MULTIPLICITIES OF EIGENVALUES AND TREE-WIDTH OF GRAPHS

YVES COLIN DE VERDIÈRE

ABSTRACT. Using multiplicities of eigenvalues of elliptic self-adjoint differential operators on graphs and transversality, we construct some new invariants of graphs which are related to tree-width.

1. INTRODUCTION AND RESULTS

In this paper, we present some extensions of our invariant $\mu(G)$ for a finite graph $G = (V, E)$ which was introduced in [3].

Let us recall that $\mu(G)$ is defined using multiplicities of the second eigenvalue λ_2 of real symmetric matrices A related to G ($A \in O_G$, i.e. $a_{i,j} < 0$ if $\{i, j\} \in E$ and $a_{i,j} = 0$ if $i \neq j$ and $\{i, j\} \notin E$). For such an operator, the first eigenvalue (ground-state, λ_1) is non-degenerate if G is connected (Perron-Frobenius).

Moreover, $\mu(G)$ is related to the genus of G: $\mu(G) \leq 3$ if and only if G is planar; $\mu(G) \leq 4$ genus$(G) + 3$. Recently, Lovasz and Schrijver [16] found that linklessly embeddable graphs are characterized by $\mu(G) \leq 4$.

What kind of extensions of these properties are valid for self-adjoint (complex) matrices related to G? Such Hermitian matrices are obtained if we discretize Schrödinger operators with magnetic fields using the method of finite elements.

The eigenvalues of such operators can be very degenerate even for a Schrödinger operator H with constant magnetic field in the plane

$$H = -(\partial_x - iBy)^2 - \partial_y^2 \ ;$$

the spectrum of H, whose elements are called Landau levels in physics, is the set of eigenvalues $\sigma(H) = \{E_n = (2n + 1)|B| \ |n \in \mathbb{N}\}$. The eigenspaces $\ker(H - E_n)$ are infinite dimensional.

Therefore, we cannot expect some upper bound for multiplicities in terms of the genus of G, see [10].

The main idea of our paper is to compare G with a tree: if T is a tree and A is some self-adjoint elliptic operator on T, it is always possible to reduce to the case where $A \in O_T$ by some gauge transformation (conjugation by some diagonal unitary matrix); then the Perron-Frobenius theorem applies and shows that the ground-state is non-degenerate.

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For the other eigenvalues λ, we can consider $B = (A-\lambda)^2$ whose ground-state is 0 with eigenspace $\ker(A-\lambda)$. Of course B is no more a differential operator on T, but it is a differential operator on the graph $T^{(2)}$ $(V(T^{(2)}) = V(T),\ E(T^{(2)}) = E(T) \cup \{\{i,j\},\ i,j \in V(T) | \text{distance}(i,j) = 2\})$ whose tree-width $tw(T^{(2)})$ (see section 6) is bounded by 3, if all vertices of T have degree at most 3.

In all cases, we see that some trees are involved !!

Let us give more precise definitions and the main statements of the paper:

if $G = (V,E)$ is a finite undirected graph, without loops or multiple edges, we write $N = |V|$ and we will often index the vertices from 1 to N. Let n be some integer ≥ 1 and $\mathcal{H} = \mathcal{H}_{G,n} = \oplus_{i\in V}\mathbb{C}^n$ with the canonical Hilbert space structure. We will often consider elements of $\mathcal{H}$ as functions from V to $\mathbb{C}^n$ and use the notation $\varphi(i)$ for $\varphi \in \mathcal{H}$ and $i \in V$.

An endomorphism A of $\mathcal{H}$ will be called an n-differential operator on G if $A = (A_{i,j}),\ (i,j) \in V \times V$ where the $A_{i,j}$'s are linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ and $A_{i,j} = 0$ if $i \neq j$ and $\{i,j\} \notin E$.

A is *elliptic* if the $A_{i,j}$ *'s*, $(\{i,j\} \in E)$ are invertible,

and *self-adjoint* if $\forall i,j,\ A^\star_{i,j} = A_{j,i}$ ($A^\star$ denotes the adjoint of A).

Definition 1. Let us denote by $M_{G,n}$ the set (manifold) of all elliptic self-adjoint n-differential operators on G and $M_G = M_{G,1}$. We will denote by $R_G \subset M_G$ the subset of A's with real coefficients and by $O_G \subset R_G$ the set of those $A = (a_{i,j})$ which satisfy

$$\forall \{i,j\} \in E,\ a_{i,j} < 0\ .$$

For any $A \in M_{G,n}$, let us denote by $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_{nN}(A)$ the ordered set of its eigenvalues repeated according to their multiplicities and by $\sigma(A) = \{\lambda_j(A),\ j = 1, \cdots, nN\}$ the spectrum of A.

If $\lambda \in \mathbb{R}$, let us denote by $d(\lambda, A)$ (or $d(\lambda)$ if no ambiguity is possible) the dimension of $\ker(A - \lambda \mathrm{Id})$.

The Perron-Frobenius theorem implies that, if G is connected, and $A \in O_G$, $d(\lambda_1(A)) = 1$.

Van der Holst proved in [13] the following extension to graphs of Cheng's theorem for manifolds [2]: if G is the 1-skeleton of some triangulation of the 2-sphere S^2 and $A \in O_G$, $d(\lambda_2(A), A) \leq 3$.

In the papers [18] and [19], Robertson and Seymour introduced the *tree-width* of a graph G, which we will denote by $tw(G)$.

We will use some slightly different definition which is more convenient for us :

Definition 2. If G is a finite graph, $la(G)$ is the smallest of those n for which G is a minor of $T \times K_n$ where T is some tree and K_n is the clique with n vertices (complete graph).

We have (see section 6):

Proposition 1. *$tw(G)$ and $la(G)$ satisfy the following inequalities:*

$$la(G) - 1 \leq tw(G) \leq 2la(G) - 1\ .$$

We prove the following results:

Theorem 1. *If $A \in M_{T,n}$ where T is a tree all of whose vertices have degree ≤ 3, $d(\lambda_1(A), A) \leq n$. Moreover, if $d(\lambda, A) \geq 2n+1$, there exists $\{i,j\} \in E(T)$ and*

$\varphi \in \ker(A-\lambda)$ *such that, if* T_1 *and* T_2 *are the two connected components of* T *with the edge* $\{i,j\}$ *deleted, then:*
(i) $\varphi(i) = \varphi(j) = 0$
(ii) there exists $a \in V(T_1)$ *and* $b \in V(T_2)$ *such that* $\varphi(a) \neq 0,\ \varphi(b) \neq 0$.

Definition 3. Let Z be some submanifold of $\mathrm{Herm}(\mathbb{C}^V)$ or $\mathrm{Sym}(\mathbb{R}^V)$. We recall that an eigenvalue λ of $A \in Z$, where $d(\lambda, A) = l$, is *Z-stable* if Z and $W_{l,\lambda}$ intersect transversally at A where $W_{l,\lambda}$ is the manifold of all matrices B in $\mathrm{Herm}(\mathbb{C}^V)$ or $\mathrm{Sym}(\mathbb{R}^V)$ such that $\dim\ker(B-\lambda) = l$.

