EQUIVARIANT CHOW GROUPS FOR TORUS ACTIONS

Michel BRION

Institut Fourier, B. P. 74
38402 Saint-Martin d'Hères Cedex, FRANCE
e-mail: mbrion@fourier.ujf-grenoble.fr

1. Introduction and statement of the main results

Equivariant Chow groups for actions of linear algebraic groups on schemes have been introduced and studied by Edidin and Graham via an algebraic version of the Borel construction in equivariant cohomology. In the present paper, we develop further the theory of equivariant Chow groups in the case of torus actions, and we apply it to (usual) intersection theory on varieties with group actions, especially to Schubert calculus and its generalizations.

Indeed, equivariant Chow groups turn out to be modules over polynomial rings, and usual Chow groups are obtained from them by killing the action of all homogeneous polynomials. Moreover, the richer structure of equivariant Chow groups make them easier to describe. This is what we do here for toric varieties, flag varieties and more generally for projective, non-singular spherical varieties.

We consider algebraic group actions on schemes over an algebraically closed field k of arbitrary characteristic. These actions are assumed to be locally linear, i.e., the schemes that we consider are covered by invariant quasi-projective open subsets (and hence by invariant affine open subsets in the case of torus actions). This assumption is fulfilled e.g. for normal schemes.

We use Edidin and Graham's definition of equivariant Chow groups (recalled in 2.1 below) and basic properties of these groups as well, see [E-G 1]. But except for §§6.6 and 6.7, the present paper is independent of Edidin and Graham's deepest results.

Let T be a torus. Denote by M the character group of T and by S the symmetric algebra over $\mathbf{Z}$ of the abelian group M; then S is the character ring of T. For a scheme X with an action of T, let $A_*^T(X)$ be the equivariant Chow group. The equivariant Chow group of a point identifies to S; more generally, $A_*^T(X)$ is an S-module. Our first result is a presentation of this module, which is reminiscent of the definition of usual Chow groups.

Theorem (2.1). *The S-module $A_*^T(X)$ is defined by generators $[Y]$ (where $Y \subset X$ is a T-invariant subvariety) and by relations $[\mathrm{div}_Y(f)] - \chi[Y]$ (where f is a non-constant rational function on Y which is an eigenvector of T of weight χ).*

Another notion of equivariant Chow groups has been proposed by Nyenhuis, see [N1] and [N2]. He considers the abelian group generated by classes $[Y]$ as above, with relations $[\mathrm{div}_Y(f)]$ for f a non-constant, *T-invariant* rational function on Y. A draw-back of this notion is its non-invariance when X is replaced by $X \times M$ for a T-module M.

By the theorem above, Edidin and Graham's group is a quotient of Nyenhuis'. Moreover, the usual Chow group is a quotient of Edidin and Graham's. More precisely:

Corollary (2.3). *The (usual) Chow group $A_*(X)$ is the quotient of $A_*^T(X)$ by its subgroup $MA_*^T(X)$.*

As in equivariant cohomology, localization at fixed points is a powerful tool for studying equivariant Chow groups, see [E-G 2] §5 where an algebraic proof of Bott's residue formula is given. Here we give a very simple proof of the localization theorem for schemes with a torus action (see 2.3). Moreover, we obtain a description of the rational equivariant Chow ring $A_T^*(X)_{\mathbf{Q}}$ in the case where X is projective and non-singular.

Theorem (3.2), (3.3). *Let X be a projective, non-singular variety with an action of T.*
(i) The $S_{\mathbf{Q}}$-module $A_T^*(X)_{\mathbf{Q}}$ is free.
(ii) The pull-back by inclusion of fixed points $i : X^T \to X$ is an injective $S_{\mathbf{Q}}$-algebra homomorphism

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}} = S_{\mathbf{Q}} \otimes A^*(X^T)$$

which is surjective after inverting all non-zero elements of M.
(iii) The image of i^* is the intersection of the images of $i_{T'}^*$ where $T' \subset T$ is any subtorus of codimension one, and where $i_{T'} : X^T \to X^{T'}$ is the inclusion.

Properties (i) and (ii) are well-known for the rational equivariant cohomology ring of a non-singular, projective T-variety; they are deduced here from the Bialynicki-Birula decomposition (recalled in 3.1) as well as statement (iii). The formulation of the latter is related to a recent result of Goresky, Kottwitz and MacPherson concerning equivariant cohomology of a topological space X with an action of a compact torus T, see [G-K-M] Theorem (6.3): They obtain an exact sequence

$$0 \to H_T^*(X, A) \to H_T^*(X^T, A) \to H_T^*(\cup_{T'} X^{T'}, X^T, A)$$

for A in the equivariant derived category of X, under certain assumptions on X and A. This result can be precised when X contains only finitely many T-invariant points and curves: then $H_T^*(X^T, \mathbf{R})$ consists in all n-tuples of polynomial functions on the Lie algebra of T, where n is the number of fixed points. Moreover, the image of $H_T^*(X, \mathbf{R})$ in $H_T^*(X^T, \mathbf{R})$ can be described by congruences involving pairs of fixed points, see [G-K-M] Theorem (1.2.2). An algebraic version of this result is as follows.

Theorem (3.4). *Let X be a projective, non-singular variety where T acts with finitely many fixed points $x_, \ldots, x_n$ and with finitely many invariant curves. Then the image of*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}} \simeq S_{\mathbf{Q}}^n$$

is the set of all $(f_1, \ldots, f_n)$ such that $f_i \equiv f_j \pmod{\chi}$ whenever x_i and x_j are connected by an invariant curve where T acts through the character χ.

Together with Corollary 2.3, this gives a complete picture of the (usual) rational Chow ring of X, which applies e.g. to flag varieties.

To study possibly singular varieties (for example, toric varieties and Schubert varieties), we develop in §4 a notion of equivariant multiplicity at a fixed point x which is *non-degenerate*, that is, all weights of T in the tangent space T_xX are non-zero. Such a notion has already appeared for X a T-module with weights in an open half-space (see [Jo] and [B-B-M]) and, more generally, for non-singular X in work of Rossmann; see [Ro]. A notion of equivariant multiplicity is studied in [N1] when X is any T-module. Here we generalize Rossmann's results and we relate them to equivariant Chow groups, as follows.

Theorem (4.2), (4.5). *Let X be a scheme with an action of T, let $x \in X$ be a non-degenerate fixed point and let $\chi_1, \ldots, \chi_n$ be the weights of T_xX.*
(i) There exists a unique S-linear map

$$e_x : A_*^T(X) \to \frac{1}{\chi_1 \cdots \chi_n} S$$

such that $e_x[x] = 1$ and that $e_x[Y] = 0$ for any T-invariant subvariety $Y \subset X$ which does not contain x.
(ii) The point x is non-singular in X if and only if

$$e_x[X] = \frac{1}{\chi_1 \cdots \chi_n} \ .$$

In this case, we have for any T-invariant subvariety $Y \subset X$:

$$e_x[X] = \frac{[Y]_x}{\chi_1 \cdots \chi_n}$$

where $[Y]_x$ denotes pull-back of $[Y]$ by inclusion of x into X. Moreover, $[Y]_x$ is Rossmann's equivariant multiplicity.

In §5 we show that equivariant multiplicities separate points in the equivariant Chow group of any toric variety. In the case where this variety is simplicial (i.e., it has quotient singularities by finite groups), this leads to the following description of its rational equivariant Chow group.

Theorem (5.4). *Let X be a toric variety such that its fan Σ consists in simplicial cones. Then $A_*^T(X)_{\mathbf{Q}}$ is isomorphic to the space of continuous, piecewise polynomial functions on Σ.*

The corresponding statement for equivariant cohomology was proved in [B-V] by another method.

In §6, we consider schemes X with an action of a connected reductive group G. Let B be a Borel subgroup of G, let T be a maximal torus of B, and let W be the Weyl group of (G, T). Then W acts on $A_*^T(X)$ compatibly with the S-module structure. It turns out (see 6.2 and 6.3) that this action extends to an action of the ring $\mathbf{D}$ of *operators of divided differences*, generated over S by the operators

$$D_\alpha = \frac{id - s_\alpha}{\alpha}$$

where α is a simple root and $s_\alpha \in W$ is the corresponding reflection. These operators were introduced by Bernstein-Gelfand-Gelfand and Demazure for studying the cohomology ring of the flag variety G/B, see [B-G-G] and [D1], [D2]. Then Arabia described *equivariant* cohomology of G/B in terms of these operators, see [Ar] and also [B-E].

In 6.4, we prove that the $\mathbf{D}$-module $A_*^T(G/B)$ is freely generated by the class of the B-fixed point in G/B. As an application, we present in 6.5 a short proof of Kumar's smoothness criterion for Schubert varieties (see [Ku] Theorem 5.5). We also obtain an equivariant version of the Chevalley formula which describes multiplication by the class of any Schubert variety of codimension one.

On the other hand, the equivariant Chow ring $A_T^*(G/B)$ turns out to be isomorphic to $A_G^*(G/B \times G/B)$ where G acts diagonally in $G/B \times G/B$. Moreover, the latter ring is isomorphic over $\mathbf{Q}$ to the tensor product $S \otimes S$ over the subring of invariants S^W. The action of $\mathbf{D}$ is then given by $D(u \otimes v) = D(u)v$, and the class of the B-fixed point in G/B identifies to the class of the diagonal in $G/B \times G/B$, see 6.6. Therefore, to relate both descriptions of $A_T^*(G/B)_{\mathbf{Q}}$, it is enough to find a representative in $S_{\mathbf{Q}} \otimes S_{\mathbf{Q}}$ of the class of the diagonal. For G a classical group, this has been done (in a different formulation) by Fulton (see [F3] and [F4]) and by Pragaz and Ratajski (see [Pr] and [Pr-Ra]). A formula for arbitrary G has been obtained by Graham, see [Gr].

For example, in the case where $G = \mathrm{GL}_n$, we have $S = \mathbf{Z}[x_1, \ldots, x_n]$ and the generators of $\mathbf{D}$ are the operators $D_1, \ldots, D_{n-1}$ such that

$$(D_i f)(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}} .$$

The T-equivariant Chow ring of G/B is the quotient of the polynomial ring

$$\mathbf{Z}[x_1, \ldots, x_n, y_1 \ldots, y_n]$$

by the ideal generated by the

$$e_i(x_1, \ldots, x_n) - e_i(y_1, \ldots, y_n)$$

where $e_1, \ldots, e_n$ are the elementary symmetric functions. The class of the B-fixed point is represented by

$$\prod_{i+j \leq n} (x_i - y_j)$$

and the classes of the Schubert varieties are represented by the corresponding Schubert polynomials, see [F3].

Back to the general case of a scheme X with an action of G, the rational G-equivariant Chow group $A_*^G(X)_{\mathbf{Q}}$ is isomorphic to the space of W-invariants $A_*^T(X)_{\mathbf{Q}}^W$, see [E-G 1] 3.2. In particular, the rational G-equivariant Chow group of the point is isomorphic to $S_{\mathbf{Q}}^W$. This connection between G- and T-equivariant Chow groups can be precised:

Theorem (6.7). *Let X be a scheme with an action of a connected reductive group G.*
(i) The map

$$\begin{array}{rccc} \gamma: & S_{\mathbf{Q}} \otimes_{S_{\mathbf{Q}}^W} A_*^G(X)_{\mathbf{Q}} & \to & A_*^T(X)_{\mathbf{Q}} \\ & u \otimes v & \mapsto & uv \end{array}$$

is an isomorphism of modules over $S_{\mathbf{Q}}$*. Moreover, for all* $D \in \mathbf{D}$*,* $u \in S$ *and* $v \in A_*^G(X)$*, we have* $D(uv) = D(u)v$*.*
(ii) The rational Chow group $A_*(X)_{\mathbf{Q}}$ *is the quotient of the rational equivariant Chow group* $A_*^G(X)_{\mathbf{Q}}$ *by its subgroup* $S_+^W A_*^G(X)_{\mathbf{Q}}$*, where* S_+^W *is the ideal of* S^W *generated by homogeneous elementsof positive degree.*
(iii) If moreover X *is projective and non-singular, then the* $S_{\mathbf{Q}}^W$*-module* $A_*^G(X)_{\mathbf{Q}}$ *is free.*

At this point, let us point out that although several results of the present paper are algebraic versions of known statements concerning equivariant cohomology, their proofs are completely different. In fact, the analogy between equivariant cohomology and equivariant intersection theory can be misleading: for example, the map $A_*^G(X) \to A_*(X)$ is always surjective over the rationals, whereas the corresponding statement in equivariant cohomology can fail, e.g. when $X = G$ where G acts by multiplication.

The final Section 7 contains applications of the previous theory to Chow groups of spherical varieties. Recall that a normal variety X with an action of a connected reductive group G is spherical if a Borel subgroup B of G has a dense orbit in X. Then it is known that G (and even B) has finitely many orbits in X. It follows that X contains only finitely many fixed points of a maximal torus $T \subset B$. If moreover X is projective and non-singular, we describe the image of

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

by congruences involving pairs, triples or quadruples of T-fixed points (Theorem 7.3). Indeed, pairs of fixed points connected by a T-invariant curve give rise to a congruence as in Theorem 3.4. Moreover, invariant curves in a smooth, projective spherical variety are isolated or occur in a one-parameter family, which sweeps out a projective plane (containing three fixed points) or a rational ruled surface (containing four fixed points).

A presentation of the G-equivariant rational cohomology ring for a class ("regular embeddings") of spherical varieties has been obtained by Bifet, De Concini and Procesi, see [B-D-P]. For wonderful compactifications of symmetric spaces, a more precise result is due to Littelmann and Procesi, see [L-P]. Our approach is quite different; it leads to a less compact but more general description, which will be developed in a subsequent paper. Both descriptions coincide in the case of the canonical equivariant completion of a semisimple adjoint group, as shown in 7.3.

Finally, we express the action of operators of divided differences on the T-equivariant Chow group of any spherical variety X, in terms of the action of the Richardson-Springer monoid on the set of B-orbits in X, see [R-S] and [Kn].

2. A presentation of equivariant Chow groups for torus actions

2.1. *Equivariant Chow groups*

First we recall Edidin and Graham's definition of these groups, see [E-G 1] 2.2. Let X be a scheme with an action of a linear algebraic group G. Let V be a finite-dimensional rational G-module, and let $U \subset V$ a G-invariant open subset such that the quotient $U \to U/G$ exists and is a principal G-bundle. Then, for the diagonal action of G on $X \times U$, the quotient $X \times U \to (X \times U)/G$ exists and is a principal G-bundle. Set $n := \dim(X)$, $l := \dim(V)$ and $d := \dim(G)$. Define the i-th equivariant Chow group $A_i^G(X)$ as the

$(i+l-d)$th Chow group of $(X \times U)/G$, if $\operatorname{codim}(V \setminus U) > n-i$ (such a pair (V,U) always exists, see [E-G 1]). Under this assumption, $A_i^G(X)$ is independent of the choice of (V,U). Finally, set $A_*^G(X) := \oplus_i A_i^G(X)$. Each closed and invariant subvariety $Y \subset X$ defines a class $[Y]$ in $A_*^G(X)$ by setting $[Y] := [(Y \times U)/G]$.

In the case where $G = T$ is a torus, the graded abelian group $A_*^T(X)$ has the structure of an S-module, see [E-G 1]. To define it, it suffices to describe multiplication by $\chi \in M$. Let $k(\chi)$ be the one-dimensional T-module with weight χ. The first projection $U \times k(\chi) \to U$ descends to a map $(U \times k(\chi))/T \to U/T$ which defines a line bundle $L(\chi)$ over U/T. Then multiplication by χ is the first Chern class of the pull-back of $L(\chi)$ to $(X \times U)/T$. So M acts in $A_*^T(X)$ by homogeneous maps of degree -1.

