THE ANDREOTTI-VESENTINI SEPARATION THEOREM AND GLOBAL HOMOTOPY REPRESENTATION

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0. Introduction and statement of the result

Let X be an n-dimensional complex manifold and E a holomorphic vector bundle over X.

Notation. We denote by $C^0_{s,r}(X,E)$ the space of continuous E-valued (s,r)-forms on X (we omit E, when E is the trivial line bundle), by $Z^0_{s,r}(X,E)$ the subspace of $\overline{\partial}$-closed forms, and by $E^0_{s,r}(X,E)$ the subspace of $\overline{\partial}$-exact forms ($E^0_{s,0}(X,E) := \{0\}$). As usual,

$$H^{s,r}(X,E) := Z^0_{s,r}(X,E)/E^0_{s,r}(X,E).$$

0.1. Definition. X will be called *q-concave-q^*-convex* where q, q^* are integers with $1 \le q \le n-1$ and $0 \le q^* \le n-1$ if there exists a real C^2 function ρ on X such that if

$$a := \inf_{\zeta\in X} \rho(\zeta) \quad \text{and} \quad b := \sup_{\zeta\in X} \rho(\zeta)$$

and $a < \alpha < \beta < b$, then the set $\{\zeta \in X \mid \alpha \le \rho(\zeta) \le \beta\}$ is compact and the following two conditions are fulfilled:

(i) There exists $\alpha_0 \in]a,b[$ such that if $\xi \in X$ with $\rho(\xi) \le \alpha_0$, then the Levi form of ρ at ξ has at least $n-q+1$ positive eigenvalues.

(ii) If $q^* = 0$, then, for all $\alpha \in]a,b[$, the set $\{\zeta \in X \mid \rho(\zeta) \ge \alpha\}$ is compact, i.e. X is q-concave in the sense of Andreotti-Grauert (resp. $(n-q)$-concave in the sense of [H-L]). If $1 \le q^* \le n-1$, then there exists $\beta_0 \in]a,b[$ such that, for all $\xi \in X$ with $\rho(\xi) \ge \beta_0$, the Levi form of ρ at ξ has at least $n - q^* + 1$ positive eigenvalues.

The starting point of this paper is the following

0.2. Theorem. *Suppose X is q-concave-q^*-convexe where $1 \le q \le n-1$ and $0 \le q^* \le n-q-1$. Then:*

(i) $E^0_{0,n-q}(X,E)$ is closed with respect to uniform convergence on the compact subsets of X.

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(ii) $\dim H^{0,r}(X, E) < \infty$ *if* $q^* \le r \le n-q-1$.

Part (ii) of this theorem is the well-known Andreotti-Grauert finiteness theorem [A-G]. Part (i), which we call the *Andreotti-Vesentini theorem*, was proved by Andreotti and Vesentini [A-V] for $q^* = 0$. The case $q^* > 0$ can be found in the paper [R] of J.-P. Ramis, where the more general situation of sheaves on complex spaces is studied.

For the sake of completeness, we begin this paper with a direct proof of Theorem 0.2 (i): Using integral operators of the Grauert-Henkin-Lieb type, we first prove that $\dim H_c^{0,q+1}(X, E) < \infty$. The separability of $H^{0,n-q}(X, E)$ then follows by Serre duality. We want to point out that, although we use integral formulas which are famous for their uniform estimates up to the boundary, in this proof, we do not use any such estimate. Working with forms with compact support, we need only simple well-known estimates for the Bochner-Martinelli operator.

0.3. Definition. An open set $\Omega \subset\subset X$ will be called *strictly q-concave-q^*-convex*, $1 \le q \le n-1, 0 \le q^* \le n-1$, if there exists a real C^3-function ρ on X such that

$$\Omega = \{\zeta \in X \mid 0 < \rho(\zeta) < 1\},$$

$d\rho(\zeta) \neq 0$ for $\zeta \in \partial\Omega$, and the following two conditions are fulfilled:

(i) For each $\xi \in \partial\Omega$ with $\rho(\xi) = 0$, the Levi form of ρ at ξ has at least $n - q + 1$ positive eigenvalues.

(ii) If $q^* = 0$, then $\rho(\zeta) < 1$ for all $\zeta \in X$, i.e. Ω is strictly q-concave in the sense of Andreotti-Grauert (resp. strictly $(n-q)$-concave in the sense of [H-L]). If $1 \le q^* \le n-1$, then, for each $\xi \in \partial\Omega$ with $\rho(\xi) = 1$, the Levi form of ρ at ξ has at least $n - q^* + 1$ positive eigenvalues.

Notation. If $D \subset\subset X$ is open, then we denote by $C^0_{s,r}(\overline{D}, E)$ the Banach space of continuous (s, r)-forms on $\overline{D}$, and $C^\alpha_{s,r}(\overline{D}, E)$, $0 < \alpha < 1$, will denote the Banach space of Hölder α-continuous (s, r)-forms on $\overline{D}$. Further, then we set

$$Z^0_{s,r}(\overline{D}, E) = C^0_{s,r}(\overline{D}, E) \cap Z^0_{s,r}(D, E),$$

$$E^{1/2\to 0}_{s,r}(\overline{D}, E) = C^0_{s,r}(\overline{D}, E) \cap \overline{\partial} C^{1/2}_{s,r-1}(\overline{D}, E) \text{ if } r \ge 1, \qquad E^{1/2\to 0}_{s,0}(\overline{D}, E) = \{0\},$$

$$H^{s,r}_{1/2\to 0}(\overline{D}, E) = Z^0_{s,r}(\overline{D}, E)/E^{1/2\to 0}_{s,r}(\overline{D}, E).$$

The spaces $Z^0_{s,r}(\overline{D}, E)$ and $E^{1/2\to 0}_{s,r}(\overline{D}, E)$ will be considered as normed spaces with the max-norm. $Z^0_{s,r}(\overline{D}, E)$ then is a Banach space.

Using now also the uniform estimates up to the boundary for the Grauert-Henkin-Lieb operators mentioned above, from Theorem 0.2 one can deduce the following version with uniform estimates:

0.4. Theorem. *Let $\Omega \subset\subset X$ be a strictly q-concave-q^*-convex open set such that $1 \le q \le n-1$ and $0 \le q^* \le n-q-1$. Then*

(i) $E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$ *is closed with respect to uniform convergence on* $\overline{\Omega}$.

(ii) $\dim H^{0,r}_{1/2\to 0}(\overline{\Omega}, E) = \dim H^{0,r}(\Omega, E) < \infty$ *if* $q^* \le r \le n-q-1$.

