Topological Boundary Values and Regular D-modules

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Résumé

La valeur au bord sur une hypersurface des solutions hyperfonctions d'un système d'équations aux dérivées partielles se définit habituellement lorsque le système est régulier le long de l'hypersurface. Ici, nous étudions ce problème d'un point de vue purement géométrique, c'est-à-dire que nous considérons un complexe de faisceaux qui n'est pas nécessairement solution d'un système et nous faisons des hypothèses sur son micro-support. Une hypothèse d'invariance par homothétie complexe permet de définir un morphisme de valeur au bord et, avec une hypothèse supplémentaire d'hyperbolicité, celui-ci est un isomorphisme. Ces résultats s'appliquent naturellement aux solutions d'un système, les conditions sur le micro-support se traduisant par des hypothèses sur la variété caractéristique. Enfin l'hypothèse de régularité permet de relier ces résultats aux cycles évanescents et d'améliorer les résultats déjà connus.

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This paper is dedicated to the memory
of our friend **Emmanuel Andronikof**

Contents

Introduction

Since the development of microlocal analysis in the seventies, several approaches of boundary values problems for hyperfunction and microfunction solutions of complex differential systems were obtained, using local, microlocal or second microlocal tools [8],[9],[20],[23],[25]. In [4] some of these results were regarded through the "$\mathcal{D}$-module" point of view, that is defining the morphism of boundary values as a functor between the hyperfunction solutions of a $\mathcal{D}$-module and the hyperfunction solutions of its vanishing and nearby cycles.

These results assumed always an hypothesis of regularity of Fuchsian type to define the morphism. In [25], Tahara considered an operator simultaneously Fuchsian and hyperbolic and proved that the morphism of boundary value is an isomorphism in this case.

In this paper, our first aim is to prove a result analogous to Tahara's in the framework of $\mathcal{D}$-module of [4]. Working with $\mathcal{D}$-module, we cannot assume that an operator is simultaneously Fuchsian and hyperbolic, so we have to separate the hypothesis, prove that if the module is Fuchsian (more precisely regular along a hypersurface) there is a boundary value morphism and if we add an hypothesis of hyperbolicity, it is an isomorphism.

In fact, it appears that the condition of regularity itself is not necessary. If we assume a geometrical condition on the characteristic variety of the $\mathcal{D}$-module we can define a boundary value morphism which is an isomorphism with the condition of hyperbolicity (which itself is a geometric condition on the characteristic variety). In this context, the morphism has values in the geometric vanishing and nearby cycles of the sheaf of solutions of $\mathcal{M}$. Now regularity condition is used only to identify these geometric vanishing cycles of solutions with solutions of a $\mathcal{D}$-module, called the vanishing cycles of the $\mathcal{D}$-module.

Once conditions have been reduced to conditions on the characteristic variety of the $\mathcal{D}$-module, it appears that we may forget completely $\mathcal{D}$-modules and use the theory of sheaves of Kashiwara-Schapira [12]. More precisely we prove a theorem for complex of sheaves of $\mathbb{C}$-vector spaces, the conditions on the characteristic variety becoming conditions on the microsupport of the complex. This boundary value morphism may be called a "topological boundary value" morphism as that of Schapira [23]. Then the result on $\mathcal{D}$-module is the special case were the complex of sheaves is the complex of holomorphic solutions of the $\mathcal{D}$-module.

Let us now explain more precisely the geometrical situation. Let M be a real analytic manifold of dimension n, N a smooth hypersurface of M, X a complexification of M and Y a smooth hypersurface of X complexifying N. Let f be a holomorphic function on X, real on M such that $f = 0$ is a local equation for Y and let :

$$M^+ = M \cap \{f > 0\}, \quad \overline{M^+} = M \cap \{f \geq 0\}$$

and $i : Y \hookrightarrow X$ the inclusion.

Let $\mathcal{F}$ be a complex of sheaves of $\mathbb{C}$-vector spaces on X. One has the classical triangle

$$\mathbb{R}\Gamma_N(\mathcal{F}) \longrightarrow \mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F})|_N \longrightarrow \mathbb{R}\Gamma_{M^+}(\mathcal{F})|_N \xrightarrow{+1} \tag{1}$$

On the other hand, one may consider the cohomology with support on N of the nearby and vanishing cycle triangle of $\mathcal{F}$ along Y :

$$\mathbb{R}\Gamma_N(\mathcal{F}) \longrightarrow \mathbb{R}\Gamma_N(\Phi_f(\mathcal{F})) \longrightarrow \mathbb{R}\Gamma_N(\Psi_f(\mathcal{F})) \xrightarrow{+1} \tag{2}$$

In this framework, the natural question is when is it possible to define a natural morphism of boundary value from (1) to (2). We recall that the microsupport of $\mathcal{F}$ was defined in [12] as a subset of T^*X, the cotangent bundle to X.

Our main result in section 1 is the following : the morphism is well defined if the normal cone $C_{T^*_YX}(SS(\mathcal{F}))$ where T^*_YX is the conormal bundle to Y is contained in some canonical hypersurface. This condition is essentially equivalent to the fact that the microlocalization of $\mathcal{F}$ is $\mathbb{C}^*$-conic. If, in addition, $SS(\mathcal{F})$ satisfies a kind of hyperbolic inequality with respect to N (which we call "near-hyperbolicity") the morphism obtained is, in fact, an isomorphism.

In section 2 we apply these results to the case $\mathcal{F} = \mathcal{S}\text{ol}(\mathcal{M})$ the complex of sheaves of holomorphic solutions of a coherent $\mathcal{D}_X$-module $\mathcal{M}$, getting a general notion of boundary morphisms for the hyperfunction solutions. Furthermore, when $\mathcal{M}$ is regular along Y, generalizing [4], we prove that the boundary values are, in fact, hyperfunction solutions of the complex of $\mathcal{D}_Y$-modules of nearby cycles, vanishing cycles of $\mathcal{M}$ along Y. Our method gives the same result for complexes of $\mathcal{D}_X$-modules with coherent cohomology without modifications of the proof. We show also that we recover the morphisms of [7], [9]. If the module $\mathcal{M}$ is given by a single differential operator P, our result is then the same than obtained by Tahara in [25].

One of the main feature of our results is that many classical problems concerning the regularity of boundary values may be reduced to the study of the $\mathcal{D}_Y$-modules of nearby and vanishing cycles of $\mathcal{M}$ along Y, which, in principle are easier to manipulate. To illustrate this, an easy application to the so-called "ideally analytic" solutions is given in the end.

1 Geometric boundary value

1.1 Complex conic sheaves

Let X be a real analytic manifold and let $D^b(X)$ denote the derived category of bounded complex of sheaves of $\mathbb{C}$-vector spaces on X. If $\mathcal{F}$ is an object of $D^b(X)$, its micro-support $SS(\mathcal{F})$ has been defined in [12] as a subset of T^*X.

Assume now that E is a fiber bundle on X. The cotangent bundle T^*E is provided with a canonical hypersurface S_E. This hypersurface has several definitions [10], [14], [12]. One may consider the canonical action of $\mathbb{R}^+$ on the fibers of E which defines an Euler vector field θ_E on E. The characteristic variety of θ_E is S_E.

Another definition is specific to fiber bundles $E \to X$ of rank one. Consider the canonical maps $p : E \to X$ and $i : X \hookrightarrow E$, they define the maps :

$$T^*X \overset{p_1}{\longleftarrow} T^*X \times_X E \overset{j_1}{\longrightarrow} T^*E$$

$$T^*X \overset{p_2}{\longleftarrow} T^*E \times_E X \overset{j_2}{\longrightarrow} T^*E$$

Then S_E is the union of $j_1(T^*X \times_X E)$ and of $j_2(T^*E \times_E X)$.

An object $\mathcal{F}$ of $D^b(E)$ is said to be $\mathbb{R}^+$conic if its cohomology groups $H^j(\mathcal{F})$ are locally constant on the orbits of the action of $\mathbb{R}^+$. It is proved in [12, Prop 5.5.3.] that $\mathcal{F}$ is $\mathbb{R}^+$conic if and only if $SS(\mathcal{F}) \subset S_E$.

The same results are still true in the complex case. If X is a complex analytic manifold and E a complex fiber bundle over X, the action of $\mathbb{C}^*$ on the fibers define an Euler vector field whose characteristic variety is a complex subvariety S_E of T^*E. An object $\mathcal{F}$ of $D^b(E)$ is $\mathbb{C}^*$-conic if and only if $SS(\mathcal{F}) \subset S_E$.

Here $\mathbb{C}^*$-conic means that the cohomology groups $H^j(\mathcal{F})$ are locally constant on the orbits of the action of $\mathbb{C}^*$. If E is a line bundle these orbits are the fibers of E and $\mathbb{C}^*$-conic is equivalent to "monodromic" as defined in [3].

To prove the equivalence between "$\mathcal{F}$ is $\mathbb{C}^*$-conic" and "$SS(\mathcal{F}) \subset S_E$" we apply [12, proposition 5.4.5.] exactly as in the real case.

Let us now consider a complex analytic manifold X with a a smooth hypersurface Y. We assume that Y has an equation $f = 0$ and denote by $i : Y \hookrightarrow X$ the inclusion.

We denote by $\tau : T_Y X \to Y$ the normal bundle to Y in X and by $\pi : \Lambda_{\mathbb{C}} = T^*_Y X \to Y$ the conormal bundle. The specialization $\nu_Y(\mathcal{F})$ and the microlocalization $\mu_Y(\mathcal{F})$ of $\mathcal{F}$ along Y are objects respectively of $D^b(T_Y X)$ and $D^b(T^*_Y X)$ defined in [12].