Let X be a scheme with an action of T, let $Y \subset X$ be a closed and invariant subvariety, and let f be a rational function on Y which is an eigenvector of T of some weight χ. Then the divisor of f defines a class $[\operatorname{div}_Y(f)]$ in $A_*^T(X)$. Observe that the equality

$$[\operatorname{div}_Y(f)] = \chi[Y]$$

holds in the S-module $A_*^T(X)$. Indeed, f can be considered as a rational section of the pull-back of $L(\chi)$ to $(X \times U)/T$, with divisor $[\operatorname{div}_Y(f)]$. This observation leads to the following description of $A_*^T(X)$.

Theorem. *The S-module $A_*^T(X)$ is defined by generators $[Y]$ (where $Y \subset X$ is a closed, invariant subvariety) and relations $[\operatorname{div}_Y(f)] - \chi[Y]$ (where f is a non-constant rational function on Y which is an eigenvector of T of weight χ).*

2.2. *Proof of Theorem 2.1*

Fix a non-negative integer i and consider the i-th equivariant Chow group $A_i^T(X)$. We begin by constructing a T-module V and an open invariant subset $U \subset V$ such that the quotient $U \to U/T$ exists and is a principal T-bundle, and that $\operatorname{codim}(V \setminus U) > n-i$.

Choose a basis $(\chi_1, \ldots, \chi_d)$ of the free abelian group M of rank d. Set $\chi_0 := -\chi_1 - \cdots - \chi_d$. Consider the T-module k^{d+1} where T acts with weights $\chi_0, \chi_1, \ldots, \chi_d$ of multiplicity one. Choose a positive integer N and consider the T-module $V = (k^{d+1})^N$. Then $V = V_0 \times V_1 \times \cdots \times V_d$, each V_j being a N-dimensional vector space where T acts through the character χ_j. Set

$$U := \{(v_0, v_1, \ldots, v_d) \mid v_j \neq 0 \ \forall\, j\} = \prod_{j=0}^{d} (V_j \setminus \{0\}) \ .$$

Then, for each $v \in U$, the orbit $T \cdot v$ is closed in V, and the isotropy group T_v is trivial. It follows that the quotient $U \to U/T$ exists and is a principal T-bundle. Moreover, the codimension of $V \setminus U$ is N. We choose N so that $N > n-i$.

Choose bases of $V_0, V_1, \ldots, V_d$ and denote by $T_j \subset GL(V_j)$ the corresponding tori of diagonal matrices. Then T embeds diagonally into the product torus

$$T_0 \times T_1 \times \ldots \times T_d := \tilde{T} \ .$$

Moreover, $\tilde{T}$ acts on V, and U is invariant under this action. This defines an action of $\tilde{T}$ on $(X \times U)/T$.

By [F-M-S-S] Theorem 1, the abelian group $A_i((X\times U)/T)$ is generated by the classes of i-dimensional, $\tilde{T}$-invariant subvarieties of $(X\times U)/T$. Moreover, the relations between these classes are consequences of relations $[\mathrm{div}(f)]=0$ where f is a rational function on an $(i+1)$-dimensional, $\tilde{T}$-invariant subvariety of $(X\times U)/T$, which is an eigenvector of $\tilde{T}$. Translating these statements into the setting of equivariant Chow groups will lead to our result, as follows.

Let Y be an i-dimensional, $\tilde{T}$-invariant subvariety of $(X\times U)/T$. Let Z be the preimage of Y in $X\times U$. Then Z is invariant by the diagonal T-action and by the $\tilde{T}$-action on U. Therefore, Z is invariant by the action of $T\times\tilde{T}$ on $X\times U$, defined as follows: $(t,\tilde{t})(x,v)=(tx,\tilde{t}v)$. But $\tilde{T}=\prod T_j$ acts on $U=\prod(V_j\setminus\{0\})$ with finitely many orbits. So, by [F-M-S-S] Lemma 3, we have $Z=Z'\times\prod Z_j$ where Z' is a closed, T-invariant subvariety of X, and where each Z_j is a closed, T_j-invariant subvariety of $V_j\setminus\{0\}$. Denote by m_j the codimension of Z_j in $V_j\setminus\{0\}$; set $Y':=(Z'\times U)/T$. Then we claim that we have in the S-module $A_*^T(X)$:

$$[Y]=\chi_0^{m_0}\cdots\chi_d^{m_d}[Y'] \ .$$

To check this formula, recall that multiplication by χ_j in $A_*^T(X)$ is the first Chern class of the line bundle on $(X\times U)/T$, pull-back of the line bundle $L(\chi_j)$ on U/T associated to the character χ_j of T. But $L(\chi_j)$ corresponds to the Cartier divisor D_j in $U/T=(\prod(V_i\setminus\{0\}))/T$, image of $(\prod_{i\neq j}(V_i\setminus\{0\}))\times(H_j\setminus\{0\})$ where H_j is a hyperplane in V_j. It follows that

$$\chi_0^{m_0}\cdots\chi_d^{m_d}[Y']=D_0^{m_0}\cdots D_d^{m_d}[(Z'\times\prod(V_j\setminus\{0\}))/T] \ .$$

Now Z_j is the transversal intersection of m_j hyperplanes in V_j, and this proves our claim.

By the claim, the S-module $A_*^T(X)$ is generated by classes of invariant subvarieties of X. We now describe the relations between these classes. Let $Y\subset(X\times U)/T$ be a $\tilde{T}$-invariant subvariety of dimension $i+1$, and let f be a rational function on Y which is an eigenvector of $\tilde{T}$. Let $Z\subset X\times U$ be the preimage of Y. We consider f as a rational function on Z, invariant under the diagonal action of T; then f is an eigenvector of $T\times\tilde{T}$. We can write as above: $Z=Z'\times\prod Z_j$. Moreover, by [F-M-S-S] Lemma 3, we have $f=f'\prod f_j$ where $f'\in R(Z')$ is an eigenvector of T of some weight χ, each $f_j\in R(Z_j)$ is an eigenvector of T_j of some weight α_j, and $(\chi+\sum\alpha_j)|T=0$ (this expresses invariance of f under the diagonal action of T). Then the $\tilde{T}$-weight of f is $\sum\alpha_j$.

Now the preimage in Z of the cycle $\mathrm{div}_Y(f)$ is the cycle

$$\mathrm{div}_Z(f)=(\mathrm{div}_{Z'}(f')\times\prod Z_j)+(Z'\times\mathrm{div}_{\prod Z_j}(\prod f_j)) \ .$$

Denoting by m_j the codimension of Z_j in $V_j\setminus\{0\}$, we then obtain in $A_*^T(X)$:

$$[\mathrm{div}_Y(f)]=\chi_0^{m_0}\cdots\chi_d^{m_d}([\mathrm{div}_{(Z'\times U)/T}(f')-\chi[(Z'\times U)/T]) \ .$$

So $\mathrm{div}_Y(f)$ belongs to the S-module generated by our relations (the latter correspond to the case where $Z=Z'\times U$).

2.3. *Some applications.*

An immediate consequence of Theorem 2.1 is the following relation between equivariant and usual Chow groups.

Corollary 1. *For any scheme X with an action of T, the map $A_*^T(X) \to A_*(X)$ vanishes on $MA_*^T(X)$, and it induces an isomorphism*

$$A_*^T(X)/MA_*^T(X) \to A_*(X) .$$

Proof. The map $A_i^T(X) \to A_i(X)$ is restriction to a fiber of the morphism $(X \times U)/T \to U/T$, see [E-G 1]. So this map vanishes on $MA_{i+1}^T(X)$ by definition of the action of M. By Theorem 2.1, the abelian group $A_*^T(X)/MA_*^T(X)$ is defined by generators $[Y]$ (where $Y \subset X$ is a T-invariant subvariety) and relations $[\mathrm{div}_Y(f)]$ (where f is a non-constant rational function on Y which is an eigenvector of T). So the statement follows from [F-M-S-S] Theorem 1.

This result will be generalized to schemes with an action of any connected reductive group, in 6.6 below.

As another application, we give a very simple proof of Edidin and Graham's localization theorem for equivariant Chow groups (see [E-G 2] for another proof, which works more generally for higher equivariant Chow groups).

Corollary 2. *Let $i : X^T \to X$ be the inclusion of the fixed point scheme. Then the S-linear map $i_* : A_*^T(X^T) \to A_*^T(X)$ is an isomorphism after inverting all non-zero elements of M.*

Proof. Let $Y \subset X$ be a closed, T-stable subvariety of positive dimension. If Y is not contained in X^T, then there exists a rational function f on Y which is an eigenvector of T for a non-zero weight χ. Indeed, Y contains a non-empty open, affine, T-stable subset U. The algebra of regular functions on U is a non-trivial rational T-module, and hence it contains an eigenvector of T with a non-zero weight. So we have after inverting χ: $[Y] = \chi^{-1}[\mathrm{div}_Y(f)]$. By induction on the dimension of Y, we obtain that i_* is surjective after inverting all non-zero elements of M.

For injectivity, assume that X is not fixed pointwise by T. Then, as before, we can find an irreducible component Y of X, and a non-constant rational function f on Y which is an eigenvector of T of non-zero weight, say χ. Denote by D the union of the support of the divisor of f in Y, and of the irreducible components of X which do not contain Y. Observe that D contains all fixed points in X. Denote by $p : (X \times U)/T \to U/T$ the projection, and consider f as a rational section of $p^*L(\chi)$. More precisely, consider the pseudo-divisor (see [F1] 2.2)

$$p^*L(\chi), (D \times U)/T, f)$$

on $(X \times U)/T$. It defines a homogeneous map of degree -1

$$j^* : A_*^T(X) \to A_*^T(D) .$$

Moreover, denoting by $j : D \to X$ the inclusion, the composition $j^* \circ j_*$ is multiplication by χ. Therefore, the map

$$j_* : A_*^T(D) \to A_*^T(X)$$

is injective after inverting χ. We conclude by Noetherian induction.

3. The equivariant Chow ring of a projective, non-singular variety

3.1. *The Bialynicki-Birula decomposition*

For a scheme X with an action of T, we denote by X^T its fixed point subscheme. Similarly, for a one-parameter subgroup λ of T, we have the fixed point subscheme $X^\lambda \supset X^T$. We call λ *generic* if $X^\lambda = X^T$. It follows easily from Sumihiro's theorem that generic one-parameter subgroups always exist.

For a subvariety Y of X^λ, we define subsets $X_+(Y,\lambda)$ and $X_-(Y,\lambda)$ by

$$X_\pm(Y,\lambda) = \{x \in X \mid \lim_{t\to 0} \lambda(t^{\pm 1})x \text{ exists and is in } Y\} \; .$$

We denote by $p_\pm : X_\pm(Y,\lambda) \to Y$ the maps $x \mapsto \lim_{t\to 0} \lambda(t^\pm)x$. Then $X_+(Y,\lambda)$, $X_-(Y,\lambda)$ are locally closed, T-invariant subvarieties of X, and p_+, p_- are T-equivariant morphisms.

Observe that any complete T-variety X is the disjoint union of locally closed subvarieties $X_+(Y,\lambda)$ where λ is a fixed generic one-parameter subgroup, and where Y runs over all connected components of X^T. If moreover X is non-singular, a much stronger result holds, due to Bialynicki-Birula (see [Bi1], [Bi2], [Bi3]). To state it, we introduce the following notation. For $x \in X^\lambda$, the tangent space T_xX is a module over the multiplicative group, via λ. We denote by $(T_xX)_0$ (resp. $(T_xX)_+$, $(T_xX)_-$) the sum of the weight subspaces of T_xX with zero (resp. positive, negative) weights. Then

$$T_xX = (T_xX)_- \oplus (T_xX)_0 \oplus (T_xX)_+ \; .$$

Theorem. *Let X be a complete, non-singular T-variety, and let λ be a generic one-parameter subgroup.*
(i) The fixed point scheme $X^\lambda = X^T$ is non-singular, and its tangent space at x is $(T_xX)_0$.
(ii) For any component Y of X^T, the maps $p_\pm : X_\pm(Y,\lambda) \to Y$ make $X_\pm(Y,\lambda)$ into an equivariant vector bundle over Y whose fiber at x is the T-module $(T_xX)_+$.

In particular, the plus stratum $X_+(\lambda, Y)$ and the minus stratum $X_-(\lambda, Y)$ are non-singular, and they intersect transversally along Y.

3.2. *Attaching strata*

Definition. A scheme X with an action of T is called *filtrable* if it satisfies both following conditions:
(i) X is the union of its plus strata $X_+(Y,\lambda)$ for some generic one-parameter subgroup λ of T.
(ii) There is an indexing $\Sigma_1, \ldots, \Sigma_n$ of the set of strata such that, for all indices i, the closure $\overline{\Sigma_i}$ is contained in the union of Σ_j with $j \geq i$.

By [Bi3], any projective scheme is filtrable. We aim at an inductive description of the equivariant Chow ring of any non-singular, filtrable scheme X with an action of T. By assumption, there exists a closed stratum $F = X_+(Y,\lambda)$, and moreover $X \setminus F$ is filtrable. We describe the ring $A_T^*(X)$ in terms of $A_T^*(F)$ and of $A_T^*(X \setminus F)$.

Denote by $j_F : F \to X$ and by $j_U : U := X \setminus F \to X$ the inclusion maps. Let d be the codimension of F in X, let N be the normal bundle to F in X, and let $c_d^T(N) \in A_T^d(F)$ be its top equivariant Chern class. Finally, let

$$q : A_T^*(F) \to A_T^*(F)/(c_d^T(N))$$

be the quotient map by the ideal generated by $c_d^T(N)$.

Proposition. *(i) Multiplication by $c_d^T(N)$ is injective in $A_T^*(F)_{\mathbf{Q}}$.*
(ii) The maps $j_{F*} : A_*^T(F) \to A_*^T(X)$ and $j_U^* : A_*^T(X) \to A_*^T(U)$ fit up into an exact sequence

$$0 \to A_*^T(F) \to A_*^T(X) \to A_*^T(U) \to 0$$

over the rationals.
(iii) For any $\beta \in A_T^*(U)$, choose $\gamma \in A_T^*(X)$ such that $j_U^*\gamma = \beta$. Then the element $q(j_F^*(\gamma))$ of $A_Y^*(F)/(c_d^T(N))$ depends only on β, and the map

$$\begin{array}{cccc} \pi : & A_T^*(U) & \to & A_T^*(F)/(c_d^T(N)) \\ & \beta & \mapsto & q(j_F^*(\gamma)) \end{array}$$

is a S-algebra homomorphism.
(iv) The algebra homomorphisms

$$(j_F^*, j_U^*) : A_T^*(X) \to A_T^*(F) \times A_T^*(U)$$

and

$$q - \pi : A_T^*(F) \times A_T^*(U) \to A_T^*(F)/(c_d^T(N))$$

define an exact sequence over $\mathbf{Q}$:

$$0 \to A_T^*(X) \to A_T^*(F) \times A_T^*(U) \to A_T^*(F)/(c_d^T(N)) \to 0 \ .$$

If moreover the abelian group $A_*(Y)$ is torsion-free, then all statements above hold over $\mathbf{Z}$.

Proof. (i) Because F is an equivariant vector bundle over Y, we have $A_T^*(F) = A_T^*(Y)$. Moreover, $A_T^*(Y) = S \otimes A^*(Y)$ by [E-G 2] Proposition 13 (alternatively, this follows from Theorem 2.1). Under the resulting identification of $A_T^*(F)$ to $S \otimes A^*(Y)$, $c_d^T(N)$ goes to

$$\prod_{i=1}^{d} (\chi_i \otimes 1 + 1 \otimes \alpha_i)$$

where $\chi_1, \ldots, \chi_d$ are the characters of $N_x = (T_xX)_-$ for any $x \in Y$, and where $\alpha_1, \ldots, \alpha_d$ are the corresponding Chern roots. Therefore, we have

$$c_d^T(N) = (\prod_{i=1}^{d} \chi_i) \otimes 1 + \nu$$

for some nilpotent $\nu \in S \otimes A^*(Y)$. It follows that $c_d^T(N)$ is not a zero-divisor in $S \otimes A^*(Y)_{\mathbf{Q}}$.