Part (ii) can be obtained using the results of subsections 15.2, 15.3, 22.2 of [H-L]; essentially it is contained already in [F-Li, Ra-Si, Ho, Li]. Part (i) follows by means of 12.4 (iii) in [H-L] (for the q^*-convex part of the boundary) and 3.1 in [La-L] (for the q-concave part of the boundary). Note that uniform estimates as in Theorem 0.4 (i) were proved for the first time in Sect. 19 of [H-L], but the conditions on Ω supposed in [H-L] are not the same as in the present paper, also the proof in [H-L] is essentially more complicated than the arguments mentioned above.

Recall that, for $q^* = n-q$, statement (i) in Theorems 0.2 and 0.4 is not true in general. This follows from the example of Rossi [Ros] (see also Sect. 24 in [H-L]) which is a strictly 1-concave-1-convex domain Ω in a 2-dimensional complex manifold with a “hole which cannot be repaired”. For this Ω, $E^{1/2\to 0}_{0,1}(\overline{\Omega})$ and $E^0_{0,1}(\Omega)$ are not closed. In fact, in Sect. 23 of [H-L] it is shown that $E^{1/2\to 0}_{0,1}(\overline{\Omega})$ cannot be closed. By the arguments mentioned above (12.4 (iii) in [H-L] and 3.1 in [La-L]), then also $E^0_{0,1}(\Omega)$ cannot be closed. However statement (i) in Theorems 0.2 and 0.4 becomes true also for $q^* = n-q$, if one supposes that “the hole can be repaired” (cf. Sect. 19 in [H-L]).

The purpose of the present paper is to prove the following global homotopy representation:

0.5. Theorem. *Let $\Omega \subset\subset X$ be a strictly q-concave-q^*-convex open set such that $1 \le q \le n-1$ and $0 \le q^* \le n-q-1$. Then there exist continuous linear operators*

$$T_{n-q} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,n-q-1}(\overline{\Omega}, E),$$
$$T_r : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r-1}(\overline{\Omega}, E), \qquad q^* \le r \le n-q-1$$

($T_r := 0$ if $r = q^* = 0$) and continuous linear projections P_r in $C^0_{0,r}(\overline{\Omega}, E)$, $q^* \le r \le n-q-1$, with

$$\dim \operatorname{Im} P_r < \infty, \quad \operatorname{Im} P_r \subseteq Z^0_{0,r}(\overline{\Omega}, E) \quad \textit{and} \quad \operatorname{Ker} P_r \supseteq E^{1/2\to 0}_{0,r}(\overline{\Omega}, E) \tag{0.1}$$

such that

$$\overline{\partial} T_r f + T_{r+1} \overline{\partial} f = f - P_r f \tag{0.2}$$

for all $f \in C^0_{0,r}(\overline{\Omega}, E)$ such that $\overline{\partial} f$ is also continuous on $\overline{\Omega}$, $q^* \le r \le n-q-1$.

Note that Theorem 0.4 is contained in Theorem 0.5. In fact, from (0.1) and (0.2) it follows that

$$Z^0_{0,r}(\overline{\Omega}, E) = E^{1/2\to 0}_{0,r}(\overline{\Omega}, E) \oplus Im\ P_r \quad \text{if} \quad q^* \le r \le n-q-1, \tag{0.3}$$

$$\overline{\partial} T_r f = f \qquad \text{if} \quad f \in E^{1/2\to 0}_{0,r}(\overline{\Omega}, E) \quad \text{and} \quad q^* \le r \le n-q. \tag{0.4}$$

Equation (0.3) shows that

$$\dim H^{0,r}_{1/2\to 0}(\overline{\Omega}, E) = \dim \operatorname{Im} P_r < \infty \qquad \text{if } q^* \le r \le n-q-1,$$

and (0.4) shows that $E^{1/2\to 0}_{0,r}(\overline{\Omega}, E)$ is closed if $q^* \le r \le n-q$.

For forms of bidegree $(0, r)$ with $\max(1, q^*) \le r \le n-q-1$, a local counterpart of the global homotopy representation (0,1), (0.2) is well-known. Namely, from [F-Li, Ra-Si, Ho, Li] (see also 7.8, 9.1, 13.6 and 14.1 in[H-L]) the following proposition follows:

0.6. Proposition. *If $\Omega \subset\subset X$ is a strictly q-concave-q^*-convex domain with $q^* \le n-q-2$, then, for each $\xi \in \partial\Omega$ and any neighborhood U_ξ of ξ there exist a neighborhood $V_\xi \subseteq U_\xi$ of ξ and continuous linear operators*

$$T^\xi_r : C^0_{0,r}(\overline{U_\xi \cap \Omega}) \longrightarrow C^{1/2}_{0,r-1}(\overline{V_\xi \cap \Omega}), \qquad \max(1, q^*) \le r \le n-q-2,$$

with

$$\overline{\partial} T^\xi_r f + T^\xi_{r+1} \overline{\partial} f = f|_{\overline{V_\xi \cap \Omega}} \tag{0.5}$$

for all $f \in C^0_{0,r}(\overline{U_\xi \cap \Omega}, E)$ such that $\overline{\partial} f$ is also continuous on $\overline{U_\xi \cap \Omega}$, $\max(1, q^*) \le r \le n-q-2$.

It is clear that for $r = q^* = 0$ there is no such local representation. But also for forms of bidegree $(0, q-1)$ such a representation cannot exist:

0.7. Proposition. *Let $\Omega \subset\subset X$ be a strictly q-concave-q^*-convex domain with the following property: If ρ is as in Def. 0.3, then there is a point $\xi \in \partial\Omega$ such that $\rho(\xi) = 0$ and the Levi form of ρ at ξ restricted to the complex tangent space of $\partial\Omega$ has $n-q$ positive and $q-1$ negative eigenvalues. Then we can find a neighborhood U_ξ of ξ such that, for no neighborhood $V_\xi \subseteq U_\xi$ of ξ, there exist continuous linear operators*

$$T^\xi_r : C^0_{0,r}(\overline{U_\xi \cap \Omega}) \longrightarrow C^0_{0,r-1}(\overline{V_\xi \cap \Omega}), \qquad r = n-q-1, n-q, \tag{0.6}$$

with

$$\overline{\partial} T^\xi_{n-q-1} f + T^\xi_{n-q} \overline{\partial} f = f|_{\overline{V_\xi \cap \Omega}} \tag{0.7}$$

for all $f \in C^0_{0,n-q-1}(\overline{U_\xi \cap \Omega})$ such that $\overline{\partial} f$ is also continuous on $\overline{U_\xi \cap \Omega}$.