Let us write $d_s(\lambda, A, Z) = d(\lambda, A)$ if $\lambda \in \sigma(A)$ is *Z-stable* and $d_s(\lambda, A, Z) = 0$ otherwise. Here $Z = M_G \subset \mathrm{Herm}(\mathbb{C}^V)$ or $Z = R_G \subset \mathrm{Sym}(\mathbb{R}^V)$.

It is possible to make the transversality condition more algebraic:

Definition 4. Let $G = (V, E)$ be a graph. If $i \in V$, define $\varepsilon_i(x) = |x_i|^2$ (quadratic form on $\mathbb{R}^V$ or Hermitian form on $\mathbb{C}^V$) and, if $\{i,j\} \in E$, define

$$\epsilon_{i,j}(x) = x_i x_j,\ \{i,j\} \in E$$

(quadratic forms) and

$$\epsilon'_{i,j}(x) = \Re(x_i \bar{x}_j),\ \varepsilon''_{i,j}(x) = \Im(x_i \bar{x}_j)$$

(Hermitian forms).

Proposition 2. *If* $F = \ker(A-\lambda)$, *the transversality condition is equivalent to the fact that the space of Hermitian forms (resp. quadratic forms) on* F *is generated over* $\mathbb{R}$ *by the restrictions to* F *of the* $|V| + 2|E|$ *forms* $\epsilon_i,\ i \in V$
and $\epsilon'_{i,j},\ \varepsilon''_{i,j},\ \{i,j\} \in E$
(resp. of the $|V| + |E|$ *forms* $\epsilon_i,\ i \in V$ *and* $\epsilon_{i,j},\ \{i,j\} \in E$*).*

We have the :

Theorem 2. *1) If* $A \in M_G$, $d_s(\lambda_1(A), A, M_G) \leq la(G)$ *and,*
$\forall k,\ d_s(\lambda_k(A), A, M_G) \leq 2la(G)$*;*
2) if $A \in R_G$, $d_s(\lambda_1(A), A, R_G) \leq la(G)$ *and,*
$\forall k,\ d_s(\lambda_k(A), A, R_G) \leq 2la(G)$.
Of course, these estimates remain valid if $A \in O_G$.

It is possible to reformulate Theorem 2 by introducing the following invariants of graphs :

Definition 5. Let us define

$$\nu_k^{\mathbb{R}}(G) = \max\{d_s(\lambda_k(A), A, R_G) | A \in R_G\}\ ,$$

$$\nu_k^{\mathbb{C}}(G)) = \max\{d_s(\lambda_k(A), A, M_G) | A \in M_G\}\ .$$

Remark : we may have defined $\mu(G)$ by the following equation :

$$\mu(G) = \max\{d_s(\lambda_2, A, O_G)\ |\ A \in O_G\}\ ,$$

and we have :

$$\mu(G) \leq \nu_2^{\mathbb{R}}(G)\ .$$

One of the main results of our paper is the following :

Theorem 3. *The invariants ν_k^K satisfy*

$$\nu_k^K(G') \le \nu_k^K(G) ,$$

for every minor G' of G.

Remark : One can show that $\nu_2^{\mathbb{R}}(K_{1,3}) = 2$ while $\nu_2^{\mathbb{C}}(K_{1,3}) = 1$.
Is it true in general that :

$$\nu_k^{\mathbb{R}}(G) \ge \nu_k^{\mathbb{C}}(G) \ ?$$

With definition 5, Theorem 2 becomes :

Theorem 4. *For $K = \mathbb{R}$ or $\mathbb{C}$, we have:*

$$(i)\ \nu_1^K(G) \le la(G) ,$$
$$(ii)\ \nu_k^K(G) \le 2\ la(G) .$$

In particular, $\mu(G) \le 2\ la(G)$.

As an easy corollary of the previous statements, we have the following characterization of forests which we prove in section 7 :

Theorem 5. *The following conditions are equivalent :*
(i) G is a forest,
(ii) $\nu_1^{\mathbb{R}}(G) = 1$,
(iii) $\nu_1^{\mathbb{C}}(G) = 1$.

It is interesting to observe that the new invariants $\nu_k^K(G)$, which we have introduced above, are not at all related to planarity :
we describe in section 7 a sequence of planar graphs G_n such that

$$\nu_1^{\mathbb{R}}(G_n) = \nu_1^{\mathbb{C}}(G_n) = n ,$$

and

$$la(G_n) = n .$$

Note *: this paper is a complete revision, including new results (in particular, concerning eigenvalues λ_k with $k > 1$) and new proofs, of my preprint* [8] *.*

Contents

2. A crucial lemma

Lemma 1. *Let $G = (V, E)$ be a finite connected graph and $\{1, 2\} \in E(G)$ such that $G' = (V, E \setminus \{1, 2\})$ is not connected. Let $A \in M_{G,n}$, $F = \ker A$, and let $r : F \to \mathbb{C}^n \oplus \mathbb{C}^n$ be given by*

$$r(\varphi) = (\varphi(1), \varphi(2)) \ .$$

Then $\dim r(F) \leq n$.

Proof. The proof is based on Green's formula. Let V_1 be the set of vertices of the connected component of 1 in G'. For any $\varphi, \psi \in F$, let us denote by φ_1, ψ_1 the truncated functions defined by $\varphi_1(i) = \varphi(i)$ if $i \in V_1$ and $\varphi_1(i) = 0$ if $i \notin V_1$; $\psi_1(i) = \psi(i)$ if $i \in V_1$ and $\psi_1(i) = 0$ if $i \notin V_1$.

Let us compute explicitly the righthand side of the following identity:

$$0 =< A\varphi_1|\psi_1 > - < \varphi_1|A\psi_1 > ,$$

using the fact that the only contribution to the scalar products is the value at vertex 1 (if $i \neq 1$, $A\varphi_1(i) = 0$ or $\psi_1(i) = 0$).

Let us compute $A\varphi_1\ (1)$ using the fact that $A\varphi\ (1) = 0$; we get

$$A\varphi_1\ (1) = -A_{1,2}(\varphi(2)) \ .$$

Hence, the expression to be evaluated reduces to

$$0 =< A_{1,2}(\varphi(2))|\psi(1) > - < \varphi(1)|A_{1,2}(\psi(2)) > ,$$

which we call Green's formula.

Let us write $B = A_{1,2}$ and denote by ω the Hermitian form on $\mathbb{C}^n \oplus \mathbb{C}^n$ given by:

$$\omega((x_1, x_2), (y_1, y_2)) = \sqrt{-1}(< Bx_2|y_1 > - < x_1|By_2 >) \ .$$

It is easy to see that ω is non degenerate, so that any isotropic subspace has dimension at most n; in particular, this is true for $r(F)$. □

There exists a modification of the above result which we state without proof:

Lemma 2. *Let $A \in M_G$, $V_1 \subset V$ and let $E_0 = \{e_j = \{a_j, b_j\} \in E,\ j = 1, \cdots, n\}$ the set of all edges $e = \{a, b\}$ of G such that $a \in V_1$ and $b \notin V_1$; let $F = \ker A$, $V_0 = \{a_j, b_j,\ j = 1, \cdots, n\}$ (we have $\#V_0 \leq 2n$).*

If $r : F \to \mathbb{C}^{V_0}$ is given by the restriction to V_0, then $\dim r(F) \leq n$.