(ii) It suffices to check that j_{F*} is injective over $\mathbf{Q}$. But composition

$$j_F^* \circ j_{F*} : A_*^T(F) \to A_{*+d}^T(F)$$

is multiplication by $c_d^T(N)$.

(iii) Let γ_1, γ_2 in $A_*^T(F)$ such that $j_U^*(\gamma_1) = j_U^*(\gamma_2) = \beta$. Then $\gamma_1 - \gamma_2 \in j_{F*}A_*^T(F)$ and hence $j_F^*(\gamma_1) - j_F^*(\gamma_2) \in (c_d^T(N))$.

(iv) By construction, we have $(q-\pi)\circ(j_F^*, j_U^*) = 0$, (j_F^*, j_U^*) is injective over $\mathbf{Q}$, and $q-\pi$ is surjective. Let $(\alpha, \beta) \in A_T^*(F) \times A_T^*(U)$ be such that $(q-\pi)(\alpha,\beta) = 0$. Write $\beta = j_U^*(\gamma)$ for some $\gamma \in A_T^*(X)$. Then $q(\alpha) = \pi(\beta) = q(j_F^*(\gamma))$ and hence $\alpha = j_F^*(\gamma + j_{F*}(\delta))$ for some $\delta \in A_T^*(F)$. So $\beta = j_U^*(\gamma + j_{F*}(\delta))$ and (α, β) is in the image of (j_F^*, j_U^*).

Corollary 1. *Let X be a non-singular, filtrable variety with an action of T.*
(i) The inclusion map $i : X^T \to X$ induces an injective S-algebra homomorphism

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

which is surjective over the quotient field of S.
(ii) The $S_{\mathbf{Q}}$-module $A_*^T(X)_{\mathbf{Q}}$ is free. If moreover the abelian group $A_*(X^T)$ is free, then the S-module $A_*^T(X)$ is free.
(iii) If X^T consists in finitely many points $x_1, \ldots, x_n$, then, for any generic one-parameter subgroup λ, the S-module $A_*^T(X)$ is freely generated by the classes of the closures of strata $X_+(\lambda, x_i)$ for $1 \le i \le n$.

Proof. Let F and Y be as above. Recall that $A_*^T(F) = A_*^T(Y) = S \otimes A_*(Y)$. Combined with statement (ii) in the Proposition, this implies our corollary, arguing by induction over the number of strata.

Consider now a non-singular *complex* algebraic variety X with an action of a complex algebraic torus T. Then there is a cycle map $cl_X : A^*(X) \to H^*(X, \mathbf{Z})$ which doubles the degree. Similarly, there is a cycle map $cl_X^T : A_T^*(X) \to H_T^*(X, \mathbf{Z})$ where $H_T^*(X, \mathbf{Z})$ denotes equivariant cohomology with integral coefficients, see [E-G 1] 2.8.

Corollary 2. *Let X be a non-singular, filtrable complex algebraic variety with an action of T. If the cycle map*

$$cl_{X^T} : A^*(X^T)_{\mathbf{Q}} \to H^*(X^T, \mathbf{Q})$$

is an isomorphism, then both cycle maps

$$cl_X^T : A_T^*(X)_{\mathbf{Q}} \to H_T^*(X, \mathbf{Q})$$

and

$$cl_X : A^*(X)_{\mathbf{Q}} \to H^*(X, \mathbf{Q})$$

are isomorphisms as well.

Proof. Observe that our inductive description of the equivariant Chow ring carries over to equivariant cohomology without any change. Moreover, our assumption implies that the cycle maps cl_Σ^T and cl_Σ are isomorphisms for any stratum Σ. Arguing by induction over the number of strata, it follows that cl_X^T is an isomorphism, and also that X has no rational cohomology in odd degree. By [G-K-M] Theorem 14.1, the $S_{\mathbf{Q}}$-module $H_T^*(X,\mathbf{Q})$ is then free, and the map

$$H_T^*(X,\mathbf{Q})/MH_T^*(X,\mathbf{Q}) \to H^*(X,\mathbf{Q})$$

is an isomorphism. This implies that cl_X is an isomorphism.

3.3. *The image of restriction to fixed points*

Let X be a non-singular, filtrable T-variety and let $i : X^T \to X$ be the inclusion map. For any subtorus $T' \subset T$, let $i_{T'} : X^T \to X^{T'}$ be the inclusion map. Because i factors through $i_{T'}$, the image of $i^* : A_T^*(X) \to A_T^*(X^T)$ is contained in the image of $i_{T'}^*$. This observation leads to the following description of the image of i^* over the rationals.

Theorem. *Let X be a non-singular, filtrable variety with an action of T. Then the image of*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

is the intersection of the images of

$$i_{T'}^* : A_T^*(X^{T'})_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

where T' runs over all subtori of codimension one in T.

Proof: By induction over the number of strata. If X is a unique stratum, then X is isomorphic to the total space of a T-equivariant vector bundle over its fixed point set. Therefore, i^* and $i_{T'}^*$ are surjective. In the general case, let $F \subset X$ be a closed stratum and let Y be the fixed point set in F. Let d, N and $c_d^T(N)$ be as in 3.2. Recall that $A_T^*(F)$ is isomorphic to $A_T^*(Y) = S \otimes A^*(Y)$ via j_Y^*. Under this isomorphism, $c_d^T(N)$ goes to $\prod_{i=1}^d (\chi_i \otimes 1 + 1 \otimes \alpha_i)$ where $\chi_1, \ldots, \chi_d$ are the weights of $(T_xX)_-$ for $x \in Y$, and where $\alpha_1, \ldots, \alpha_d$ are the corresponding Chern roots of N. Decompose this product as

$$\prod_{i=1}^d (\chi_i \otimes 1 + 1 \otimes \alpha_i) = \prod_{\chi} c_\chi$$

where χ runs over all characters of T which are primitive (i.e., not divisible in M), and where c_χ denotes the product of the $\chi_i \otimes 1 + 1 \otimes \alpha_i$ such that χ_i is a multiple of χ. For χ as above, the kernel of χ is a subtorus of codimension one of T, and c_χ is the top equivariant Chern class of the normal bundle to $F^{\ker(\chi)}$ in $X^{\ker(\chi)}$. Observe that any subtorus of codimension one of T can be written as $\ker(\chi)$ for a primitive character χ of T, uniquely determined up to sign.

Let $\gamma \in A_T^*(X^T)_{\mathbf{Q}}$ be in the image of $i_{T'}^*$ for all subtori T' of codimension one. By the induction hypothesis applied to U, the class $j_{U^T}^*\gamma$ is in the image of the map $A_T^*(U)_{\mathbf{Q}} \to A_T^*(U^T)_{\mathbf{Q}}$. Recall that $U^T = X^T \setminus Y$. Because $j_U^* : A_T^*(X) \to A_T^*(U)$ is surjective, we can find $\alpha \in A_T^*(Y)_{\mathbf{Q}}$ and $\beta \in A_T^*(X)_{\mathbf{Q}}$ such that $\gamma = \alpha + i^*\beta$.

Let χ be a primitive character of T. Then $\alpha = \gamma - i^*\beta$ is in the image of $i_{\ker(\chi)}^*$. By Proposition 3.2 applied to the component of $X^{\ker(\chi)}$ which contains Y, it follows that

α is divisible by c_χ in $A_T^*(F^{\ker(\chi)})_{\mathbf{Q}} = A_T^*(Y)_{\mathbf{Q}}$. Hence, by the lemma below (applied to $A = A^*(Y)$), α is divisible by $\prod_\chi c_\chi = c_d^T(N)$. So $\gamma - i^*\beta$ is in $c_d^T(N)A_T^*(Y)_{\mathbf{Q}} = i^*(j_{F*}A_*^T(F)_{\mathbf{Q}})$ and γ is in $i^*A_T^*(X)_{\mathbf{Q}}$.

Lemma. *Let $A = \oplus_{n=0}^\infty A_n$ be a graded ring with $A_0 = \mathbf{Q}$ and $A_n = 0$ for n large enough. Set $B := A \otimes S_{\mathbf{Q}}$ and endow B with the grading $B = \oplus_{n=0}^\infty A_n \otimes S_{\mathbf{Q}}$. Let f, g, h in B such that:*

(i) f is divisible by g and by h in B, and

(ii) g_0 and h_0 are non-zero and coprime in $S_{\mathbf{Q}}$.

Then f is divisible by gh in B.

Proof. Let N be the smallest integer such that $A_n = 0$ for any $n > N$. We argue by induction over N, the case where $N = 0$ being trivial. We have $f = gu = hv$ for some u, v in B. By the induction hypothesis applied to the ring A/A_N, there exists $w \in B$ such that $f - ghw$ is in B_N. We may assume that the component of degree N of w is zero. For homogeneous components, we have $f_N = (gu)_N = (hv)_N$ and moreover $u_n = (hw)_n$, $v_n = (gw)_n$ for all $n < N$. It follows easily that

$$g_0u_N + h_0(g_1w_{N-1} + g_2w_{N-2} + \cdots + g_Nw_0) = h_0v_N + g_0(h_1w_{N-1} + \cdots + h_Nw_0)$$

an equation in the free $S_{\mathbf{Q}}$-module B_N. Because g_0 and h_0 are coprime, this implies

$$u_N = h_0w_N + h_1w_{N-1} + \cdots + h_Nw_0$$

for some w_N in B_N. Then $f = gh(w + w_N)$.

3.4. *A structure theorem for the equivariant Chow ring*

We will deduce from Theorem 3.3 a complete description of the ring $A_T^*(X)_{\mathbf{Q}}$ in the case where X is projective, non-singular and contains finitely many invariant points and curves. Other applications of Theorem 3.3 will be given in §7.

Theorem. *Let X be a non-singular, filtrable variety where T acts with finitely many fixed points $x_1, \ldots, x_n$ and with finitely many invariant curves. Then the image of*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

is the set of all $(f_1, \ldots, f_n) \in S_{\mathbf{Q}}^n$ such that $f_i \equiv f_j \pmod{\chi}$ whenever x_i, x_j are connected by an invariant curve where T acts through the weight χ. If moreover all such weights χ are primitive in M, then the statement holds over the integers.

Proof. Let π be a primitive character of T. Then the space $X^{\ker(\pi)}$ is at most one-dimensional, because X contains finitely many invariant curves. Moreover, $X^{\ker(\pi)}$ is non-singular, and hence it consists in a disjoint union of points and non-singular, irreducible curves; let C be such a curve.

If C contains a unique fixed point x, then $i_x^* : A_T^*(C) \to A_T^*(x) = S$ is an isomorphism. Otherwise, C is isomorphic to projective line. It follows that C contains two distinct fixed points x, y. Moreover, the image of

$$i_C^* : A_T^*(C) \to A_T^*(C^T) = S \times S$$

consists in all pairs (f, g) such that $f \equiv g \pmod{\chi}$ where T acts on C through the weight χ (a multiple of π). Indeed, this image is the S-module generated by $i_C^*[x] = (\chi, 0)$, $i_C^*[y] = (0, -\chi)$ and $i_C^*[C] = (1, 1)$. Now apply Theorem 3.3 to obtain the statement over the rationals.

In the case where all such χ are primitive, the proof of Theorem 3.3 adapts to yield the statement over the integers. Indeed, the lemma is replaced by the following observation: if $u \in S$ is divisible (in S) by pairwise distinct primitive characters $\chi_1, \ldots, \chi_n$, then u is divisible (in S) by $\prod_{i=1}^n \chi_i$.

4. Equivariant multiplicities at non-degenerate fixed points

4.1. *Non-degenerate fixed points*

Let X be a scheme with an action of T. Call a fixed point $x \in X$ *non-degenerate* if the tangent space $T_x X$ contains no non-zero fixed point. Equivalently, 0 is not a weight of the T-module $T_x X$. The set of weights (with multiplicities) of this module will be called the weights of x.

Observe that a fixed point in a non-singular T-variety is non-degenerate if and only if it is isolated; indeed, we have $T_x(X^T) = (T_x X)_0$.

Proposition. *Let $x \in X$ be a non-degenerate fixed point with weights $\chi_1, \ldots, \chi_n$. Then there exists an open affine T-invariant neighborhood U of x such that:*
(i) The map $i_* : A_*^T(x) = S \to A_*^T(U)$ is injective, where i is inclusion of x in X.
(ii) The image of i_* contains $\chi_1 \cdots \chi_n A_*^T(U)$.

Proof. We may assume that X is affine. Then there exist regular functions $f_1, \ldots, f_n$ on X which are eigenvectors of T of weights $\chi_1, \ldots, \chi_n$, such that $f_1, \ldots, f_n$ vanish at x and that the differentials $df_1(x), \ldots, df_n(x)$ are a basis of $T_x X$. We can assume furthermore that x is the unique common zero to $f_1, \ldots, f_n$; then x is the unique fixed point in X.

For any T-invariant subvariety $Y \subset X$, denote by $j(Y)$ the smallest integer j such that $f_j \neq 0$ on Y. We claim that

$$\Big(\prod_{j \geq j(Y)} \chi_j\Big)[Y] \in i_* A_*^T(X) \ .$$

This claim is checked by induction on the dimension of Y. Indeed, if this dimension is zero, then $Y = \{x\}$ and there is nothing to prove. For positive-dimensional Y, we have

$$\chi_{j(Y)}[Y] = [\mathrm{div}_Y(f_j)]$$

and the latter is a combination of T-invariant subvarieties Z with $\dim(Z) = \dim(Y) - 1$ and $j(Z) > j(Y)$. So

$$\Big(\prod_{j > j(Y)} \chi_j\Big)[\mathrm{div}_Y(f_j)] \in i_* A_*^T(X)$$

by the induction hypothesis.

Now assertion (ii) follows from the claim, whereas (i) is a consequence of the localization theorem.

4.2. *Equivariant multiplicities*

Let $\mathcal{Q}$ be the quotient field of S; let $N = \mathrm{Hom}(M, \mathbf{Z})$ be the dual lattice to M, and let $N_{\mathbf{Q}} := N \otimes_{\mathbf{Z}} \mathbf{Q}$ be the associated rational vector space. Then $\mathcal{Q}$ is the field of rational functions on $N_{\mathbf{Q}}$ with rational coefficients.

Theorem. *Let $x \in X$ be a non-degenerate fixed point with weights $\chi_1, \ldots, \chi_n$.*
(i) There exists a unique S-linear map $e_{x,X} : A_*^T(X) \to \mathcal{Q}$ such that $e_{x,X}[x] = 1$ and that $e_{x,X}[Y] = 0$ for any T-invariant subvariety $Y \subset X$ which does not contain x. Moreover, the image of $e_{x,X}$ is contained in $(1/\chi_1 \cdots \chi_n)S$.
(ii) For any T-invariant subvariety $Y \subset X$, the rational function $e_{x,X}[Y]$ is homogeneous of degree $-\dim(Y)$ and it coincides with $e_{x,Y}[Y]$.
(iii) The point x is non-singular in X if and only if $(\chi_1 \cdots \chi_n)\, e_{x,X}X = 1$.

Proof. (i) Let $U \subset X$ as in Proposition 4.1. Denote by $j : U \to X$ the inclusion. By this proposition, for any $\alpha \in A_*^T(X)$, there exists a unique $\beta \in S$ such that

$$(\chi_1 \cdots \chi_n) j^* \alpha = i_* \beta \ .$$

Define $e_{x,X}$ by

$$e_{x,X}(\alpha) := \frac{\beta}{\chi_1 \cdots \chi_n} \ .$$

Then $e_{x,X}$ has the required properties.