Let us recall the definition of $\nu_Y(\mathcal{F})$ to fix the notations. We denote by $\widetilde{X_Y}$ the (real) normal deformation of Y with the maps $t : \widetilde{X_Y} \to \mathbb{R}$ and $p : \widetilde{X_Y} \to X$, by $\Omega = t^{-1}(\mathbb{R}^+)$ and by $\tilde{p}$ the restriction of p to Ω. We have the diagram (4.1.5.) of [12] :

$$\begin{array}{ccccc} T_Y X & \xrightarrow{\sigma} & \widetilde{X_Y} & \xleftarrow[j]{} & \Omega \\ {\scriptstyle\tau}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle\tilde{p}}\downarrow \\ Y & \xrightarrow{i} & X & = & X \end{array} \tag{1.1.1}$$

Then, by definition,

$$\nu_Y(\mathcal{F}) = \sigma^{-1}\mathbb{R}j_*\tilde{p}^{-1}\mathcal{F}$$

and $\mu_Y(\mathcal{F})$ is the Fourier transform of $\nu_Y(\mathcal{F})$.

It is clear from the definition that $\nu_Y(\mathcal{F})$ and $\mu_Y(\mathcal{F})$ are $\mathbb{R}^+$-conic.

The normal cone $C_{\Lambda_{\mathbb{C}}}(SS(\mathcal{F}))$ to the micro-support $SS(\mathcal{F})$ along $\Lambda_{\mathbb{C}}$ is a subset of the normal bundle $T_{\Lambda_{\mathbb{C}}}(T^*X)$ which may be identified to $T^*\Lambda_{\mathbb{C}}$ by the Hamiltonian isomorphism (see (6.2.2.) in [12]).

It is proved in [12, Theorem 6.4.1.] that $SS(\mu_Y(\mathcal{F})) = SS(\nu_Y(\mathcal{F})) \subset C_{\Lambda_{\mathbb{C}}}(SS(\mathcal{F}))$.

We will now consider an object $\mathcal{F}$ of $D^b(X)$ satisfying :

$$C_{\Lambda_{\mathbb{C}}}(SS(\mathcal{F})) \subset S_{\Lambda_{\mathbb{C}}} \tag{1.1.2}$$

We deduce from the previous results that :

Lemma 1.1.1. *If $\mathcal{F}$ satisfies condition (1.1.2), the localization $\nu_Y(\mathcal{F})$ and microlocalization $\mu_Y(\mathcal{F})$ of $\mathcal{F}$ are $\mathbb{C}^*$-conic.*

The differential *df* defines a map $\tilde{f} : T_Y X \to \mathbb{C}$ and a bijection $\gamma : T_Y X \to Y \times \mathbb{C}$ hence a section $s : Y \to T_Y X$ by $s(x) = \gamma^{-1}(x, 1)$. In the same way, a section s' of $T^*_Y X$ is given by $s'(x) = ({}^t d_x f)(1)$.

Definition 1.1.2. We define two sheaves on Y by the following :

$$\Psi'_f(\mathcal{F}) = s^{-1}\nu_Y(\mathcal{F}) \quad \text{and} \quad \Phi'_f(\mathcal{F}) = s'^{-1}\mu_Y(\mathcal{F})[1]$$

Lemma 1.1.1 shows that $\nu_Y(\mathcal{F}) \simeq \tau^{-1}\Psi'_f(\mathcal{F})$ on a neighborhood of $s(Y)$ and that $\mu_Y(\mathcal{F})[1] \simeq \pi^{-1}\Phi'_f(\mathcal{F})$ on a neighborhood of $s'(Y)$.

If $\mathcal{F}$ is a complex with $\mathbb{C}$-constructible cohomology, proposition 8.6.3. of [12] shows that what we called $\Psi'_f(\mathcal{F})$ and $\Phi'_f(\mathcal{F})$ are the classical complex of nearby and vanishing cycles, that is $\Psi'_f(\mathcal{F}) \simeq \Psi_f(\mathcal{F})$ and $\Phi'_f(\mathcal{F}) \simeq \Phi_f(\mathcal{F})$.

Remark 1.1.3. The definition that we have taken here for the vanishing cycles is the classical definition of Deligne. It differs from the definition of [12] by a shift, that is the sheaf of vanishing cycles of [12] is denoted here by $\Phi_f(\mathcal{F})[-1]$. The definition adopted here suits better with the definition of vanishing cycles of a $\mathcal{D}_X$-module in proposition 2.2.1.

Without this hypothesis, we can still use the proof of proposition 8.6.3. of loc. cit. , to get morphisms :

$$\Psi_f(\mathcal{F}) \longrightarrow \Psi_{\tilde{f}}(\nu_Y(\mathcal{F}))$$
$$\Phi_f(\mathcal{F}) \longrightarrow \Phi_{\tilde{f}}(\nu_Y(\mathcal{F}))$$

with $\tilde{f} : T_Y X \to Y$ canonical projection. And when $\nu_Y(\mathcal{F})$ and $\mu_Y(\mathcal{F})$ are $\mathbb{C}^*$-conic, the same proof gives :

$$\Psi'_f(\mathcal{F}) \simeq \Psi_{\tilde{f}}(\nu_Y(\mathcal{F}))$$
$$\Phi'_f(\mathcal{F}) \simeq \Phi_{\tilde{f}}(\nu_Y(\mathcal{F}))$$

In fact, the proof shows that for any $\mathbb{C}^*$-conic subsheaf $\mathcal{G}$ of $T_Y X$ with Fourier transform $\widehat{\mathcal{G}}$, we have isomorphisms :

$$s^{-1}\mathcal{G} \simeq \Psi_{\tilde{f}}(\mathcal{G}) \text{ and } s'^{-1}\widehat{\mathcal{G}}[1] \simeq \Phi_{\tilde{f}}(\mathcal{G})$$

The classical triangle

$$\mathbb{R}\Gamma_Y(\mathcal{F})[1] \longrightarrow \Phi_f(\mathcal{F}) \xrightarrow{var} \Psi_f(\mathcal{F}) \xrightarrow{+1}$$

gives

$$\mathbb{R}\Gamma_Y(\mathcal{F})[1] \longrightarrow \Phi_{\tilde{f}}(\nu_Y(\mathcal{F})) \xrightarrow{var} \Psi_{\tilde{f}}(\nu_Y(\mathcal{F})) \xrightarrow{+1}$$

because $\mathbb{R}\Gamma_Y(\nu_Y(\mathcal{F})) = \mathbb{R}\Gamma_Y(\mathcal{F})$ hence a triangle

$$\mathbb{R}\Gamma_Y(\mathcal{F})[1] \longrightarrow \Phi'_f(\mathcal{F}) \xrightarrow{var} \Psi'_f(\mathcal{F}) \xrightarrow{+1} \tag{1.1.3}$$

Let us still remark that when $\mathcal{F}$ is the solution complex of a coherent $\mathcal{D}_X$-module and satisfy the hypothesis of lemma 1.1.1, we will see that we always have $\Psi'_f(\mathcal{F}) \simeq \Psi_f(\mathcal{F})$ and $\Phi'_f(\mathcal{F}) \simeq \Phi_f(\mathcal{F})$.

Consider maps j_1, j_2, p_1 and p_2 associated to the fiber bundle $T^*_Y X \to Y$ which where defined previously.

Proposition 1.1.4. *We assume that $SS(\mathcal{F})$ is a real analytic subset of T^*X and that it satisfies condition (1.1.2), then the tangent cone splits into :*

$$C_{\Lambda_\mathbb{C}}(SS(\mathcal{F})) = j_1 p_1^{-1} C_1 \cup j_2 p_2^{-1} C_2$$

where C_1 and C_2 are two involutive homogeneous subvarieties of T^*Y and we have :

$$SS(\Psi'_f(\mathcal{F})) \subset C_1 \quad \text{and} \quad SS(\Phi'_f(\mathcal{F})) \subset C_2$$

Proof. By theorem 6.5.4. of [12], $SS(\mathcal{F})$ is involutive, and we assumed that it is a real analytic subset, so the same is true for $C_{\Lambda_\mathbb{C}}(SS(\mathcal{F}))$. It is easy to show that any involutive subvariety of $S_{\Lambda_\mathbb{C}}$ splits in $j_1 p_1^{-1} C_1 \cup j_2 p_2^{-1} C_2$ (see [16, Lemme 4.5.1.] for the details).

As $SS(\mu_Y(\mathcal{F})) \subset C_{\Lambda_\mathbb{C}}(SS(\mathcal{F})) \subset S_{\Lambda_\mathbb{C}}$, the map s' is non characteristic for $\mu_Y(\mathcal{F})$ hence, using [12, Prop 5.4.13] we get

$$SS(\Phi'_f(\mathcal{F})) = SS({s'}^{-1}\mu_Y(\mathcal{F})) \subset C_2$$

The same proof works for $SS(\Psi'_f(\mathcal{F}))$. □

1.2 The real case

Let M be a real analytic manifold of dimension n, N a smooth hypersurface of M, X be a complexification of M and Y a smooth hypersurface of X complexified of N. We assume that Y has an equation $f = 0$ which is real on N and denote by

$$M^+ = M \cap \{f > 0\}, \quad \overline{M^+} = M \cap \{f \geq 0\}$$

and by $j : M^+ \hookrightarrow M$ the inclusion.

Let $\mathcal{G}$ be an object of $D^b(M)$. As $M^+ = \overline{M^+} - N$ we have a triangle :

$$\mathbb{R}\Gamma_N(\mathcal{G}) \longrightarrow \mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{G})|_N \longrightarrow \mathbb{R}j_* j^{-1}\mathcal{G}|_N \xrightarrow{+1}$$

If $\mathcal{F}$ is an object of $D^b(X)$, we may apply this to $\mathbb{R}\Gamma_M(\mathcal{F})$ and get :

$$\mathbb{R}\Gamma_N(\mathcal{F}) \longrightarrow \mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F})|_N \longrightarrow \mathbb{R}\Gamma_{M^+}(\mathcal{F})|_N \xrightarrow{+1} \tag{1.2.1}$$

where $\mathbb{R}\Gamma_{M^+}(\mathcal{F}) = \mathbb{R}j_* j^{-1}\mathbb{R}\Gamma_M(\mathcal{F})$.