3. Proof of Theorem 1

Let T be a finite tree with vertices of degree ≤ 3.

For each edge $\{i, j\} \in E(T)$, let us denote by $T_{i,j}$ and $T_{j,i}$ the two subtrees of T obtained by deleting the edge $\{i, j\}$ and such that $i \in V(T_{i,j})$ and $j \in V(T_{j,i})$.

Let $n \geq 1$ be some integer and $A \in M_{T,n}$.

Let us write $F = \ker A$. For each $\{i, j\} \in E(T)$, let us denote by $F_{i,j} \subset F$ the vector space of those $\varphi \in F$ which vanish at i and j and whose support lies inside $V(T_{i,j})$.

If $r = r_{i,j} : F \to \mathbb{C}^n \oplus \mathbb{C}^n$ is defined by $r(\varphi) = (\varphi(i), \varphi(j))$ it is easy to check that:

$$\ker r_{i,j} = F_{i,j} \oplus F_{j,i} \ .$$

We have then the :

Lemma 3. *If $\dim F > n$, there exists $\{i, j\} \in E(T)$ such that*

(i) $F_{i,j}$ is not 0,

(ii) degree$(i) = 3$,

(iii) the maps $\epsilon_\alpha : F_{i,j} \to \mathbb{C}^n$ defined by $\phi \to \phi(\alpha)$, where α is one of the neighbours of i with $\alpha \neq j$, are injective. In particular, $1 \leq \dim F_{i,j} \leq n$.

Corollary 1. *If $\dim F > 2n$, there exists $\{i, j\} \in E(T)$ such that*

$$\dim F_{i,j} \geq 1, \ \dim F_{j,i} \geq 1 \ .$$

Corollary 2. *If $A \geq 0$ (i.e. the quadratic form associated to A is positive), $\dim F \leq n$.*

Proof. (*Corollary 1*)

Choose $\{i, j\}$ according to the lemma and put $F_o = \ker(r_{i,j})$, we have: $\dim F_o \geq n + 1$ (lemma 1), and, because $F_o = F_{i,j} \oplus F_{j,i}$, and $\dim F_{i,j} \leq n$, $F_{j,i}$ is not trivial. □

Proof. (*Corollary 2*)

If $\dim F > n$, by (i), there exists $\varphi \in F_{i,j} \setminus 0$, and by (iii) $\varphi(\alpha) \neq 0$, where $\alpha \neq j$ is any neighboor of i. Define ψ by $\psi(k) = \varphi(k)$ for $k \in V(T_{\alpha,i})$ and $\psi(k) = 0$ otherwise. Then $\psi \in \ker A$:

$$(A\psi|\psi) = 0 \ ,$$

because $A\psi$ vanishes where ψ does not.

Set $Q(f) = (Af|f)$ and let δ be the numerical function on $V(T)$ which is defined by $\delta(i) = 1$ and $\delta(k) = 0$ if $k \neq i$.

Evaluating $Q(\psi + \epsilon\delta v)$ for $v \in \mathbb{C}^n$ and $\epsilon > 0$, we find:

$$Q(\psi + \epsilon\delta v) = 2\epsilon\Re(v|A_{i,\alpha}(\psi(\alpha))) + O(\epsilon^2) \ .$$

It is always possible to choose v such that

$$Q(\psi + \epsilon\delta v) < 0$$

for $\epsilon > 0$ small enough (because $A_{i,\alpha}$ is non singular and $\psi(\alpha) \neq 0$).

This gives a contradiction with the fact that $A \geq 0$. □

Proof. (*Lemma 3*)

Choose first arbitrarily some edge $\{i_1, j_1\}$ of T. By lemma 1, we may then assume that F_{i_1,j_1} is not trivial (otherwise permute i_1 and j_1).

It is clear that i_1 is of degree 2 or 3.

If this degree is 2 and if $\{\alpha, j_1\}$ is the set of neighbours of i_1, $\forall\varphi \in F_{i_1,j_1}$, $\varphi(\alpha) = 0$. We take then $i_2 = \alpha,\ j_2 = i_1$ and iterate.

If this degree is 3 and if $\{j_1, \alpha, \beta\}$ is the set of neighbours of i_1, then, if the map $\varphi \to \varphi(\alpha)$ is not injective on F_{i_1,j_1}, one of the spaces F_{α,i_1} or F_{β,i_1} is not trivial; we may assume that the space F_{α,i_1} is not trivial, set $i_2 = \alpha,\ j_2 = i_1$ and iterate.

This process will stop, and yields a solution. □

Reducing to $\lambda_1 = 0$ and $\lambda = 0$, Theorem 1 is an easy reformulation of corollaries 1 and 2.

4. Lagrangian compactification

4.1. **Introduction.** We want to stabilize the multiplicities of eigenvalues with respect to minors.

Let us explain what this means.

Consider the graphs $W_n = P_{2n+1} \times K_2$ (P_l is the path with l vertices) and S_n with $V(S_n) = \{1, 2, \cdots, 2n+1\}$ and $E(S_n) = \{\{i, i+1\},\ \{2j-1, 2j+1\} | i = 1, \cdots 2n,\ j = 1, \cdots, n\}$. For any $A \in M_{W_n}$, $\dim \ker A \leq 2$: if $\varphi \in \ker A$ vanishes at the two vertices in $\{a\} \times V(K_2)$ where a is an end of the path, it is easy to see that φ vanishes identically.

There exists some $A_o \in R_{S_n}$ which is ≥ 0 and $\dim \ker A_o = n + 1$. S_n is the union of n triangles and we define the quadratic form q associated with A_o by $q(x) = \sum_t (x_i + x_j + x_k)^2$ where the sum is on the n triangles $t = \{i, j, k\}$ of S_n.

It is easy to check that S_n is a minor of W_n.

In some sense, the small dimension of $\dim \ker A$, for $A \in M_{W_n}$, is not *stable* with respect to minors.

If W is a submanifold of some manifold M and $j : N \to M$ is a smooth map whose differential is injective, we will say that j is *transversal* to W at $x_o \in N$ if $j(x_o) \in W$ and

$$T_{j(x_o)} M = T_{j(x_o)} W + j'(x_o)(T_{x_o} N) \ .$$

We want ot use the basic property of *transversality*, see [12] p. 27 :

if j_ε (ε small real number) is a smooth map from N to M converging to j in the C^1-topology near x_o to j (it means that j_ε and its first order derivatives converge uniformly to j on some neighbourhood of x_o), then, for ε small enough, there exists $x(\varepsilon) \in N$ such that j_ε is transversal to W at $x(\varepsilon)$.

