On fundamental groups of elliptically connected surfaces

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Abstract

A compact complex manifold X is called *elliptically connected* if any pair of points in X can be connected by a chain of elliptic or rational curves. We prove that the fundamental group of an elliptically connected compact complex surface is almost abelian. This confirms a conjecture which states that the fundamental group of an elliptically connected Kähler manifold must be almost abelian.

Introduction

We use below the following

0.1. Definition. Let X be a compact complex space. We say that X is *elliptically* (resp. *torically*) *connected* if any two points $x', x'' \in X$ can be joined by a finite chain of (possibly, singular) rational or elliptic curves (resp. of holomorphic images of complex tori). Here we discuss the following

0.2. Conjecture [Z]. *Let X be a compact Kähler manifold. If X is torically connected, then the fundamental group $\pi_1(X)$ is almost abelian (or, for a weaker form, almost nilpotent).*

A group G is called *almost abelian*[1] (resp. *almost nilpotent, almost solvable, etc.*) if it contains an abelian (resp. nilpotent, solvable, etc.) subgroup of finite index. Obviously, each of these properties is stable under finite extensions.

More generally, we may ask whether a Kähler variety connected by means of chains of subvarieties with almost abelian (resp. almost nilpotent, almost solvable, etc.) fundamental groups has itself such a fundamental group.

[1] or *virtually abelian*, or *abelian-by-finite*.

It is known that a rationally connected (i.e. connected by means of chains of rational curves) compact Kähler manifold is simply connected [KMM, Cam1 (3.5), Cam2 (5.7), Cam3 (2.4′), Cam4 (5.2.3)] (see also [Se]). Moreover, it follows from [Cam1 (2.2), Cam2 (5.2), (5.4); Cam4 (5.2.4.1)] that

a) if a compact Kähler manifold X is connected by means of chains of holomorphic images of simply connected varieties, then $\pi_1(X)$ is a finite group;

b) if X as above can be covered by holomorphic images of complex tori passing through a point $x_0 \in X$, then the group $\pi_1(X)$ is almost abelian.

This gives a motivation for the above conjecture. Another kind of motivation is provided by the following function–theoretic consideration. We introduce the next

0.3. Definition. We say that a complex space X is *sub–Liouville* if its universal covering space U_X is *Liouville*, i.e. if any bounded holomorphic function on U_X is constant.

The complex tori yield examples of sub–Liouville compact manifolds. By a theorem of Lin [Li], any quasi–compact complex variety X with an almost nilpotent fundamental group $\pi_1(X)$ is a sub–Liouville one. It is easily seen that any complex space with countable topology, connected by means of chains of sub–Liouville subspaces, is itself sub–Liouville [DZ, (2.3)]. In particular, any torically connected variety is sub–Liouville. Thus, the question arises whether such a variety should also satisfy the assumption of Lin's Theorem, which is just Conjecture 0.2 in its weaker form.

0.4. Note that even in its weaker form the conjecture fails for non–Kählerian compact complex manifolds. An example (communicated by J. Winkelmann[2]) is a complex 3-fold which is a quotient of $SL_2(\mathbb{C})$ by a discrete cocompact subgroup (for details see Appendix below).

In this note we consider the simplest case of complex surfaces. We prove the following

0.5. Theorem. *Let S be a smooth compact complex surface. If S is torically connected, then the group $\pi_1(X)$ is almost abelian.*

0.6. The above conjecture can be also formulated for non–compact Kähler manifolds, in particular, for smooth quasi–projective varieties. To this point, in Definition 0.1 one should consider, instead of chains of rational or elliptic curves (resp. compact complex tori), the chains of non–hyperbolic quasi–projective curves[3] (resp. products

[2] we are thankful to J. Winkelmann for a kind permission to mention it here.

[3] i.e. those with abelian fundamental groups

of compact tori and factors $(\mathbb{C}^*)^m$, $m \in \mathbb{N}$). L. Haddak[4] has checked that Theorem 0.5 holds true for smooth quasi–projective surfaces. The proof is based on the Fujita classification results for open surfaces [Fu].