If $\mathcal{F}$ satisfies condition (1.1.2), we may apply the functor $\mathbb{R}\Gamma_N(.)$ to the triangle (1.1.3) and get

$$\mathbb{R}\Gamma_N(\mathcal{F})[1] \longrightarrow \mathbb{R}\Gamma_N\left(\Phi'_f(\mathcal{F})\right) \xrightarrow{var} \mathbb{R}\Gamma_N\left(\Psi'_f(\mathcal{F})\right) \xrightarrow{+1} \tag{1.2.2}$$

The aim of this section is to define a morphism of triangles from (1.2.1) to (1.2.2).

We denote by $\Lambda_\mathbb{C} = T^*_Y X$ the conormal bundle to Y in X and by $\pi_\mathbb{C} : T^*_Y X \to Y$ the projection. The conormal bundle to M in X is denoted by $T^*_M X = iT^*M$ and is identified to the cotangent bundle T^*M. We denote by $\Lambda = iT^*_N M = T^*_Y X \cap iT^*M$ the normal bundle to N and by $\pi : iT^*_N M \to N$ the projection. The orientation sheaf of N in M is denoted by $or_{N|M}$.

We also denote by $T_Y X$ the normal bundle to Y in X and by $\tau_\mathbb{C} : T_Y X \to Y$ the projection. The normal bundle to N in M is $T_N M$ with $\tau : T_N M \to N$. The differential df defines a function $df : T_N M \to N$ and we set :

$$T_N M^+ = T_N M \cap \{df > 0\}, \quad \overline{T_N M^+} = T_N M \cap \{df \geq 0\}$$

As in the complex case, f defines sections $s : N \to T_N M$ and $s' : N \to iT^*_N M$.

Lemma 1.2.1. *Let $\mathcal{G}$ be an object of $D^b(T_YX)$ which is $\mathbb{R}^+$-conic, then*

$$\mathbb{R}\tau_*\mathbb{R}\Gamma_{T_NM^+}(\mathcal{G}) \simeq s^{-1}\mathbb{R}\Gamma_{T_NM^+}(\mathcal{G}) \simeq s^{-1}\mathbb{R}\Gamma_{T_NM}(\mathcal{G})$$

Proof. The second inequality is clear because $T_NM = T_NM^+$ near s(N). Let U be an open subset of N, W be a neighborhood of $s(U)$ in T_YX such that $W \cap T_NM = s(U)$ and V be the positive real cone generated by W. We have $V \cap T_NM = \tau^{-1}U \cap T_NM^+$ hence by definition of $\mathbb{R}\Gamma_{T_NM^+}$:

$$\mathbb{R}\Gamma(U, \mathbb{R}\tau_*\mathbb{R}\Gamma_{T_NM^+}(\mathcal{G})) = \mathbb{R}\Gamma(\tau^{-1}U, \mathbb{R}\Gamma_{T_NM^+}(\mathcal{G})) = \mathbb{R}\Gamma(V, \mathbb{R}\Gamma_{T_NM}(\mathcal{G}))$$

As $\mathcal{G}$ is $\mathbb{R}^+$-conic we have :

$$\mathbb{R}\Gamma(V, \mathbb{R}\Gamma_{T_NM}(\mathcal{G})) = \mathbb{R}\Gamma(W, \mathbb{R}\Gamma_{T_NM}(\mathcal{G})) = \mathbb{R}\Gamma(U, s^{-1}\mathbb{R}\Gamma_{T_NM}(\mathcal{G}))$$

□

Proposition 1.2.2. *Let $\mathcal{G}$ be an object of $D^b(T_YX)$ which is $\mathbb{R}^+$-conic, there is a canonical morphism :*

$$\mathbb{R}\tau_*\mathbb{R}\Gamma_{\overline{T_NM^+}}(\mathcal{G}) \longrightarrow s^{-1}\mathbb{R}\Gamma_{iT^*_NM}(\widehat{\mathcal{G}})[+1]$$

which is an isomorphism if $\mathcal{G}$ is $\mathbb{C}^*$-conic.

Proof. The manifold $\Lambda = iT^*_NM$ is a line bundle on N with a non zero section s' hence we may define Λ^+ as $\mathbb{R}^+s'(N)$, the positive cone generated by the image of s'. Identifying Y with the zero section of T^*_YX, we set $K = Y \cup \overline{\Lambda^+}$ and we have $K - Y = \Lambda^+$.

Let K^o be the polar of K. It is an open subset of T_YX equal to $T_YX - H$ where H is the closed subset of $(T_YX) \times_Y N$ where the imaginary part of df is ≤ 0.

To prove this we may choose local coordinates :

Let (x,t) be a local coordinate system of M such that $f(x,t) \equiv t$ and let $(x+iy, t+is)$ be local coordinates of X, so that $f \equiv t + is$.

We have $N = \{\,(x,t) \in M \mid t = 0\,\}$, $Y = \{\,(x+iy, t+is) \in X \mid t+is = 0\,\}$, $M = \{\,(x+iy, t+is) \in X \mid x = 0, s = 0\,\}$ and $M^+ = \{\,(x,t) \in M \mid t > 0\,\}$.

We denote by $(x+iy, \tilde{t}+i\tilde{s})$ the local coordinates of T_YX so that $T_NM = \{\,(x+iy, \tilde{t}+i\tilde{s}) \in T_YX \mid y = 0, \tilde{s} = 0\,\}$ and $\overline{T_NM^+} = \{\,(x+iy, \tilde{t}+i\tilde{s}) \in T_YX \mid y = 0, \tilde{s} = 0, \tilde{t} > 0\,\}$. We have also $H = \{\,(x+iy, \tilde{t}+i\tilde{s}) \in T_YX \mid \tilde{s} \geq 0\,\}$.

On the dual space, the coordinates of T^*_YX are $(x+iy, \tau+i\sigma)$,$iT^*_NM = \{\,(x+iy, \tau+i\sigma) \in T^*_YX \mid y = 0, \tau = 0\,\}$, $\Lambda^+ = \{\,(x, i\sigma) \in iT^*_NM \mid \sigma > 0\,\}$ and $K = \{\,(x+iy, \tau+i\sigma) \in T^*_YX \mid \tau = \sigma = 0$ or $y = 0, \tau = 0, \sigma > 0\,\}$.

Then it is clear that K^o is the complementary to H in T_YX.

Let U be an open subset of Y. Proposition 3.7.12.(ii) of [12] gives isomorphisms :

$$\mathbb{R}\Gamma_K(\pi^{-1}(U), \widehat{\mathcal{G}}) \xrightarrow{\sim} \mathbb{R}\Gamma(K^o \cap \tau^{-1}(U), \mathcal{G})[-2] \tag{1.2.3}$$

$$\mathbb{R}\Gamma_Y(\pi^{-1}(U), \widehat{\mathcal{G}}) \xrightarrow{\sim} \mathbb{R}\Gamma(\tau^{-1}(U), \mathcal{G})[-2] \tag{1.2.4}$$

As $K^o = T_Y X - H$ we have a triangle :

$$\mathbb{R}\Gamma_H(\tau^{-1}(U),\mathcal{G}) \longrightarrow \mathbb{R}\Gamma(\tau^{-1}(U),\mathcal{G}) \longrightarrow \mathbb{R}\Gamma(K^o \cap \tau^{-1}(U),\mathcal{G}) \xrightarrow{+1} \tag{1.2.5}$$

On the other hand, as $K - Y = \Lambda^+$ we have a triangle :

$$\mathbb{R}\Gamma_Y(\pi^{-1}(U),\widehat{\mathcal{G}}) \longrightarrow \mathbb{R}\Gamma_K(\pi^{-1}(U),\widehat{\mathcal{G}}) \longrightarrow \mathbb{R}\Gamma_\Lambda(V \cap \pi^{-1}(U),\widehat{\mathcal{G}}) \xrightarrow{+1} \tag{1.2.6}$$

where V is any open subset of $T^*_Y X$ such that $\Lambda^+ = V \cap \Lambda$.

Comparing triangles (1.2.5)and (1.2.6) gives an isomorphism :

$$\mathbb{R}\Gamma_H(\tau^{-1}(U),\mathcal{G}) \xrightarrow{\sim} \mathbb{R}\Gamma_\Lambda(V \cap \pi^{-1}(U),\widehat{\mathcal{G}})[+1] \tag{1.2.7}$$

As $\widehat{\mathcal{G}}$ is $\mathbb{R}^+$-conic we have (cf. proof of lemma 1.2.1) :

$$\mathbb{R}\Gamma_\Lambda\left(V \cap \pi^{-1}(U),\widehat{\mathcal{G}}\right) = \mathbb{R}\Gamma\left(U, s'^{-1}\mathbb{R}\Gamma_\Lambda(\widehat{\mathcal{G}})\right)$$

hence the isomorphism (1.2.7) is equal to :

$$\mathbb{R}\Gamma\left(U, \mathbb{R}\tau_* \mathbb{R}\Gamma_H(\mathcal{G})\right) \xrightarrow{\sim} \mathbb{R}\Gamma\left(U, s'^{-1}\mathbb{R}\Gamma_\Lambda(\widehat{\mathcal{G}})\right)[+1]$$

and this isomorphism being compatible with the restriction on $U' \subset U$, we get an isomorphism of sheaves :

$$\mathbb{R}\tau_* \mathbb{R}\Gamma_H(\mathcal{G}) \xrightarrow{\sim} s'^{-1}\mathbb{R}\Gamma_\Lambda(\widehat{\mathcal{G}})[+1] \tag{1.2.8}$$

As $\overline{T_N M^+}$ is the boundary of H we have a morphism

$$\mathbb{R}\Gamma_{\overline{T_N M^+}}(\mathcal{G}) \longrightarrow \mathbb{R}\Gamma_H(\mathcal{G}) \tag{1.2.9}$$

which composed to (1.2.8) gives

$$\mathbb{R}\Gamma_{\overline{T_N M^+}}(\mathcal{G}) \longrightarrow s'^{-1}\mathbb{R}\Gamma_\Lambda(\widehat{\mathcal{G}})[+1] \tag{1.2.10}$$

Assume now that $\mathcal{G}$ is $\mathbb{C}^*$-conic. The fibers of $T_Y X - \overline{T_N M^+}$ and of $T_Y X - H$ being contractible, the restriction morphism

$$\mathbb{R}\Gamma\left(\tau^{-1}(U) - \overline{T_N M^+}, \mathcal{G}\right) \to \mathbb{R}\Gamma\left(\tau^{-1}(U) - H, \mathcal{G}\right)$$

is an isomorphism, hence (1.2.9) and (1.2.10) are isomorphisms. □

Proposition 1.2.3. *Let $\mathcal{F}$ be an object of $D^b(X)$, there is a morphism of triangles :*

$$\begin{array}{ccccccc}
\mathbb{R}\Gamma_N(\mathcal{F}) & \longrightarrow & \mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F})|_N & \longrightarrow & \mathbb{R}\Gamma_{M^+}(\mathcal{F})|_N & \xrightarrow{+1} & \\
\| & & \downarrow & & \downarrow & & \\
\mathbb{R}\Gamma_N(\mathcal{F}) & \longrightarrow & \mathbb{R}\tau_*\mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F})) & \longrightarrow & s^{-1}\mathbb{R}\Gamma_{T_N M}(\nu_Y(\mathcal{F})) & \xrightarrow{+1} &
\end{array} \tag{1.2.11}$$

Proof. (This diagram was defined in [5]).