Uniqueness of $e_{x,X}$ follows from the localization theorem.

(ii) The assertion on the degree of $e_{x,X}Y$ follows from the definition of $e_{x,X}$ given above. Denote by $i_Y : Y \to X$ the inclusion map. Then it follows from (i) that composition

$$e_{x,X} \circ (i_Y)_* : A_*^T(X) \to \mathcal{Q}$$

coincides with $e_{x,Y}$.

(iii) If $e_{x,X}X = 1/\chi_1 \cdots \chi_n$ then $\dim_x(X) = n$ by (ii). But $\dim(T_xX) = n$ and hence x is regular. Conversely, if x is regular, then we can find rational functions $f_1, \ldots, f_n$ which are defined at x, eigenvectors of T of weights $\chi_1, \ldots, \chi_n$ and such that the divisors $\mathrm{div}(f_1), \ldots, \mathrm{div}(f_n)$ intersect transversally at x. Then we have

$$\chi_1 \cdots \chi_n [U] = [x]$$

in $A_*^T(U)$, and hence $\chi_1 \cdots \chi_n e_{x,X}[X] = 1$.

For any T-invariant subvariety $Y \subset X$, we set $e_{x,X}[Y] := e_x[Y]$ (this makes sense because of (ii)) and we call $e_x[Y]$ the *equivariant multiplicity* of Y at x.

Corollary. *Let X be a scheme with an action of T such that all fixed points in X are non-degenerate. Then, for any $\alpha \in A_*^T(X)$, we have in $A_*^T(X) \otimes_S \mathcal{Q}$:*

$$\alpha = \sum_{x \in X^T} e_x(\alpha)[x] \ .$$

Proof. By the localization theorem, i_* is surjective over $\mathcal{Q}$. Therefore, we may assume that $\alpha = [x]$ for some $x \in X^T$. Then the statement follows from (i) above.

4.3. *The behaviour of equivariant multiplicities under proper morphisms*

The following easy result will be used in §6 to compute equivariant multiplicities of Schubert varieties.

Proposition. *Let X, X' be schemes with an action of T, and let $\pi : X' \to X$ be a proper, surjective, T-equivariant morphism of finite degree d. Let $x \in X$ be a non-degenerate fixed point such that all fixed points in the fiber $\pi^{-1}(x)$ are non-degenerate in X'. Then we have*

$$e_x[X] = \frac{1}{d} \sum_{x' \in X'^T, \pi(x')=x} e_{x'}[X'] \ .$$

Proof. We may replace X by any T-invariant neighborhood of x. Therefore, we may assume that x is the unique fixed point in X. Then all fixed points in X' map to x by π. So we have by corollary 4.2:

$$[X'] = \sum_{x' \in X'^T} e_{x'}[X'][x'] \ .$$

Applying π_* to this equation, we obtain

$$d[X] = (\sum_{x' \in X'^T} e_{x'}[X'])[x] \ .$$

On the other hand, we have $[X] = (e_x[X])[x]$. Together with Proposition 4.1 (i), this gives our formula.

4.4. *The case of an attractive fixed point.*

Let X be a scheme with an action of T. Call a fixed point $x \in X$ *attractive* if all weights in the tangent space T_xX are contained in some open half-space of $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$. We denote by $\chi_1, \ldots, \chi_n$ these weights, and we set

$$\sigma_x := \{\lambda \in N_{\mathbf{R}} \mid \langle \lambda, \chi_i \rangle \geq 0 \text{ for } 1 \leq i \leq n\} \ .$$

Then σ_x is a rational polyhedral convex cone in $N_{\mathbf{R}}$ with a non-empty interior σ_x^0.

Proposition. *Let $x \in X$ be an attractive fixed point with weights $\chi_1, \ldots, \chi_n$ and let $\lambda \in \sigma_x^0$.*
(i) The set $X_x := X_+(\lambda, x)$ is independent of λ, and this set is the unique affine, T-invariant open neighborhood of x in X.
(ii) The rational function $e_x[X]$ is defined at λ, and its value is the multiplicity of the algebra of regular functions on X_x graded via the action of λ.

Recall that the multiplicity of a finitely generated, graded k-algebra

$$A = \oplus_{n=0}^{\infty} A_n$$

is the unique rational number e such that

$$\sum_{m=0}^{n} \dim_k(A_m) \sim_{n\to\infty} e\frac{n^d}{d!}$$

where d is the dimension of A, see [Sm]. For other approaches to equivariant multiplicity in this setting, see [Jo] and [B-B-M].

Proof. (i) Let U be an open T-invariant affine neighborhood of x in X. Because all weights of T_xX lie in the open half-space where $\lambda > 0$, the set U contains $X_+(\lambda, x)$ as an open T-invariant subset. Moreover, any T-invariant regular function on $X_+(\lambda, x)$ is constant, and hence the same holds for U. It follows that x is the unique closed orbit in U. If $U \neq X_+(\lambda, x)$ then the closed, T-invariant subset $U \setminus X_+(\lambda, x)$ contains a closed orbit and does not contain x, a contradiction.

(ii) Let $Y \subset X_x$ be a T-invariant subvariety. Denote by $(\varepsilon_x Y)(\lambda)$ the multiplicity of the algebra of regular functions $k[Y \cap X_x]$. Then it is easily checked that for any $f \in k[Y \cap X_x)$ which is an eigenvector of T of weight χ, we have

$$\langle\chi, \lambda\rangle(\varepsilon_x Y)(\lambda) = \sum_D \mathrm{ord}_D(f)(\varepsilon_x D)(\lambda)$$

(sum over all irreducible, T-invariant divisors $D \subset Y$). By induction over the dimension of Y, it follows that the function

$$\begin{array}{ccc} \sigma_x^0 & \to & \mathbf{Q} \\ \lambda & \mapsto & (\varepsilon_x Y)(\lambda) \end{array}$$

extends uniquely to a rational function on $N_{\mathbf{Q}}$, that is, to an element $\varepsilon_x Y$ of $\mathcal{Q}$. Moreover, for any rational function f on Y which is an eigenvector of T of weight χ, we have

$$\chi\, \varepsilon_x Y = \sum_D \mathrm{ord}_D(f)\, \varepsilon_x D\ .$$

So, by Theorem 2.1, the assignement $Y \to \varepsilon_x Y$ induces a S-linear map $\varepsilon_x : A_*^T(X) \to \mathcal{Q}$ such that $\varepsilon_x[x] = 1$. By Proposition 2.2, we conclude that $\varepsilon_x = e_x$.

4.5. *The connection with Rossmann's equivariant multiplicity*

Let X be a non-singular variety with an action of T, let $x \in X$ be an isolated fixed point with weights $\chi_1, \ldots, \chi_n$, and let $Y \subset X$ be a T-invariant subvariety. An equivariant multiplicity $\rho_x Y$, with values in S, has been defined by Rossmann (see [Ro]; Rossmann's notation has been changed here). In fact, this notion is equivalent with ours, as shown by the following

Theorem. *For any isolated fixed point x in a non-singular T-variety X, and for any T-invariant subvariety $Y \subset X$, we have*

$$[Y]_x = \rho_x Y = \chi_1 \cdots \chi_n e_x[Y]$$

where $\chi_1, \ldots, \chi_n$ are the weights of x.

Proof. Restriction to x induces an S-linear map $\varepsilon_x : A_*^T(X) \to S$ such that $\varepsilon_x[x] = \chi_1 \ldots \chi_n$ and that $\varepsilon_x[Y] = 0$ for any T-invariant subvariety $Y \subset X$ which does not contain x. By Proposition 4.2 (i), we then have $\varepsilon_x = \chi_1 \ldots \chi_n e_x$.

So it suffices to prove that $\rho_x Y = \chi_1 \ldots \chi_n e_x[Y]$. For a T-invariant subvariety $Y \subset X$, denote by $C_x Y \subset T_x X$ its tangent cone at x. By Rossmann's definition, we have $\rho_x Y = \rho_0(C_x Y)$. We claim that $e_x[Y] = e_0[C_x Y]$. Indeed, let $\tilde{X}$ the space obtained from the blow-up of $X \times \mathbf{A}^1$ (resp. $Y \times \mathbf{A}^1$) at $(x, 0)$ by removing the projectivization of $T_x X$ (resp. of $C_x Y$). Let T act on $X \times \mathbf{A}^1$ by acting trivially on $\mathbf{A}^1$. This defines an action of T on $\tilde{X}$, such that $\tilde{Y}$ is a T-invariant subvariety. Moreover, we have a flat, T-invariant morphism $p : \tilde{Y} \to \mathbf{A}^1$ such that $p^{-1}(0) \simeq C_x Y$ and that $p^{-1}(t) \simeq Y$ for all $t \neq 0$. It follows that $[Y] = [C_x Y]$ in $A_*^T(\tilde{Y})$. Intersecting in $\tilde{X}$ with the fixed point scheme (which is the strict transform of $x \times \mathbf{A}^1$), we obtain $[Y]_x = [C_x Y]_0$ which implies our claim.

So we may assume that X is a T-module with weights $\chi_1, \ldots, \chi_n$, that x is the origin, and that Y is invariant under scalar multiplication. Set $T' := T \times \mathrm{G}_m$, denote by θ the character $(t, u) \mapsto u$ of T', and let T' act on X with weights $\chi_1 + \theta, \ldots, \chi_n + \theta$. Then the origin is an attractive fixed point for this action. Using Proposition 4.4 (ii) and [R] p. 316, we then obtain

$$(\chi_1 + \theta) \cdots (\chi_n + \theta) e'_x[Y] = \rho'_x Y$$

where e', ρ' denote equivariant multiplicities with respect to T'. Now we conclude by the following easy consequence of Proposition 4.2.

Lemma. *Let $x \in X$ be a non-degenerate T-fixed point. Assume that the action of T on X extends to an action of a torus $T' \supset T$ which fixes x. Then x is non-degenerate for the T'-action. Moreover, for any T'-invariant subvariety $Y \subset X$, the T'-equivariant multiplicity $e'_x[Y]$ specializes to $e_x[Y]$ under the map $S' \to S$.*

5. Equivariant Chow groups of toric varieties

5.1. *Toric varieties and fans*

Let X be a toric variety, that is, X is normal and T acts on X with a dense orbit isomorphic to T. Recall that X is determined by its fan Σ in $N_{\mathbf{R}}$, see e.g. [F2]. The cones of Σ parametrize the orbits in X; we denote by $\sigma \to \Omega_\sigma$ this parametrization, and by $V(\sigma)$ the closure of Ω_σ in X. Then $\Omega_\sigma = T/T_\sigma$ where T_σ is the subtorus of T with character lattice $M/M \cap \sigma^\perp$ and with lattice of one-parameter subgroups N_σ (the subgroup of N generated by $N \cap \sigma$). In particular, the dimension of Ω_σ is the codimension of σ.

Proposition. *Let X be a toric variety with fan Σ. Then the S-module $A_*^T(X)$ is defined by generators $F_\sigma = [V(\sigma)]$ (where $\sigma \in \Sigma$) and relations*

$$\chi F_\sigma - \sum_\tau \langle \chi, n_{\sigma\tau} \rangle F_\tau$$

where $\chi \in M \cap \sigma^\perp$; the summation is over all $\tau \in \Sigma$ which contain σ as a face of codimension one, and $n_{\sigma\tau} \in N/N_\sigma$ is the unique generator of the semigroup $(\tau \cap N)/N_\sigma$.

Proof. The T-invariant subvarieties in X are the orbit closures $V(\sigma)$. Moreover, any rational function f on $V(\sigma)$ which is an eigenvector of T is determined up to scalar multiplication by its weight χ: a character of T which vanishes identically on σ. By [F2] p. 61, the divisor of f on $V(\sigma)$ is then $\sum_\tau \langle \chi, n_{\sigma\tau}\rangle F_\tau$. We conclude by Theorem 2.1.

5.2. *Equivariant multiplicities of toric varieties*

For a closed convex cone σ in $N_{\mathbf{R}}$, we denote by

$$\sigma^\vee = \{x \in M_{\mathbf{R}} \mid \langle \lambda, x\rangle \geq 0 \ \forall \ \lambda \in \sigma\}$$

its dual cone. Moreover, for $\lambda \in N_{\mathbf{R}}$, we set

$$P_\sigma(\lambda) := \{x \in \sigma^\vee \mid \langle \lambda, x\rangle \leq 1 \ \} \ .$$

If λ is in σ^0, then $P_\sigma(\lambda)$ is a convex polytope.

Proposition. *Let X be a toric variety with a fixed point x, and let σ be the corresponding d-dimensional cone in Σ. Then, notation being as in 4.4, x is attractive, and $\sigma = \sigma_x$. Moreover, for any $\lambda \in \sigma^0$, the equivariant multiplicity $e_x[X](\lambda)$ is $d!$ times the volume of $P_\sigma(\lambda)$.*

Proof. Recall that x is contained in a unique T-invariant open affine subset X_σ of X. Moreover, the set of weights of T in the algebra of regular functions on X_σ is the intersection of M with $\sigma^\vee$. By Proposition 4.4 (i), it follows that x is attractive and that $\sigma_x = \sigma$. For any $\lambda \in \sigma^0$, we have for the grading of $k[X_\sigma]$ defined by λ:

$$\sum_{m=0}^{n} \dim k[X_\sigma]_m = \mathrm{card}\{\chi \in \sigma^\vee \mid \langle \chi, \lambda\rangle \leq n\} \ .$$

This function of n grows like n^d times the volume of $P_\sigma(\lambda)$.

For x and σ as above, we denote $e_x : A_*^T(X) \to \mathcal{Q}$ by e_σ. More generally, for any $\sigma \in \Sigma$, we will define an S-linear map $e_\sigma : A_*^T(X) \to \mathcal{Q}_\sigma$ where $\mathcal{Q}_\sigma$ is the field of rational functions on $(N_\sigma)_{\mathbf{Q}}$.

Let X_σ be the unique T-invariant open affine subset of X which contains Ω_σ as a closed subset. Then there exists a unique T_σ-toric variety S_σ such that X_σ is equivariantly isomorphic to $T \times_{T_\sigma} S_\sigma$. Moreover, S_σ is affine and contains a fixed point of T_σ. We define e_σ as composition

$$A_*^T(X) \to A_*^{T_\sigma}(S_\sigma) \to \mathcal{Q}_\sigma$$

where the first arrow is restriction to S_σ, and the second one is T_σ-equivariant multiplicity for S_σ.

Corollary. *Let σ, τ be cones in Σ.*

(i) If τ is not contained in σ, then $e_\sigma F_\tau = 0$.

(ii) If τ is contained in σ, then, for any λ in the relative interior of σ, the value at λ of $e_\sigma F_\tau$ is $(\dim(\sigma) - \dim(\tau))!$ times the volume of the convex polytope in $\tau^\perp/\sigma^\perp$, image of the set $P_\sigma(\lambda) \cap \tau^\perp$. In particular, $e_\sigma F_\sigma = 1$.

Proof. Apply the proposition above to the affine toric variety $S_\sigma \cap V(\tau)$ for the torus T_σ/T_τ. Then the associated cone is $(\sigma^\vee \cap \tau^\perp)/\sigma^\perp$ in the linear space $\tau^\perp/\sigma^\perp$.

5.3. *An embedding of the equivariant Chow group*

We will show that the equivariant multiplicities constructed in 5.2 separate points in the equivariant Chow group of any toric variety. A complete description of this group will be given in 5.4 in the simplicial case; the general case is still open.

Proposition. *(i) For any toric variety X with fan Σ, the S-linear map*

$$\prod_{\sigma\in\Sigma} e_\sigma : A_*^T(X) \to \prod_{\sigma\in\Sigma} \mathcal{Q}_\sigma$$

is injective. Moreover, this map induces an isomorphism over $\mathcal{Q}$.