Proof of Theorem 0.5

In the proof we use the following two lemmas.

1.1. Lemma [Fu, Thm. 2.12; No, Lemma 1.5.C]. *Let X and Y be connected compact complex manifolds, and let $f : X \to Y$ be a dominant holomorphic mapping. Then $f_*\pi_1(X) \subset \pi_1(Y)$ is a subgroup of finite index. In particular, if the group $\pi_1(X)$ is almost abelian (resp. almost nilpotent, almost solvable), then so is $\pi_1(Y)$.*

1.2. Lemma. *Every elliptically connected smooth compact complex surface S is projective.*

Proof. If the algebraic dimension $a(S)$ were zero, then S would have only a finite number of irreducible curves [BPV, IV.6.2] and hence, it would not be elliptically connected. In the case when $a(S) = 1$, S is not elliptically connected, either. Indeed, such an S is an elliptic surface [BPV, VI.4.1], and any irreducible curve on it is contained in a fibre of the elliptic fibration $\pi : S \to B$, where B is a smooth curve (because, if an irreducible curve $E \subset S$ were not contained in a fibre of π, then one would have $E \cdot F > 0$, where F is a generic fibre of π, and hence $(E + nF)^2 > 0$ for n large enough, which would imply that S is projective [BPV, IV.5.2], a contradiction). Thus, $a(S) = 2$, and therefore, S is projective (see e.g. [BPV, IV.5.7]). □

1.3. *Proof of Theorem 0.5.* Let S be a smooth compact complex surface. Suppose that S is torically connected. Then either S itself is dominated by a complex torus, and then, by Lemma 1.1.*c*), the group $\pi_1(S)$ is almost abelian, or S is elliptically connected. Consider the latter case. Due to the bimeromorphic invariance of the fundamental group, we may assume S being minimal. S being elliptically connected, by Lemma 1.2 it is a projective surface with a rational or elliptic curve passing through each point of S. Certainly, the Kodaira dimension $k(S) \leq 1$. According to the possible values of $k(S)$, consider the following cases.

a) Let $k(S) = -\infty$. Then S is either a rational surface or a non–rational ruled surface over a curve E. In the first case, S is simply connected. In the second one, E should

[4] unpublished

be an elliptic curve. Indeed, since S is elliptically connected, E is dominated by a rational or elliptic curve $C \setminus S$, and therefore it is itself rational or elliptic. The surface S being non–rational, E must be elliptic. Thus, we have a relatively minimal ruling $\pi : S \to E$, which is a smooth fibre bundle with a fibre $\mathbb{P}^1$. From the exact sequence

$$\mathbf{1} = \pi_2(E) \to \mathbf{1} = \pi_1(\mathbb{P}^1) \to \pi_1(S) \to \pi_1(E) \to \mathbf{1}$$

we obtain $\pi_1(S) \cong \pi_1(E) \cong \mathbb{Z}^2$.

b) Let $k(S) = 0$. By the Enriques–Kodaira classification (see e.g. [GH, p.590] or [BPV, Ch. VI]), there are the following four possibilities:

* S is a K3–surface, and then $\pi_1(S) = \mathbf{1}$.

* S is an Enriques surface, and then $\pi_1(S) \cong \mathbb{Z}/2\mathbb{Z}$.

* S is an abelian surface, and then $\pi_1(S) \cong \mathbb{Z}^4$.

* S is a hyperelliptic surface, and then, being a finite non–abelian extension of $\mathbb{Z}^4$, the group $\pi_1(S)$ is almost abelian.

Note that in the last two cases S is dominated by a torus.

c) Suppose further that $k(S) = 1$. Then S is an elliptic surface [GH, p. 574]; let $\pi_S : S \to B$ be an elliptic fibration. Since S is elliptically connected, the base B is dominated by a rational or elliptic curve $C \subset S$. Hence, B itself is rational or elliptic.