We have $\overline{M^+} - M^+ = N$ which gives the first line and $\overline{T_N M^+} - T_N M^+ = N$ which gives :

$$\mathbb{R}\Gamma_N(\nu_Y(\mathcal{F})) \longrightarrow \mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F})) \longrightarrow \mathbb{R}\Gamma_{T_N M^+}(\nu_Y(\mathcal{F})) \xrightarrow{+1}$$

But $\mathbb{R}\Gamma_N(\nu_Y(\mathcal{F})) = \mathbb{R}\tau_* \mathbb{R}\Gamma_N \mathbb{R}\Gamma_Y(\nu_Y(\mathcal{F})) = \mathbb{R}\Gamma_N(\mathcal{F})$ which together with lemma 1.2.1 gives :

$$\mathbb{R}\Gamma_N(\mathcal{F}) \longrightarrow \mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F})) \longrightarrow s^{-1}\mathbb{R}\Gamma_{T_N M}(\nu_Y(\mathcal{F})) \xrightarrow{+1}$$

Let us now consider the diagram (1.1.1). Let $Z = \tilde{p}^{-1}(\overline{M^+})$ and $\bar{Z}$ the closure of Z in $\widetilde{X_Y}$. We have $j^{-1}(Z) = T_N M^+$, $j^{-1}(\bar{Z}) = \overline{T_N M^+}$ and, by definition

$$\nu_Y(\mathcal{F}) = \sigma^{-1}\mathbb{R}j_*\tilde{p}^{-1}\mathcal{F}$$

so we have a sequence of morphisms :

$$\begin{aligned} \sigma^{-1}p^{-1}\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) &\to \sigma^{-1}\mathbb{R}j_*\tilde{p}^{-1}\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) \to \sigma^{-1}\mathbb{R}j_*\mathbb{R}\Gamma_Z(\tilde{p}^{-1}\mathcal{F}) \\ &\simeq \sigma^{-1}\mathbb{R}\Gamma_{\bar{Z}}(\mathbb{R}j_*\tilde{p}^{-1}\mathcal{F}) \to \mathbb{R}\Gamma_{\overline{T_N M^+}}(\sigma^{-1}\mathbb{R}j_*\tilde{p}^{-1}\mathcal{F}) = \mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F})) \end{aligned}$$

As $p_o\sigma_o = \tau$, we get a morphism :

$$\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) \to \mathbb{R}\tau_*\mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F}))$$

The same proof gives $\mathbb{R}\Gamma_{M^+}(\mathcal{F}) \to \mathbb{R}\tau_*\mathbb{R}\Gamma_{T_N M^+}(\nu_Y(\mathcal{F}))$ and, by construction, they commute with the canonical morphisms $\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) \to \mathbb{R}\Gamma_{M^+}(\mathcal{F})$ and $\mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F})) \to \mathbb{R}\Gamma_{T_N M^+}(\nu_Y(\mathcal{F}))$. □

Remark 1.2.4. Morphisms $\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) \to \mathbb{R}\tau_*\mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F}))$ and prop. 1.2.2 give a morphism :

$$\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) \to s'^{-1}\mathbb{R}\Gamma_{iT_N^*M}(\mu_Y(\mathcal{F}))[+1]$$

which is identical to the morphism of proposition (2.4.1.) in [13].

To see this we may look at the fibers at one point. In this case, we refer to [5] where the calculation has been made when $\mathcal{F}$ is the sheaf of holomorphic functions on X but the proof is identical for general $\mathcal{F}$.

Lemma 1.2.5. *Assume that $\mathcal{F}$ in $D^b(X)$ satisfies condition (1.1.2), then :*

$$s'^{-1}\mathbb{R}\Gamma_{iT_N^*M}(\mu_Y(\mathcal{F})) = \mathbb{R}\Gamma_N(\Phi'_f(\mathcal{F})) \otimes or_{N|M}[-2] \quad (1.2.12)$$

$$s^{-1}\mathbb{R}\Gamma_{T_N M}(\nu_Y(\mathcal{F})) = \mathbb{R}\Gamma_N(\Psi'_f(\mathcal{F})) \otimes or_{N|M}[-1] \quad (1.2.13)$$

Proof. In a neighborhood of the image of s' we have $\mu_Y(\mathcal{F}) = \pi_{\mathbb{C}}^{-1}(\Phi'_f(\mathcal{F}))[-1]$ where $\pi_{\mathbb{C}}$ is the projection $T_Y^*X \to Y$ hence

$$s'^{-1}\mathbb{R}\Gamma_{iT_N^*M}(\mu_Y(\mathcal{F})) = s'^{-1}\mathbb{R}\Gamma_{iT_N^*M}\left(\pi_{\mathbb{C}}^{-1}(\Phi'_f(\mathcal{F}))\right)[-1]$$

We have $\pi_{\mathbb{C}}^{-1}(N) = N \times_Y T^*_Y X$ and $\pi_{\mathbb{C}}$ satisfies the condition of lemma 2.2.4. Ch I of [22] hence :

$$\begin{aligned}\mathbb{R}\Gamma_{iT^*_N M}\left(\pi_{\mathbb{C}}^{-1}(\Phi'_f(\mathcal{F}))\right) &= \mathbb{R}\Gamma_{iT^*_N M}\mathbb{R}\Gamma_{N\times_Y T^*_Y X}\left(\pi_{\mathbb{C}}^{-1}(\Phi'_f(\mathcal{F}))\right)\\ &= \mathbb{R}\Gamma_{iT^*_N M}\pi_{\mathbb{C}}^{-1}\mathbb{R}\Gamma_N\left(\Phi'_f(\mathcal{F})\right)\end{aligned}$$

Now we apply [12, Prop 3.3.4.(iii)] :

$$\mathbb{R}\Gamma_{iT^*_N M}\pi_{\mathbb{C}}^{-1}\mathbb{R}\Gamma_N\big(\Phi'_f(\mathcal{F})\big) = \pi^{-1}\mathbb{R}\Gamma_N\big(\Phi'_f(\mathcal{F})\big) \otimes or_{iT^*_N M|N} \otimes or_{N\times_Y T^*_Y X|N}[-1]$$

But formula (4.2.4) of [12] gives :

$$or_{iT^*_N M|N} \otimes or_{N\times_Y T^*_Y X|N} = or_{N|M} \otimes or_{N|Y} \otimes or_{N|X} = or_{N|M}$$

which proves the first part of the lemma and the proof of the second part is the same. □

Remark 1.2.6. In this paper, we always assume that an equation f of N in M is given. Such an equation defines a canonical isomorphism between $or_{N|M}$ and the constant sheaf $\mathbb{C}_N$.

Theorem 1.2.7. *Let $\mathcal{F}$ be an object of $D^b(X)$ which satisfies condition (1.1.2), there are canonical morphisms of triangles :*

$$\begin{array}{ccccccc}
\mathbb{R}\Gamma_N(\mathcal{F}) & \longrightarrow & \mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F})|_N & \longrightarrow & \mathbb{R}\Gamma_{M^+}(\mathcal{F})|_N & \xrightarrow{+1} & \\
\| & & \downarrow & & \downarrow & & \\
\mathbb{R}\Gamma_N(\mathcal{F}) & \longrightarrow & s'^{-1}\mathbb{R}\Gamma_\Lambda(\mu_Y(\mathcal{F}))[1] & \longrightarrow & s^{-1}\mathbb{R}\Gamma_{T_N M}(\nu_Y(\mathcal{F})) & \xrightarrow{+1} & \\
\| & & \wr\downarrow & & \wr\downarrow & & \\
\mathbb{R}\Gamma_N(\mathcal{F}) & \longrightarrow & \mathbb{R}\Gamma_N(\Phi'_f(\mathcal{F}))[-1] & \longrightarrow & \mathbb{R}\Gamma_N(\Psi'_f(\mathcal{F}))[-1] & \xrightarrow{+1} &
\end{array} \qquad (1.2.14)$$

Proof. This theorem is the direct consequence of proposition 1.2.3, proposition 1.2.2 and lemma 1.2.5.