Let $\lambda_k = 0$ be the k-th eigenvalue having multiplicity l of $A \in M_{G'}$; we can think of this situation as follows : A belongs to the intersection of two submanifolds in $\mathrm{Herm}(\mathbb{C}^{V'})$: the manifold $j(M_{G'})$ (where j is the embedding of $M_{G'}$ into $\mathrm{Herm}(\mathbb{C}^{V'})$) and the manifold W_l of matrices whose kernel has dimension l.

If G' is obtained from G by deleting the edge $\{1, 2\}$, we will denote this by $G' = D_{1,2}(G)$ and we have $V = V'$. We can consider the maps $j_\varepsilon : M_{G'} \to \mathrm{Herm}(\mathbb{C}^{V'})$ defined by $j_\varepsilon(q) = j(q) + \varepsilon |x_1 - x_2|^2$.

$d_s(0, A_o, M_{G'}) = l$ is equivalent to "j is transversal to W_l at A_o".

As $j_\varepsilon(M_{G'}) \subset M_G$, the *basic property of transversality* shows that $\nu_k^{\mathbb{C}}(G') \leq \nu_k^{\mathbb{C}}(G)$.

If G' is obtained from G by contracting the edge $\{1,2\}$ (we shall write $G' = C_{1,2}(G)$), we need to embedd $M_{G'}$ and M_G as submanifolds into the same manifold : this is possible using appropriate Grassmann manifolds.

We will now describe the appropriate general tools in order to make stabilization ; of course, the same kind of proofs as in [3] applies, but we want to have a more *natural* setting, even if this seems to imply more geometric material !!

4.2. **Lagrangian Grassmann manifolds and quadratic forms.** In the following, X is a real N-dimensional vector space. In fact, up to obvious changes, everything extends to the complex case.

Proofs will be given only for the real case.

For applications to graphs, X will be $\mathbb{R}^V$ (or $\mathbb{C}^V$).

Let us denote by Z the space $T^\star(X) = X \oplus X^\star$, where $X^\star$ is the dual of X, endowed with the canonical symplectic form ω defined by

$$\omega((x,\xi),(x',\xi')) = \xi(x') - \xi'(x) \ .$$

We denote by $\mathcal{L}_X$ (or $\mathcal{L}$ if no ambiguity arises) the Grassmann manifold of the Lagrangian subspaces of Z. Let us recall that a Lagrangian subspace of Z is a maximal subspace which is ω-isotropic: such subspaces are of dimension N and $\mathcal{L}_X$ is a real analytic compact manifold of dimension $N(N+1)/2$, cf Duistermaat [11].

Remark: in the complex case, we need to consider the canonical Hermitian form $\omega_{\mathbb{C}}$ on $X \oplus X^\star$, where $X^\star$ is the antidual of X, given by

$$\omega_{\mathbb{C}}((x,\xi),(x',\xi')) = \sqrt{-1}(\xi(x') - \bar{\xi}'(x)) \ ,$$

and the corresponding Grassmann manifold which is of dimension N^2.

Denote by $\mathcal{Q}(X)$ the vector space of all (real) quadratic forms on X (or all Hermitian forms on X in the complex case).

Every quadratic form $q(x) = (Ax|x)_{X^\star,X}$ on X can be identified with the symmetric linear map A from X to $X^\star$ and this defines an embedding $J : \mathcal{Q}(X) \to \mathcal{L}_X$ where $J(q)$ is the graph of the linear map A.

We give the following

Definition 6. $\rho = (q,F)$ will be called a *generalized quadratic form* on X if F is a subspace of X and $q \in \mathcal{Q}(F)$.

To each generalized quadratic form $\rho = (q,F)$, we associate some Lagrangian space:

$$J(\rho) = \{(x,\xi) | x \in F \text{ and } \forall y \in F, C_q(x,y) = \xi(y)\} \ ,$$

where C_q is the symmetric bilinear form associated with q. In other words, if $B_q : F \to F^\star$ is the linear map associated with q

$$J(\rho) = \{(x,\xi) \in F \times X^\star | \ \xi_{|F} = B_q x\} \ .$$

Conversely, if L is a *Lagrangian subspace*, we associate with it a generalized quadratic form $K(L) = \rho = (q,F)$ where F is the projection of L onto X and $\forall x,y \in F$, $C_q(x,y) = \xi(y)$ where $(x,\xi) \in L$. The fact that $\xi(y)$ is independent of the choice of $(x,\xi) \in L$ comes from the fact that L is a Lagrangian: if (x,ξ) and (x,ξ') are in L, then $(0,\xi-\xi') \in L$ and, for $(y,\eta) \in L$,

$$0 = \omega((0,\xi-\xi'),(y,\eta)) = \xi(y) - \xi'(y) \ .$$

Using $\omega((x,\xi),(y,\eta))=0$, it is clear that C_q is symmetric.

It is easy to check that J and K are inverse maps.

In this way, we have a bijection of $\mathcal{L}$ with the set of all generalized quadratic forms. Since $\mathcal{L}$ is a compact manifold, we have also a compactification of $\mathcal{Q}(X)$. *The corresponding topology on generalized quadratic forms will be called the Lagrangian topology.*

Given a Lagrangian space L_0, it is possible to identify the tangent space of $\mathcal{L}$ at L_0 with the space $\mathcal{Q}(L_0)$ in the following way: there exist an open dense set in $\mathcal{L}$ of Lagrangian spaces L_1 such that $Z = L_0 \oplus L_1$ and ω identifies L_1 with the dual of L_0. Lagrangian spaces L which are close enough to L_0 can be considered as graphs of linear maps from L_0 to L_1 and these map are symmetric once L_1 is identified, using ω, with the dual $L_0^\star$ of L_0.

In this way, we get charts of $\mathcal{L}$ near L_0.

The following proposition is proved in Duistermaat [11] :

Proposition 3. *All these charts give rise to the same identification of the tangent space at L_0 with $\mathcal{Q}(L_0)$.*

4.3. **Some examples of singular limits.** We consider a family of symmetric operators from X to $X^\star$ of the following type :

$$A(\varepsilon) = A_0 + \frac{1}{\varepsilon}A_1 \ , \varepsilon \neq 0 \ .$$

Proposition 4. *In the manifold $\mathcal{L}$, the graph of $A(\varepsilon)$ has a limit for $\varepsilon \to 0$ which is the generalized quadratic form $\Phi(A_o) = (q, F)$ where $F = \ker A_1$ and q is the restriction to F of the quadratic form associated with A_0.*

Moreover the maps Φ_ε from $\mathrm{Sym}(X)$ to $\mathcal{L}$ defined by $\Phi_\varepsilon(A_o) = J(A_o + \frac{A_1}{\varepsilon})$ converge in the C^1 topology to Φ.