Proof. By a theorem of Andreotti and Hill [A-Hi] (see also Theorem 18.5 in [H-L]), there exist a neighborhood Θ_ξ of ξ and a form $g \in Z^0_{0,q}(\overline{\Theta_\xi \cap \Omega})$ such that there is no neighborhood V_ξ of ξ for which $g|_{V_\xi \cap \Omega}$ is $\overline{\partial}$-exact. From the approximation theorem 10.1 in [H-L] then we obtain a neighborhood $U_\xi \subseteq \Theta_\xi$ of ξ and a sequence $g_j \in Z^0_{0,n-q}(U_\xi)$ converging to g, uniformly on $\overline{U_\xi \cap \Omega}$. We may assume that U_ξ is pseudoconvex and hence $g_j = \overline{\partial} u_j$ for some $u_j \in C^0_{0,n-q-1}(U_\xi)$.

Assume now that there exist a neighborhood $V_\xi \subseteq U_\xi$ of ξ and continuous linear operators (0.6) satisfying (0.7). Then

$$\overline{\partial} T^\xi_{n-q-1} u_j + T^\xi_{n-q} g_j = u_j|_{V_\xi \cap \Omega}$$

and hence $\overline{\partial} T^\xi_q g_j = g_j|_{V_\xi \cap \Omega}$ for all j. Since $g_j \to g$ and $T^\xi_q g_j \to T^\xi_q g$, uniformly on $V_\xi \cap \Omega$, this implies that $\overline{\partial} T^\xi_q g = g|_{V_\xi \cap \Omega}$ which is impossible, for $g|_{V_\xi \cap \Omega}$ is not $\overline{\partial}$-exact. □

Note that, essentially, Proposition 0.7 is already contained in Remark 18.7 of [H-L]. Note also that Proposition 0.7 is closely related to the observation of Nagel and Rosay [Na-Ro] (see also [T]) that on certain CR submanifolds of $\mathbb{C}^n$ there is a maximal "small" degree for which the tangential Cauchy-Riemann equation admits local right inverses but no local homotopy representations.

Local integral representations of the Grauert-Henkin-Lieb type form the main tool of this paper. Since such representations exist also on certain CR manifolds, the constructions of the present paper admit corresponding generalizations to CR manifolds. Statements of the Andreotti-Vesentini type on CR manifolds are already known. Henkin [H 1] obtained an Andreotti-Vesentini theorem for imbedded compact q-concave CR manifolds. Then Hill and Nacinovich [Hi-N] proved this for the abstract situation. In the survey article [H 2] of Henkin, one can find also an Andreotti-Vesentini theorem for manifolds of the form $M \setminus \overline{\Omega}$ where M is an imbedded compact q-concave CR manifold and Ω is a strictly pseudoconvex domain in the ambient complex manifold. The method of Sect. 1 below can be used to prove this result of Henkin (and certain generalization of it).

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1. Proof of the Andreotti-Vesentini theorem

Here we prove Theorem 0.2 (i). By Serre duality [S] (with respect to Dolbeault cohomology, see also Sect. 20 in [H-L]) for that it is sufficient to prove the following

1.1. Theorem. *Let X be an n-dimensional q-concave-$(n-q-1)$-convex complex manifold, $1 \le q \le n-1$. Then, for all holomorphic vector bundles E over X,*

$$\dim H^{0,q+1}_c(X,E) < \infty \tag{1.1}$$

where $H^{0,q+1}_c(X,E)$ denotes the Dolbeault group of bidegree $(0,q+1)$ for E-valued forms with compact support in X.

First we introduce some notations and prove some lemmas.

Let X be a complex manifold and E a holomorphic vector bundle over X. If $D \subset\subset X$ is open, then we denote by $C^\alpha_{s,r}(\overline{D}; X, E)$, $0 \le \alpha < 1$, the Banach space of forms $f \in C^0_{s,r}(X,E)$ with

$$\operatorname{supp} f \subseteq \overline{D} \qquad \text{and} \qquad f|_{\overline{D}} \in C^\alpha_{s,r}(\overline{D}, E).$$

$C^\alpha_{s,r}(X,E)$ denotes the Fréchet space of forms $f \in C^0_{s,r}(X,E)$ such that $f|_{\overline{D}} \in C^\alpha_{s,r}(\overline{D},E)$ for each open $D \subset\subset X$, endowed with the topology of convergence in each $C^\alpha_{s,r}(\overline{D},E)$.

If Y is an arbitrary subset of X, then we denote by $C^\alpha_{s,r}(Y;X,E)$ the subspace of all $f \in C^\alpha_{s,r}(X,E)$ with $\operatorname{supp} f \subseteq Y$ endowed with the Fréchet topology of $C^\alpha_{s,r}(X,E)$. We set

$$Z^\alpha_{s,r}(Y;X,E) = Z^\alpha_{s,r}(X,E) \cap C^\alpha_{s,r}(Y;X,E).$$

$Z^\alpha_{s,r}(Y;X,E)$ will be considered also as Fréchet space endowed with the topology of $C^\alpha_{s,r}(X,E)$. Note that if Y is compact, then $C^\alpha_{s,r}(Y;X,E)$ and $Z^\alpha_{s,r}(Y;X,E)$ are Banach spaces.

$C^\alpha_{s,r}(c;X,E)$ denotes the linear subspace of $C^\alpha_{s,r}(X,E)$ which consists of the forms with compact support. Set

$$Z^\alpha_{s,r}(c;X,E) = Z^\alpha_{s,r}(X,E) \cap C^\alpha_{s,r}(c;X,E).$$

These two spaces will be considered as topological vector spaces endowed with the inductive limit topology of the spaces $C^\alpha_{s,r}(K;X,E)$, $K \subset\subset X$ compact.

Further, we denote by $E^0_{s,r}(c;X,E)$ the space of all $\varphi \in C^0_{s,r}(c;X,E)$ of the form $\varphi = \overline{\partial}\psi$ with $\psi \in C^0_{s,r-1}(c;X,E)$ if $r > 0$, and we set $E^0_{s,r}(c;X,E) = \{0\}$ if $r = 0$. With these notations, $H^{s,r}_c(X,E) = Z^0_{s,r}(c;X,E)/E^0_{s,r}(c;X,E)$.

1.2. Lemma. *Let X be an n-dimensional complex manifold, E a holomorphic vector bundle over X, and ρ a real C^2 function on X whose Levi form has at least $n-q+1$ positive eigenvalues everywhere on X, $1 \le q \le n-1$, such that if*

$$a := \inf_{\zeta \in X} \rho(\zeta) \qquad \text{and} \qquad b := \sup_{\zeta \in X} \rho(\zeta),$$

then, for all $\alpha, \beta \in]a,b[$, the set $\{\alpha \le \rho \le \beta\}$ is compact. Then, for all $\alpha, \beta \in]a,b[$ with $\alpha < \beta$ and for any $\delta > 0$, the following two assertions hold:

(i) There exists a continuous linear operator

$$T^\alpha_{q+1} : Z^0_{0,q+1}(\{\alpha \le \rho\};X,E) \longrightarrow \bigcap_{\varepsilon > 0} C^{1-\varepsilon}_{0,q}(\{\alpha - \delta \le \rho \le \beta + \delta\};X,E)$$

such that

$$\overline{\partial} T^\alpha_{q+1} f = f \qquad \text{on} \quad \{\rho < \beta\}$$

for all $f \in Z^0_{0,q+1}(\{\alpha \le \rho\};X,E)$.