We have to remark that our proof of proposition 1.2.2 is very similar to the proof of the identity between $s'^{-1}\mu_Y(\mathcal{F})$ and $\Phi_{\tilde{f}}(\nu_Y(\mathcal{F})$ in [12] Proposition 8.6.3. Both use the same Fourier transform and it is not difficult to see that they are compatible, that is the low-right square in (1.2.14) is commutative. □

Remark 1.2.8. The morphism $\mathbb{R}\Gamma_{M^+}(\mathcal{F})|_N \longrightarrow \mathbb{R}\Gamma_N(\mathcal{F})[1]$ given by the triangle (1.2.1) is equal to the "topological boundary value" defined by Schapira in [23],[24]. The diagram (1.2.14) shows that it factors through :

$$\mathbb{R}\Gamma_{M^+}(\mathcal{F})|_N \longrightarrow \mathbb{R}\Gamma_N(\Psi'_f(\mathcal{F}))[-1] \longrightarrow \mathbb{R}\Gamma_N(\mathcal{F})[1]$$

1.3 The hyperbolic case

We will show in this section that under a suitable hyperbolicity condition, the morphisms of triangles in theorem 1.2.7 are isomorphisms.

Let us fix local coordinates as in the proof of theorem 1.2.3 :

Let (x,t) be a local coordinate system of M such that $f(x,t) \equiv t$ and let $(x+iy, t+is)$ be local coordinates of X, so that $f \equiv t+is$.

We have $N = \{\,(x,t) \in M \mid t=0\,\}$, $Y = \{\,(x+iy,t+is) \in X \mid t+is=0\,\}$, $M = \{\,(x+iy,t+is) \in X \mid x=0, s=0\,\}$ and $M^+ = \{\,(x,t) \in M \mid t>0\,\}$.

The coordinates of $T^*_Y X$ are $(x+iy, t+is, \xi+i\eta, \tau+i\sigma)$, $T^*_Y X = \{\,(x+iy,t+is,\xi+i\eta,\tau+i\sigma) \in T^*X \mid t+is=0, \xi+i\eta=0\,\}$ and $iT^*_N M = \{\,(x+iy,\tau+i\sigma) \in T^*_Y X \mid y=0, \tau=0\,\}$.

Definition 1.3.1. Let $\mathcal{F}$ be an object of $D^b(X)$, we will say that $\mathcal{F}$ is "near-hyperbolic" at 0 along N if there exists some $\varepsilon > 0$ and $C > 0$ which satisfies :

$$SS(\mathcal{F}) \cap \{\,|x|<\varepsilon, |y|<\varepsilon, 0<t<\varepsilon, |s|<\varepsilon\,\} \subset \{\,|\tau| \le C|\eta|(|y|+|s|) + C|\xi|\,\}$$

It is not difficult to verify that this condition is independant of local coordinates.

As usual, a function $p : T^*X \to \mathbb{C}$ will be said to be a homogeneous hyperbolic polynomial at $x_0 \in M$ in the direction $\alpha \in iT^*_{x_0}M$ if p is homogeneous polynomial in the fibers of T^*X and if for any $\xi \in iT^*_{x_0}M$, we have :

$$p(\xi + i\varepsilon\alpha) \neq 0 \text{ for } \varepsilon \neq 0$$

Lemma 1.3.2. *Let p be a homogeneous hyperbolic polynomial at $x_0 \in M$ in the directions of $iT^*_N M$ and $\mathcal{F}$ be an object of $D^b(X)$ such that $SS(\mathcal{F}) \subset \{(fp)^{-1}(0)\}$ then $\mathcal{F}$ is "near-hyperbolic" at 0 along N.*

Proof. From theorem 2.3. of [2], we know that there exists some $C>0$ and $\varepsilon>0$ such that :

$$\{p^{-1}(0)\} \cap \{\,|x|<\varepsilon, |y|<\varepsilon, |t|<\varepsilon, |s|<\varepsilon\,\} \subset \{\,|\sigma| \le C|\xi|(|y|+|s|) + C|\eta|\,\}$$

and because p is homogeneous we may replace p by ip to get definition 1.3.1. □

This lemma shows that the condition of near-hyperbolicity is weaker than hyperbolicity at 0. But the constant C in the definition 1.3.1 is independant of ε, hence near-hyperbolicity is stronger than hyperbolicity on $t>0$ near 0.

Proposition 1.3.3. *Let $\mathcal{F}$ be an object of $D^b(X)$ which is "near-hyperbolic" at 0 along N. Then there is an isomorphism*

$$\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) \xrightarrow{\sim} \mathbb{R}\tau_* \mathbb{R}\Gamma_{\overline{T_N M^+}}(\nu_Y(\mathcal{F}))$$

Proof. The morphism has been defined in the previous section and to prove that it is an isomorphism, we have to prove that there is a neighborhood U_0 of 0 in X such that for any neighborhood $U \subset U_0$ of 0 and any $k \ge 0$, we have :

$$H^k(\tau^{-1}(U) - \overline{T_N M^+}, \nu_Y(\mathcal{F})) \xrightarrow{\sim} H^k(U - \overline{M^+}, \mathcal{F})$$

By the definition of the specialization we have [12, th 4.2.3] :

$$H^k(\tau^{-1}(U) - \overline{T_N M^+}, \nu_Y(\mathcal{F})) = \varinjlim H^k(V, \mathcal{F}) \tag{1.3.1}$$

where the limit is taken on the family of open subsets V of U such that $C_Y(U - V) \subset \overline{T_N M^+}$.

Let G be a closed subset of X such that $C_Y(G) \subset \overline{T_N M^+}$. By definition $\overline{T_N M^+} = \{ (x + iy, \tilde{t} + i\tilde{s}) \in T_Y X \mid y = 0, \tilde{s} = 0, \tilde{t} \geq 0 \}$.

When $y \neq 0$, we have $C_Y(G) = \emptyset$ hence $G \cap Y \subset N$. This implies as in the proof of proposition 3.2.3. in [7] :

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, G \cap \{ (x + iy, t + is) \in X \mid |y| \geq \varepsilon, |s| + |t| \leq \delta_\varepsilon \} = \emptyset$$

There exists a function $a : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^\infty$ such that $a(\varepsilon) \leq \delta\varepsilon$ and which satisfy $a(0) = a'(0) = 0$, $0 < a'(r) < a(r)$ if $r \neq 0$. We have :

$$G \subset \{ (x + iy, t + is) \in X \mid |t| + |s| \geq a(|y|) \}$$

hence

$$G \subset \{ (x + iy, t + is) \in X \mid t \geq a(|y|) \}$$

When $y = 0$, we have $C_Y(G) \subset \{ \tilde{s} = 0, \tilde{t} \geq 0 \}$ hence :

$$\forall \varepsilon > 0, \exists \delta > 0, G \cap \{ (x + iy, t + is) \in X \mid |y| \leq \delta, |s| \leq \delta, |t| \leq \delta \} \subset \{ \varepsilon t \geq |s| \}$$

This implies that there exists a function a_1 satisfying the same conditions than a and such that

$$G \subset \{ (x + iy, t + is) \in X \mid t \geq a_1(|\frac{s}{t}|) \}$$

and thus there exists a function $b : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^\infty$ on $\mathbb{R} - \{0\}$ such that $b(0) = 0$, $b(r) > 0$ if $r \neq 0$, $b'(r) > 0$ if $r \neq 0$ and $\lim_{r \to 0} b'(r) = +\infty$ and

$$G \subset \{ (x + iy, t + is) \in X \mid t \geq b(|s|) \}$$

Taking the supremum of a and b and smoothing it we get a function $\varphi(r, s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^\infty$ except at $(0, 0)$ such that

$$G \subset \{ (x + iy, t + is) \in X \mid t \geq \varphi(|y|, s) \}$$

and which satisfy the following conditions :

(i) $\varphi(0, 0) = 0$ and $\varphi(r, s) > 0$ if $(r, s) \neq (0, 0)$.

(ii) $r \mapsto \varphi(r, s)$ is of class $\mathcal{C}^\infty$ everywhere, $\varphi'_r(0, 0) = 0$ and $0 < \varphi'_r(r, s) < \varphi(r, s)$ if $(r, s) \neq (0, 0)$.

(iii) $\varphi'_s(r, s) > 0$ if $(y, s) \neq (0, 0)$ and $\lim_{(r,s) \to (0,0)} \varphi'_s(r, s) = +\infty$.

Conversely, if a function φ satisfy theses conditions, the closed set $G_\varphi = \{\, t \geq \varphi(|y|, s) \,\}$ satisfy $C_Y(G_\varphi) \subset \overline{T_N M^+}$.

Any G such that $C_Y(G) \subset \overline{T_N M^+}$ is contained in some G_φ hence the inductive limit in the equation 1.3.1 may be taken on these sets that is :

$$H^k(\tau^{-1}(U) - \overline{T_N M^+}, \nu_Y(\mathcal{F})) = \varinjlim H^k(U - G_\varphi, \mathcal{F}) \tag{1.3.2}$$

where the limit is taken on the neighborhoods U of Y in X and on the functions φ satisfying (i),(ii) and (iii).

To prove the lemma, we have now to prove that :

$$H^k(U - G_\varphi, \mathcal{F}) = H^k(U - \overline{M^+}, \mathcal{F}) \tag{1.3.3}$$

So, we fix U and φ and define for any $c \geq 0$:

$$\Omega_c = \{\, (x + iy, t + is) \in U \mid t < c\varphi(|y|, s) \,\}$$

We have $\Omega_1 = U - G_\varphi$ and

$$\Omega = \cup_{c \geq 1} \Omega_c = U - \overline{M^+}$$

To prove formula 1.3.3 we will use proposition 2.7.2. of [12].

To do it we have to check that

$$\mathbb{R}\Gamma_{\Omega - \Omega_c}(\mathcal{F})_x = 0$$

for any x in $Z_c = \cap_{c' > c} \overline{(\Omega_{c'} - \Omega_c)}$.

(In fact the exact condition required by [12] is $\mathbb{R}\Gamma_{X - \Omega_c}(\mathcal{F})_x = 0$ but it is easy to see that the proof still works here with Ω instead of X).

By the definition of the micro-support $SS(\mathcal{F})$ we have to verify that the varieties

$$Z_c = \{\, (x + iy, t + is) \in U \mid t = c\varphi(|y|, s) \,\}$$

are non characteristic for $SS(\mathcal{F})$ if $(y, s) \neq 0$.

