On characteristic cones, clusters and chains of infinitely near points

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1 Introduction

Let $\pi : Z \to X$ be a projective birational morphism of smooth surfaces. Assume π is an isomorphism outside $\pi^{-1}(Q)$ for some closed point $Q \in X$. For each factorization $Z \to Y \to X$, where Y is a normal surface, one has that Y is the blowing up of a complete (i.e. integrally closed) ideal I on X with $I\mathcal{O}_Z$ invertible and support at Q. Since π is the composition of the successive blowing ups of a finite set $\mathcal{C}$ of infinitely near points (we will call $\mathcal{C}$ a constellation) to Q, the theorem of Zariski on unique factorization of complete ideals allow to describe all these sandwiched surfaces Y as well as the contractions $Z \to Y$. Such a contraction becomes the minimal resolution of the singularities of Y (a class of rational singularities called sandwiched).

Zariski asked the question of extending to higher dimensions the theory of complete ideals. Unique factorization is not true in general. Lipman in [7] has showed that one has unique factorization with integer (maybe negative) exponents if the morphism π is obtained by blowing up a constellation of i.n.p. to a smooth point in dimension $d \geq 3$. In this case complete ideals are said to be finitely supported and they are studied in [1] by means of clusters, i.e. integer weighted constellations. For toric constellations, i.e. when X is a toric variety and the points of $\mathcal{C}$ are closed orbits, all the sandwiched varieties can be described in terms of $\mathcal{C}$. If, furthermore, the toric constellation is a chain, i.e. a sequence of i.n.p. such that each one is in the exceptional divisor created by its predecesor, then one has unique factorization with non negative exponents and Zariski's result and conclusions follow in the same way.

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Cutkosky [2] approaches the general case (of projective birational morphisms $\pi : Z \to X$ where Z, X are normal) by means of the characteristic cone $\tilde{P}(Z/X)$, i.e. the cone in $N^1(Z/X) \bigotimes_{\mathbf{Z}} \mathbf{R}$ spanned by the classes of the π-generated line bundles. Since sandwiched varieties correspond one to one to topological cells of $\tilde{P}(Z/X)$ (a result which follows from Kleiman [3]), one can study factorizations properties of complete ideals from the cone structure of $\tilde{P}(Z/X)$ as, according to [6], complete ideals are nothing but sets of global sections of π-generated line bundles. In Zariski's situation the cone $\tilde{P}(Z/X)$ is regular polyhedral (a consequence of unique factorization). Cutkosky gives two examples showing that the cone $\tilde{P}(Z/X)$ could be non polyhedral and even non closed. Both examples are cases of non chain constellations with respective cardinalities 10 and 17. In the first case one has infinitely many sandwiched varieties.

Two natural questions arise. First, to give conditions in order that the number of sandwiched varieties is finite. Second, in the case of constellations which are chains, to investigate if Zariski result holds.

This paper deals with the above two questions. In Section2 we prove that the number of sandwiched varieties is finite if the cone $NE(Z/X)$ of relative nef curves is (finite) polyhedral. For the case of constellations, we prove, in Section3, that $NE(Z/X)$ is polyhedral if $NE(E_i)$ is so for every component E_i of the exceptional divisor of π. If $d = 3$, we derive that $NE(E_i)$ is polyhedral if the set of points blown up to create E_i from the projective plane is either toric or it has cardinality at most 8. Finally in Section4, we show examples of chains proving that the characteristic cone can be polyhedral non simplicial, or regular with semigroup of complete ideals smaller than that of lattice points of $\tilde{P}(Z/X)$, or even non polyhedral and non closed (the cardinalities of involved constellations is at most 10). Thus, Zariski's results of dimension 2 fail in higher dimension even for chains.

2 Characteristic cones, complete ideals and sandwiched varieties

Consider a projective birational morphism $T \to S$ where T and S are normal algebraic varieties over an algebraically closed field k. Let $Q \in S$ be a closed point and set $R = \mathcal{O}_{S,Q}$, $X = SpecR$, $Z = T \times_S X$ and $\pi : Z \to X$ the induced projective birational morphism. Denote by $N_1(Z/X)$ (resp. $N^1(Z/X)$) the abelian group of 1-dimensional cycles on Z whose support contracts to Q (resp. Cartier divisors on Z) modulo numerical equivalence.

Here a one dimensional cycle C (resp. Cartier divisor D) is numerically equivalent to 0 iff one has $C \cdot D = 0$ for all Cartier divisors D (resp. all complete curves contracted to Q) on Z.