(ii) There exists a continuous linear operator

$$T^{\beta}_{n-q} : Z^0_{0,n-q}(\{\rho \le \beta\}; X, E) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-q-1}(\{\alpha - \delta \le \rho \le \beta + \delta\}; X, E)$$

such that

$$\overline{\partial} T^{\beta}_{n-q} f = f \qquad on \quad \{\rho > \alpha\}$$

for all $f \in Z^0_{0,n-q}(\{\rho \le \beta\}; X, E)$.

Proof. Part (i): Lemmas 12.3 and 12.4 (iii) in [H-L] immediately imply the following statement: If $f \in Z^0_{0,q+1}(\{\alpha \le \rho\}; X, E)$, then there exists $u \in C^0_{0,q}(X, E)$ with $\overline{\partial} u = f$ on $\{\rho < \beta\}$. Moreover, the proof of Lemma 12.4 (iii) in [H-L] (see page 107 in [H-L]) shows that this solution can be given by an operator T^{α}_{q+1} as required. Part (ii): Theorem 16.1 in [H-L] says that, for each $f \in Z^0_{0,n-q}(\{\rho \le \beta\}; X, E)$, there exists $u \in C^0_{0,n-q-1}(\{\rho \le \beta + \delta\}; X, E)$ with $\overline{\partial} u = f$ on $\{\rho > \alpha\}$. The proof of this Theorem (see page 145 in [H-L]) shows that this solution can be given by an operator T^{β}_{n-q} as required . $\square$

1.3. Theorem. *Let X be an n-dimensional q-concave-$(n-q-1)$-convex complex manifold, $1 \le q \le n-1$, E a holomorphic vector bundle over X, and let ρ, a, b be as in Def. 0.1. Further, let $\alpha, \alpha_0, \beta_0, \beta \in \mathbb{R} \cup \{\infty\}$ be given such that: $a < \alpha < \alpha_0$ and α_0 is as in condition (i) of Def. 0.1, $\beta_0 = \beta = \infty$ if $q = n-1$, $\alpha_0 < \beta_0 < \beta < b$ and β_0 is as in condition (ii) of Def. 0.1 if $q \le n-1$.*

Then, for all $\delta > 0$, there exist continuous linear operators

$$T_{q+1} : Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,q}(\{\alpha - \delta \le \rho \le \beta + \delta\}; X, E)$$

and

$$K_{q+1} : Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E) \longrightarrow \bigcap_{\varepsilon>0} Z^{1-\varepsilon}_{0,q+1}(\{\alpha_0 \le \rho \le \beta_0\}; X, E)$$

such that

$$\overline{\partial} T_{q+1} f = f + K_{q+1} f$$

for all $f \in Z^0_{0,q+1}(\{\alpha < \rho < \beta\}; X, E)$.

Proof. We may assume that δ is sufficiently small. Then

$$\alpha_0 + \delta < \beta_0 - \delta$$

and, by Lemma 1.2, there exist continuous linear operators

$$T^{\alpha}_{q+1} : Z^0_{0,q+1}(\{\alpha < \rho\}; X, E) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,q}(\{\alpha - \delta < \rho < \alpha_0 + \delta\}; X, E)$$

and

$$T^{\beta}_{q+1} : Z^0_{0,q+1}(\{\rho < \beta\}; X, E) \longrightarrow \bigcap_{\varepsilon > 0} C^{1-\varepsilon}_{0,q}(\{\beta_0 - \delta < \rho < \beta + \delta\}; X, E)$$

such that

$$\overline{\partial} T^{\alpha}_{q+1} f = f \qquad \text{on} \quad \left\{\rho < \alpha_0 + \frac{\delta}{2}\right\}$$

and

$$\overline{\partial} T^{\beta}_{q+1} f = f \qquad \text{on} \quad \left\{\beta_0 - \frac{\delta}{2} < \rho\right\}.$$

Take a C^∞ partition of unity $\chi_\alpha, \chi_\beta, \chi_1, \dots, \chi_N$ on X such that

(a) $\chi_\alpha \equiv 1$ in a neighborhood of $\{\rho \le \alpha_0\}$ and $\chi_\alpha \equiv 0$ in a neighborhood of $\{\alpha_0 + \delta/2 \le \rho\}$;

(b) $\chi_\beta \equiv 1$ in a neighborhood of $\{\beta_0 \le \rho\}$ and $\chi_\beta \equiv 0$ in a neighborhood of $\{\rho \le \beta_0 - \delta/2\}$;

(c) for $1 \le j \le N$, the support of χ_j is contained in certain open ball $U_j \subset\subset \{\alpha_0 < \rho < \beta_0\}$ (with respect to some holomorphic coordinates in a neighborhood of $\overline{U_j}$).

Since the U_j are balls, we have continuous linear operators

$$T^{j}_{q+1} : Z^0_{0,q+1}(X, E) \longrightarrow \bigcap_{\varepsilon > 0} C^{1-\varepsilon}_{0,q}(U_j, E), \qquad 1 \le j \le N,$$

such that

$$\overline{\partial} T^{j}_{q+1} f = f|_{U_j}$$

for all $f \in Z^0_{0,q+1}(X, E)$ (see, e.g., 2.11 in [H-L]).