Proof: By induction over the number of cones in Σ. Choose a maximal cone Σ and consider the commutative diagram

$$\begin{array}{ccccccccc} & & A_*^T(\Omega_\sigma) & \to & A_*^T(X) & \to & A_*^T(X\setminus\Omega_\sigma) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{Q}_\sigma & \to & \prod_{\tau\in\Sigma}\mathcal{Q}_\tau & \to & \prod_{\tau\in\Sigma\setminus\{\sigma\}}\mathcal{Q}_\tau & \to & 0 \end{array}$$

which induces a map $A_*^T(\Omega_\sigma) \to \mathcal{Q}_\sigma$. This map is S-linear and sends $[\Omega_\sigma] = F_\sigma$ to $e_\sigma F_\sigma = 1$. Moreover, $A_*^T(\Omega_\sigma) = A_*^T(T/T_\sigma)$ is the symmetric algebra over $M/M\cap\sigma^\perp$. Therefore, our map $A_T^*(\Omega_\sigma) \to \mathcal{Q}_\sigma$ identifies with inclusion of this symmetric algebra into its quotient field $\mathcal{Q}_\sigma$.

Remark. This description shows that the abelian group $A_*^T(X)$ is torsion-free. On the other hand, there exist projective toric surfaces X such that $A_1(X)$ has torsion, see [F2] p. 65. For such X, the S-module $A_*^T(X)$ cannot be free: the assumption of non-singularity is necessary in corollary 3.2.1.

5.4. *The case of simplicial toric varieties*

Recall that the toric variety X is *simplicial* if each cone of its fan is generated by linearly independent vectors; equivalently, X has quotient singularities by finite groups. In this case, we will describe the equivariant Chow group $A_*^T(X)$ in terms of piecewise polynomial functions.

Let $\Sigma(1)$ be the set of one-dimensional cones in Σ. For any $\rho\in\Sigma(1)$, the semigroup $\rho\cap N$ has a unique generator n_ρ. Because Σ is simplicial, any continuous and piecewise linear function on the support of Σ is uniquely defined by its values at the n_ρ for $\rho\in\Sigma(1)$. In particular, there is a unique continuous, piecewise linear function φ_ρ such that $\varphi_\rho(n_\rho)=1$ and that $\varphi_\rho(n_{\rho'})=0$ for all $\rho'\in\Sigma(1)$, $\rho'\neq\rho$. For $\sigma\in\Sigma$, we set

$$\varphi_\sigma := \mathrm{mult}(\sigma)\prod_{\rho\in\sigma(1)}\varphi_\rho$$

where $\mathrm{mult}(\sigma) = [N_\sigma : \sum_{\rho\in\sigma(1)} \mathbf{Z}n_\rho]$ is the *multiplicity* of σ. Then φ_σ is a continuous, piecewise polynomial function which is homogeneous of degree $\dim(\sigma) = \mathrm{codim}_X V(\sigma)$ and which vanishes outside the star of σ in Σ.

We denote by R_Σ the ring of continuous, piecewise polynomial functions on the support of Σ which take rational values on N. Then R_Σ is a $S_{\mathbf{Q}}$-algebra; let A_Σ be the S-submodule of R_Σ generated by the φ_σ, $\sigma \in \Sigma$. Observe that A_Σ is not a subalgebra of R_Σ in general; but it generates R_Σ as a $\mathbf{Q}$-vector space, see e.g. [Br1] Corollary 1.2.

Theorem. *Let X be a simplicial toric variety.*
(i) For any cones σ, τ in Σ, restriction to σ^0 of $\varphi_\tau/\varphi_\sigma$ is defined and equal to $e_\sigma F_\tau$. In particular, $1/\varphi_\sigma$ coincides with $e_\sigma X$ on σ^0.
(ii) The S-linear map

$$\begin{array}{ccc} A_*^T(X) & \to & \prod_{\sigma\in\Sigma} \mathcal{Q}_\sigma \\ u & \mapsto & (e_\sigma u/e_\sigma X)_{\sigma\in\Sigma} \end{array}$$

is injective, and its image is A_Σ.

Proof. (i) is a straightforward computation, each polytope $P_\sigma(\lambda)$ being a simplex.

(ii) By Proposition 5.3, our map is injective; by Proposition 5.1 and (i), its image is the S-module generated by the $(e_\sigma F_\tau/e_\sigma X)_{\sigma\in\Sigma} = \varphi_\tau$ $(\tau \in \Sigma)$.

Remark. For X as above, the rational equivariant Chow group $A_*^T(X)_{\mathbf{Q}}$ has a ring structure, and the map $A_*^T(X)_{\mathbf{Q}} \to R_\Sigma$ is a ring isomorphism. Indeed, X is a quotient of a smooth toric variety with respect to another torus $\tilde{T}$, by a subtorus of $\tilde{T}$ which acts with finite isotropy groups (see e.g. [B-V]), so the assertion follows from [E-G 1] Theorem 4.

6. Equivariant Chow groups for actions of connected reductive groups

6.1. *A refined presentation of equivariant Chow groups*

We obtain a refinement of Theorem 2.1 for schemes with a torus action which extends to an action of a larger group.

Proposition. *Let X be a scheme with an action of a connected solvable linear algebraic group Γ, and let T be a maximal torus of Γ. Then:*
(i) The equivariant Chow group $A_*^T(X)$ is generated as an S-module by the classes $[Y]$ where $Y \subset X$ is a Γ-invariant subvariety.
(ii) If moreover the S-module $A_*^T(X)$ is free, then the S-module of relations between these classes is generated by the $[\mathrm{div}_Y(f)] - \chi[Y]$ where $Y \subset X$ is a Γ-invariant subvariety, and where f is a non-constant rational function on Y which is an eigenvector of Γ of weight χ.

We ignore whether (ii) holds in full generality.

Proof. Define a S-module $A_*^{(\Gamma)}(X)$ by generators $[Y]$ and relations $[\mathrm{div}_Y(f)] - \chi[Y]$ as above. Then $A_*^{(\Gamma)}(X)$ is graded, where the degree of $[Y]$ is the dimension of Y. Consider the natural S-linear map

$$u : A_*^{(\Gamma)}(X) \to A_*^T(X)$$

which is homogeneous of degree zero. It induces a map

$$\overline{u} : A_*^{(\Gamma)}(X)/MA_*^{(\Gamma)}(X) \to A_*^T(X)/MA_*^T(X) \ .$$

The right-hand side is $A_*(X)$ by corollary 2.3.1, whereas the left-hand side is the abelian group defined by generators $[Y]$ and relations $[\mathrm{div}_Y(f)]$ for $Y \subset X$ invariant by Γ, and f a non-constant, Γ-semi-invariant rational function on Y. By [F-M-S-S] Theorem 1, the map

$\overline{u}$ is an isomorphism. We conclude by the graded Nakayama lemma, which can be applied because the degrees in $A_*^{(\Gamma)}(X)$ and $A_*^T(X)$ are bounded from above by the dimension of X.

6.2. *The action of the Weyl group on the equivariant Chow group*

Let G be a connected reductive group. Choose a Borel subgroup $B \subset G$ and a maximal torus T of B. Denote by W the Weyl group and by R the root system of (G,T). We have the set R_+ of positive roots and its subset Σ of simple roots. For $\alpha \in \Sigma$, we denote by $s_\alpha \in W$ the corresponding simple reflection and by $P_\alpha := B \cup Bs_\alpha B$ the corresponding minimal parabolic subgroup. Recall that the group W is generated by the s_α, $\alpha \in \Sigma$.

Let X be a scheme with an action of G. Then W acts on the equivariant Chow group $A_*^T(X)$ (this follows e.g. from the presentation of this group given in 2.1). To describe this action, it suffices to calculate the action of a simple reflection s_α on the class of a B-invariant subvariety $Y \subset X$. This is what we do in our next proposition, after introducing more notation.

Let $P \subset G$ be a parabolic subgroup, and let $Y \subset X$ be a B-invariant subvariety. Denote by $P \times_B Y$ the quotient of $P \times Y$ by the B-action given by $b \cdot (p,y) := (pb^{-1}, by)$. Then the map $P \times Y \to X : (p,y) \mapsto py$ factors through a proper morphism $P \times_B Y \to PY$. In particular, PY is a closed subvariety of X. If moreover the parabolic subgroup P is minimal and moves Y in X, then $\dim(P \times_B Y) = \dim(Y) + 1 = \dim(PY)$ and hence the morphism $P \times_B Y \to PY$ is generically finite.

Proposition. *Let X be a scheme with an action of G, let $Y \subset X$ be a B-invariant subvariety, and let α be a simple root with associated minimal parabolic subgroup $P = P_\alpha$.*
(i) If Y is P-invariant, then $s_\alpha[Y] = [Y]$.
(ii) If Y is not P-invariant, then

$$s_\alpha[Y] = [Y] - d(Y,\alpha)\alpha[PY]$$

where $d(Y,\alpha)$ is the degree of the morphism $P \times_B Y \to PY$.
(iii) If moreover PY contains a dense B-orbit, then denoting by P_Y the isotropy subgroup in P of a general point in PY, we have: $d(Y,\alpha) = 2$ if the image of P_Y in $\mathrm{Aut}(P/B) \simeq \mathrm{PGL}_2$ is the normalizer of a maximal torus in PGL_2, and $d(Y,\alpha) = 1$ otherwise.

Proof. (i) Because $PY = Y$, we have $s_\alpha Y = Y$ and hence $s_\alpha[Y] = [Y]$.
(ii) Set $Z := P \times_B Y$ with inclusion map $i : Y \to Z$ and projection $\pi : Z \to P/B$. Then P/B is isomorphic to projective line where B acts by the character $-\alpha$. So π can be seen as a B-semi-invariant rational function on Z with divisor $-[i(Y)] + [s_\alpha i(Y)]$. Therefore, we have in $A_*^T(Z)$:

$$-\alpha[Z] = -[i(Y)] + s_\alpha[i(Y)] \ .$$

Now consider the proper, surjective morphism

$$f : Z = P \times_B Y \to PY$$

and apply f_* to the identity above. Then we obtain our formula, because f is P-equivariant and maps $i(Y)$ isomorphically to Y.

(iii) is implicit in [R-S] §4 and in [Kn] §3; it can be checked as follows. Choose $y \in Y$ such that By is dense in Y. Then the space $P \times_B Y$ contains $P \times_B By$ as a dense P-orbit. This orbit is mapped by f onto $Py = P/P_y$. Therefore, the degree of f is $[P_y : B_y] = [P_y : P_y \cap B]$. Now $P_y = P_Y$ acts on P/B with a dense orbit (because B has a dense orbit in Py) and with a finite orbit $P_y/P_y \cap B$. So our statement follows by inspection of groups acting on projective line with a dense orbit.

Corollary. *Let X, Y and α be as above, and let $x \in X$ be a non-degenerate fixed point. Then $s_\alpha x$ is non-degenerate, and we have:*

$$e_{s_\alpha x}[Y] = \begin{cases} s_\alpha e_x[Y] & \text{if } PY = Y \\ s_\alpha(e_x[Y] - d(Y,\alpha)\alpha e_x[PY]) & \text{otherwise.} \end{cases}$$

Proof. By uniqueness of $e_x : A_*^T(X) \to \mathcal{Q}$, we have

$$e_{s_\alpha x}[Y] = s_\alpha e_x[s_\alpha Y] \ .$$

This formula implies both statements. Alternatively, one may apply apply Proposition 4.3 to the morphism $f : P \times_B Y \to PY$. Then the fixed points above x are $i(x)$ and $s_\alpha i(x)$. Both are non-degenerate, and we have

$$e_{i(x)}[P \times_B Y] = -\alpha^{-1} e_x[Y], \ e_{s_\alpha i(x)}[P \times_B Y] = \alpha^{-1} s_\alpha(e_x[Y]) \ .$$

6.3. *The action of operators of divided differences*

Let $\mathcal{Q}[W]$ be the twisted group ring of W with coefficients in $\mathcal{Q}$ (the fraction field of S), that is, $\mathcal{Q}[W]$ is the $\mathcal{Q}$-vector space with basis W and multiplication

$$(\sum_{u\in W} f_u\, u)(\sum_{v\in W} g_v\, v) = \sum_{w\in W} (\sum_{w=uv} f_u u(g_v))\, w \ .$$

Let α be a simple root. Following [D1], define an *operator of divided differences* $D_\alpha \in \mathcal{Q}[W]$ by

$$D_\alpha = \frac{id - s_\alpha}{\alpha} \ .$$

Then D_α acts on $\mathcal{Q}$. Observe that

$$D_\alpha(uv) = uD_\alpha(v) + D_\alpha(u)s_\alpha(v) \qquad \forall u, v \in \mathcal{Q}$$

and that

$$D_\alpha(\chi) = \langle \chi, \alpha^\vee \rangle \qquad \forall \chi \in M$$

It follows that D_α leaves S invariant.

Theorem. *Let X be a scheme with an action of G. Then there exists a unique action of D_α on $A_*^T(X)$ such that:*

(i) For all $u \in S$ and $v \in A_*^T(X)$, we have $D_\alpha(uv) = uD_\alpha(v) + D_\alpha(u)s_\alpha(v)$.

(ii) For any B-invariant subvariety Y of X, we have

$$D_\alpha[Y] = \begin{cases} 0 & \text{if } P_\alpha Y = Y \\ d(Y,\alpha)[P_\alpha Y] & \text{if } P_\alpha Y \neq Y \end{cases}$$

where $d(Y,\alpha)$ denotes the degree of the map $P_\alpha \times_B Y \to P_\alpha Y$, see 6.2. Moreover, we have for all $u \in A_*^T(X)$:

$$\alpha D_\alpha(u) = u - s_\alpha(u) \ .$$

Finally, D_α commutes with G-equivariant proper push-forwards, flat pull-backs and Gysin morphisms associated to l.c.i. morphisms.

Proof. Uniqueness of D_α follows from Proposition 6.1. For existence, we fix an index i and we construct an operator $D_\alpha : A_i^T(X) \to A_{i+1}^T(X)$ as follows. Let V and U be as in 2.1, and set $Z := X \times U$. Then the quotient map $Z \to Z/G$ factors through $p : Z/T \to Z/B$ followed by $q : Z/B \to Z/P$ where $P := P_\alpha$. By the lemma below, we can identify $A_*(Z/T)$ with $A_*(Z/B)$ via p^*. Define

$$\begin{array}{rccc} D_\alpha : & A_i(Z/B) & \to & A_{i+1}(Z/B) \\ & u & \mapsto & q^*(q_* u). \end{array}$$

Arguing as in the proof of Proposition 1 in [E-G 1], we see that D_α is independent of the choice of V. To prove that (i) holds, observe that $q : Z/B \to Z/P$ is the projective line bundle associated to the action of P on P/B. It follows that

$$A_*(Z/B) = q^* A_*(Z/P) \oplus c \cap q^* A_*(Z/P)$$

where $c \in A^1(Z/B)$ identifies to multiplication by α. Moreover, $q^*A_*(Z/P)$ consists of fixed points of s_α in $A_*(Z/B)$. So, using the projection formula, it suffices to check (i) for $v = [Z/B]$ and for $v = \alpha[Z/B]$. But both cases reduce to the well-known formula

$$q^* q_* u = \frac{u - s_\alpha(u)}{\alpha}$$

in S (a consequence of Propositions 3 and 4 in [D2]).