Set $A_1(Z/X) = N_1(Z/X) \bigotimes_{\mathbf{Z}} \mathbf{R}$, $A^1(Z/X) = N^1(Z/X) \bigotimes_{\mathbf{Z}} \mathbf{R}$. Since $\pi^{-1}(Q)$ is a projective scheme over k and the vector space $A^1(Z/X)$ maps injectively into $A^1(\pi^{-1}(Q))$, the dimension $\rho(Z/X)$ of $A^1(Z/X)$ is finite and, therefore, the intersection pairing makes $A_1(Z/X)$ and $A^1(Z/X)$ dual vector spaces.

Let $NE(Z/X)$ be the convex cone in $A_1(Z/X)$ generated by the classes of the effective curves in Z which contract to Q. In the dual space $A^1(Z/X)$ we will consider two cones $P(Z/X)$ and $\tilde{P}(Z/X)$. $P(Z/X)$ is the dual cone of $-NE(Z/X)$, i.e. the cone consisting of the vectors l such that $c \cdot l \leq 0$ for every class c of a contracted effective curve in Z. In other words the cone $P(Z/X)$ is minus the semiample relative cone for the morphism π at Q (see [3]). $\tilde{P}(Z/X)$ is the convex cone generated by the classes of the Cartier semiample divisors D such that $\mathcal{O}_Z(-D)$ is generated by their global sections. The cone $\tilde{P}(Z/X)$ is called the characteristic cone for π at Q, the terminology being due to Hironaka.

Since a divisor whose ideal sheaf is generated by its global sections is numerically effective, one has $\tilde{P}(Z/X) \subset P(Z/X)$. On the other hand, according to [3], one has that the topological interior $P^\circ(Z/X)$ of $P(Z/X)$ is minus the relative ample cone (i.e., the convex cone generated by the classes of divisors such that $\mathcal{O}_Z(-D)$ is ample or, equivalently such that $C \cdot D < 0$ for every effective contracted to Q curve C on Z). Since some multiple of an ample divisor is generated by its global sections, it follows that $P^\circ(Z/X) \subset \tilde{P}(Z/X)$ and hence $P^\circ(Z/X) = \tilde{P}^\circ(Z/X)$.

The characteristic cone can also be seen as the convex cone generated by the Cartier divisors D such that $\mathcal{O}_Z(-D) = I\mathcal{O}_Z$ for some ideal $I \subset R$ (see [6]). Among the ideals I with the above property there is a largest one, namely the integral closure $\bar{I}$ of I. Integrally closed ideals are also called complete ideals since Zariski showed that they are the local analogous to complete linear systems. Thus, $\tilde{P}(Z/X)$ can be understood as the convex cone generated by the divisor classes corresponding (by the correspondence $I \mapsto D$ given by $I\mathcal{O}_Z = \mathcal{O}_Z(-D)$) to the complete ideals of R such that $I\mathcal{O}_Z$ is invertible. Notice that the set of such complete ideals is a semigroup for the $*$-operation given by $I * J = \overline{IJ}$ and that the above correspondence takes the operation $*$ to the summation of divisors. Denote this semigroup, or its additive image in $A^1(Z/X)$, by $S(Z/X)$. If $\tilde{S}(Z/X)$ is the additive semigroup of lattice points of $\tilde{P}(Z/X)$, i.e. $\tilde{S}(Z/X) = \tilde{P}(Z/X) \cap N^1(Z/X)$, one obviously has $S(Z/X) \subset \tilde{S}(Z/X)$ and both semigroups generate the cone $\tilde{P}(Z/X)$.

For a sandwiched variety we will mean a normal scheme for which π factorizes as a product of birational projective maps $Z \to Y \to X$. From Kleiman [3], it follows that there is a one to one correspondence between sandwiched varieties and topological cells of the characteristic cone (see [2, theorem 13]). Moreover, for two sandwiched varieties Y and Y', the cell associated to Y' is included in the one associated to Y if and only if there is a birational morphism $Y' \to Y$. The correspondence works as follows: the relative interior of the cell associated to Y contains exactly the classes of those divisors corresponding to complete ideals I such that Y is the normalized blowing up of I.