Now the operators

$$T_{q+1} := \chi_\alpha T^{\alpha}_{q+1} + \chi_\beta T^{\beta}_{q+1} + \sum_{j=1}^{N} \chi_j T^{j}_{q+1}$$

and

$$K_{q+1} := \overline{\partial}\chi_\alpha \wedge T^{\alpha}_{q+1} + \overline{\partial}\chi_\beta \wedge T^{\beta}_{q+1} + \sum_{j=1}^{N} \overline{\partial}\chi_j \wedge T^{j}_{q+1}$$

have the required properties. □

Proof of Theorem 1.1. Let ρ, a, b, α, α_0, β_0, β, δ, T_r, K_r be as is Theorem 1.4 where δ is so small that $a < \alpha - \delta$ and $\beta + \delta < b$. Then $Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E)$ is a Banach space,

$$T_{q+1}\left(Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E)\right) \subseteq C^0_{0,q}(c; X, E) \tag{1.2}$$

and, by Ascoli's theorem, K_{q+1} is compact as an operator acting from $Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E)$ into itself. Since $\overline{\partial} T_{q+1} = id + K_{q+1}$ on this space, it follows that $\overline{\partial} T_{q+1}$ is a Fredholm operator in $Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E)$. Hence

$$\overline{\partial} T_{q+1} \left(Z^0_{0,r}(\{\alpha \le \rho \le \beta\}; X, E) \right)$$

is of finite codimension in $Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E)$. Since, by (1.2),

$$\overline{\partial} T_{q+1} \left(Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E) \right) \subseteq E^0_{0,q+1}(c; X, E),$$

this implies that

(1.3) $$\dim \left[Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E) \Big/ \left(E^0_{0,q+1}(c; X, E) \cap Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E) \right) \right] < \infty.$$

Moreover, Theorem 1.3 implies the relation

(1.4) $$Z^0_{0,q+1}(c; X, E) = \text{linear hull of} \quad E^0_{0,q+1}(c; X, E) \cup Z^0_{0,q+1}(\{\alpha \le \rho \le \beta\}; X, E).$$

(1.1) now follows from (1.3) and (1.4). □

2. Proof of the global homotopy representation (Theorem 0.5)

In this section, E is a holomorphic vector bundle over an n-dimensional complex manifold X and $\Omega \subset\subset X$ is a relatively compact C^3 domain in X which is strictly q-concave-q^*-convex where $1 \le q \le n-1$ and $0 \le q^* \le n-q-1$, i.e. the hypotheses of Theorem 0.5 are fulfilled. Further we assume that ρ is as in Def. 0.3, and we set

$$\partial_0 \Omega = \partial\Omega \cap \{\rho = 0\} \quad \text{and} \quad \partial_1 \Omega = \partial\Omega \cap \{\rho = 1\}.$$

An open set $D \subset\subset X$ will be called a *local q-concave domain* (cf. Def. 2.1.1 in [La-L] - note that there q is on the place of $n-q$) if there exists an open set $U \subset\subset X$ and a real C^3 function φ on U such that, with respect to some holomorphic coordinates $z_1, \dots, z_n$ in a neighborhood of $\overline{U}$, the following holds:

(i) U is convex with respect to the underlying real coordinates;

(ii) $D = \{0 < \varphi < 1\} \subset\subset U$, $\partial D = \{\varphi = 0\} \cup \{\varphi = 1\}$, $\{\varphi = 0\} \ne \emptyset$, $\{\varphi = 1\} \ne \emptyset$, and $d\varphi(\zeta) \ne 0$ for all $\zeta \in \partial D$;

(iii) on $\{1 \le \varphi\}$, φ is strictly convex with respect to the underlying real coordinates of $z_1, \dots, z_n$ and, everywhere on U, φ is strictly convex with respect to the underlying real coordinates of $z_1, \dots, z_{n-q+1}$.

If $D \subset\subset X$ is a local q-concave domain, then we set

$$\partial_0 D = \{\varphi = 0\} \qquad \text{and} \qquad \partial_1 D = \{\varphi = 1\}.$$

A collection $[D; U, V, V', V'', \lambda]$ will be called a $\partial_0\Omega$-*adapted* local q-concave domain if $D \subset\subset X$ is a local q-concave domain, $V \subset\subset V' \subset\subset V'' \subset\subset X$ are open sets, and λ is a real C^∞-function on X such that

$$
\begin{aligned}
& V \cap \partial D = V \cap \partial_0 D = V \cap \partial_0\Omega, \quad \overline{V} \cap \overline{D} = \overline{V} \cap (\Omega \cup \partial_0\Omega), \\
& \overline{D} \cap \overline{\Omega} \subseteq (\partial_0\Omega \cup \Omega), \quad (\overline{V''} \setminus V') \cap \overline{D} \subset\subset \Omega, \\
& \operatorname{supp} \lambda \subset\subset V'', \quad \lambda \equiv 1 \quad \text{in a neighborhood of} \quad \overline{V'}.
\end{aligned} \tag{2.1}
$$

It is clear that a local q-concave domain remains such a domain after a pertubation of the boundary which is sufficiently small in the C^3-topology. Therefore from Lemma 2.1.4 in [La-L] we obtain the following

2.1. Lemma. *For each point $\xi \in \partial_0\Omega$ there exists a $\partial_0\Omega$-adapted local q-concave domain $[D; U, V, V', V'', \lambda]$ with $\xi \in V$.*

2.2. Lemma. *For each local q-convex domain $D \subset\subset X$, there exist continuous linear operators*

$$
\begin{aligned}
& T_r : C^0_{0,r}(\overline{D}) \longrightarrow C^{1/2}_{0,r-1}(\overline{D}), \quad 0 \le r \le n-1, \quad (T_0 := 0) \\
& L_r : C^0_{0,r}(\overline{D}) \longrightarrow \bigcap_{\varepsilon > 0} C^{1-\varepsilon}_{0,r}(D), \quad 0 \le r \le n-1,
\end{aligned}
$$

such that the following asssertions hold:

(i) If $f \in C^0_{0,r}(\overline{D})$ such that also $\overline{\partial} f$ is continuous on $\overline{D}$, then

$$
f = L_r f + \overline{\partial} T_r f + T_{r+1} \overline{\partial} f \quad \text{on } D, \qquad 0 \le r \le n.
$$

(ii) If K is a closed subset of $\partial_0 D$ and $f_j \in C^0_{0,r}(\overline{D})$ is a sequence with $f_j = 0$ on $\partial_0 D \setminus K$ which converges uniformly on K to some $f \in C^0_{0,r}(\overline{D})$, then, for each open $U \subseteq D$ with $\overline{U} \subset \overline{D} \setminus K$, $L_r f_j|_{\overline{U}}$ converges to $L_r f|_{\overline{U}}$, in the topology of each $C^{1-\varepsilon}_{0,r}(\overline{U})$, $\varepsilon > 0, 1 \le r \le n$.

(iii) If $1 \le r \le n-q-1$, then $L_r f = 0$ for all $f \in C^0_{0,r}(\overline{D})$. Moreover, $L_0 f = 0$ for all $f \in C^0_{0,0}(\overline{D})$ with $f = 0$ on $\partial_1 D$.

(iv) $L_{n-q} \overline{\partial} u = 0$ if $u \in C^0_{0,n-q-1}(\overline{D})$ such that $\overline{\partial} u$ is also continuous on $\overline{D}$.