To check (ii), let $Y \subset X$ be a B-stable subvariety and set $Y' := (Y \times U)/B$. If $PY = Y$, then $\dim(q(Y')) = \dim(Y') + 1$. Therefore, $q_*[Y'] = 0$ and $D_\alpha[Y] = 0$. On the other hand, if $PY \neq Y$, then $q|_{Y'} : Y' \to q(Y')$ is generically finite of degree $d(Y,\alpha)$. Indeed, we have $d(Y,\alpha) = [P_y : B_y]$ for general $y \in Y$ (see the proof of Proposition 6.2) and hence $d(Y,\alpha)$ is the cardinality of the set

$$\{p \in P \mid py \in By\}/B \ .$$

But this set identifies with the fiber of q at $(y,v)B$ for any $v \in U$. It follows that

$$D_\alpha[Y] = d(Y,\alpha)[q^{-1}(q(Y'))] = d(Y,\alpha)[PY] \ .$$

By Proposition 6.2, we have $[Y] - s_\alpha[Y] = \alpha D_\alpha[Y]$ for any B-invariant subvariety Y of X. Using Proposition 6.1 and (i), it follows that $u - s_\alpha u = \alpha D_\alpha u$ for any $u \in A_*^T(X)$.

Finally, we check that D_α commutes with G-equivariant proper push-forward. Let X' be a scheme with a G-action and let $f : X' \to X$ be a proper, G-equivariant morphism. Set $Z' := X' \times U$; consider the induced maps $f_B : Z'/B \to Z/B$, $f_P : Z'/P \to Z/P$ and $q' : Z'/B \to Z'/P$. Then we have a Cartesian square

$$\begin{array}{ccc} Z'/B & \to & Z'/P \\ \downarrow & & \downarrow \\ Z/B & \to & Z/P \end{array}$$

with flat horizontal arrows. It follows that $(f_P)_* q'^* = q^*(f_B)_*$ and hence that $(f_B)_* q'^* q'_* = q^* q_* (f_B)_*$ as required. The proofs of the other assertions are similar.

Lemma. *Let B be connected solvable linear algebraic group, and let $T \subset B$ be a maximal torus. Let Z be a scheme with an action of B and a quotient $\pi : Z \to Z/B$ which is a principal B-bundle. Then π factors through a smooth map $p : Z/T \to Z/B$ which induces an isomorphism $p^* : A_*(Z/B) \to A_{*+N}(Z/T)$ where N is the dimension of B/T.*

Proof. We can choose a sequence of connected subgroups

$$B_0 = T \subset B_1 \subset \cdots \subset B_N = B$$

such that $\dim(B_i) = \dim(T) + i$. Then π factors through the quotient map $Z \to Z/T$ followed by composition of maps $p_i : Z/B_i \to Z/B_{i+1}$. Observe that

$$Z/B_i = Z \times_{B_{i+1}} B_{i+1}/B_i$$

and that B_{i+1}/B_i is the affine line where B_{i+1} acts either by the group of translations, or by the full affine group. This identifies Z/B_i with the total space of a projective line bundle over Z/B_{i+1}, minus a section. Therefore, $p_i^* : A_*(Z/B_{i+1}) \to A_{*+1}(Z/B_i)$ is an isomorphism (see e.g. [Vi] Lemma 1.4).

6.4. *The ring of operators of divided differences*

Following [D1], denote by $\mathbf{D}$ the subring of $\mathcal{Q}[W]$ generated by $S[W]$ and by the operators D_α for all simple roots α. We call $\mathbf{D}$ the *ring of operators of divided differences.* We have

$$S[W] \subset \mathbf{D} \subset \mathcal{Q}[W] .$$

Moreover, $\mathbf{D}$ can be seen as the ring of endomorphisms of the abelian group S which is generated by the operators D_α and by arbitrary multiplications by elements of S. Observe that S^W-linear endomorphisms, where S^W denotes the ring of W-invariants in S.

For any scheme X with an action of G, the ring $S[W]$ acts on the equivariant Chow group $A_*^T(X)$. By Theorem 6.3, this action extends to an action of the ring $\mathbf{D}$. We will describe the latter action in the case where $X = G/B$ is the flag variety of G. For this, we introduce the following notation.

For any $w \in W$, we choose a reduced decomposition

$$w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l}$$

where $l = l(w)$ is the length of w. We define $D_w \in \mathbf{D}$ by

$$D_w := D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_l} \ .$$

By [D1], the operator D_w is independent of the choice of the reduced decomposition of w. Moreover, the D_w ($w \in W$) are a basis of $\mathbf{D}$ as a left S-module.

Proposition. *The* $\mathbf{D}$*-module* $A_*^T(G/B)$ *is freely generated by the class of the* B*-fixed point* x. *Moreover,* $D_w[x]$ *is the class of the Schubert variety* $\overline{BwB}/B$ *in* $A_*^T(G/B)$. *Finally, denoting by* $\int_{G/B} : A_T^*(G/B) \to S$ *the push-forward for the structural morphism, we have for all* $D \in \mathbf{D}$:

$$\int_{G/B} D[x] = D(1) \ .$$

Proof. First observe that the S-module $A_*^T(G/B)$ is free, e.g. by Corollary 3.2. We construct a basis of this module as follows. For $w \in W$, denote by $C(w)$ the Schubert cell BwB/B and by $X(w)$ the closure of $C(w)$ in G/B. Then, by the Bruhat decomposition, each $C(w)$ is an affine space of dimension $l(w)$, and G/B is the disjoint union of the $C(w)$ ($w \in W$). In fact, the $C(w)$ are the Bialynicki-Birula cells associated to a one-parameter subgroup in the interior of the positive Weyl chamber. So, by Corollary 3.2, the classes $[X(w)]$ are a basis of the S-module $A_*^T(G/B)$.

Let w be a non-trivial element of W. Write $w = s_\alpha \tau$ with α simple and $l(\tau) = l(w)-1$. Then the map $P_\alpha \times_B X(\tau) \to X(w)$ is birational. Using Theorem 6.3, it follows that $D_\alpha[X(\tau)] = [X(w)]$ and hence that $[X(w)] = D_w[x]$ in $A_*^T(G/B)$. Therefore, we have $A_*^T(G/B) = \mathbf{D}[x]$. Furthermore, the S-module $\mathbf{D}$ is torsion-free of rank $|W|$, which is the rank of the S-module $A_*^T(G/B)$. It follows that the map

$$\begin{array}{ccc} \mathbf{D} & \to & A_*^T(G/B) \\ D & \mapsto & D[x] \end{array}$$

is an isomorphism.

Finally, for any $w \in W$, we have

$$\int_{G/B} D_w[x] = \int_{G/B} [X(w)] = \begin{cases} 1 & \text{if } w = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, our formula for $\int_{G/B}$ holds when $D = D_w$. By linearity, it holds for all $D \in \mathbf{D}$.

Remark. The proof above shows by geometric arguments that the D_w are independent of the choice of reduced decompositions, and that they form a basis of the left S-module $\mathbf{D}$.

6.5. *Equivariant multiplicities of Schubert varieties*

Denote by $i : (G/B)^T \to G/B$ the inclusion of the fixed point set, and identify $(G/B)^T$ with W. Then $A_*^T((G/B)^T)$ identifies with the ring $S[W]$ as a S-algebra with a compatible action of W.

Proposition. *(i) The image of*

$$i^* : A_T^*(G/B) \to S[W]$$

consists in all $\sum_{w\in W} f_w\, w$ *such that* $f_w \equiv f_{s_\alpha w}$ *(mod* α*) whenever* $w \in W$ *and* $\alpha \in R^+$.
(ii) We have

$$i^*[X(w)] = (-1)^N (\prod_{\alpha\in R^+} \alpha) \sum_{\tau\le w} \det(\tau)\, e_\tau[X(w)]\, \tau$$

where N *is the number of positive roots, and where* e_τ *denotes equivariant multiplicity at* τ *in* G/B*, see §4. Moreover,* e_τ *is uniquely defined by:*

$$e_1[x] = 1,\;\; e_\tau[x] = 0 \text{ for all } \tau \neq 1$$

where x *is the* B*-fixed point in* G/B*, and by the recursive formula*

$$e_\tau[X(s_\alpha w)] = \frac{e_\tau[X(w)] - s_\alpha(e_{s_\alpha\tau}[X(w)])}{\alpha}$$

for all simple roots α *such that* $l(s_\alpha w) = l(w) + 1$.

Proof. (i) follows from 3.4 combined with the description of all T-invariant curves in G/B. This description can be found in [Ca]; we recall it for completeness. Let $C \subset G/B$ be a T-invariant curve with weight $\alpha \in M$. Then the kernel of α is a singular torus in the sense of [Bo] (13.2), and hence α is a root. Let $G_\alpha \subset G$ be the centralizer of the kernel of α. Then G_α is a reductive group of semisimple rank one. Moreover, C is equal to $G_\alpha wB/B$ for some $w \in W$. So $C^T = \{w, s_\alpha w\}$. Now we conclude by 3.4.

(ii) Observe that any $\tau \in W$ is an attractive fixed point in G/B with weights $-w(\alpha)$ ($\alpha \in R^+$). Using 4.5, it follows that we have for all $u \in A_T^*(G/B)$:

$$i^*u = \sum_{\tau\in W} u_\tau\, \tau = \sum_{\tau\in W} e_\tau(u)\, (\prod_{\alpha\in R^+} -\tau(\alpha))\, \tau = (-1)^N (\prod_{\alpha\in R^+} \alpha) \sum_{\tau\in W} \det(\tau) e_\tau(u) \tau\ .$$

Finally, the recursive formula follows from corollary 6.2.

As an application, here is a short proof of a smoothness criterion for Schubert varieties, due to S. Kumar (see [Ku] Theorem 5.5). Let $\tau, w \in W$ such that $\tau \le w$. Then τ is a non-singular point of $X(w)$ if and only if

$$(K) \qquad e_\tau[X(w)] = (-1)^{l(w)} \prod_{\alpha\in R^+, s_\alpha\tau\le w} \alpha^{-1}\ .$$

Indeed, recall that the T-invariant curves through τ in G/B are the $G_\alpha \tau B/B$ where α is a positive root, and where $G_\alpha \subset G$ is the corresponding reductive subgroup of semisimple rank one. Moreover, $G_\alpha\tau B/B$ is contained in $X(w)$ if and only if $s_\alpha\tau \le w$.

If τ is a non-singular point of $X(w)$, then the weights of $T_\tau X(w)$ are the weights of the T-invariant curves through τ, i.e., the opposites of positive roots α such that $s_\alpha\tau \le w$. The number of such weights is $\dim X(w) = l(w)$. So (K) follows from 4.2.

Conversely, assume that (K) holds. Consider the unique open affine T-invariant neighborhood $X(w)_\tau$ of τ in $X(w)$ defined in 4.4, and denote by A the algebra of regular functions on $X(w)_\tau$. By the proof of Proposition 2.2 in [Po], or by the proof of Proposition 5.2

in [Ku], for any $\alpha \in R^+$ such that $s_\alpha \tau \leq w$, there exists $f_\alpha \in A$ which is an eigenvector of T of weight $-\alpha$. Moreover, the unique common zero of the f_α's is τ.

Because the degree of the rational function $e_\tau[X(w)]$ is $-\dim X(w) = -l(w)$, equation (K) implies that the number of f_α's is the dimension of $X(w)$. Therefore, the f_α's generate a polynomial subring R of A and moreover A is a finite R-module. Moreover, because the equivariant multiplicities of R and of A are equal, the rank of the R-module A is one. It follows that $A = R$, i.e., $X(w)_\tau$ is an affine space.

6.6. *The rational equivariant Chow ring of the flag variety*

The results in 6.4 and 6.5 give a picture of the equivariant Chow group $A_*^T(G/B)$ as an S-module. In this section, we will describe the rational equivariant Chow ring $A_T^*(G/B)_{\mathbf{Q}}$, the action of $\mathbf{D}$ on this ring, and its relation to the previous picture and to the rational G-equivariant Chow ring $A_G^*(G/B \times G/B)_{\mathbf{Q}}$ as well.

For any $\chi \in M$, consider the G-equivariant line bundle $G \times_B k(\chi)$ over G/B and denote by $c^T(\chi)$ its T-equivariant Chern class. Then $c^T(\chi)$ is in $A_T^1(G/B)$. The additive map

$$\begin{array}{ccc} M & \to & A_T^1(G/B) \\ \chi & \mapsto & c^T(\chi) \end{array}$$

extends to a ring homomorphism: the *characteristic homomorphism*

$$c^T : S \to A_T^*(G/B) \ .$$

Proposition. *(i) The map*

$$\begin{array}{ccc} S \times S & \to & A_T^*(G/B) \\ (f,g) & \mapsto & fc^T(g) \end{array}$$

is S^W-bilinear.
(ii) The induced map

$$\gamma : S \otimes_{S^W} S \to A_T^*(G/B)$$

is an isomorphism over the rationals. If moreover G is special, then γ is an isomorphism.
(iii) For all $D \in \mathbf{D}$ and f, g in S, we have

$$D(fc^T(g)) = D(f)c^T(g)$$

and moreover

$$\int_{G/B} f\, c^T(g) = (-1)^N (\prod_{\alpha \in R^+} \alpha^{-1})\, f \sum_{w \in W} det(w)\, w(g) \ .$$

(iv) For any character χ of T, we have in $A_T^*(G/B)$:

$$c^T(\chi)[X(w)] = w(\chi)[X(w)] + \sum_\beta \langle \chi, \check\beta \rangle [X(ws_\beta)]$$

(sum over all positive roots β such that $l(ws_\beta) = l(w) - 1$). In particular,

$$c^T(\chi) = w_0(\chi)[G/B] + \sum_\alpha \langle \chi, \check\alpha \rangle [X(w_0 s_\alpha)]$$

(sum over all simple roots α), where w_0 is the longest element in W.

Proof. (i) Observe that $i^*(fc^T(g)) = \sum_{w\in W} f\, w(g)\, w$. It follows that the map

$$\begin{array}{ccc} S\times S & \to & S[W] \\ (f,g) & \mapsto & i^*(fc^T(g)) \end{array}$$

is S^W-bilinear. Now (i) follows from injectivity of i^*.

(ii) The map γ induces a map

$$\overline{\gamma} : (S\otimes_{S^W} S)/M(S\otimes_{S^W} S) = S/S^W_+ S \to A^*_T(G/B)/MA^*_T(G/B) = A^*(G/B)$$

where S^W_+ denotes the ideal of S^W generated by homogeneous elements of positive degree. By [D1], the map $\overline{\gamma}$ is an isomorphism over the rationals; if moreover G is special, then $\overline{\gamma}$ is an isomorphism. Therefore, γ is surjective over the rationals, by Nakayama's lemma. But $S_{\mathbf{Q}}$ is a free module over $S^W_{\mathbf{Q}}$ and hence $S_{\mathbf{Q}}\otimes_{S^W_{\mathbf{Q}}} S_{\mathbf{Q}}$ is a free module over $S_{\mathbf{Q}}$. It follows that γ is an isomorphism over the rationals (resp. an isomorphism if G is special).

(iii) For any simple root α, we have

$$(id - s_\alpha)(fc^T(g)) = (f - s_\alpha(f))c^T(g)$$

because $c^T(g)$ is W-invariant. Moreover, the S-module $A^*_T(G/B)$ is free, and therefore

$$D_\alpha(fc^T(g)) = D_\alpha(f)c^T(g)\ .$$

Finally, we have by Bott's residue formula (see [E-G 2] §5, or use Corollary 4.2):

$$\int_{G/B} f\, c^T(g) = \sum_{w\in W} f\, w(g)\, e_w[G/B] = f \sum_{w\in W} w(g) \prod_{\alpha\in R^+}(-w(\alpha))$$

which implies our second formula.