Fix a dominant morphism $g : E \to C$ from a smooth elliptic curve E. Set $f = \pi_S \circ g : E \to B$, and consider the product $X = S \times_B E$. The elliptic fibration $\pi_X : X \to E$ obtained from $\pi_S : S \to B$ by the base change $f : E \to B$ has a regular section $\sigma : E \ni e \longmapsto (e, g(e)) \in X = E \times_B S$. Passing to a normalization and a minimal resolution of singularities $X' \to X$ we obtain a smooth surface X' with an elliptic fibration $\pi_{X'} : X' \to E$ and a section $\sigma' : E \to X'$. Thus, $\pi_{X'}$ has no multiple fibre. Replacing X' by a birationally equivalent model we may also assume this fibration to be relatively minimal.

If it were no singular fibre, then $\pi_{X'}$ would be a smooth morphism, and so $\chi(X') = \chi(F)\chi(E) = 0$, where F denotes the generic fibre of $\pi_{X'}$. The formula for the canonical class of a relatively minimal elliptic surface [GH, p.572] implies that $K_{X'} = \pi^*_{X'}(L)$, where L is a line bundle over E. Hence, $c_2(X') = K^2_{X'} = 0$, and by the Noether formula,

$$\chi(\mathcal{O}_{X'}) = \frac{c_1(X')^2 + c_2(X')}{12} = \frac{K^2_{X'} + \chi(X')}{12} = 0\,.$$

Thus, $\deg L = 2g(E) - 2 + \chi(O_{X'}) = 0$. Therefore, the line bundle $K_{X'}$ is trivial, and so $k(X') = 0$, in contradiction with our assumption (indeed, since X' rationally dominates S and $k(S) = 1$, we have $k(X') \geq 1$).

Hence, $\pi_{X'} : X' \to E$ is a minimal elliptic fibration with a singular fibre. By Proposition 2.1 in [FM, Ch. II], we have $\pi_1(X') \cong \pi_1(E) \cong \mathbb{Z}^2$. Since S is dominated by a surface birationally equivalent to X', whose fundamental group is isomorphic to those of X', by Lemma 1.1.c), the group $\pi_1(S)$ is almost abelian. This completes the proof of Theorem 0.5. $\square$

1.4. *Remark.* For explicite examples of smooth elliptic surfaces $\pi_S : S \to \mathbb{P}^1$ with a section $\sigma : \mathbb{P}^1 \to S$ ($\pi_S \circ \sigma = \mathrm{id}_{\mathbb{P}^1}$) of Kodaira dimension 1, one may consider a resolution of a surface in $\mathbb{P}^2 \times \mathbb{P}^1$ given as an appropriate Weierstrass fibration over $\mathbb{P}^1$ (see e.g. [Ka, Mi]). In the same way, replacing $\mathbb{P}^1$ by $\mathbb{P}^n$, one can construct examples of elliptically connected smooth projective varieties X of Kodaira dimension $k(X) = \dim X - 1$.

APPENDIX: Winkelmann's example

We present here the example mentioned in (0.4) above, of an elliptically connected smooth compact non-Kählerian 3-fold X such that the group $\pi_1(X)$ contains a non–abelian free subgroup, and hence, is not even almost solvable. We are grateful to D. Akhiezer for the detailed exposition reproduced below.

Let $\Gamma \subset SL_2(\mathbb{C})$ be a discrete cocompact subgroup. Due to Selberg's Lemma, there exists a torsion free subgroup of Γ of finite index. Replacing Γ by this subgroup we may assume Γ being torsion free. By the Borel Density Theorem (see [Ra, 5.16]), Γ is Zariski dense in $SL_2(\mathbb{C})$.

Set $X = SL_2(\mathbb{C})/\Gamma$. Thus, X is a (non–Kählerian) compact homogeneous 3-fold with the fundamental group $\pi_1(X) \cong \Gamma$. Suppose that Γ has a solvable subgroup $\Gamma' \subset \Gamma$ of finite index. Then Γ' being Zariski dense in $SL_2(\mathbb{C})$, we would have that $SL_2(\mathbb{C})$ is solvable, too. $SL_2(\mathbb{C})$ being simple, Γ cannot be almost solvable. By Tits' alternative [Ti], Γ must contain a non–abelian free subgroup.