Proof. Take for T_r, L_r the operators for D constructed in Sect. 2 of [La-L]. (The fact that $\overline{D}$ is of class C^∞ assumed in [La-L] is not used there for the construction of these operators.) Then the estimates claimed for these operators can be proved as usual (cf., e.g., the proofs of Theorems 9.1 and 14.1 in [H-L]), and (i) holds by Theorem 2.2.4 in [La-L]. By Lemma 2.2.5 (i) in [La-L],

$$
L_r f(z) = \int_{\partial_0 D} f(\zeta) \wedge \Lambda_r(z, \zeta) \tag{2.2}
$$

where the kernel $\Lambda_r(z,\zeta)$ is of class C^1 for $z \ne \zeta$. This implies (ii). Moreover, by Lemma 2.2.5 (iii) and (iv) in [La-L], $\Lambda_r(z,\zeta) \equiv 0$ if $0 \le r \le n-q-1$ and $\overline{\partial}_\zeta \Lambda_q(z,\zeta) \equiv 0$, which implies (iii) and (iv). □

2.3. Lemma. *Let $[D; U, V, V', V'', \lambda]$ be a $\partial_0\Omega$-adapted local q-concave domain, and let L_{n-q} be the operator from Lemma 2.2 for D. Then, by (2.1), we have a continuous linear operator*

$$L_{n-q}^{V\cap\Omega}(\lambda\,\cdot) : C^0_{0,n-q}(\overline{\Omega}) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-q}(V\cap\Omega)$$

defined by $\left[L_{n-q}^{V\cap\Omega}(\lambda\,\cdot)\right](f) = L_{n-q}(\lambda f)|_{V\cap\Omega}$. This operator has the following additional property:

If $f \in E^{1/2\to 0}_{0,n-q}(\overline{\Omega})$, then $L_{n-q}^{V\cap\Omega}(\lambda f) \in \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-q}(\overline{V\cap\Omega})$ and if $E^{1/2\to 0}_{0,n-q}(\overline{\Omega})$ is considered as normed space endowed with the max-norm[1]*, then the operator*

$$L_{n-q}^{V\cap\Omega}(\lambda\,\cdot) : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-q}(\overline{V\cap\Omega})$$

is continuous.

Proof. Let $f_j \in E^{1/2\to 0}_{0,n-q}(\overline{\Omega})$ be a sequence which converges uniformly on $\overline{\Omega}$. Since, by the Andreotti-Vesentini theorem 0.2 (i), $E^0_{0,n-q}(\Omega)$ is closed with respect to uniform convergence on compact sets, then, by Banach's open mapping theorem, there is a sequence $u_j \in C^0_{0,n-q-1}(\Omega)$ which is uniformly convergent on compact sets such that $\overline{\partial} u_j = f_j$ on Ω. By Lemma 2.2 (iv),

$$L_{n-q}(\lambda f_j) = -L_{n-q}(\overline{\partial}\lambda \wedge u_j). \tag{2.3}$$

Set $K = (\overline{V}'' \setminus V') \cap \partial_0 D$. Since, by (2.1), $K \subset\subset \Omega$, $\overline{V\cap\Omega} \subseteq \overline{D}\setminus K$ and $\overline{\partial}\lambda \equiv 0$ outside K, it follows from (2.3) and Lemma 2.2 (ii) that the sequence $L_{n-q}(\lambda f_j)|_{V\cap\Omega}$ converges in $\bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,q}(\overline{V\cap\Omega})$. $\square$

2.4. Lemma. *For each point $\xi \in \overline{\Omega}$, there exist a neighborhood U of ξ and continuous linear operators ($E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$ is considered as normed space endowed with the max-norm)*

$$\begin{aligned}
&A_{n-q} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,n-q-1}(\overline{U\cap\Omega}, E),\\
&K_{n-q} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,n-q}(\overline{U\cap\Omega}, E),\\
&A_r : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r-1}(\overline{U\cap\Omega}, E), \quad q^* \le r \le n-q-1,\\
&K_r : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r}(\overline{U\cap\Omega}, E) \quad q^* \le r \le n-q-1,
\end{aligned} \tag{2.4}$$

($A_0 := 0$ if $q^* = 0$) such that

$$\overline{\partial} A_{n-q} f = f + K_{n-q} f \quad \text{on} \quad \overline{U\cap\Omega} \tag{2.5}$$

[1] We do not use here that $E^{1/2\to 0}_{0,n-q}(\overline{\Omega})$ is a Banach space, as we know from Theorem 0.4 (i). In the proof below we use only that, by Theorem 0.2 (i), $E^0_{0,n-q}(\Omega)$ is a Fréchet space. The fact that $E^{1/2\to 0}_{0,n-q}(\overline{\Omega})$ is a Banach space then follows (see Corollary 2.6 below).

for all $f \in E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$, *and*

$$\overline{\partial} A_r f + A_{r+1}\overline{\partial} f = f + K_r f \quad \text{on} \quad \overline{U \cap \Omega}, \qquad q^* \leq r \leq n-q-1, \tag{2.6}$$

for all $f \in C^0_{0,r}(\overline{\Omega}, E)$ *such that also* $\overline{\partial} f$ *is continuous on* $\overline{\Omega}$.

Proof. If $\xi \in \Omega$, this folows from the Bochner-Martinelli-Koppelman formula. If $\xi \in \partial_1\Omega$, one can use fhe fact that $\partial_1\Omega$ is strictly q^*-convex in the sense of Andreotti-Grauert (see, e.g., Theorems 7.1 and 9.1 in [H-L]).

Now let $\xi \in \partial_0\Omega$. By Lemma 2.1 we have a $\partial_0\Omega$-adapted local q-concave domain $[D; U, V, V', V'', \lambda]$ with $\xi \in V$. Let L_r and T_r be the operators from Lemma 2.2 for D. Set $U = V$. Then it follows from (2.1), the corresponding continuity properties of the operators T_r and from Lemma 2.3 that, by setting

$$A_{n-q} f = T_{n-q}(\lambda f)|_{\overline{U \cap \Omega}} \quad \text{and}$$
$$K_{n-q} f = -T_{n-q+1}(\overline{\partial}\lambda f)|_{\overline{U \cap \Omega}} - L_{n-q}(\lambda f)|_{\overline{U \cap \Omega}} \qquad \text{for} \quad f \in E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E),$$
$$A_r f = T_r(\lambda f)|_{\overline{U \cap \Omega}} \quad \text{and}$$
$$K_r f = -T_{r+1}(\overline{\partial}\lambda f)|_{\overline{U \cap \Omega}} \qquad \text{for} \quad f \in C^0_{0,r}(\overline{\Omega}, E), \quad q^* \leq r \leq n-q-1,$$

one obtains continuous linear operators as in (2.4). Relations (2.5) and (2.6) follow from Lemma 2.2 (i) and (iii). □