The normal vector to the variety $t = c\varphi(|y|, s)$ is $(\xi = 0, \eta = c\varphi'_r(|y|, s)y/|y|, \tau = -1, \sigma = -\varphi'_s(|y|, s)$ thus using definition 1.3.1 we have to prove that :

$$Cc|\varphi|'_r(|y|, s)(|y| + |s|) < 1$$

and using condition (ii) we have to prove that $C|t|(|y| + |s|) < 1$.

This condition does not depend on c and so it is true if U is small. □

Corollary 1.3.4. *Let $\mathcal{F}$ be an object of $D^b(X)$ which satisfies condition (1.1.2) and is near-hyperbolic along N then the morphisms of triangles of theorem 1.2.7 are isomorphisms.*

2 Boundary Values for $\mathcal{D}$-modules

2.1 General case

We keep the notations of section 1.2, that is we consider a real analytic manifold M of dimension n, N a smooth hypersurface of M, X be a complexification of M and Y a smooth hypersurface of X complexified of N. We assume that N has an equation $f=0$ and denote by

$$M^+ = M \cap \{f > 0\}, \quad \overline{M^+} = M \cap \{f \geq 0\}$$

and by $j : M^+ \hookrightarrow M$ the inclusion. An equation of Y is $f_{\mathbb{C}}$, the complexification of f.

We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by $\mathcal{D}_X$ the sheaf of differential operators on X with holomorphic coefficients.

In section 2, $\mathcal{M}$ will denote a coherent $\mathcal{D}_X$-module or an object of $D_c(\mathcal{D}_X)$, the derived category of bounded complexes of $\mathcal{D}_X$-modules with coherent cohomology. We will apply the results of section 1 to the complex $\mathcal{F}$ of solutions of $\mathcal{M}$ that is

$$\mathcal{F} = \mathcal{S}\mathrm{ol}(\mathcal{M}) := \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

The sheaf of hyperfunctions on M is, by definition :

$$\mathcal{B}_M = \mathcal{H}^n_M(\mathcal{O}_X) \otimes or_M$$

As $\mathcal{H}^k_M(\mathcal{O}_X) = 0$ for $k \neq n$ and as $\mathcal{B}_M$ is a flabby sheaf we have :

$$\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{O}_X) \otimes or_M[n] = \mathbb{R}\Gamma_{\overline{M^+}}(\mathbb{R}\Gamma_M(\mathcal{O}_X) \otimes or_M[n]) = \mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{B}_M) = \Gamma_{\overline{M^+}}(\mathcal{B}_M)$$

This means that $\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{O}_X)[n]$ is the sheaf of hyperfunctions on M with support in $\overline{M^+}$. The same result apply with N or M^+ replacing $\overline{M^+}$.

We have

$$\begin{aligned}\mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{F}) = \mathbb{R}\Gamma_{\overline{M^+}}\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{\overline{M^+}}(\mathcal{O}_X)) \\ &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\overline{M^+}}(\mathcal{B}_M)) \otimes or_M[-n]\end{aligned}$$

The microlocalization of $\mathcal{O}_X$ along Y was first defined in [22] and is generally denoted by $\mathcal{C}^{\mathbb{R}}_{Y|X}$. As the complex $\mu_Y(\mathcal{O}_X)$ is concentrated in degree 1 (the codimension of Y) we have :

$$\mathcal{C}^{\mathbb{R}}_{Y|X} = \mu_Y(\mathcal{O}_X)[1] = \mathcal{H}^1(\mu_Y(\mathcal{O}_X))$$

The sheaf of 2-hyperfunctions on $\Lambda = iT^*_N M$ has been defined in [7] where it is denoted by $\widetilde{\mathcal{B}}_\Lambda$ and in [13] where it is denoted by $\mathcal{B}^2_\Lambda$. By definition :

$$\mathcal{B}^2_\Lambda = \mathcal{H}^n_\Lambda(\mathcal{C}^{\mathbb{R}}_{Y|X}) \otimes or_N$$

As $\mathcal{H}^k_\Lambda(\mathcal{C}^{\mathbb{R}}_{Y|X}) = 0$ for $k \neq n$ and as the microlocalization commutes with the functor $\mathbb{R}\mathcal{H}\mathrm{om}$ we have

$$\mathbb{R}\Gamma_{iT^*_N M}(\mu_Y(\mathcal{F})) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\Gamma_{iT^*_N M}(\mu_Y(\mathcal{O}_X))) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^2_\Lambda) \otimes or_N[-n-1]$$

The next thing to remark is that the micro-support $SS(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ of the complex of solutions is equal to the characteristic variety $Ch(\mathcal{M})$ of the module $\mathcal{M}$ [12, theorem 11.3.3]. If $\mathcal{M}$ is an object of $D_c(\mathcal{D}_X)$, the same is true if $Ch(\mathcal{M})$ denotes the union of the characteristic varieties of all the cohomology groups of $\mathcal{M}$ [11, theorem 10.1.1].

In particular, $\mathcal{M}$ will be said "near-hyperbolic" if $Ch(\mathcal{M})$ satisfies the same condition than $SS(\mathcal{F})$ in definition 1.3.1. Lemma 1.3.2 shows that $\mathcal{M}$ is near-hyperbolic if it is hyperbolic.

We may now apply theorem 1.2.7 to the present situation and get :

Theorem 2.1.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module or an object of $D_c(\mathcal{D}_X)$ such that :*

$$C_{\Lambda_{\mathbb{C}}}(Ch(\mathcal{M})) \subset S_{\Lambda_{\mathbb{C}}} \tag{2.1.1}$$

then there are canonical morphisms of triangles :

$$\begin{array}{ccccccc}
\mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_N(\mathcal{B}_M)) & \longrightarrow & \mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_{\overline{M^+}}(\mathcal{B}_M))|_N & \longrightarrow & \mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_{M^+}(\mathcal{B}_M))|_N & \xrightarrow{+1} \\
\| & & \downarrow & & \downarrow & \\
\mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_N(\mathcal{B}_M)) & \longrightarrow & s'^{-1}\,\mathcal{S}\mathrm{ol}(\mathcal{M},\mathcal{B}^2_{\Lambda}) & \longrightarrow & s^{-1}\,\mathcal{S}\mathrm{ol}(\mathcal{M},\mathcal{G}_N) & \xrightarrow{+1} \\
\| & & \wr\downarrow & & \wr\downarrow & \\
\mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_N(\mathcal{B}_M)) & \longrightarrow & \mathbb{R}\Gamma_N(\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M})))[n-1] & \longrightarrow & \mathbb{R}\Gamma_N(\Psi_f(\mathcal{S}\mathrm{ol}(\mathcal{M})))[n-1] & \xrightarrow{+1}
\end{array}$$

If $\mathcal{M}$ is near-hyperbolic, these morphisms are isomorphisms.

In this formula, $\mathcal{G}_N = \mathbb{R}\Gamma_{T_N M}(\nu_Y(\mathcal{O}_X))[n] \otimes or_N$, $\mathcal{S}\mathrm{ol}(\mathcal{M},.) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M},.)$ and we have omitted $\otimes or_N$ in the last line.

Remark 2.1.2. If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, it always satisfies condition (2.1.1).

Proof. If we apply theorem 1.2.7 and corollary 1.3.4 we get the result except the fact that the classical sheaf of vanishing cycles $\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ has been used instead of $\Phi'_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$. The equality between $\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ and $\Phi'_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ will come from the following theorem of [17] :

Theorem 2.1.3. *Let $\mathcal{M}$ be an object of $D_c(\mathcal{D}_X)$ defined in a neighborhood of Y whose characteristic variety satisfies $C_{\Lambda_{\mathbb{C}}}(Ch(\mathcal{M})) \subset S_{\Lambda_{\mathbb{C}}}$.*

Let $\pi' : \dot{T}^*_Y X \to Y$ be the canonical projection from $\dot{T}^*_Y X = T^*_Y X - Y$ to Y. There exists a complex of $\pi'^{-1}\mathcal{D}^\infty_Y$ module $\widetilde{\Phi}\{1\}(\mathcal{M})$ such that :

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}\left(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}\right) \xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\pi'^{-1}\mathcal{D}^\infty_Y}\left(\widetilde{\Phi}\{1\}(\mathcal{M}), \pi'^{-1}\mathcal{O}_Y\right)$$

The complex $\widetilde{\Phi}\{1\}(\mathcal{M})$ of vanishing cycles of type $\{1\}$ has been defined algebraically in [17] and we will not recall its definition here.

We just use the fact that on $\dot{T}^*_Y X$, it has resolutions by free $\pi'^{-1}\mathcal{D}^\infty_Y$-modules of finite type, hence it is locally constant on the fibers of π'.

The proof that $\Phi'_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ is isomorphic to $\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ being local, we may assume that $X = T^*_Y X = Y \times \mathbb{C}$. Let $\widetilde{\mathbb{C}^*}$ be the universal covering of $\mathbb{C}^*$ and $\pi : Y \times \widetilde{\mathbb{C}^*} \to Y \times \mathbb{C}^*$ the canonical projection.

Let $x \in Y \times \mathbb{C}^*$ and U a small neighborhood of x. We have $\pi^{-1}(U) \simeq U \times \mathbb{Z}$ and because of its finite type resolution, the sheaf $\mathcal{G} = \widetilde{\Phi}\{1\}(\mathcal{M})$ satisfies :

$$\mathbb{R}\Gamma(U, \pi_* \pi^{-1}\mathcal{G}) = \mathbb{R}\Gamma(\pi^{-1}(U), \pi^{-1}\mathcal{G}) = \mathbb{R}\Gamma(U, \mathcal{G})^{\mathbb{Z}}$$

The same property is then true for $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X} \mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}) = \mu_Y(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ and the proof of theorem 8.6.3. in [12] may be used. This shows the result. □

Remark 2.1.4. It is proved in [13] that the canonical morphism $\Gamma_{\overline{M^+}}(\mathcal{B}_M) \longrightarrow \mathcal{B}^2_\Lambda$ is injective. This shows that if $\mathcal{M}$ is a $\mathcal{D}_X$-module

$$\mathcal{H}\mathrm{om}(\mathcal{M}, \Gamma_{\overline{M^+}}(\mathcal{B}_M))|_N \longrightarrow s'^{-1}\,\mathcal{H}\mathrm{om}(\mathcal{M}, \mathcal{B}^2_\Lambda)$$

is injective even without hyperbolicity.