For a convex cone K generating a vector space $\mathbf{R}^m$, the cells are defined in the following way. The only m-dimensional cell is K and, by descending induction, the other cells are those of the maximal convex cones contained in $K \setminus K^\circ$, where the upper index $^\circ$ means the relative interior. For the cones $P(Z/X)$ and $\tilde{P}(Z/X)$, which both generate $A^1(Z/X)$, the only $\rho(Z/X)$-dimensional relative interior of a cell is $P^\circ(Z/X)$. The cells are uniquely determined by their relative interior.

Theorem 1 *The inclusion of relative interiors of cells gives an injective map from the set of cells of $\tilde{P}(Z/X)$ into that of $P(Z/X)$.*

Proof: Let $\tilde{U}$ be a cell of $\tilde{P}(Z/X)$ with associated sandwiched variety Y. Then the morphism $Z \to Y$ contracts exactly those curves C such that $C \cdot D = 0$ where D is any divisor with class in $\tilde{U}^\circ$. Thus, all such divisors D have classes contained in the same cell U of $P(Z/X)$ and, therefore since those divisor classes generate $\tilde{U}^\circ$, one has $\tilde{U}^\circ \subset U^\circ$ and the map is well defined. Assume that for a second cell $\tilde{U}'$ of $\tilde{P}(Z/X)$ one has $\tilde{U}'^\circ \subset U^\circ$ and take complete ideals I, J with respective divisor classes in $\tilde{U}^\circ, \tilde{U}'^\circ$. Then the divisor class of $I * J$ is in $\tilde{U}''^\circ \subset \tilde{U}^\circ$for a third cell $\tilde{U}''$. For the associated sandwiched varieties Y, Y', Y'' one has birational morphisms $Y'' \to Y$ and $Y'' \to Y'$. Moreover, the curves of Z in $\pi^{-1}(Q)$ contracted in the three varieties Y, Y', Y'' are exactly the same. On the other hand, since $Y'' \neq Y$ and both Y'', Y are normal, there exists a complete curve C'' in Y'' which is contracted in Y. Take a curve C in the inverse image of C'' in Z dominating C'' (it always exists because we are dealing with algebraic varieties). Now, C is contracted in Y but it is not in Y'' which is a contradiction. This completes the proof.

Corollary 1 *If the cone $NE(Z/X)$ is polyhedral then the set of sandwiched varieties relative to π is finite.*

Proof: Since $-NE(Z/X)$ is polyhedral, its dual cone $P(Z/X)$ is also polyhedral, so it has finitely many cells and, hence, there are finitely many sandwiched varieties.

3 Cones and constellations of infinitely near points

From now on, we will consider the case in which X is smooth and π is the composition of a sequence of blowing ups at closed points. The semigroup $S(Z/X)$ is studied in [7] and [1]. We will use here the description in [1].

Assume $X = SpecR$ is smooth and $dimX = d \geq 2$. For a constellation of infinitely near points (i.n.p. in short) to Q we mean a set $\mathcal{C} = \{Q_0, Q_1, \ldots, Q_n\}$ where $Q_0 = Q$ and each Q_i is a closed point in the blown up variety of the variety containing Q_{i-1} with center at Q_{i-1} which maps to Q in X. Let $\pi : Z \to X$ be the composition of the successive blowing ups of the points of $\mathcal{C}$. Denote by B_i the exceptional divisor of the blowing up with center at Q_i, by E_i (resp. E_i^*) the strict (resp. total) transform of B_i in Z.

Both, the classes of $\{E_0, E_1, \ldots, E_n\}$ and those of $\{E_0^*, E_1^*, \ldots, E_n^*\}$ are basis of the lattice $N^1(Z/X)$. The basis change is given by

$$E_i = E_i^* - \sum_{j \to i} E_j^*$$

where $j \to i$ means that Q_j is proximate to Q_i, i.e. that zkJQ_j belongs to the strict transform of B_i in the variety containing Q_j.

For each i, one has E_i dominates B_i and the restriction $\pi : E_i \to B_i$ is a map obtained by composition of the successive blowing ups at the points of the set $\mathcal{C}_i$ of proximate points to Q_i ($\mathcal{C}_i$ can be considered as union of (d-1)-dimensional constellations). Since $Pic(Z/X) \to Pic(E_i)$ is surjective, one has an injective linear map $A_1(E_i) \to A_1(Z/X)$, where $A_1(E_i) = N_1(E_i) \bigotimes_{\mathbf{Z}} \mathbf{R}$ and $N_1(E_i)$ is the group of 1-cycles modulo numerical equivalence on E_i. The cone $NE(E_i)$ generated by the classes in $N_1(E_i)$ of effective curves on E_i is mapped, by the above linear map, into the cone $NE(Z/X)$. It is clear that $NE(Z/X)$ is nothing but the convex sum of the images of the cones $NE(E_i)$ in $A_1(Z/X)$.