2.5. Lemma. *There exist continuous linear operators (* $E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$ *is considered as normed space endowed with the max-norm)*

$$A_{n-q} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,n-q-1}(\overline{\Omega}, E),$$
$$K_{n-q} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,n-q}(\overline{\Omega}, E),$$
$$A_r : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r-1}(\overline{\Omega}, E), \quad q^* \leq r \leq n-q-1, \quad (A_0 := 0 \text{ if } q^* = 0),$$
$$K_r : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r}(\overline{\Omega}, E) \quad q^* \leq r \leq n-q-1,$$

such that

$$\overline{\partial} A_{n-q} f = f + K_{n-q} f \quad \text{on} \quad \overline{\Omega} \tag{2.7}$$

for all $f \in E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$, *and*

$$\overline{\partial} A_r f + A_{r+1}\overline{\partial} f = f + K_r f \quad \text{on} \quad \overline{\Omega}, \qquad q^* \leq r \leq n-q-1, \tag{2.8}$$

for all $f \in C^0_{0,r}(\overline{\Omega}, E)$ *such that also* $\overline{\partial} f$ *is continuous on* $\overline{\Omega}$.

Proof. By Lemma 2.4 there is a finite number of open sets $U_1, \ldots, U_N \subseteq X$ such that $\overline{\Omega} \subseteq U_1 \cup \ldots \cup U_N$ and, for each $j \in \{1, \ldots, N\}$, we have continuous linear operators

$$A_{n-q,j} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,n-q-1}(\overline{U_j \cap \Omega}, E),$$
$$K_{n-q,j} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,n-q}(\overline{U_j \cap \Omega}, E),$$
$$A_{r,j} : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r-1}(\overline{U_j \cap \Omega}, E), \quad q^* \leq r \leq n-q-1,$$
$$K_{r,j} : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r}(\overline{U_j \cap \Omega}, E) \quad q^* \leq r \leq n-q-1,$$

($A_0 := 0$ if $q^* = 0$) such that

$$\overline{\partial} A_{n-q,j} f = f + K_{n-q,j} f \quad \text{on} \quad \overline{U_j \cap \Omega}$$

for all $f \in E^{1/2 \to 0}_{0,n-q}(\overline{\Omega}, E)$, and

$$\overline{\partial} A_r f + A_{r+1} \overline{\partial} f = f + K_r f \quad \text{on} \quad \overline{U_j \cap \Omega}, \qquad q^* \le r \le n-q-1,$$

for all $f \in C^0_{0,r}(\overline{\Omega}, E)$ such that also $\overline{\partial} f$ is continuous on $\overline{\Omega}$.

Take C^∞-functions $\chi_1, \dots, \chi_N$ on X with $\operatorname{supp} \chi_j \subset\subset U_j$ such that $\chi_1 + \ldots + \chi_N \equiv 1$ in a neighborhood of $\overline{\Omega}$, and set

$$A_r = \sum_{j=1}^{N} \chi_j A_{r,j} \qquad \text{and}$$

$$K_r = -\sum_{j=1}^{N} \overline{\partial} \chi_j \wedge A_{r,j}, \qquad q^* \le r \le n-q-1. \quad \square$$

2.6. Corollary. *The spaces $E^{1/2\to 0}_{0,r}(\overline{\Omega}, E)$, $q^* \le r \le n-q$, are closed with respect to uniform convergence on $\overline{\Omega}$, and, for $q^* \le r \le n-q-1$, these spaces are of finite codimension in $Z^0_{0,r}(\overline{\Omega}, E)$. (If $q^* = 0$ this means in particular that the space of holomorphic sections $Z^0_{0,0}(\overline{\Omega}, E)$ is finite dimensional.)*

Proof. Let A_r and K_r, $q^* \le r \le n-q$, be the operators from Lemma 2.5. First we consider the case $r = n-q$. Denote by $\overline{E}^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$ the closure of $E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$ with respect to uniform convergence on $\overline{\Omega}$. Let $\overline{A}_{n-q}$ and $\overline{K}_{n-q}$ be the continuous extensions of A_{n-q} and K_{n-q} to $\overline{E}^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$, respectively. Since $\overline{\partial}$ is closed, then it follows from (2.7) that

$$\overline{\partial}\, \overline{A}_{n-q} f = f + \overline{K}_{n-q} f \quad \text{on} \quad \overline{\Omega} \qquad \text{for all} \quad f \in \overline{E}^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E).$$

Hence

$$\operatorname{Im}(I + \overline{K}_{n-q}) \subseteq E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E).$$

Since, by Ascoli's theorem, $\overline{K}_{n-q}$ is compact as operator in $\overline{E}^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$ and therefore $\operatorname{Im}(I + \overline{K}_{n-q})$ is closed and of finite codimension in $\overline{E}^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$, this implies that $E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) = \overline{E}^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$.

Now let $q^* \le r \le n-q-1$. Then it follows from (2.8) that

$$\overline{\partial} A_r f = f + K_r f \qquad \text{for all} \quad f \in Z^0_{0,r}(\overline{\Omega}, E).$$

Since K_r is compact in $Z^0_{0,r}(\overline{\Omega}, E)$, this implies that $\dim Z^0_{0,0}(\overline{\Omega}, E) < \infty$ if $q^* = 0$, and $E^{1/2\to 0}_{0,r}(\overline{\Omega}, E)$ is of finite codimension in $Z^0_{0,r}(\overline{\Omega}, E)$ if $r \ge 1$. $\square$

Proof of Theorem 0.5. Let $\operatorname{dom}\overline{\partial}_r$ be the space of all $u \in C^0_{0,r}(\overline{\Omega}, E)$ such that $\overline{\partial} u$ is continuous on $\overline{\Omega}$, set $D_r = \operatorname{dom}\overline{\partial}_r \cap C^{1/2}_{0,r}(\overline{\Omega}, E)$ and $D'_r = C^0_{n,n-r}(c; \Omega, E^*)$ where E^* is the dual of E, $q^* \le r \le n-q-1$. Denote by $D'_r \otimes D_{r-1}$, $q^* \le r \le n-q$ the space of operators

$$S : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^{1/2}_{0,r-1}(\overline{\Omega}, E)$$

of the form

$$Sf = \sum_{j=1}^{N} \left[\int_\Omega f \wedge \varphi_j \right] u_j, \qquad f \in C^0_{0,r}(\overline{\Omega}, E),$$

where $u_1, \dots, u_N \in D_{r-1}$, $\varphi_1, \dots, \varphi_N \in D'_r$, $N < \infty$ ($D'_0 \otimes D_{-1} := \{0\}$ if $q^* = 0$).