2.2 Regular $\mathcal{D}$-modules

The sheaf $\mathcal{D}_X|_Y$ of differential operators defined in a neighborhood of the submanifold Y of X is provided with two filtrations. The first one is the usual filtration by the order and will be denoted by $(\mathcal{D}_{X\,k})_{k\in\mathbb{N}}$. The second one is the V-filtration defined in [6] by :

$$V^j\mathcal{D}_X = \{\, P \in \mathcal{D}_X|_Y \mid \forall j \in \mathbb{Z}, \quad P\mathcal{I}^j_Y \subset \mathcal{I}^{j-k}_Y \,\}$$

where $\mathcal{I}_Y$ is the defining ideal of Y in X and $\mathcal{I}^j_Y = \mathcal{O}_X$ if $f \leq 0$.

The graduate sheaf $gr_V\mathcal{D}_X$ associated to this filtration is isomorphic to the direct image $\tau_*\mathcal{D}_{[T_Y X]}$ of the sheaf of differential operators on $T_Y X$ with polynomial coefficients in the fibers of $\tau : T_Y X \to Y$.

We denote by θ the Euler vector field of the normal bundle $T_Y X$ and by Θ a section of $V^0\mathcal{D}_X$ whose class in $gr^0_V\mathcal{D}_X$ is θ.

We recall that a coherent $\mathcal{D}_X$-module is said to be specializable along Y if for any local section u of $\mathcal{M}$, there exists some polynomial b and some differential operator Q in $V^{-1}\mathcal{D}_X$ such that $b(\Theta)u + Qu = 0$.

If moreover such b and Q may always be found with Q of order less than the degree of b, $\mathcal{M}$ is said to be 1-specializable [15] or regular-specializable along Y.

An object of $D_c(\mathcal{D}_X)$ is specializable along Y or regular-specializable along Y if all its cohomology groups have the property.

We recall that a holonomic module is specializable along any submanifold Y of X and that a regular holonomic module is regular-specializable along any submanifold Y [7].

If we fix local coordinates (x, t) of X such that $Y = \{\, t = 0 \,\}$, we may choose $\Theta = tD_t$ and Q is in $V^{-1}\mathcal{D}_X$ if it is of the form $tQ_1(x, t, D_x, tD_t)$.

We remark that the principal symbol of $P = b(tD_t) + tQ_1(x, t, D_x, tD_t)$ where b is of degree m and Q_1 of order $\leq m$ is $(t\tau)^m + tq(x, t, \xi, t\tau)$ and thus the tangent cone along

$\Lambda_{\mathbb{C}}$ of the characteristic variety of P is contained in $\tilde{t}\tau = 0$ that is $S_{\Lambda_{\mathbb{C}}}$. This shows that if $\mathcal{M}$ is regular-specializable, then it satisfies condition (2.1.1).

Let $i : Y \hookrightarrow X$ the canonical immersion and $i^!\mathcal{M}$ the extraordinary inverse image of $\mathcal{M}$. Recall that $i^!\mathcal{M} = \mathbb{D}_Y i^* \mathbb{D}_X \mathcal{M}$ where $i^*\mathcal{M}$ is the ordinary inverse image and $\mathbb{D}_X$ the duality of $\mathcal{D}_X$-modules, that is $\mathbb{D}_X \mathcal{M} = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$.

The sheaves $\Phi_Y(\mathcal{M})$ of vanishing cycles and $\Psi_Y(\mathcal{M})$ of nearby cycles of the specializable $\mathcal{D}_X$-module $\mathcal{M}$ where defined in [6]. Other definitions which extend to the derived category $D_c(\mathcal{D}_X)$ were given in [21], [16], [19]. We refer to these papers for the precise definitions and the proof of the following proposition :

Proposition 2.2.1. *If $\mathcal{M}$ is a an object of $D_c(\mathcal{D}_X)$ specializable along Y there are triangles in the category $D_c(\mathcal{D}_Y)$:*

$$\Psi_Y(\mathcal{M}) \xrightarrow{can} \Phi_Y(\mathcal{M}) \longrightarrow i^!\mathcal{M} \xrightarrow{+1}$$
$$\Phi_Y(\mathcal{M}) \xrightarrow{var} \Psi_Y(\mathcal{M}) \longrightarrow i^*\mathcal{M} \xrightarrow{+1}$$

If $\mathcal{M}$ is regular-specializable, there are isomorphisms :

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(i^!\mathcal{M}, \mathcal{O}_Y) &\simeq \mathbb{R}\Gamma_Y(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))[+1] \\ \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{O}_Y) &\simeq \Psi_{f_{\mathbb{C}}}(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \\ \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{O}_Y) &\simeq \Phi_{f_{\mathbb{C}}}(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \end{aligned}$$

This proposition gives an isomorphism of triangles :

$$\begin{array}{ccccccc} \mathbb{R}\Gamma_Y(\mathcal{S}ol(\mathcal{M}))[+1] & \longrightarrow & \Phi_{f_{\mathbb{C}}}(\mathcal{S}ol(\mathcal{M})) & \longrightarrow & \Psi_{f_{\mathbb{C}}}\mathcal{S}ol(\mathcal{M})) & \xrightarrow{+1} \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & \\ \mathcal{S}ol(i^!\mathcal{M}) & \longrightarrow & \mathcal{S}ol(\Phi_Y(\mathcal{M})) & \longrightarrow & \mathcal{S}ol(\Psi_Y(\mathcal{M})) & \xrightarrow{+1} \end{array}$$

Remark 2.2.2. The isomorphism $\mathcal{S}ol(i^!\mathcal{M}) \simeq \mathbb{R}\Gamma_Y(\mathcal{S}ol(\mathcal{M}))[1]$ has been proved in a more general case (Fuchsian modules) in [18, theorem 3.2.4.]

We have

$$\mathcal{B}_N = \mathbb{R}\Gamma_N(\mathcal{O}_Y)[n-1] \text{ and } \Gamma_N(\mathcal{B}_M) = \mathbb{R}\Gamma_N\mathbb{R}\Gamma_M(\mathcal{O}_X)[n] = \mathbb{R}\Gamma_N\mathbb{R}\Gamma_Y(\mathcal{O}_X)[n]$$

so, if we apply the functor $\mathbb{R}\Gamma_N(.)$ to the morphism of triangles, we get isomorphisms :

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N(\mathcal{B}_M)) &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(i^!\mathcal{M}, \mathcal{B}_N) \\ \mathbb{R}\Gamma_N(\Phi_f(\mathcal{S}ol(\mathcal{M}))) &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{B}_N) \\ \mathbb{R}\Gamma_N(\Psi_f(\mathcal{S}ol(\mathcal{M}))) &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{B}_N) \end{aligned}$$

As regular-specializable $\mathcal{D}_X$-modules satisfy condition 2.1.1, we may combine this result with theorem 2.1.1 and get :

Theorem 2.2.3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module or an object of $D_c(\mathcal{D}_X)$ which is regular-specializable along Y, then there is a morphism of triangles :*

$$\begin{array}{ccccccc}
\mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_N(\mathcal{B}_M)) & \longrightarrow & \mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_{\overline{M^+}}(\mathcal{B}_M))|_N & \longrightarrow & \mathcal{S}\mathrm{ol}(\mathcal{M},\Gamma_{M^+}(\mathcal{B}_M))|_N & \xrightarrow{+1} \\
\wr\downarrow & & \ell\downarrow & & \beta\downarrow & \\
\mathcal{S}\mathrm{ol}(i^!\mathcal{M},\mathcal{B}_N) & \longrightarrow & \mathcal{S}\mathrm{ol}(\Phi_Y(\mathcal{M}),\mathcal{B}_N) & \longrightarrow & \mathcal{S}\mathrm{ol}(\Psi_Y(\mathcal{M}),\mathcal{B}_N) & \xrightarrow{+1}
\end{array}$$

which is an isomorphism if $\mathcal{M}$ is near-hyperbolic along Y.

[$\mathcal{S}\mathrm{ol}(.,.)$ means $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(.,.)$ in the first line and $\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(.,.)$ in the second.]

Remark 2.2.4. Using remark (2.1.4) we can see that the following morphism is always injective if $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module :

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_{\overline{M^+}}(\mathcal{B}_M))|_N \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}),\mathcal{B}_N)$$

2.3 "Classical" boundary value theorems

In this section, we want to compare the morphisms of theorem 2.2.3 with older results.

Proposition 2.3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is regular-specializable along Y. The morphism*

$$\beta' : \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_{M^+}(\mathcal{B}_M)) \longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}),\mathcal{B}_N)$$

defined in [5] is the same than the morphism β of theorem 2.2.3.

Proof. Both morphisms are defined as the composition of the same one

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_{M^+}(\mathcal{B}_M)) \longrightarrow \mathbb{R}\tau_* \,\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathbb{R}\Gamma_{T_N M}(\nu_Y(\mathcal{O}_X))[-n]\otimes or_N)$$

and morphisms β and $\tilde{\beta}'$

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathbb{R}\Gamma_{T_N M}(\nu_Y(\mathcal{O}_X))[-n]\otimes or_N) \xrightarrow{\tilde{\beta},\tilde{\beta}'} \mathbb{R}\tau_* \,\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}),\mathcal{B}_N)$$

so, we have to prove the identity of $\tilde{\beta}$ and $\tilde{\beta}'$.

As both are functorial we may as in the proof of theorem 2.2.1. in [16] replace $\mathcal{M}$ by a resolution, fix local coordinates (x,t) and assume that $\mathcal{M}$ is defined by a matrix $b(tD_t)I_N - A$ where b is a polynomial, I_N the $N\times N$-identity matrix and A a matrix satisfying suitable hypothesis. Let $\mathcal{M}_0$ be the module defined by $b(tD_t)I_N$.