(iv) Let $u\in A^T_*(G/B)$. Using the formula

$$D_\alpha(\chi u) = s_\alpha(\chi)D_\alpha(u) + \langle\chi,\check{\alpha}\rangle u$$

(valid for any simple root α) and induction over the length of w, we obtain

$$D_w(\chi u) = w(\chi)D_w(u) + \sum_\beta \langle\chi,\check{\beta}\rangle D_{ws_\beta}(u)\ .$$

In particular, taking $u = [x]$, we obtain

$$D_w(\chi[x]) = w(\chi)[X(w)] + \sum_\beta \langle\chi,\check{\beta}\rangle [X(ws_\beta)]\ .$$

But $\chi[x] = c^T(\chi)[x]$ (as can be checked by restriction to fixed points). Moreover, by (iii), multiplication by $c^T(\chi)$ commutes with D_w. So we obtain

$$D_w(\chi[X]) = c^T(\chi)D_w[x] = c^T(\chi)[X(w)]$$

which proves the first formula. The second one follows by taking $w = w_0$; then the positive roots β such that $l(ws_\beta) = l(w) - 1$ are exactly the simple roots.

Statement (iv) implies readily an equivariant version of the Chevalley formula which describes multiplication by the class of a Schubert subvariety of codimension one in G/B. Recall that these varieties are the $X(w_0 s_\alpha)$ (where α is a simple root) and that their classes generate the S-algebra $A_*^T(G/B)$; so the following result describes (in theory) multiplication of classes of all Schubert varieties.

Corollary. *For any simple root α, we have in $A_T^*(G/B)$:*

$$[X(w_0 s_\alpha)][X(w)] = (w(\omega_\alpha) - w_0(\omega_\alpha))[X(w)] + \sum_\beta \langle \omega_\alpha, \check{\beta} \rangle [X(ws_\beta)]$$

(sum over all positive roots β such that $l(ws_\beta) = l(w) - 1$), where ω_α is the fundamental weight which is not orthogonal to α.

Finally, we interpret the T-equivariant Chow group of G/B as the G-equivariant Chow group of $G/B \times G/B$. Let V be a G-module, and let $U \subset V$ be an open G-invariant subset such that the quotient $U \to U/G$ exists and is a principal G-bundle. For any scheme X with an action of B, denote by $G \times_B X$ the quotient of $G \times X$ by the diagonal action of B. Then the maps

$$(X \times U)/T \to (X \times U)/B \simeq ((G \times_B X) \times U)/G$$

induce an isomorphism of degree N

$$A_*^G(G \times_B X) \to A_*^T(X) \ .$$

If moreover $X = G/B$, then the map

$$\begin{array}{ccc} G \times_B X & \to & G/B \times G/B \\ (g,u)B & \mapsto & (gB, guB) \end{array}$$

is a G-equivariant isomorphism, where G acts diagonally on $G/B \times G/B$. It follows that there is an isomorphism

$$A_T^*(G/B) \to A_G^*(G/B \times G/B)$$

which maps each $[X(w)]$ to $[G(x, X(w))]$. In particular, the class of the B-fixed point x is mapped to the class of the diagonal in $G/B \times G/B$. Moreover, γ is identified to the characteristic homomorphism of $G/B \times G/B$.

Similarly, taking for X a point, we obtain that the characteristic homomorphism $S = A_T^*(pt) \to A_G^*(G/B)$ is an isomorphism.

6.7. *The module structure of equivariant Chow groups*

By the results of 6.6, there is an isomorphism

$$S_{\mathbf{Q}} \otimes_{S^W_{\mathbf{Q}}} A^G_*(G/B)_{\mathbf{Q}} \to A^T_*(G/B)_{\mathbf{Q}}.$$

Moreover, the rational Chow group $A_*(G/B)_{\mathbf{Q}}$ is the quotient of $A^G_*(G/B)_{\mathbf{Q}}$ by its subgroup $S^W_+ A^G_*(G/B)_{\mathbf{Q}}$. In this section, we will show that both results extend to any scheme X with an action of G.

Recall the isomorphism (see [E-G 1] Proposition 6)

$$A^G_*(X)_{\mathbf{Q}} \simeq A^T_*(X)^W_{\mathbf{Q}}$$

(If moreover G is special, then this statement holds over the integers.) In particular, the rational G-equivariant Chow group of the point is isomorphic to $S^W_{\mathbf{Q}}$. Therefore, the S-module structure on $A^T_*(X)$ restricts to the structure of a $S^W_{\mathbf{Q}}$-module on $A^G_*(X)_{\mathbf{Q}}$ together with a map

$$\gamma : S_{\mathbf{Q}} \otimes_{S^W_{\mathbf{Q}}} A^G_*(X)_{\mathbf{Q}} \to A^T_*(X)_{\mathbf{Q}} \ .$$

Observe that the left-hand side is a $\mathbf{D}$-module via $D(u \otimes v) = D(u) \otimes v$ (this makes sense because $\mathbf{D}$ consists in S^W-linear endomorphisms of S). By Theorem 6.3, the right-hand side is a $\mathbf{D}$-module, too.

Theorem. *Let X be a scheme with an action of a connected reductive group G with maximal torus T. Then the map*

$$\gamma : S_{\mathbf{Q}} \otimes_{S^W_{\mathbf{Q}}} A^G_*(X)_{\mathbf{Q}} \to A^T_*(X)_{\mathbf{Q}}$$

is an isomorphism of $\mathbf{D}$*-modules. If moreover G is special, then the statement holds over the integers.*

Proof. As in the proof of Proposition 6.3, it is enough to check that the map

$$\begin{array}{rccc} \gamma : & S_{\mathbf{Q}} \otimes_{S^W_{\mathbf{Q}}} A_*(Z/G)_{\mathbf{Q}} & \to & A_*(Z/B)_{\mathbf{Q}} \\ & u \otimes v & \mapsto & c(u) \cap \pi^* v \end{array}$$

is an isomorphism of $\mathbf{D}$-modules. Here $Z \to Z/G$ is a principal G-bundle, $\pi : Z/B \to Z/G$ is the associated complete flag bundle, and c is the characteristic homomorphism, defined as follows: a character $\chi \in M$ acts by multiplication by the first Chern class of the line bundle $Z \times_B k(\chi)$ over Z/B.

It has been shown by Vistoli that γ is an isomorphism (see [Vi] Theorem 2.3). A somewhat simpler proof is as follows. The case where $Z = G$ (with the left action of G) has been treated in 6.6. In the general case, we claim first that γ is surjective, i.e.,

$$A_*(Z/B)_{\mathbf{Q}} = c(S_{\mathbf{Q}}) \cap \pi^* A_*(Z/G)_{\mathbf{Q}} \ .$$

Indeed, we can find an open subset $U \subset Z/G$ and a finite map $\varphi : U' \to U$ such that pull-back by φ of the bundle $p : Z \to Z/G$ is trivial. Then pull-back by φ of $\pi : Z/B \to Z/G$ is the trivial flag bundle $G/B \times U' \to U'$. Therefore,

$$A_*(\pi^{-1}(U') \times_U U')_{\mathbf{Q}} = A_*(G/B)_{\mathbf{Q}} \otimes \pi^* A_*(U')_{\mathbf{Q}} = c(S_{\mathbf{Q}}) \cap \pi^* A_*(U')_{\mathbf{Q}}$$

where the first equality holds e.g. by [F-M-S-S] Theorem 2. This implies that

$$A_*(\pi^{-1}(U))_{\mathbf{Q}} = c(S_{\mathbf{Q}}) \cap \pi^* A_*(U)_{\mathbf{Q}} \ .$$

On the other hand, we may assume by Noetherian induction that

$$A_*(\pi^{-1}(Z/B \setminus U)_{\mathbf{Q}} = c(S_{\mathbf{Q}} \cap \pi^* A_*(Z/G \setminus U)_{\mathbf{Q}} \ .$$

This implies our claim.

Now we check that γ is injective. By a theorem of Chevalley, we can find a basis $(a_w)_{w\in W}$ of the $S_{\mathbf{Q}}^W$-module $S_{\mathbf{Q}}$ consisting of homogeneous elements. Consider the map

$$\begin{array}{ccc} S_{\mathbf{Q}} \times S_{\mathbf{Q}} & \to & \mathcal{Q} \\ (f,g) & \mapsto & f \cdot g := (-1)^N (\prod_{\alpha\in R^+} \alpha^{-1}) \sum_{w\in W} \det(w) w(fg) \ . \end{array}$$

Clearly, this map takes values in $S_{\mathbf{Q}}^W$ and is $S_{\mathbf{Q}}^W$-bilinear. We claim that it is non-degenerate. Indeed, by 6.6, the induced map

$$(S_{\mathbf{Q}}/S_+^W S_{\mathbf{Q}}) \times (S_{\mathbf{Q}}/S_+^W S_{\mathbf{Q}}) \to \mathbf{Q}$$

coincides with the intersection pairing

$$\begin{array}{ccc} A^*(G/B)_{\mathbf{Q}} \times A^*(G/B)_{\mathbf{Q}} & \to & \mathbf{Q} \\ (f,g) & \mapsto & \int_{G/B} fg \end{array}$$

and the latter is non-degenerate. We denote by $(b_w)_{w\in W}$ the dual basis of (a_w) for this dot pairing. Now let $(u_w)_{w\in W}$ be a family in $A_*(Z/G)_{\mathbf{Q}}$ such that

$$\sum_{w\in W} c(a_w) \cap \pi^* u_w = 0 \ .$$

Fix $\tau \in W$ and apply $\pi^*\pi_* c(b_\tau)$ to the relation above, to obtain

$$\sum_{w\in W} \pi^*\pi_* c(a_w b_\tau) \cap \pi^* u_w = 0$$

by using the projection formula. But $c(a \cdot b) = \pi^*\pi_* c(ab)$ for all a, b in $S_{\mathbf{Q}}$ (this follows from 6.6; see also [Br2] Proposition 1.1), so we obtain $a_\tau = 0$.

Finally, we check that γ is D_α-linear for each simple root α. By the proof of Proposition 6.3, we have $D_\alpha = q^* q_*$ where $q : Z/B \to Z/P_\alpha$ is induced by $\pi : Z \to Z/G$. But $q_*(\pi^* v) = 0$ for all $v \in A_*(Z/G)$ and hence we have for all $u \in S$:

$$D_\alpha(c(u) \cap \pi^* v) = (q^* q_* c(u)) \cap \pi^* v = c(D_\alpha u) \cap \pi^* v \ .$$

Now we combine this theorem with our previous results for tori, to study the module structure of $A_*^G(X)_{\mathbf{Q}}$ over the polynomial ring $S_{\mathbf{Q}}^W$.

Corollary. *Let X be a scheme with an action of G.*
(i) The rational Chow group $A_*(X)_{\mathbf{Q}}$ is the quotient of the rational equivariant Chow group $A_*^G(X)_{\mathbf{Q}}$ by its subgroup $S_+^W A_*^G(X)_{\mathbf{Q}}$ where S_+^W denotes the ideal of S^W generated by all homogeneous elements of positive degree.
(ii) If moreover X is projective and non-singular, then the $S_{\mathbf{Q}}^W$-module $A_*^G(X)_{\mathbf{Q}}$ is free.

Proof. (i) follows immediately from the theorem above, together with Corollary 2.3.

(ii) By corollary 3.2, the $S_{\mathbf{Q}}$-module $A_*^T(X)_{\mathbf{Q}}$ is free. Moreover, the $S_{\mathbf{Q}}^W$-module $S_{\mathbf{Q}}$ is free, and hence the $S_{\mathbf{Q}}^W$-module $A_*^T(X)_{\mathbf{Q}}$ is free, too. Now the $S_{\mathbf{Q}}^W$-module $A_*^T(X)_{\mathbf{Q}}^W$ is a direct summand of $A_*^T(X)_{\mathbf{Q}}$, and hence it is projective. Moreover, $A_*^T(X)_{\mathbf{Q}}^W$ is graded, with degrees bounded from above. Therefore, it is free over $S_{\mathbf{Q}}^W$.

7. Equivariant Chow groups of spherical varieties

7.1. *Fixed points of codimension one tori in spherical varieties*

Let G be a connected reductive group, $B \subset G$ a Borel subgoup and $T \subset B$ a maximal torus. Let X be a projective, non-singular G-variety. Assuming that X is spherical (i.e., B has a dense orbit in X), we will apply Theorem 3.3 to the description of the rational equivariant Chow ring $A_T^*(X)_{\mathbf{Q}}$. For this, we study fixed points of codimension one subtori of T.

Recall that a subtorus $T' \subset T$ is *regular* if its centralizer $C_G(T')$ is equal to T; otherwise T' is *singular*. A subtorus of codimension one is singular if and only if it is the kernel of some positive root α. Then α is unique, and the group $C_G(T')$ is the product of T' with a subgroup S isomorphic to SL_2 or to PSL_2. Observe that the fixed point set of T' in any G-variety inherits an action of the group $C_G(T')/T'$, a quotient of S.

Proposition. *Let X be a spherical G-variety, and let $T' \subset T$ be a subtorus of codimension one.*
(i) If T' is regular, then the fixed point set $X^{T'}$ is at most one-dimensional.
(ii) If T' is singular, then $X^{T'}$ is at most two-dimensional. If moreover X is complete and non-singular, then any two-dimensional connected component of $X^{T'}$ is (up to a finite, purely inseparable equivariant morphism) either a rational ruled surface

$$\mathbf{F}_n = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$$

where $C_G(T')$ acts through the natural action of SL_2, or the projective plane where $C_G(T')$ acts through the projectivization of a non-trivial SL_2-module of dimension three.

Proof. Let Y be an irreducible component of $X^{T'}$. By a result of Luna (personal communication), the $C_G(T')$-variety Y is spherical. Luna's proof is as follows: Let λ be a generic one-parameter subgroup of T'; then $C_G(T') = C_G(\lambda)$. Denote by $G(\lambda)$ the set of all $g \in G$ such that $\lim_{t\to 0} \lambda(t)g\lambda(t^{-1})$ exists. Recall that $G(\lambda)$ is a parabolic subgroup of G, with Levi subgroup $C_G(\lambda)$. Moreover, the Bialynicki-Birula stratum $X_+(Y,\lambda)$ is invariant under $G(\lambda)$, and the map $p_+ : X_+(Y,\lambda) \to Y$ is a $G(\lambda)$-equivariant retraction, where $G(\lambda)$ acts on Y through its quotient $C_G(\lambda)$. Because X is spherical, it contains only finitely many orbits of any Borel subgroup of G. Therefore, a Borel subgroup of $G(\lambda)$ has finitely many orbits in $X_+(Y,\lambda)$, and finally a Borel subgroup of $C_G(\lambda)$ has finitely many orbits in Y.

If T' is regular, then $C_G(T')$ acts on $X^{T'}$ through the one-dimensional torus T/T', whence (i). If T' is singular, then Y is a spherical S-variety. So the dimension of Y is at most the dimension of a Borel subgroup of S, whence the first statement of (ii).

If moreover X is complete and non-singular, then the same holds for Y. Choose a point y such that the orbit $S \cdot y$ is open in Y, and denote by H the preimage in SL_2 of the isotropy group S_y. Then the map

$$\begin{array}{ccc} \mathrm{SL}_2/H & \to & Y \\ gH & \mapsto & g \cdot y \end{array}$$

is dominant and purely inseparable.

Because H is a spherical subgroup of SL_2, three cases can occur:
(1) H is a one-dimensional torus. Then the homogeneous space SL_2/H admits $\mathbf{P}^1 \times \mathbf{P}^1$ as an equivariant completion, with boundary the diagonal Δ. The rational, S-equivariant map

$$f : \mathbf{P}^1 \times \mathbf{P}^1 - - \to Y$$

is defined at some point of Δ, and hence everywhere because Δ is a unique S-orbit. Moreover, f cannot contract Δ, and therefore f is finite.
(2) H is the normalizer of a one-dimensional torus. Then SL_2/H admits $\mathbf{P}(sl_2)$ (the projectivization of the Lie algebra of SL_2) as an equivariant completion, with boundary the conic of nilpotent matrices. By the argument above, the rational equivariant map

$$f : Y - - \to \mathbf{P}(sl_2)$$

is everywhere defined and finite.
(3) H is the semi-direct product of the subgroup $U \subset \mathrm{SL}_2$ of unipotent matrices, by the cyclic group Z_n of diagonal matrices with eigenvalues (ζ, ζ^{-1}) where ζ is a n-th root of unity.