Proposition 1 *If $NE(E_i)$ is a polyhedral cone for every i, then the number of sandwiched varieties relative to π is finite.*

Proof: The convex sum $NE(Z/X)$ is polyhedral, so the result follows from Corollary1.

For each index i, $NE(E_i)$ is a polyhedral cone in each of the two following cases. First, if the set $\mathcal{C}_i$ is toric, i.e. if it consists of i.n.p. which are 0-dimensional orbits of some structure of toric variety on the projective space $B_i \cong \mathbf{P}^{d-1}$, then $NE(E_i)$ is the cone generated by 1-dimensional orbits, hence it is rational polyhedral (see [5], [1]). Second, one can apply Kawamata's theorem [4] which guarantees that $NE(E_i)$ is rational polyhedral if it is contained in the half space $c \cdot K_{E_i} < 0$ where K_{E_i} is the class of the canonical divisor of E_i, i.e. if the anticanonical bundle of E_i is ample. Thus one gets the following result.

Corollary 2 *If for each i one has either $\mathcal{C}_i$ is toric or the anticanonic bundle of E_i is ample, then the number of sandwiched varieties relative to π is finite.*

In particular, if the whole constellation $\mathcal{C}$ is toric (i.e. if Q_0 is a closed orbit of a toric structure on the affine d-dimensional space and all the i.n.p. Q_i are also closed orbits for the derived toric structures on the blow up spaces) the cones $NE(Z/X)$ and $P(Z/X)$ are rational polyhedral. Moreover, as shown in [1], in this case one has $P(Z/X) = \tilde{P}(Z/X)$, $S(Z/X) = P(Z/X) \cap N^1(Z/X)$, and the extremal rays in $NE(Z/X)$ can be described explicitly in terms of the combinatorics of the constellation. Thus, one can characterize [1, 2.20] those toric constellations for which the cone $NE(Z/X)$ is simplicial. One sees that in this case $NE(Z/X)$, and so also $P(Z/X) = \tilde{P}(Z/X)$, is a regular cone and $S(Z/X)$ is a free semigroup. This characterization includes the case of chains, i.e. constellations such that $Q_{i+1} \in B_i$ for each $i \geq 0$. Later on, we will show with some examples that these results are not true in general for non toric chains.

One can use Kawamata's theorem with some weaker assumptions than in Corollary2. For fixed i and $j \to i$, denote by $E_{ij} = E_i \cap E_j$, $E^*_{ij} = E_i \cap E^*_j$ (here $\cap$ means the cycle given by the proper intersection). The canonical divisor of E_i is given by $-dH^*_i + (d-2)\sum_{j \to i} E^*_{ij}$, where H^*_i is the total transform in E_i of a general hyperplane in B_i. Assume that the linear system of effective divisors F' on B_i such that $\pi^*_i F' \geq (d-2)\sum_{j \to i} E^*_{ij}$ has a base point set $\mathcal{S}_i$ of dimension at most one. Then, if $C' \subset B_i$ is an irreducible curve not contained in $supp(\mathcal{S}_i)$, it follows from Bezout's theorem (applied to C' and some convenient member of the above linear system) that the class c in $N_1(E_i)$ of the strict transform of C' in E_i satisfies $c \cdot K_{E_i} \leq 0$. This means that $NE(E_i)$ is generated by the curves in the region $c \cdot K_{E_i} \leq 0$, the classes of the curves in $supp(\mathcal{S}_i)$ and the exceptional curves in the region $c \cdot K_{E_i} > 0$ (exceptional means contracted by π_i). Kawamata's theorem gives information on the intersection of $N_1(E_i)$ with the region $c \cdot K_{E_i} < 0$ (the set of extremal rays in this region is discrete). Thus, since the classes of the exceptional curves will appear also in others $NE(E_j)$, it is possible to know the contribution of $NE(E_i)$ to $NE(Z/X)$ if one controls the curves with class in the hyperplane $c \cdot K_{E_i} = 0$.