By Corollary 2.6, we have continuous projections Q_r in $C^0_{0,r}(\overline{\Omega}, E)$ such that

$$\dim \operatorname{Im} Q_r < \infty, \qquad \operatorname{Im} Q_r \oplus E^{1/2\to 0}_{0,r}(\overline{\Omega}, E) = Z^0_{0,r}(\overline{\Omega}, E), \qquad q^* \le r \le n-q-1.$$

Let A_r and K_r, $q^* \le r \le n-q$, be the operators from Lemma 2.5.

Now it follows from the theorem in Sect. 1 of [L] that there exist operators $S_j \in D'_r \otimes D_{r-1}$, $q^* \le r \le n-q$, such that the following holds:

Set

$$\tilde{K}_r = K_r - \overline{\partial} \circ S_r - S_{r+1} \circ \overline{\partial} - Q_r \quad \text{for} \quad q^* \le r \le n-q-1,$$
$$\tilde{K}_{n-q} = K_{n-q} - \overline{\partial} \circ S_{n-q}.$$
$$M_r = I - \tilde{K}_r, \quad \text{for} \quad q^* \le r \le n-q.$$

Then M_{n-q} is an isomorphism of $E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$, the operators M_r, $q^* \le r \le n-q-1$, are isomorphisms of $C^0_{0,r}(\overline{\Omega}, E)$ with $M_r(\operatorname{dom}\overline{\partial}_r) = \operatorname{dom}\overline{\partial}_r$, and, setting

$$T_r = (A_r + S_r) M_r^{-1}, \quad q^* \le r \le n-q, \text{and} \quad P_r = Q_r M_r^{-1}, \qquad q^* \le r \le n-q-1,$$

we obtain continuous linear operators

$$T_{n-q} : E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E) \longrightarrow C^0_{n-q-1}(\overline{\Omega}) \qquad \text{and}$$
$$T_r : C^0_{0,r}(\overline{\Omega}, E) \longrightarrow C^0_{0,r-1}(\overline{\Omega}), \qquad q^* \le r \le n-q-1,$$

and continuous linear projections P_r in $C^0_{0,r}(\overline{\Omega}, E)$, $q^* \le r \le n-q-1$, such that (0.1) and (0.2) hold.

That the operators T_r admit the Hölder 1/2-estimates required in Theorem 0.5 follows from the equation $T_r = (A_r + S_r) M_r^{-1}$ and the fact that A_r and S_r admit these estimates, M_r^{-1}, $q^* \le r \le n-q-1$, is continuous as operator in $C^0_{0,r}(\overline{\Omega}, E)$, and M^{-1}_{n-q} is continuous as operator in $E^{1/2\to 0}_{0,n-q}(\overline{\Omega}, E)$ endowed with the max-norm. □

References

[A-G] A. Andreotti, H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. math. France **90** (1962), 193-259.

[A-Hi] A. Andreotti, C.D. Hill, *E. E. Levi onvexity and the Hans Lewy problem*, Ann. Scuola Norm. Sup. Pisa **26** (1972), no. 2: 325-363, no. 4: 747-806.

[A-V] A. Andreotti, E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Publ. Math. I.H.E.S. **25** (1965), 81-130.

[F-Li] W. Fischer, I. Lieb, *Lokale Kerne und beschränkte Lösungen für den $\overline{\partial}$-Operator auf q-konvexen Gebieten*, Bull. Soc. math. France **208** (1974), 249-265.

[H 1] G. Henkin, *Solution des équations de Cauchy-Riemann tangentielles sur des variétés de Cauchy-Riemann q-concaves*, C. R. Acad. Sc. Paris **292** (1981), 27-30.

[H 2] G. Henkin, *The Method of Integral Representations in Complex Analysis*, Encyclopaedia of Mathematikcal Sciences, Vol. 7, Several Complex Variables I, edited by A.G. Vitushkin, Springer 1990 (1990), 19-116.

[H-L] G. Henkin, J. Leiterer, *Andreotti-Grauert Theory by Integral Formulas*, Birkhäuser, Progress in Mathematics 74, 1988.

[Hi-N] C.D. Hill, M. Nacinovich, *Pseudoconcave CR Manifolds*, Universita di Pisa, Dipartimento di matematica, Sezione di geometria e algebra, Preprint 1.76 (723) Febbraio, 1993.

[Ho] M. Hortmann, *Über die Lösbarkeit der $\overline{\partial}$-Gleichung auf Ringgebieten mit Hilfe von L^p-, $\mathcal{C}^k$- und $\mathcal{D}'$-stetigen Integraloperatoren*, Math Ann. **223** (1976), 139-156.

[La-L] C. Laurent-Thiébaut, J. Leiterer, *The Andreotti-Vesentini Separation Theorem with C^k Estimates and Extension of CR-forms*, Several Complex Variables, Proceedings of the Mittag-Leffler Institute, 1987-1988, edited by John Eric Fornaess (1993), 416-439.

[L] J. Leiterer, *From Local to Global Homotopy Formulas for $\overline{\partial}$ and $\overline{\partial}_b$*, Geometric Complex Analysis, edited by J. Noguchi et al., World Scientific Publishing Co. (1996), 385-391.

[Li] I. Lieb, *Beschränkte Lösungen der Cauchy-Riemannschen Differentialgleichungen auf q-konkaven Gebieten*, manusscripta math **26** (1979), 387-409.

[Na-Ro] A. Nagel, J.-P. Rosay, *Non-existence of Homottopy Formulas for $(0,1)$-Forms on Hypersurfaces in $\mathbb{C}^3$*, Duke J. **58** (1989), 823-827.

[R] J.-P. Ramis, *Théorèmes de séparation et de finitude pour l'homologie et la cohomologie des espaces (p,q)-convexes-concaves*, Ann. Scuola Norm. Super. Pisa **27** (1973), 933-997.

[Ra-Si] R. M. Range, Y.-T. Siu, *C^k-Approximation by Holomorphic Functions and $\overline{\partial}$-Closed forms on C^k-Submanifolds of a Complex Manifold*, Math. Ann. **210** (1974), 105-122.

[Ros] H. Rossi, *Attaching Analytic Spaces to an Analytic Space along a Pseudoconcave Boundary*, Proceedings of the Conference on Complex Analysis, Minneapolis 1964, edited by A. Aeppli et al., Springer-Verlag (1965), 242-256.

[S] G. Schmalz, *Solution of the $\overline{\partial}$-equation on Non-Smooth Strictly q-Concave Domains with Hölder Estimates and the Andreotti-Vesentini Separation Theorem*, Ann. Scuola. Norm. Pisa **18** (1991), 67-82.

[T] F. Trèves, *Homotopy Formulas in the Tangential Cauchy-Riemann Complex*, Memoirs of the AMS **87** (1990).

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