Proof. Let us consider the decompositions $X = U \oplus V$, with $U = \ker A_1$ and $A_1(V) \subset V^\star$, and $X^\star = U^\star \oplus V^\star$. We describe then the graph of $A(\varepsilon)$ in the following way. For $u \in U$, $v \in V$, let us write $A(\varepsilon)(u,v) = (\xi, \eta)$ with $\xi \in U^\star$ and $\eta \in V^\star$, then we have :

$$\xi = B(u,v),\ \ \eta = C(u) + D(v) + \frac{1}{\varepsilon}Gv \ ,$$

(here $B : X \to U^\star$, $C : U \to V^\star$, D and $G : V \to V^\star$ are linear maps and G is non-singular) which may be rewritten as

$$\xi = B(u,v),\ \ (G + \varepsilon D)(v) = \varepsilon(\eta - C(u)) \ .$$

For ε small, $G + \varepsilon D$ is close to G and hence invertible ; from the second equation, we obtain, for ε small enough :

$$v = \varepsilon K(\varepsilon)(\eta, u) \ ,$$

where $K(\varepsilon) : V^\star \oplus U \to V$ is linear, and inserting into the first one :

$$\xi = L(\varepsilon)(\eta, u) \ ,$$

where $L(\varepsilon) : V^\star \oplus U \to U^\star$ is linear. This shows that the graphs L_ε of $A(\varepsilon)$ admit a limit L_o as ε goes to 0 which is the graph of the map

$$(\eta, u) \to (v = 0, \xi = B(u, 0))$$

from $V^\star \oplus U$ into $V \oplus U^\star$.

It remains to check that $K(L_o)$ is the *generalized quadratic form* $\rho = (q, U)$ where q is the restriction to U of the quadratic form associated with A_o. This comes from the definition of B.

The C^1 convergence of Φ_ε to Φ comes from the fact that $D \to (G + \varepsilon D)^{-1}$ is C^1.

□

More generally, we can prove the following result:

Proposition 5. *Any meromorphic map from some open set* $\Omega \subset \mathbb{C}$ *into* $\mathrm{Sym}(X)$ *extends to a holomorphic map into* $\mathcal{L}_X$.

4.4. **Stratification of the Lagrangian Grassmann manifold.** Fix some Lagrangian space $K_0 \in \mathcal{L}$ and denote by W_l the set of all Lagrangian spaces L such that $\dim(L \cap K_0) = l$.

If $Z = X \oplus X^\star$, $K_0 = X \oplus 0$ and if $\rho = (q, F)$ is a generalized quadratic form, it is equivalent to say that $J(\rho) \in W_l$ and that $\dim \ker q = l$. this definition of W_l is the natural extension to the generalized quadratic forms of the definition of $W_{l,0}$ given in definition 3.

The following theorem is proved in Duistermaat [11] :

Theorem 6. W_l *is a (non closed) submanifold of* $\mathcal{L}$ *whose tangent space at* L *is the set of quadratic forms on* L *which vanish identically on* $L \cap L_0$.

Comments: this result is strongly related to the perturbation theory of degenerate eigenvalues. If $Z = X \oplus X^\star$ and $L_0 = X \oplus 0$, for any $A \in \mathrm{Sym}(X)$ whose graph is L_A, we have :

$$\dim \ker A = \dim(L_0 \cap L_A) \ .$$

Moreover, if this dimension is ≥ 1, eigenvalues close to 0 of $A_\varepsilon = A + \varepsilon B$ are very close to eigenvalues of the quadratic form associated to B restricted to $\ker A$.

5. Monotony for minors

In this section, we will show how to stabilize bounds on multiplicities with respect to minors, using the previous tools (transversality, Lagrangian Grassmann manifolds).

5.1. **Minors.** We will say that G' is a minor of G if G' is obtained from G by a sequence of the following three operations :

(D) which consists in deleting one edge.

(C) which consists in contracting one edge and identifying its two end vertices.

(R) which consists in deleting one isolated vertex.

It is possible to describe this in a more global way :

let us give a partition $V = \cup_{\alpha \in W} V_\alpha$ of the vertex set of G into connected subsets. $V(G')$ is a subset of W, and $E' = E(G')$ satisfies :

$$(\{\alpha, \beta\} \in E') \Rightarrow (\exists i \in V_\alpha, j \in V_\beta,\ \{i,j\} \in E) \ .$$

If we study some property (P) of graphs which is hereditary with respect to minors (for instance, the existence of an embedding in a given surface), a deep and difficult result (Wagner's conjecture, proved by Robertson and Seymour in a series of papers in JCTB) states that this property is characterized by a *finite* number of excluded minors : there exists a *finite* list of graphs such that (P) is equivalent to the property of *having no minor in this list.*

The simplest example is the characterization of forests by excluding the triangles as a minor.

Some further classical example is Kuratowski's characterization of planar graphs by excluding K_5 and $K_{3,3}$ as minors.

5.2. **Monotony.** We will now prove Theorem 3.

Proof.

We will give the proof in the real case (i.e. $K = \mathbb{R}$).

It is enough to show the result for $G' = D_{1,2}(G)$ or $G' = C_{1,2}(G)$ where $\{1,2\}$ is some edge of G. The most difficult case is the contraction of an edge since then $\mathbb{R}^{V'}$ is not equal to $\mathbb{R}^V$. We will only give the proof for this case.

Let us denote by 0 the vertex of G' which is obtained by contracting the edge $\{1,2\} \in E(G)$, by B the set of vertices of G which are adjacent to 1, but not to 2, by C the set of vertices which are adjacent to 2, but not to 1, and by D the set of vertices which are adjacent to 1 and 2.

We will define maps j_ε, $\varepsilon > 0$, from $R_{G'}$ to $R_G \subset \mathrm{Sym}(\mathbb{R}^V) \subset \mathcal{L}_V$. Let us denote by k_0 the embedding of $\mathrm{Sym}(\mathbb{R}^{V'})$ into $\mathcal{L}_V$, which associates to the quadratic form q on $\mathbb{R}^{V'}$, the Lagrangian space $J(\rho)$ where $\rho = (Q, D_{1,2})$ on $\mathbb{R}^V$ is defined in the following way : its domain $D_{1,2}$ is the subspace defined by the equation $\{x_1 = x_2\}$ and Q is defined by transferring q to $D_{1,2}$, using the bijection $\varphi : \mathbb{R}^{V'} \to D_{1,2}$ defined by : $\varphi(x_0, x_3, \cdots, x_N) = (x_0, x_0, x_3, \cdots, x_N)$. We will then prove, using proposition 4, that j_ε converges in the C^1 topology to j_0, the restriction of k_0 to $R_{G'}$.

Let $q(x_0, x_3, \cdots, x_N)$ be some quadratic form in $R_{G'}$, we associate to it $j_\varepsilon(q) = J(q_\varepsilon)$ in the following way :

let us write

$$q(x_0, x_3, \cdots, x_N) = W_0 x_0^2 + \sum_{j \sim 0} c_{0,j}(x_j - x_0)^2 + r(x_3, \cdots, x_N) \ .$$

We define :

$$q_\varepsilon(x_1, \cdots, x_N) = \frac{1}{\varepsilon}(x_1 - x_2)^2 + W_0 x_1^2 + \sum_{j \in B} c_{0,j}(x_1 - x_j)^2 + \sum_{j \in C} c_{0,j}(x_2 - x_j)^2$$
$$+ \sum_{j \in D} \frac{c_{0,j}}{2}((x_1 - x_j)^2 + (x_2 - x_j)^2) + r(x_3, \cdots, x_N) \ .$$

With this definition, $q_\varepsilon \in R_G$ and the restriction of q_ε to $D_{1,2}$ is the same as $q(x_1, x_3, \cdots, x_N)$.