We will precise the above situation for $d = 3$. Assume that the linear system $\mathcal{F}_i$ of curves F' in $B_i \cong \mathbf{P}^2$ with $\pi_i^* F' \geq \sum_{j \to i} E_{ij}^*$ is non-empty (notice that this is always true if $card(\mathcal{C}_i) \leq 9$), i.e. one has $dim(\mathcal{S}_i) \leq 1$. Thus $NE(E_i)$ is generated by its intersection with $c \cdot K_{E_i} \leq 0$ and finitely many more classes of curves, namely those in $supp(\mathcal{S}_i)$ and the exceptional ones. Furthermore, if the linear system F_i contains a pencil (e.g. if $card(\mathcal{C}_i) \leq 8$), then $NE(E_i)$ is generated by its intersection with $c \cdot K_{E_i} < 0$ and finitely many more classes, namely those of the exceptional curves and those of the irreducible components of the members of the pencil. Notice that this set of classes is finite as all the general curves of the pencil have the same class in $N_1(E_i)$ and, hence, this also happens for the classes of their irreducible components.

Now, the extremal rays of $N_1(E_i)$ in the region $c \cdot K_{E_i} < 0$ are those corresponding to irreducible curves $C \subset E_i$ which can be contracted on a smooth surface, i.e. those irreducible curves such that $p_a(C) = 0$ and $C \cdot C = -1$, or equivalently $C \cdot C = C \cdot K_{E_i} = -1$. Consider on the lattice $N_1(E_i)$ the basis given by the classes of the cycles $-H_i^*$ and $-E_{ij}^*$ for $j \to i$. Thus, if the class of C has coordinates $(-n, \{e_j\}_{j \to i})$ in the above basis, then the conditions $C \cdot C = C \cdot K_{E_i} = -1$ are written in the following way

$$\sum_{j \to i} e_j^2 = n^2 + 1, \quad \sum_{j \to i} e_j = 3n - 1.$$

Notice that, if the irreducible curve C is not exceptional then n is the degree of its image C' in B_i and e_j is the multiplicity at Q_j of the strict transform of C'. If C is exceptional, then $n = 0$ and C, being irreducible should be one of the curves E_{ij} with j maximal (i.e. such that there is no index l with $l \to i$ and $l \to j$).

Lemma 1 *With notations as above, keep the assumption $d = 3$. For each i denote by $\mathcal{R}_i$ the set of rays in $NE(E_i)$ which are either extremal for $NE(E_i)$ in the region $c \cdot K_{E_i} < 0$ or generated by classes of irreducible curves in the hyperplane $c \cdot K_{E_i} = 0$. Then one has:*

(i) If $card(\mathcal{C}_i) \leq 8$ the set $\mathcal{R}_i$ is finite.

(ii) If $card(\mathcal{C}_i) = 9$ the set $\mathcal{R}_i$ has at most one limit point, namely the ray generated by the class of $\mathcal{C}_0 = 3H_i^* - \sum_{j\to i} E_{ij}^*$.

Proof: Any ray in $\mathcal{R}_i$ is generated by a vector of coordinates $(-n, \{e_j\}_{j\to i})$ where either $\sum_{j\to i} e_j = 3n - 1$ and $\sum_{j\to i} e_j^2 = n^2 + 1$ (extremal rays in $c \cdot K_{E_i} < 0$) or $\sum_{j\to i} e_j = 3n$ and $\sum_{j\to i} e_j^2 = n^2 + 2$ (classes of curves with $C \cdot K_{E_i} = 0$ and $p_a(C) \geq 0$). Since for any value of n there are only finitely many possible values of $\{e_j\}_{j\to i}$ fitting in one of two above arithmetical situations, any limit ray of $\mathcal{R}_i$ should be a limit of rays generated by vectors as above with $n \to \infty$. Such a limit is generated by a vector of type $(-1, \{x_j\}_{j\to i})$ with $x_j \geq 0$, $\sum_{j\to i} x_j = 3$ and $\sum_{j\to i} x_j^2 = 1$. Now, if $h = card(\mathcal{C}_i)$, the h-variable function $\sum_{j\to i} x_j^2$ has an absolute minimum at $x_j = 3/h$ for every $j \to i$, the minimum value being $9/h$. Thus, if $h \leq 8$ the equality $\sum_{j\to i} x_j^2 = 1$ is impossible and therefore the set $\mathcal{R}_i$ is discrete and hence finite. If $h = 9$, the equality $\sum_{j\to i} x_j^2 = 1$ implies that $x_j = 1/3$ for each $j \to i$, so the ray generated by $\mathcal{C}_0$ is the only possible limit point of $\mathcal{R}_i$.