Let $x \sim y$ mean that the points x and y in X can be connected in X by a chain of rational or elliptic curves. To show that X is elliptically connected, it is enough to check this locally. That is to say, to show the existence of a neighborhood U_0 of the point $x_0 := \mathbf{e} \cdot \Gamma$ in $X = SL_2(\mathbb{C})/\Gamma$ such that $x \sim x_0$ for any point $x \in U_0$.

Suppose we can find three one–dimensional algebraic tori (i.e. one–parametric subgroups isomorphic to $G_m \cong \mathbb{C}^*$) A_0, A_1, $A_2 \subset SL_2(\mathbb{C})$ such that

(i) $A_i/(A_i \cap \Gamma)$ is compact, and therefore, the image E_i of A_i in X is a smooth elliptic curve, $i = 0, 1, 2$;

(ii) the Lie subalgebras $\mathfrak{a}_i \subset \mathfrak{sl}_2(\mathbb{C})$, $i = 0, 1, 2$, span $\mathfrak{sl}_2(\mathbb{C})$.

Then, by (ii), any point x in a small enough neighborhood U_0 of the point $x_0 \in X$ can be presented as $a_0 a_1 a_2 \cdot x_0$ with some $a_i \in A_i$, $i = 0, 1, 2$. Hence, by (i), x and

x_0 are joined in X by the chain of elliptic curves $E_0,\ E_1' := a_0E_1,\ E_2' := a_0a_1E_2$. Indeed, we have

$$x_0,\ a_0 \cdot x_0 \in A_0 \cdot x_0 = E_0\ ,$$

$$a_0 \cdot x_0,\ a_0a_1 \cdot x_0 \in a_0A_1 \cdot x_0 = E_1'\ ,$$

$$a_0a_1 \cdot x_0,\ x = a_0a_1a_2 \cdot x_0 \in a_0a_1A_2 \cdot x_0 = E_2'\ .$$

This proves that X is elliptically connected.

To find three tori $A_0,\ A_1,\ A_2$ in $SL_2(\mathbb{C})$ with properties (i) and (ii) note that the Zariski dense torsion free subgroup $\Gamma \subset SL_2(\mathbb{C})$ must contain at least one semisimple element $\gamma \neq \mathbf{e}$. Indeed, there exists a Zariski open subset $\Omega \subset G$ such that all elements of Ω are semisimple. We may assume that $\mathbf{e}$ is not in Ω. Since Γ is Zariski dense in G, Γ can not be contained in $G \setminus \Omega$. Thus, there is a semisimple element $\gamma \neq \mathbf{e}$ in Γ.

Let $A_0 \subset SL_2(\mathbb{C})$ be a torus which contains γ, and let $v \in \mathfrak{a}_0,\ v \neq 0$. In view of the Zariski density of Γ, the orbit of v by the adjoint action of Γ on $\mathfrak{sl}_2(\mathbb{C})$ generates $\mathfrak{sl}_2(\mathbb{C})$. Hence, we can find $\gamma_1,\ \gamma_2 \in \Gamma$ such that v, $\mathrm{Ad}\,(\gamma_1) \cdot v$ and $\mathrm{Ad}\,\gamma_2 \cdot v$ form a basis of $\mathfrak{sl}_2(\mathbb{C})$. Then for A_i we can take the torus $\gamma_i A_0 \gamma_i^{-1}$ through $\gamma_i\gamma_0\gamma_i^{-1},\ i = 1, 2$. Clearly, (ii) is fulfilled and (i) follows from the fact that Γ has no torsion.

Finally, $X = G/\Gamma$ is non-Kähler. Indeed, by a theorem of Borel and Remmert (see [Ak, 3.9]), a complex compact homogeneous Kähler manifold is a product of a simply connected projective variety and a torus. Thus, it has an abelian fundamental group. Here Γ is certainly non–abelian.

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