Using proposition 4, we see that j_ε converges smoothly to j_0.

Let us denote by $W_l' \subset \mathrm{Sym}(\mathbb{R}^{V'})$ the set of matrices whose kernel is of dimension l. If $d_s(0, A_0, R_{G'}) = l$, by definition

$$\mathrm{Sym}(\mathbb{R}^{V'}) = T_{A_0} W_l' + T_{A_0} R_{G'} \ .$$

If $Z = k_0(\mathrm{Sym}(\mathbb{R}^{V'}))$ and $Y = j_0(R_{G'})$, and using the fact that $k_0(W_l') = W_l \cap Z$, we observe that, writing $L_0 = j_0(A_0)$, $T_{L_0} Z = T_{L_0} Y + T_{L_0}(W_l \cap Z)$.

We have then the :

Lemma 4. *Using the same notations as before, $j_0 : R_{G'} \to \mathcal{L}_V$ is transversal to W_l at A_0 : absolute and relative transversality are identical.*

Proof. We begin with the following observation:

$$T_{L_0}(\mathcal{L}_V) = T_{L_0} W_l + T_{L_0} Z \ .$$

Indeed, using proposition 3 and theorem 6 and writing $H = L_0 \cap (\mathbb{R}^V \oplus 0) = \ker A_0$, we have : $T_{L_0}(\mathcal{L}_V) = \mathcal{Q}(L_0)$, $T_{L_0} W_l = \{q \in \mathcal{Q}(L_0) | q_{|H} = 0\}$ and $T_{L_0}(Z) = \{S \circ \pi\}$

where π is the projection from L_0 to $D_{1,2}$ and $S \in \mathcal{Q}(D_{1,2})$. The observation follows then from the fact that $H \subset D_{1,2}$.

We use now the fact that

$$T_{L_0}(Z) = T_{L_0}(W_l \cap Z) + j_0'(T_{A_0}(R_{G'})) ,$$

to concluse that :

$$T_{L_0}(\mathcal{L}_V) = T_{L_0}(W_l) + j_0'(T_{A_0}(R_{G'})) .$$

□

Let q_0 be the quadratic form associated with $A_0 \in R_{G'}$ such that $\lambda_k(A_0) = 0$ and

$$d_s(0, A_o, R_{G'}) = l .$$

We have seen that j_ε converges smoothly to j_0. By the basic property of transversality, for $\varepsilon > 0$ small enough, there exists some $A_\varepsilon \in j_\varepsilon(R_{G'})$ such that $\lambda_k(A_\varepsilon) = 0$ and $d_s(0, A_\varepsilon, j_\varepsilon(R_{G'})) = l$. Then, because $j_\varepsilon(R_{G'}) \subset R_G$, $d_s(0, A_\varepsilon, R_G) = l$.

This completes the proof of theorem 3.

5.3. **Proof of Theorem 2.** First, define the product $K = G \times H$ of two graphs G and H by:

$$V(K) = V(G) \times V(H) ,$$

and

$$\{(g_1, h_1), (g_2, h_2)\} \in E(K) \text{ if and only if } g_1 = g_2 \textit{ and } \{h_1, h_2\} \in E(H)$$

$$\text{or } h_1 = h_2 \textit{ and } \{g_1, g_2\} \in E(G) .$$

Let us now prove the second part of Theorem 2 (i.e. the real case):

Proof. Let G be a graph such that $la(G) = n$; then, there exists some tree T with vertices of degree ≤ 3 such that G is a minor of $T \times K_n$. Hence

$$\nu_k^{\mathbb{R}}(G) \leq \nu_k^{\mathbb{R}}(T \times K_n) .$$

We will use the natural identification of $\mathbb{C}^{V(T \times K_n)}$ with the space of maps from $V(T)$ into $\mathbb{C}^n$.

Using this identification, every scalar elliptic self-adjoint operator A on $T \times K_n$ becomes an elliptic self-adjoint $n-$differential operator on T.

k=1: in this case, by Theorem 1, the multiplicity of the ground state of $T \times K_n$ is always $\leq n$.

k arbitrary: if $\nu_k^{\mathbb{R}}(T \times K_n) > 2n$, there exists $A \in R_{T \times K_n}$ such that $d_s(\lambda_k, A, R_{T \times K_n}) > 2n$. Applying Theorem 1 (and using the notations there), let us denote by φ_i, $i = 1, 2$ the restrictions of φ to $V(T_i)$ extended by 0 outside T_i. Then $\varphi_i \in \ker(A - \lambda_k)$, for any $\alpha \in V(T \times K_n)$, $\varepsilon_\alpha(\varphi_1, \varphi_2) = 0$ (we identify here ε_α with the associated bilinear form) because supports are disjoint, and for any $\{\alpha, \beta\} \in E(T \times K_n)$, $\varepsilon_{\alpha,\beta}(\varphi_1, \varphi_2) = 0$, because there is no edge $\{\alpha, \beta\}$ for which $\varphi_1(\alpha)\varphi_2(\beta) \neq 0$.

This shows that transversality does not hold.

□

6. Tree-widths

In [18], N. Robertson and P. Seymour give the following definition for the *tree-width* $tw(G)$ of a graph G: we define a *tree-like decomposition* of G as a pair $(T, \mathcal{X})$ where T is a tree and where $\mathcal{X} = \{X_t | t \in V(T)\}$ is a family of subsets of $V(G)$ indexed by $t \in V(T)$, such that the following conditions hold :

$$V(G) = \cup_{t \in V(T)} X_t \tag{6.1}$$

$$\forall e = \{a, b\} \in E(G),\ \exists t \text{ such that } a, b \in X_t \tag{6.2}$$

$$\forall x, y \in V(T), \forall z \in]x, y[,\ X_x \cap X_y \subset X_z \tag{6.3}$$

Here $]x, y[$ denotes the set of interior vertices of the unique path between x and y.

We define then the width of $(T, \mathcal{X})$ by :

$$w(T, \mathcal{X}) = \max |X_t| - 1\ ,$$

and

$$tw(G) = \min w(T, \mathcal{X})$$

where the min is among all tree-like decompositions of G.

On the other hand, we define some closely related invariant $la(G)$ as follows.

We define $la(G)$ as the minimum N such that G is a minor of some product $T \times K_N$ where T is a tree and K_N is the clique with N vertices.

We want to prove proposition 1.

Remark: in [14] p. 91, Hein van der Holst proves a sharper inequality:

$$tw(G) \leq la(G)\ .$$

Proof. $(tw(G) \leq 2la(G) - 1)$

By definition, there exists a tree T such that G is a minor of $T \times K_N$ where $N = la(G)$. Since tw is monotonous with respect to the minor relation we have $tw(G) \leq tw(T \times K_N)$ and it is hence enough to prove the inequality $tw(T \times K_N) \leq 2N - 1$.