Theorem 2 *Let Q be a smooth closed point of a 3-dimensional variety and $\pi : Z \to X$ a morphism obtained by blowing up a constellation $\mathcal{C}$ of i.n.p. to Q. Assume that for every $Q_i \in \mathcal{C}$ either Q_i is toric or $card(\mathcal{C}_i) \leq 8$. Then The cone $NE(Z/X)$ is polyhedral and the number of sandwiched varieties relative to π is finite.*

Proof: If $\mathcal{C}_i$ is toric, the cone $NE(E_i)$ is polyhedral. If $card(\mathcal{C}_i) \leq 8$, then by Lemma1 $NE(E_i)$ is generated by the finite set $\mathcal{R}_i$ and finitely many other curves (the linear system F_i contains a pencil in this case), so the cone $NE(E_i)$ is also polyhedral. Thus $NE(Z/X)$ is also a polyhedral cone and, hence, by Corollary1 the set of sandwiched varieties is finite.

Remark 1 If $card(\mathcal{C}_i) = 9$, the cone $NE(E_i)$ could be non polyhedral as shown, for instance, in example 1 [2, p. 37] when $\mathcal{C} = \{Q_0, Q_1, \ldots, Q_9\}$ and $Q_1, \ldots, Q_9$ are the intersection points of two general cubics in B_0. The method to discuss the examples in next section shows us how in practice, even for nine points, in many cases, one can decide if the cone $NE(E_i)$ is polyhedral or not.

4 Clusters and chains of infinitely near points

Let $\pi : Z \to X = Spec(R)$ the morphism obtained by blowing up a constellation $\mathcal{C}$ of i.n.p. to the smooth closed point $Q \in Spec(R)$. The classes of the divisors E_i^* are a basis for the lattice $N^1(Z/X) = Pic(Z/X)$. Thus to give a relative divisor $D = \sum m_i E_i^*$ is equivalent to give an integer weight on the points of $\mathcal{C}$ by assigning to Q_i the weight m_i. Such a weighted constellation is called a cluster. A cluster is called idealistic if the divisor D comes from a complete ideal I such that $I\mathcal{O}_Z$ is invertible, i.e. if it belongs to the semigroup $S(Z/X)$. Thus, $S(Z/X)$ can be considered as the additive semigroup of the idealistic clusters and $\tilde{P}(Z/X)$ as the cone generated by those clusters. The cone $P(Z/X)$ is given by the so called proximity inequalities (see [1]), i.e., for each irreducible exceptional curve C and i the only index such that $C \subset E_i$ and its image C' in B_i is not a point, the inequality

$$deg(C')m_i \geq \sum_{j \to i} e_j(C')m_j$$

where $deg(C')$ is the degree of C' in the projective space B_i and $e_j(C')$ the multiplicity at Q_j of the strict transform of C. Corollary2 and Theorem2 give conditions under which the cone $P(Z/X)$ is given by finitely many proximity inequalities.

A classic result by Zariski, which has given rise to the theory of complete ideals, asserts that if $d = 2$ the cone $\tilde{P}(Z/X)$ is polyhedral regular and that one has $\tilde{P}(Z/X) = P(Z/X)$ and $S(Z/X) = P(Z/X) \cap N^1(Z/X)$. This follows from the obvious fact that $NE(Z/X)$ is the regular cone generated by the classes of the curves E_i and the fact that any cluster satisfying the proximity inequalities is idealistic. In [1] it is shown that the same is true if $d \geq 3$ and the constellation is toric and it is a chain. By a chain we mean that $\mathcal{C} = \{Q_0, Q_1, \ldots, Q_n\}$ and $Q_{i+1} \in B_i$ for every $i \geq 0$. The example quoted in Remark1 shows that $\tilde{P}(Z/X)$ could be non polyhedral for a suitable constellation and therefore $S(Z/X)$ is not a finitely generated semigroup. Even $\tilde{P}(Z/X)$ could be non closed, and hence $\tilde{P}(Z/X) \neq P(Z/X)$ as shown in example 3 in [2, p. 37], where $\mathcal{C} = \{Q_0, Q_1, \ldots, Q_{16}\}$ the sixteen last points being in general position in B_0.

