In each of sumands $F_1' \times F_2''$ has trivial normal bundle, so we may chose the (symplectically embedded) neighborhoods $F_1' \times F_1'' \times \mathbf{D}_\epsilon$ in each. For writing down van Campen's diagram, we consider the open sets:

$U_1 := W_1^* \times F_1' - F_1'' \times F_1' \times \mathbf{D}_{3\epsilon/4} = (W_1^* - F_1'' \times \mathbf{D}_{3\epsilon/4}) \times F_1'$
$U_2 := S_{1,1} \times F_1'' - F_1' \times F_2'' \times \mathbf{D}_{3\epsilon/4} = (S_{1,1} - F_1' \times \mathbf{D}_{3\epsilon/4}) \times F_1''$

the intersection being $U_1 \cap U_2 = F_1' \times F_1'' \times \mathbf{A}$ with $\mathbf{A} := \mathbf{D}(\epsilon/4, 3\epsilon/4)$ an annulus. We have the following commuting diagram:

$$\begin{array}{ccccc} & & \pi_1(U_1) & & \\ & i'_* \nearrow & & \searrow j'_* & \\ \pi_1(F_1') \oplus \pi_1(F_1'') \oplus \pi_1(\mathbf{A}) & & & & \pi_1(\Sigma) \\ & i''_* \searrow & & \nearrow j''_* & \\ & & \pi_1(U_2) & & \end{array}$$

$\pi_1(\Sigma)$ is generated by the images of $\pi_1(U_1)$ and $\pi_1(U_2)$. We will prove that these images are trivial.

$\pi_1(U_1) = \pi_1(W_1^* - F_1'' \times \mathbf{D}_{3\epsilon/4}) \oplus \pi_1(F_1') = 0 \oplus \pi_1(F_1')$

$\pi_1(U_2) = \pi_1(S_{1,1} - F_1' \times \mathbf{D}_{3\epsilon/4}) \oplus \pi_1(F_2'') = 0 \oplus \pi_1(F_1'')$

To write down these equalities we used the fact that in W_1^* and $S_{1,1}$ there are 2-spheres which cut transversally in one point F_1'' and F_1' respectively ; after we applied proposition 1.2. Let us prove that the images of $\pi_1(U_1)$ and $\pi_1(U_2)$ cancel in $\pi_1(\Sigma)$:

$$j'_*\pi_1(U_1) = j'_*i'_*\pi_1(F_1') = j''_*i''_*\pi_1(F_1') = j''_*0 = 0$$

An annalogous reasonement shows the canceling of $j''_*\pi_1(U_2)$. Therefore Σ is 1-connected and has obviously $c_1c_2 = 0$. It satisfies the conditons of the lemma and we are done. □

<u>$c_1c_2/24 = 1$</u> Take $\Sigma := P_1 \times \mathbf{P}^1$. The generic projective line passing through the node of the considered quartic meet it two more times. Blowing-up the node we will have a projective line (fix one) which meet the proper transform of the quartic in two points. Now, blow-up these two points and ten more times to obtain in P_1 the F_2 curve of self-intersection 0 and the (-2) projective line E. We are again in the conditions of the lemma with $\alpha = 2$. □

<u>$c_1c_2/24 = -1$</u> Just take:

$$\Sigma := W_1^* \times F_2 \sharp_{F_1 \times F_2} P_1 \times F_1$$

The result is 1-connected and satisfy the lemma with $\alpha = 1$. □

<u>$c_1c_2/24 \geq 2$</u> We are just considering the product $\Sigma := \mathbf{X} \times \mathbf{P}^1$; its Chern numbers are $(0, 24(m+2), 2(12m+24))$. This is a 1-connected symplectic manifold. In Σ there is the projective line e_m which lies in W_m^* and has the normal bundle $\mathcal{O} \oplus \mathcal{O}(-1)$. One can see that the α in the lemma above is -1. There is also the curve F_2 on Σ having trivial normal bundle. Blowing-up r times along e_m, x points and z times along F_2 we obtain a symplectic manifold having the Chern numbers:

$$c_1^3 = 2(-2r - 4x + 3z)$$
$$c_1c_2 = 24(m+2)$$
$$c_3 = 2(12m + r + x - z + 24)$$

Let us impose now that $(c_1^3, c_1c_2, c_3) = (2a, 24b, 2c)$. We will obtain the following restrictions on the parameters r, x, z:
- $r - x = a - 36b + 3c$;
- $m = b - 2$;
- $2r - z = a - 48b + 4c$.

One can see immediately that this system can be solved under the restriction that all parameters are positive if and only if $b \geq 2$. $\square$

$\underline{c_1c_2/24 \leq -2}$ Let us consider now the product consider $\mathbf{X} \times F_2$; the Chern numbers are: $(0, -24(m+2), -2(12m+24))$. But now we may see that this symplectic manifold is not 1-connected ; $\pi_1(\Sigma) = \mathbf{Z}^{\oplus 4}$ which comes from the F_2 factor. We cancel this fundamental group taking the symplectic connected sum (of pairs):

$$\Sigma := \mathbf{X} \times F_2 \sharp_{F_1 \times F_2} S_{1,1} \times F_1$$

Σ is symplectic and using proposition 1.3 we deduce that its Chern numbers are $(0, -24(m+2), -2(12m+24))$. A completely similar argument as in the $c_1c_2 = 0$ case shows that Σ is 1-connected. Doing again the blow-up of x points, r curves e_m and z curves F_2 we obtain a symplectic manifold having the Chern numbers

$$c_1^3 = 2(-2r - 4x + 3z)$$
$$c_1c_2 = -24(m+2)$$
$$c_3 = 2(-12m + r + x - z - 24)$$

Imposing $(c_1^3, c_1c_2, c_3) = (2a, 24b, 2c)$ we obtain the following restrictions:
- $r - x = a - 36b + 3c$
- $m = -b - 2$
- $2r - z = a - 48b + 4c$

This system is solvable iff $b \leq -2$. $\square$

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