We will construct a tree-like decomposition $(T, \mathcal{X})$ of $T \times K_N$: orient first T from some choosen root α and write $X_t = \{t_-, t\} \times V(K_N)$ where t_- is the unique predecessor of t (for $t \neq \alpha$) and $X_\alpha = \{\alpha\} \times V(K_N)$.

It is clear that we get in this way a tree-like decomposition of $T \times K_N$ whose width is $2N - 1$. □

Proof. $(la(G) \leq tw(G) + 1.)$

Let $(T, \mathcal{X})$, $\mathcal{X} = \{X_t | t \in T\}$, with $tw(G) = w(T, \mathcal{X}) = N - 1$, be a tree-like decomposition of G.

Let G' be the graph whose vertices are the pairs (t, x) with $t \in V(T)$, $x \in X_t$, and whose edges are of the form $\{(t, x), (t, x')\}$ with $\{x, x'\} \in E$ and of the form $\{(t, x), (t', x)\}$ where $\{t, t'\}$ is an edge of T and $x \in X_t \cap X_{t'}$.

Then G is a minor of G' : contract the edges of the form $\{(t, x), (t', x)\}$ and use the fact that $A_x = \{t | x \in X_t\}$ induces a connected subgraph of T (a reformulation of property (6.3) of a tree-like decomposition) to embed the resulting vertex set in $V(G)$. This vertex set is actually $V(G)$ by (6.1) and all edges of G are present by (6.2).

And G' is a minor of $T \times K_N$: to see this, it is enough to construct some injective map

$$j : V(G') \to V(T) \times \{1, \cdots, N\}$$

which satisfies

1) $j(t,x) = (t, n(t,x))$

2) for any $x \in X_t \cap X_{t'}$, $n(t,x) = n(t',x)$.

We construct j starting from some root α of T: we choose an arbitrary numbering of X_α and propagate it along the edges of T using the condition 2).

$\square$

7. The graphs G_n

Here, we will give an explicit family of planar graphs $G_n = (V_n, E_n)$ such that :
(i) $\nu_1^K(G_n) = n$ for $K = \mathbb{R}$ and for $K = \mathbb{C}$,
(ii) $la(G_n) = n$.

Remark : I do not know a proof of $la(G_n) \geq n$ not using spectral methods !!

G_n can be considered as the 1-skeleton of the regular subdivision of an equilateral triangle into $(n-1)^2$ small equilateral triangles. Each edge of the big triangle is divided into $n-1$ edges belonging to some small triangles. We may describe vertices of G_n by their Cartesian coordinates in the basis (e, f) of $\mathbb{R}^2 = \mathbb{C}$ ($e = (1,0)$, $f = (\frac{1}{2}, \frac{\sqrt{3}}{2})$) :

$$V_n = \{S_{m,k} = me + kf | 0 \leq m \leq n-1,\ 0 \leq k \leq n-1,\ m+k \leq n-1\} \ .$$

It is easy to check that G_n is a minor of $P_{3n} \times P_n$, where P_k is the path with k vertices ; this shows that $la(G_n) \leq n$. We will prove that $\nu_1^{\mathbb{C}}(G_n) = n$; the same kind of proof works for $K = \mathbb{R}$. By Theorem 4, it shows that $la(G_n) \geq n$.

First, for any $A \in M_{G_n}$, $\dim(\ker(A)) \leq n$: otherwise there exists a nonzero function in $\ker(A)$ which vanishes on the n vertices $S_{0,0}$, $S_{1,0}, \cdots, S_{n-1,0}$. It is clear that such a function φ vanishes identically because we can compute (using $A\varphi = 0$) by induction on k its values on the vertices $S_{.,k}$ from its values on the vertices $S_{.,k'}$, $k' < k$.

For the converse, we exhibit an element $A \in M_{G_n}$. The most simple one has real coefficients :

$$A\varphi(z) = \sum_{z' \sim z} \varphi(z') + \frac{d(z)}{2}\varphi(z) \ ,$$

where $d(z)$ is the degree of z ($d(z) = 2, 4$ or 6 depending on the position of z). It is easier to define A by his quadratic form $q_A(x) = < Ax | x >$.

Call a triangle of G_n *black* if it is of the form $(z, z+e, z+f)$. Then we have :

$$q_A(x) = \sum_{T=\{i,j,k\}} (x_i + x_j + x_k)^2 \ ,$$

where the summation is on all black triangles T.

It is easy to check the following facts :

(i) $A \in R_{G_n}$ because each edge of G_n is (in a unique way) an edge of some black triangle,

(ii) A is non negative,

(iii) the dimension of the kernel F of A is n because q_A is written as a sum of $|V(G_n)| - n$ (number of black triangles) squares of independent linear forms.

More precisely, there exists functions

$$\varphi_l \ (l = 0, \cdots, n-1)$$

where $\varphi_l(S_{i,0}) = \delta_{i,l}$ which form a basis of F.

What remains to be done is to check transversality.

Proof. We need first the

Lemma 5. *The support of φ_l consists of the $S_{m,k}$ in V_n which satisfy:*

$$l - k \leq m \leq l \ .$$

The lemma follows from the relations :

$$\varphi_l(S_{m,k}) = -(\varphi_l(S_{m,k-1}) + \varphi_l(S_{m+1,k-1}))$$

which are very close to the relations between binomial coefficients and can be solved explicitly :

$$\varphi_l(S_{m,k}) = (-1)^k C_k^{m+k-l} \ .$$

We will use proposition 2.

Let us introduce some notations. F is identified with $\mathbb{C}^{\{0,1,\cdots,n-1\}}$ using the basis φ_l. We denote by $H = \mathrm{Herm}(F)$ the set of Hermitian forms on F and introduce a filtration

$$H_0 \subset H_1 \subset \cdots \subset H_{n-1} = H$$

in the following way : H_l is the set of Hermitian matrices whose entries $h_{i,j}$ vanish for $|i - j| > l$.

We introduce the space $Q \subset \mathrm{Herm}(\mathbb{C}^{V_n})$ which is generated by the n^2 independent forms (using the notations of definition 4) :

$$\varepsilon_{m,0} = \varepsilon_{S_{m,0}}, \ m = 0, \cdots, n-1 \ ,$$

$$\varepsilon'_{m,k} = \varepsilon'_{z,z-f}, \ \varepsilon''_{m,k} = \varepsilon''_{z,z-f} \ ,$$

for $z = S_{m,k}, \ k \geq 1$.

We introduce the filtration $Q_0 \subset \cdots \subset Q_{n-1} = Q$ where Q_0 is generated by the $\varepsilon_{m,0}$, and Q_l, for $l \geq 1$, is generated by Q_0 and the $\varepsilon'_{m,k}$ and $\varepsilon''_{m,k}$ with $k \leq l$.

It is enough to prove that, if $\rho : Q \to H$ is the restriction to F, ρ is an isomorphism.