Then proposition 2.2.1. of [16] shows that $\mathbb{R}\mathcal{H}om(\mathcal{M},\mathcal{C}^{\mathbb{R}}_{Y|X}) = \mathbb{R}\mathcal{H}om(\mathcal{M}_0,\mathcal{C}^{\mathbb{R}}_{Y|X})$ and a Fourier transform shows that the same is true for $\nu_Y(\mathcal{O}_X)$.

So we may assume from now on that $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X b(tD_t)$. Splitting b into prime factors we may assume that $b(\theta) = (\theta - a)^p$ and by induction on p that $p = 1$.

We have to prove the proposition in the very simple case where $\mathcal{M}$ is the $\mathcal{D}_X$-module $\mathcal{M}_\alpha = \mathcal{D}_X/\mathcal{D}_X(tD_t - \alpha)$. We will make a detailed proof because it is the basic example which explain all the construction. We have to distinguish three cases, that is α non integer, α nonnegative integer and α negative integer.

Let us first remark that in all cases, $\Phi_Y(\mathcal{M})$ and $\Psi_Y(\mathcal{M})$ are isomorphic to $\mathcal{D}_Y$ as $\mathcal{D}_Y$-module but the morphisms "var" and "can" depend on α.

a) $\underline{\alpha \in \mathbb{C} - \mathbb{Z}}$: We have $i^*\mathcal{M} = 0$ and $i^!\mathcal{M} = 0$ and thus the morphisms $\Psi_Y(\mathcal{M}) \xrightarrow{can} \Phi_Y(\mathcal{M})$ and $\Phi_Y(\mathcal{M}) \xrightarrow{var} \Psi_Y(\mathcal{M})$ are isomorphisms of $\mathcal{D}_Y$-modules.

If we fix a determination of $\log t$ and denote by $p : X \to Y$ the projection (given by local coordinates), we have $Sol(\mathcal{M})|_{X-Y} \simeq p^{-1}\mathcal{O}_Y \otimes t^\alpha$ hence

$$\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M})) \simeq \Psi_f(\mathcal{S}\mathrm{ol}(\mathcal{M})) \simeq \mathcal{O}_Y$$
$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\overline{M^+}}(\mathcal{B}_M)) \simeq \mathcal{B}_N \otimes t_+^\alpha$$
$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M^+}(\mathcal{B}_M)) \simeq \mathcal{B}_N \otimes t_{(+)}^\alpha$$

where t_+ is the function on $\mathbb{R}$ equal to t if $t > 0$ and to 0 if $t \leq 0$ and $t_{(+)}$ its restriction to $t > 0$.

By definition the morphism β' associate to $u(x)t^\alpha$ the hyperfuntion $u(x) \in \mathcal{B}_N$. It is clear that morphisms β defined here is the same.

Let us remark that the morphism $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\overline{M^+}}(\mathcal{B}_M)) \to \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{B}_N)$ is slightly more complicated. It associates first to $u(x)t_+^\alpha$ the class of the function whose boundary value is $u(x)t_+^\alpha$ that is the class of $u(x)t^\alpha/k_\alpha$ with $k_\alpha = e^{2i\pi\alpha} - 1$ and then the boundary value, that is $u(x)/k_\alpha$.

Of course, this is compatible with the morphism of variation :

$$\mathcal{S}\mathrm{ol}(\Phi_Y(\mathcal{M})) \xrightarrow{var} \mathcal{S}\mathrm{ol}(\Psi_Y(\mathcal{M}))$$

which associate to $u(x)t^\alpha$ its variation $u(x)((te^{2i\pi})^\alpha - t^\alpha) = k_\alpha u(x)$

b) $\underline{\alpha \in \mathbb{Z},\ \alpha < 0}$: We have $i^*\mathcal{M} = 0$ and $\Psi_Y(\mathcal{M}) \xrightarrow{can} \Phi_Y(\mathcal{M})$ is an isomorphism of $\mathcal{D}_Y$-modules. On the contrary, the morphism $\Phi_Y(\mathcal{M}) \xrightarrow{var} \Psi_Y(\mathcal{M})$ is zero while $i^!\mathcal{M}$ has two non zero cohomology groups, both isomorphic to $\mathcal{D}_Y$. We have :

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\overline{M^+}}(\mathcal{B}_M)) \simeq \mathcal{B}_N \otimes \delta^{(-\alpha-1)}(t)$$
$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M^+}(\mathcal{B}_M)) \simeq \mathcal{B}_N \otimes t_{(+)}^\alpha$$

and the morphism between them is zero. ($\delta^{(j)}(t)$ is the n-th derivative of the Dirac distribution).

The morphisms β and β' are the same than in the previous case.

c) $\underline{\alpha \in \mathbb{Z},\ \alpha \geq 0}$: The situation is dual to the previous one, we have $i^!\mathcal{M} = 0$ and $\Phi_Y(\mathcal{M}) \xrightarrow{var} \Psi_Y(\mathcal{M})$ is an isomorphism of $\mathcal{D}_Y$-modules while $\Psi_Y(\mathcal{M}) \xrightarrow{can} \Phi_Y(\mathcal{M})$ is zero and $i^*\mathcal{M}$ has two non zero cohomology groups, both isomorphic to $\mathcal{D}_Y$. We have :

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\overline{M^+}}(\mathcal{B}_M)) \simeq \mathcal{B}_N \otimes t_+^\alpha$$
$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{M^+}(\mathcal{B}_M)) \simeq \mathcal{B}_N \otimes t_{(+)}^\alpha$$

and the morphism between them is zero. The morphisms β and β' are the same than before. □

Let us now recall the construction of Kashiwara-Oshima [9] and Kashiwara-Kawai [7]. In [9] the authors considered the case where $\mathcal{M}$ is defined by an operator P with regular singularities along Y with simple characteristic exponents α_i such that $\alpha_i - \alpha_j \notin \mathbb{Z}$ if $i \neq j$. Then proposition 4.4. of [9] is equivalent to the following isomorphism :

$$s'^{-1}\,\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_M) \xrightarrow{\sim} s'^{-1}\,\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}^2_\Lambda)$$

Considering the factorization :

$$\Gamma_{\overline{M^+}}(\mathcal{B}_M) \hookrightarrow s'^{-1}\mathcal{C}_M \hookrightarrow s'^{-1}\mathcal{B}^2_\Lambda$$

the morphism of boundary value in [9] is the factorization of ℓ :

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_{\overline{M^+}}(\mathcal{B}_M)) &\longrightarrow s'^{-1}\,\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_M)\\ &\xrightarrow{\sim} s'^{-1}\,\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}^2_\Lambda) \longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}),\mathcal{B}_N),\end{aligned}$$

while in [7] the authors considered the case of multiple characteristic exponents, and their morphism is a particular case of ℓ, as shown in proposition 2.3.1.

Using the identification of proposition 2.3.1, we can see that if we restrict theorem 2.2.3 to the case $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ for some differential operator P, we recover exactly a result of Tahara [25]. The hypothesis of Tahara was that P is "Fuchsian hyperbolic" which means in our notations that it is regular-specializable and satisfy the condition of lemma 1.3.2 hence is near-hyperbolic.

2.4 Application

A natural question which arises in boundary value problems is the one of knowing under which assumptions the boundary values of the solutions of a given system $\mathcal{M}$ are analytic functions on the boundary N (see [20], where such solutions are called ideally analytic).

By theorem 2.2.3 this assumption should be read in $\Psi_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ and $\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$, on the grounds that they have to be elliptic systems on N.

Example 2.4.1. Let $M = \mathbb{R}^n$, $X = \mathbb{C}^n$, with the coordinates $(x_1, \ldots, x_{n-1}, t)$, real on M, and $Y = \{t = 0\}$, $N = Y \cap M$. Let $\mathcal{M} = \frac{\mathcal{D}_X}{\mathcal{L}}$ be regular along Y, and assume that the Laplacian

$$\triangle_Y = \sum_{i=1}^{n-1}\left(\frac{\partial}{\partial x_i}\right)^2 \in \mathcal{L}$$

Then $\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$, $\Phi_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))$ are elliptic on N, hence the boundary values are analytic.

More generally, denoting $\Lambda = iT^*_N M$ and $\Lambda_{\mathbb{C}} = T^*_Y X$, we have :

Proposition 2.4.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module regular-specializable along Y.*

Then the boundary values of the solutions are analytic if the characteristic variety of $\mathcal{M}$ satisfy :

$$C_{\Lambda_{\mathbb{C}}}(Ch(\mathcal{M})) \cap T^*_\Lambda \Lambda_{\mathbb{C}} \subset T^*_{\Lambda_{\mathbb{C}}}\Lambda_{\mathbb{C}} \cup T^*_Y \Lambda_{\mathbb{C}}$$

Proof. Proposition 1.1.4 shows that

$$C_{\Lambda_{\mathbb{C}}}(Ch(\mathcal{M})) = j_1 p_1^{-1} C_1 \cup j_2 p_2^{-1} C_2$$

where C_1 and C_2 are two involutive homogeneous subvarieties of T^*Y and that :

$$SS(\Psi'_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))) \subset C_1 \quad \text{and} \quad SS(\Phi'_f(\mathcal{S}\mathrm{ol}(\mathcal{M}))) \subset C_2$$

The condition of the proposition is equivalent to :

$$C_1 \cap T_N^*Y \subset T_Y^*Y \quad \text{and} \quad C_2 \cap T_N^*Y \subset T_Y^*Y$$

If $\mathcal{M}$ is regular-specializable along Y, $\Phi'_f(\mathcal{S}\mathrm{ol}(\mathcal{M})) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{B}_N)$ and thus

$$Ch(\Phi_Y(\mathcal{M})) \cap T_N^*Y \subset T_Y^*Y$$

that is $\Phi_Y(\mathcal{M})$ is elliptic and the same is true for $\Psi_Y(\mathcal{M})$. There solutions are thus analytic. □

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