The result of Zariski in dimension two implies that, in this case, the semigroup $S(Z/X)$ is free, i.e. that one has unique factorization with non negative exponents in terms of the irreducible elements. Zariski proposed to extend to higher dimensions this kind of results. The discussion in terms of the structure of the various cones can provide several types of generalizations of the Zariski's above factorization property. Thus, to be $\tilde{P}(Z/X)$ a simplicial cone means that one has semiunique factorization, i.e. unique factorization with rational exponents in terms of the primitive extremal vectors of $S(Z/X)$. To be $\tilde{P}(Z/X)$ polyhedral but not simplicial means non unique semifactorization and $\tilde{P}(Z/X) \neq P(Z/X)$ or $P(Z/X)$ non polyhedral which means non unique semifactorization in terms of infinitely many primitive extremals. Lipman in [7] showed that for any constellation, the semigroup $S(Z/X)$ contains a concrete lattice basis of $N^1(Z/X)$, so that in terms of the basis one has unique factorization with integral exponents.

A natural question is to ask if Zariski's result is true for constellations which are chains. Next examples show that this question has a negative answer. For the all three examples we assume $d = 3$.

Example 1 Consider the chain $\mathcal{C} = \{Q_0, Q_1, \ldots, Q_5\}$ where $Q_1, \ldots, Q_5$ are five consecutive points on a smooth conic G in B_0, i.e. $Q_{i+1} \in B_i$ for $i \geq 0$ and Q_1 is on G and Q_i on the strict transform of G for $i \geq 2$. In particular $i \to 0$ for $i \geq 1$, the embedding $N_1(E_0) \to N_1(Z/X)$ is an isomorphism and it takes the cone $NE(E_0)$ to $NE(Z/X)$. Take the basis $\{-H_0^*, -E_{01}^*, \ldots, -E_{05}^*\}$ on $N_1(E_0)$ and represent the vectors in $A_1(E_0)$ by their 6-uple of coordinates.

The linear system $\mathcal{F}_0$ contains the pencil generated by the cubics $G + L$, $G + L'$, where L, L' are generic lines in B_0. Thus after the comments in Section3, the cone $NE(E_0)$ is generated by the class $g = (-2, 1, 1, 1, 1, 1)$, the exceptional classes $f_1 = (0, -1, 1, 0, 0, 0)$, $f_2 = (0, 0, -1, 1, 0, 0)$, $f_3 = (0, 0, 0, -1, 1, 0)$, $f_4 = (0, 0, 0, 0, -1, 1)$, $f_5 = (0, 0, 0, 0, 0, -1)$ and the vectors $(-n, e_1, \ldots, e_5)$ with $\sum_{i=1}^{5} e_i^2 = n^2 + 1$, $\sum_{i=1}^{5} e_i = 3n - 1$ and $n > 0$ (all these vectors are classes of effective curves in E_0, may be non irreducible ones, as the number of imposed conditions by the multiplicities e_i is $(1/2)\sum e_i(e_i + 1)$ which is one unit less than the dimension of the space of n forms). Since $card(\mathcal{C}_0) = 5$, the only possibilities for these vectors are $l = (-1, 1, 1, 0, 0, 0)$ and $g = (-2, 1, 1, 1, 1, 1)$. Thus $NE(E_0)$ is generated by the seven vectors $l, f_1, \ldots, f_5, g$ and it is not a simplicial cone.

The dual cone $P(Z/X)$ of $-NE(Z/X)$ is given by the following proximity inequalities $m_0 \geq m_1 + m_2$, $2m_0 \geq m_1 + \cdots + m_5$, $m_1 \geq m_2 \geq m_3 \geq m_4 \geq m_5 \geq 0$. By looking to solutions with equality at least in 5 of the above inequalities one gets the following 9 extremal vectors: $(1, 0, 0, 0, 0, 0)$, $(1, 1, 0, 0, 0, 0)$, $(2, 1, 1, 0, 0, 0)$, $(2, 1, 1, 1, 0, 0)$, $(2, 1, 1, 1, 1, 0)$, $(3, 2, 1, 1, 1, 1)$, $(4, 2, 2, 2, 1, 1)$, $(5, 2, 2, 2, 2, 2)$, $(6, 3, 3, 2, 2, 2)$, the six first ones being the Lipman basis.

Since in this case one has $\tilde{P}(Z/X) = P(Z/X)$ one has non unique semi-factorization: The cone $\tilde{P}(Z/X)$ is not simplicial, hence it is not regular.

Example 2 Consider the chain $\mathcal{C} = \{Q_0, Q_1, \ldots, Q_9\}$ where $Q_1, \ldots, Q_9$ are consecutive points on an inflection point Q_1 of a rational cubic $\mathcal{C}_0$ in B_0. As above one has $NE(E_0) = NE(Z/X)$ and the vectors $A_1(E_0)$ can be represented by a 10-uple of coordinates.