First consider the case where $n = 1$. Then $\mathrm{SL}_2/H = k^2 \setminus \{0\}$ as an SL_2-variety. Arguing as before, we see that any non-singular completion of SL_2/H is isomorphic to $\mathbf{P}(k^2 \oplus k)$ or to its blow-up at the origin, i.e. to $\mathbf{F}_1$.

Finally, for any integer n, the homogeneous space SL_2/UZ_n admits $\mathbf{F}_n$ as an equivariant completion. Moreover, the boundary consists in the curves $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus 0)$ and $\mathbf{P}(0 \oplus \mathcal{O}_{\mathbf{P}^1}(n))$. Both curves are homogeneous, and therefore the inclusion $\mathrm{SL}_2/H \to Y$ extends to a morphism $f : \mathbf{F}_n \to Y$. For $n > 1$, no boundary curve can be contracted to yield a non-singular surface, and hence f is finite.

7.2. *Equivariant Chow rings of rational ruled surfaces*

Let D be the torus of diagonal matrices in SL_2 and let α be the character of D given by

$$\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^2 \ .$$

We will identify the rational character ring of D with $\mathbf{Q}[\alpha]$.

Consider a rational ruled surface $\mathbf{F}_n$ with ruling $\pi : \mathbf{F}_n \to \mathbf{P}^1$. Observe that $\mathbf{F}_n$ contains exactly four fixed points x, y, z, t of D where x, y (resp. z, t) are mapped to 0 (resp. ∞) by π. Moreover, we may assume that y and z lie in one G-invariant section of π, and that x and t lie in the other G-invariant section. This ordering of the fixed points identifies $A_T^*(\mathbf{F}_n^T)$ with $\mathbf{Q}[\alpha]^4$.

On the other hand, denote by $\mathbf{P}(V)$ the projectivization of a non-trivial SL_2-module V of dimension three. The weights of D in V are either $-\alpha$, 0, α (in the case where $V = sl_2$) or $-\frac{\alpha}{2}$, 0, $\frac{\alpha}{2}$ (in the case where $V = k^2 \oplus k$). We denote by x, y, z the corresponding fixed points of D in $\mathbf{P}(V)$, and we identify $A_D^*(\mathbf{P}(V)^T)$ with $\mathbf{Q}[\alpha]^3$.

Proposition. *Notation being as above, the image of*

$$i^* : A_D^*(\mathbf{F}_n)_{\mathbf{Q}} \to S_{\mathbf{Q}}^4$$

consists in all (f_x, f_y, f_z, f_t) *such that* $f_x \equiv f_y \equiv f_z \equiv f_t \pmod{\alpha}$ *and that* $f_x - f_y + f_z - f_t \equiv 0 \pmod{\alpha^2}$. *Moreover, the image of*

$$i^* : A_D^*(\mathbf{P}(V))_{\mathbf{Q}} \to S_{\mathbf{Q}}^4$$

consists in all (f_x, f_y, f_z) *such that* $f_x \equiv f_y \equiv f_z \pmod{\alpha}$ *and that* $f_x - 2f_y + f_z \equiv 0 \pmod{\alpha^2}$.

Proof. First we consider the case of $\mathbf{P}(V)$. The closures of the Bialynicki-Birula cells are then: the point z, the line (yz) and the whole $\mathbf{P}(V)$. The classes of these closures are mapped by i^* to

$$(0, 0, 2\alpha^2),\ (0, \alpha, 2\alpha),\ (1, 1, 1)$$

in the case where $V = sl_2$, and to

$$(0, 0, \frac{\alpha^2}{2}),\ (0, \frac{\alpha}{2}, \alpha),\ (1, 1, 1)$$

in the case where $V = k^2 \oplus k$. By Corollary 3.2 (iii), the image of i^* is generated as an S-module by images of closures of cells. This implies easily our statement.

The proof for $\mathbf{F}_n$ is similar; it is enough to check the result for $\mathbf{F}_0$ and $\mathbf{F}_1$ (indeed, for any positive n, the surface $\mathbf{F}_n$ is the quotient of $\mathbf{F}_1$ by the action of a cyclic group of order n which commutes with the action of $\mathbf{D}$).

7.3. *Equivariant Chow rings of projective, non-singular spherical varieties*

Recall that any spherical G-variety contains only finitely many orbits of G, and therefore only finitely many fixed points of T. Combining Theorem 3.3 wih the results of 7.1 and 7.2, we obtain immediately the following

Theorem. *For any projective, non-singular, spherical G-variety X, the map*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

is injective. Moreover, the image of i^* consists in all families $(f_x)_{x \in X^T}$ such that:
(i) $f_x \equiv f_y \pmod{\chi}$ whenever x and y are connected by a T-invariant curve with weight χ.

(ii) $f_x - 2f_y + f_z \equiv 0 \pmod{\alpha^2}$ *whenever* α *is a positive root,* x*,* y*,* z *lie in a component of* $X^{\ker(\alpha)}$ *isomorphic to* $\mathbf{P}^2$*, and* x*,* y*,* z *are ordered as in 7.2.*
(iii) $f_x - f_y + f_z - f_t \equiv 0 \pmod{\alpha^2}$ *whenever* α *is a positive root,* x*,* y*,* z*,* t *lie in a component of* $X^{\ker(\alpha)}$ *isomorphic to a rational ruled surface, and* x*,* y z*,* t *are ordered as in 7.2.*

This approach to equivariant Chow rings of spherical varieties will be pursued in a subsequent paper. Here we observe that case (ii) occurs e.g. when X is the space of complete conics; then $X^{\ker(\alpha)}$ is either the space of pairs of lines through a given point, or the space of pairs of points on a given line. An example where case (iii) occurs is the blow-up of the diagonal in $\mathbf{P}^2 \times \mathbf{P}^2$; then $X^{\ker(\alpha)}$ is the strict transform of $\ell \times \ell$ where ℓ is a line in $\mathbf{P}^2$. Finally, here is an example where cases (ii) and (iii) do not occur.

Let G be a connected semisimple adjoint group of rank r. Consider G as a $G \times G$-variety for the action given by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$ (this variety is spherical, by the Bruhat decomposition). There exists a canonical smooth equivariant completion $G \subset \overline{G}$; its boundary $\overline{G} \setminus G$ consists in r smooth irreducible divisors intersecting transversally along an orbit of $G \times G$. The construction of $\overline{G}$ is due to De Concini and Procesi over $\mathbf{C}$ (as a special case of their construction of canonical compactifications of adjoint symmetric spaces); it was extended by Strickland to arbitrary characteristic, see [St].

Let T be a maximal torus of G, with normalizer N and Weyl group $W = N/T$. Then the closure $\overline{N}$ of N in $\overline{G}$ is smooth, and is the disjoint union of $|W|$ copies of $\overline{T}$; moreover, $\overline{N}$ contains all $T \times T$-fixed points in X, see [L-P] 4.1. It is easy to see that $\overline{G}$ contains only finitely many $T \times T$-invariant curves, and that all such curves are contained in $\overline{N}$. Therefore, using Thorem 3.4, we see that the restriction map

$$A^*_{T\times T}(\overline{G})_{\mathbf{Q}} \to A^*_{T\times T}(\overline{N})_{\mathbf{Q}}$$

is an isomorphism. It follows that composition

$$A^*_{G\times G}(\overline{G})_{\mathbf{Q}} = A^*_{T\times T}(\overline{G})^{W\times W}_{\mathbf{Q}} \to A^*_{T\times T}(\overline{N})^{W\times W}_{\mathbf{Q}} = (S_{\mathbf{Q}} \otimes A^*_T(\overline{T}))^W$$

is an isomorphism. This was proved in [L-P] 2.3 for equivariant cohomology and $k = \mathbf{C}$.

7.4. *The action of operators of divided differences*

Let X be a spherical G-variety. Then X contains only finitely many B-orbits. Equivalently, the set $\mathcal{B}(X)$ of B-invariant, closed, irreducible subvarieties of X is finite. A short proof of this result was given by Knop (see [Kn] Corollary 2.6), based on the action on $\mathcal{B}(X)$ of a monoid W^* defined as follows: W^* is the set W endowed with the product $*$ such that

$$\overline{BwB} * \overline{B\tau B} = \overline{BwB\tau B}$$

in G. This monoid had already appeared in Richardson and Springer's work on B-orbits in symmetric spaces, see [R-S]. Its action on $\mathcal{B}(X)$ is defined by

$$w * Y := \overline{BwY} \; .$$

We will relate this action to the action of $\mathbf{D}$ on $A^T_*(X)$. For this, we associate to Y and w as above, an integer $d(Y, w)$: If the map $BwB \times_B Y \to BwY$ is generically finite (i.e., if

$\dim(BwY) = \dim(Y) + l(w))$, then $d(Y, w)$ is its degree; otherwise, $d(Y, w) = 0$. Observe that $d(Y, s_\alpha) = d(Y, \alpha)$ with notation as in 6.1.

Proposition. *Let X be a spherical variety. Then, for any $w \in W$ and $Y \in \mathcal{B}(X)$, the integer $d(Y, w)$ is 0 or a power of 2. Moreover, we have in $A_*^T(X)$:*

$$D_w[Y] = d(Y, w)[w * Y] \ .$$

Proof. Choose a reduced decomposition $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l}$. Then the map

$$\overline{Bs_{\alpha_1}B} \times_B \overline{Bs_{\alpha_2}B} \times_B \cdots \times_B \overline{Bs_{\alpha_l}B} \to \overline{BwB}$$

is birational. It follows that

$$d(Y, w) = d(Y, \alpha_1)\, d(Y, \alpha_2) \cdots d(Y, \alpha_l)$$

which implies the first statement, using Proposition 6.2 (iii). The second statement follows from Theorem 6.3 (ii).

Remark. Call $Y \in \mathcal{B}(X)$ *induced* if there exists $w \in W$ and $Z \in \mathcal{B}(X)$, such that $Y = w * Z$ and that $Z \neq Y$. If Y is not induced, call it *cuspidal*. By the proposition above, the **D**-module $A_*^T(X)$ is generated by classes of cuspidal B-orbit closures. This raises the question of their description. In the case where X is a unique orbit of G, observe that any closed B-orbit is cuspidal. The converse holds in G/B by the Bruhat decomposition, and also in symmetric spaces by [R-S] Theorem 4.6, but not in general.

References

[Ar] A. Arabia: *Cycles de Schubert et cohomologie équivariante de K/T*, Invent. math. **85** (1986), 39-52.

[B-G-G] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand: *Schubert cells and cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), 1-26.

[Bi1] A. Bialynicki-Birula: *Some theorems on actions of algebraic groups*, Ann. Math. **98** (1973), 480-497.

[Bi2] A. Bialynicki-Birula: *On fixed points of torus actions on projective varieties*, Bull. Acad. Polon. Sci. Séri. Sci. Math. Astronom. Phys. **22** (1974), 1097-1101.

[Bi3] A. Bialynicki-Birula: *Some properties of the decomposition of algebraic varieties determined by the action of a torus*, Bull. Acad. Polon. Sci. Séri. Sci. Math. Astronom. Phys. **24** (1976), 667-674.

[B-D-P] E. Bifet, C. De Concini and C. Procesi: *Cohomology of regular embeddings*, Adv. Math. **82** (1990), 1-34.

[Bo] A. Borel: *Linear Algebraic Groups*, second enlarged edition, Springer-Verlag, New York 1991.

[B-B-M] W. Borho, J-L. Brylinski and R. MacPherson: *Nilpotent Orbits, Primitive Ideals, and Characteristic Classes*, Progress in Math. **78**, Birkhäuser, Boston 1989.

[B-E] P. Bressler and S. Evens: *The Schubert calculus, braid relations, and generalized cohomology*, Trans. AMS **317** (1990), p. 799-811.

[Br1] M. Brion: *Piecewise polynomial functions, convex polytopes and enumerative geometry*, in: Parameter spaces, Banach Center Publications **36** (1996), p. 25-44.

[Br2] M. Brion: *The push-forward and Todd class for flag bundles*, in: Parameter spaces, Banach Center Publications **36** (1996), p. 45-50.

[B-V] M. Brion and M. Vergne: *An equivariant Riemann-Roch theorem for complete, simplicial toric varieties*, preprint 1995.

[Ca] J. B. Carrell: *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, in: Proc. Symp. in Pure Math. **56** (1994), Part 1, p. 53-31.

[D1] M. Demazure: *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. math. **21** (1973), p. 287-301.

[D2] M. Demazure: *Désingularisation des variétés de Schubert généralisées*, Ann. Scient. Éc. Norm. Sup. **7** (1974), p. 53-88.

[E-G 1] D. Edidin and W. Graham: *Equivariant intersection theory*, preprint 1996.

[E-G 2] D. Edidin and W. Graham: *Localization in equivariant intersection theory and the Bott residue formula*, preprint 1996.

[F1] W. Fulton: *Intersection Theory*, Ergebnisse der Math. **2**, Springer-Verlag, New York 1984.

[F2] W. Fulton: *Introduction to Toric Varieties*, Annals of Math. Studies **131**, Princeton University Press, 1993.

[F3] W. Fulton: *Flags, Schubert polynomials, degeneracy loci and determinantal formulas*, Duke Math. J. **65** (1992), 381-420.

[F4] W. Fulton: *Schubert varieties in flag bundles for classical groups*, in: Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry, Bar-Ilam University 1996, p. 241-262.

[F-M-S-S] W. Fulton, R. MacPherson, F. Sottile and B. Sturmfels: *Intersection theory on spherical varieties*, J. Alg. Geom. **4** (1995), 181-193.

[G-K-M] M. Goresky, R. Kottwitz and R. MacPherson: *Equivariant cohomology, Koszul duality, and the localization theorem*, preprint 1996.

[Gr] W. Graham: *The class of the diagonal in flag bundles*, preprint 1996.

[Jo] A. Joseph: *On the variety of a highest weight module*, J. Algebra **88** (1984), 238-278.

[Kn] F. Knop: *On the Set of Orbits for a Borel Subgroup*, Comment. Math. Helv. **70** (1995), 285-309.

[Ku] S. Kumar: *The nil Hecke ring and singularity of Schubert varieties*, Invent. math. **123** (1996), 471-506.

[L-P] P. Littelmann and C. Procesi: *Equivariant Cohomology of Wonderful Compactifications*, in: Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Birkhäuser, Basel 1990.

[N1] M. Nyenhuis: *Equivariant Chow Groups and Multiplicities*, preprint 1996.

[N2] M. Nyenhuis: *Equivariant Chow Groups and Equivariant Chern Classes*, preprint 1996.

[Po] P. Polo: *On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar*, Indag. Math. **5** (1994), p. 483-493.

[Pr] P. Pragacz: *Symmetric polynomials and divided differences in formulas of intersection theory*, in: Parameter Spaces, Banach Center Publications **36** (1996).

[Pr-Ra] P. Pragacz and J. Ratajski: *Formulas for Lagrangian and orthogonal degeneracy loci: the $\tilde{Q}$-polynomials approach*, Composition Math., to appear.

[R-S] R. W. Richardson and T. A. Springer: *The Bruhat order on symmetric varieties*, Geometriae Dedicata **35** (1990), 389-436.

[Ro] W. Rossmann: *Equivariant multiplicities on complex varieties*, Astérisque **173-174** (1989), 313-330.

[Sm] W. Smoke: *Dimension and multiplicity for graded algebras*, J. Algebra **21** (1972), 149-173.

[St] E. Strickland: *A vanishing theorem for group compactifications*, Math. Ann. **277** (1987), 165-171.

[Vi] A. Vistoli: *Characteristic classes of principal bundles in algebraic intersection theory*, Duke Math. J. **58** (1989), 299-315.