In fact, ρ is compatible with the filtrations :

$$\rho(Q_l) \subset H_l \ .$$

For example, we have:

$$\rho(\varepsilon'_{m,k})(\varphi_i, \varphi_j) = \frac{1}{2}(\varphi_i(S_{m,k})\varphi_j(S_{m,k-1}) + \varphi_i(S_{m,k-1})\varphi_j(S_{m,k})) \ ,$$

which vanishes if $|i - j| > k$ by Lemma 5.

We shall check that :

$$\rho_l : \frac{Q_l}{Q_{l-1}} \to \frac{H_l}{H_{l-1}}$$

is an isomorphism for $l \geq 0$ (setting $Q_{-1} = H_{-1} = 0$).

Both spaces have the same dimension (n if $l = 0$ and $2(n-l)$ if $l \geq 1$).

Let us compute

$$B = \rho(\epsilon'_{m,k})(\sum x_i \varphi_i, \sum x_i \varphi_i) \ ;$$

we find:

$$B = \Re(\sum x_i \bar{x}_j \varphi_i(S_{m,k}) \varphi_j(S_{m,k-1})) \ ,$$

and the product $\varphi_i(S_{m,k})\varphi_j(S_{m,k-1})$ vanishes if $|i-j| > k$ and, if $|i-j| = k$, it vanishes too unless $j = m, \ i = m+k$. This shows that ρ_l ($l > 0$) has a diagonal non-singular matrix with respect to the basis

$$\varepsilon'_{m,l}, \ \varepsilon''_{m,l}$$

for Q_l/Q_{l-1} and the basis of H_l/H_{l-1} consisting of elementary Hermitian matrices with nonzero entries at places where $|i-j| = l$.

□

Remark : we started with a (slighly) more complicated example which is gauge equivalent to this one.

Let us define a *holomorphic function* on $V(G_n)$ by the condition that the image of any direct black triangle has to be a direct equilateral triangle.

Define $B \in M_{G_n}$ by the associated Hermitian form

$$q_B(\varphi) = \sum_z |\varphi(z+f) - \varphi(z) - e^{i\pi/3}(\varphi(z+e) - \varphi(z))|^2 \ ,$$

where the summation is on the $z = S_{m,k}$ with $m + k < n - 1$. Then the kernel of B is the space of holomorphic functions on G_n and B is unitarily equivalent to A by the gauge transformation :

$$\varphi(S_{m,k}) = e^{(2m+k)\frac{2i\pi}{3}} \varphi_1(S_{m,k}) \ ,$$

i.e. $q_B(\varphi) = q_A(\varphi_1)$.

It remains to prove Theorem 5. In one direction, it follows from Theorem 2. In the other one, it follows from the fact $\nu_1^K(G_2) = 2$ and from the characterization of forests as graphes whose G_2 is not a minor.

8. Questions

Here is a selection of open questions which were presented at a CWI seminar.

1. Computability questions

Find algorithms computing $\mu(G)$ and $\nu_k^K(G)$ for a given graph G. Theoretically, there exist algorithms because everything can be expressed in terms of intersections of algebraic manifolds. Of course, it would be nice to have a computer program which computes these numbers.

2. Maximizing the gap

Let us come back to the real case. For many purposes it is interesting to have matrices A in O_Γ with a large gap (gap(A)=$\lambda_2 - \lambda_1$). The problem is to find an appropriate normalization condition which insures that the problem is well posed. Moreover, it seems reasonable to think that the multiplicity of $\lambda_2(A)$ is the largest possible if A maximizes the gap. Compare with [17] for the continuous case.

3. $\nu_k^K(G)$ and $tw(G)$

From general results by Robertson and Seymour, there exist functions $F_k^K : \mathbb{N} \to \mathbb{N}$ such that

$$tw(G) \leq F_k^K(\nu_k^K(G))$$

holds for planar graphs G. The question is to find some explicit functions $F_k^K : \mathbb{N} \to \mathbb{N}$, in other words to find explicit upper bounds for $tw(G)$ in terms of $\nu_k^K(G)$.

4. Higher dimensional complexes

The question is to extend the invariants considered in this paper to higher dimensional complexes, and find the relationship with Hodge-de Rham Laplace operators on forms.

5. Chromatic number

This problem is the most exciting: prove or disprove

$$\chi(G) \leq \mu(G) + 1 \ ,$$

where $\chi(G)$ is the chromatic number of G. This would imply the 4-color theorem and is weaker than the Hadwiger conjecture.

6.Prescribing spectras

Describe all possible spectra for $A \in O_G$ or $A \in R_G$ or $A \in M_G$. Already for special graphs like trees this problem is not yet solved. It is solved for paths and for cycles.

For the cycle on N vertices C_N, we have the following set of inequalities for any $A \in O_{C_N}$:

$$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \cdots \ .$$

It is known that for any graph G with N vertices and any subset $\sigma = \{\lambda_1 < \lambda_2 < \cdots < \lambda_N\}$ of $\mathbb{R}$, there exists $A \in O_G$ such that

$$\text{Spectrum}(A) = \sigma \ .$$

There is a general question: is it always true that the restrictions on possible spectra are given by restrictions on the multiplicities of eigenvalues ? More precisely, if there exists $A_o \in O_G$ whose spectrum is $\{\lambda_1 < \lambda_2 < \cdots < \lambda_N\}$ with multiplicity$(\lambda_i) = n_i, \ 1 \leq i \leq N$, does there exists for any given $\mu_1 < \cdots < \mu_N$ some $A \in O_G$ whose spectrum is $\{\mu_1 < \cdots < \mu_N\}$ with multiplicity$(\mu_i) = n_i, \ 1 \leq i \leq N$?

7. Lex Schrijver's question

Is it always true that

$$\mu(G) = \min_{G \text{ minor of } G'} m(G') \ ,$$

where $m(G')$ is the maximal multiplicity of the second eigenvalue for $A \in O_{G'}$?

This is true for example for planar graphs because $\mu(G) \leq 3$ if G is planar and we can use the characterisations given in [3] of graphs with $\mu(G) = 1, 2, 3$.

Same question for $\nu_k^K(G)$.

8. Bounds on multiplicities using fluxes

Given some $A \in M_G$, we may define the flux of the magnetic field through each cycle of G as a number in $\mathbb{R}/2\pi\mathbb{Z}$: if $\gamma = (a_1, a_2, a_N)$ with $\forall i \ (1 \leq i \leq N)$, $\{a_i, a_{i+1}\} \in E(G)$ $(a_{N+1} = a_1$, the flux of the magnetic field associated with A is the argument of the product $\Pi_{i=1}^N A_{i,i+1}$.

Question: is there any upper bound of $\dim(\ker A)$ for $A \in M_G$ in terms of information on the flux?

For this problem, it is interesting to compare with the paper of Lieb and Loss [15].

9. Critical graphs

Find critical graphs for $\mu(G)$ and $\nu_k^K(G)$; G is *critical* for ν if every strict minor G' of G satisfies $\nu(G') < \nu(G)$.

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Institut Fourier, UMR 5582 (UJF-CNRS), BP 74, F-38402-St Martin d'Hères Cedex
E-mail address: ycolver@fourier.ujf-grenoble.fr