The linear system $\mathcal{F}_0$ contains the pencil generated by $\mathcal{C}_0$ and $3L'$ where L' is the tangent line to $\mathcal{C}_0$ at Q_1. From Section3, $NE(E_0)$ is generated by the class $l = (-1, 1, 1, 1, 0, 0, 0, 0, 0, 0)$ and the exceptional classes $f_1, \ldots, f_9$ as above (i.e., f_i has -1 as i-th entry, 1 as $(i+1)$-th entry for $i \leq 8$ and 0 as entry otherwise). In fact, notice that the class of the transform of $\mathcal{C}_0$ and those of the effective curves with $C \cdot C = C \cdot K_{E_0} = -1$ are in the cone generated by $l, f_1, \ldots, f_9$ (since $C \cdot (C - K_{E_0}) = 0$ and $L \cdot (C - K_{E_0}) < 0$, where L is the strict transform of L', it follows from Bezout theorem that L should be a component of C, so C is not irreducible).

One has $\tilde{P}(Z/X) = P(Z/X)$ so $\tilde{P}(Z/X)$ is a regular cone. If Q_1 is the origin of the curve $y = x^3$, then there is no cubic having intersection multiplicity 8 with $\mathcal{C}_0$ at Q_1, so the cluster with weight $m = (3, 1, 1, 1, 1, 1, 1, 1, 1, 0)$ satisfies the proximity relations but it is not idealistic (otherwise the tangent cone of the hypersurface given by a general element of the ideal would achieve the intersection multiplicity 8). Thus one has $S(Z/X) \neq \tilde{S}(Z/X)$. One has unique semifactorization and the Lipman basis contains the vector $(4, 1, 1, 1, 1, 1, 1, 1, 1, 0)$, so in Lipman factorization there are clusters (for instance the cluster with weight $2m$) with negative exponents. The semigroup $S(Z/X)$ has more than 10 irreducible elements, so if one wants non negative integral coefficients one has non unique factorization.

Finally, if we consider only 8 points Q_i instead of 9, one gets an alternative example with identical characteristics.

Example 3 Consider the chain $\mathcal{C} = \{Q_0, Q_1, \ldots, Q_9\}$ where $Q_1, \ldots, Q_9$ are consecutive points on a non inflection smooth point Q_1 of a rational cubic $\mathcal{C}_0$ in B_0. Take, for instance, Q_1 the origin of $y = x^2 + x^3$.

Since the only irreducible curve A in B_0 having intersection multiplicity with $\mathcal{C}_0$ greater or equal than $3deg(A)$ is the same curve $\mathcal{C}_0$, it follows that $NE(E_0)$ is included in $c \cdot K_{E_0} \leq 0$ and its intersection with $c \cdot K_{E_0} = 0$ is the cone generated by the classes $c_0 = (-3, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ and $f_1, \ldots, f_8$ (as in Example2). Thus $NE(E_0)$ is generated besides $c_0, f_1, \ldots, f_8$ by f_9 and those $c = (-n, e_1, \ldots, e_9)$ such that $n > 0$, $\sum_{i=1}^{9} e_i^2 = n^2+1$, $\sum_{i=1}^{9} e_i = 3n-1$

and $e_1 \geq e_2 \geq \cdots \geq e_9 \geq 0$. One can see that each such a c is the class of an irreducible curve, so those c are extremal vectors for $NE(E_0)$. Finally, notice that there are infinitely many values of c (take, for instance the sequence $(-(3t^2+3), t^2+t, t^2+2, t^2+1, t^2+1, t^2+1, t^2+1, t^2+1, t^2+1, t^2-t))$, so $NE(E_0)$ is not a polyhedral cone.

Thus the dual cone $P(Z/X)$ is also not a polyhedral cone. Moreover, since $NE(E_0)$ is included in $c \cdot K_{E_0} \leq 0$, the cluster with weight $m = (3,1,1,1,1,1,1,1,1,1)$ satisfies the proximity inequalities but sm is not an idealistic cluster for every $s \geq 1$ (otherwise the tangent cone to a general element of the ideal will be the curve of degree $3s$ with intersection multiplicity $9s$ with $\mathcal{C}_0$ at Q_1 and not containing $\mathcal{C}_0$ in its support). It follows that one has $P(Z/X) \neq \tilde{P}(Z/X)$ and $S(Z/X)$ is a non finitely generated semigroup.

Finally, we remark that this is an example of non closed characteristic cone obtained by blowing up only ten points (in a chain).

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