NON-ABELIAN DUALITY ON STEIN SPACES

by Siegmund KOSAREW

There seems to be a general interest in developing a reasonable (non linear) duality theory for complex varieties. Some aspects of this problem will be treated in an other (forthcoming) paper ([K]). Here we restrict to the following special case which is interesting for itself: let X be a Stein complex space. We want to prove a (bi)duality theorem for X, very similar to the fact that X can be recovered as the set of continuous characters of the $\mathbb{C}$-algebra $\Gamma(X,\mathcal{O}_X)$ of global holomorphic functions on X. This is the content of the theorem of Igusa-Remmert-Forster (see [F]). Replacing the algebra $\mathbb{C}$ by the complex Lie group $G\ell_n(\mathbb{C})$, one obtains the following result.

THEOREM. — *Let $n \geq 2$ be a natural number and consider the following "Gelfand" evaluation map*

$$\begin{aligned} X &\longrightarrow \mathrm{Hom}_e(\mathrm{Mor}(X,G\ell_n(\mathbb{C})),G\ell_n(\mathbb{C})) \\ x &\longmapsto \phi_x \end{aligned} \tag{0.1}$$

with $\phi_x(f) := f(x)$, where $\mathrm{Mor}(X,G\ell_n(\mathbb{C}))) =: G$ is the topological group of holomorphic maps from X to $G\ell_n(\mathbb{C})$ and $\mathrm{Hom}_e(G,G\ell_n(\mathbb{C}))$ denotes the set of continuous group homomorphisms from G to $G\ell_n(\mathbb{C})$ which are in addition equivariant with respect to the left and right $G\ell_n(\mathbb{C})$-actions on both sides.

The Gelfand map in injective and, moreover, for any $\chi : G \to G\ell_n(\mathbb{C})$ in $\mathrm{Hom}_e(G,G\ell_n(\mathbb{C}))$ there are

i) a point $x_0 \in X$,

ii) a continuous group homomorphism

$$\delta : \Gamma(X,\mathcal{O}_X^*) \longrightarrow \mathbb{C}^*$$

with $\delta(c) = 1$ for any locally constant holomorphic function c.

(iii) a homomorphism

$$\alpha : G/G^0 \longrightarrow \mathbb{C}^*$$

where G^0 is the connected component of unity, such that

$$\chi(A) = A(x_0) \cdot \delta(\det(A)) \cdot \alpha(A \cdot G^0). \tag{0.2}$$

Moreover, x_0, δ and α determine χ uniquely. We have $\delta \equiv 1$ if χ commutes with the operations on matrices given by interchanging two arbitrary entries (if the resulting matrices are again invertible).

In the above sense, X may be in particular recovered from the non-linear $G\ell_n(\mathbb{C})$-spectrum which should be, by definition, the right hand-side of the Gelfand map.

Remarks.

1. If X is a contractible Stein space then $G = G^0$ by the Oka-Grauert principle. So α is trivial in this case.

2. We may replace $G\ell_n(\mathbb{C})$ by $S\ell_n(\mathbb{C})$ in the theorem above. Then δ disappears. Especially the Gelfand map for $S\ell_n(\mathbb{C})$, $n \geq 2$, is bijective, if X is contractible for example. In this case we can speak of a proper biduality result.

3. Both sides of the Gelfand map (0.1) are obviously fonctorial in X.

4. In general one cannot replace above $G\ell_n(\mathbb{C})$ by an arbitrary complex Lie group, as is shown by the example $(\mathbb{C},+)$.

The proof of this theorem is furnished in section 2 and 3, first for the case $n = 2$ which is already the crucial one. An important ingredient is the Bruhat decomposition for $G\ell_n(\mathbb{C})$, see for instance [B] IV.14, [H] 28, [S] 10.2. The strategy is to extract from χ in a canonical way a (continuous) character of the $\mathbb{C}$-algebra $\Gamma(X,\mathcal{O}_X)$ which gives the point x_0 of X by Igusa-Remmert-Forster.

From our point of view, the Lie group $G\ell_n(\mathbb{C})$ or $S\ell_n(\mathbb{C})$ serves as a dualizing object in the category of Stein spaces. So $G = \mathrm{Mor}(X, G\ell_n(\mathbb{C}))$ should be considered as a dual and $\mathrm{Hom}_e(G, G\ell_n(\mathbb{C}))$ as the corresponding bidual space. In order to define a structure sheaf on the last space, one could adapt the method of O. Forster, described in [F]. But here we want to restrict to the question of bijectivity only.

A question has been posed to the author by G. Henkin and E. Oeljeklaus if this method of duality would imply new insights for finding hulls of holomorphy (so X is there assumed to be not Stein). This does not seem to be the case here.

1. Some general remarks on biduality in the Stein (and affine) case

1.1. — Let X be a Stein space and R a (topological) $\mathbb{C}$-algebra, not necessarily commutative. Then there is a Gelfand mapping

$$\begin{aligned} X &\longrightarrow \mathrm{Hom}_{R-\mathrm{alg}}(\mathrm{Mor}(X,R),R) \\ x &\longmapsto \phi_x \end{aligned}$$

where $\mathrm{Mor}(X,R)$ is the set of holomorphic maps from X to R and $\mathrm{Hom}_{R-\mathrm{alg}}(-,-)$ denotes the (continuous R-algebra) homomorphisms. As usual $\mathrm{Mor}(X,R)$ is endowed with the compact-open topology.

Clearly this Gelfand map is injective if the holomorphic maps $f: X \to R$ separate points (which is true here).

Now for the surjectivity, let $\chi; \mathrm{Mor}(X,R) \to R$ be a R-algebra homomorphism and $I := \mathrm{Ker}(\chi)$. Evidently $\mathrm{Mor}(X,R)/I \simeq R$. We put

$$V(I) := \left\{ x \in I \mid f(x) = 0 \ \text{ for all } \ f \in I \right\}.$$

which is clearly an analytic subset of X. Let us assume that there are points $x_0, x_1 \in V(I)$ with $x_0 \neq x_1$, so $f(x_0) = f(x_1) = 0$ for all $f \in I$. Now take any holomorphic $f: X \to R$. Then $f - \chi(f)$ is in I. Consequently

$$0 = f(x_0) - \chi(f) = f(x_1) - \chi(f).$$

But this is impossible, since x_0, x_1 can be separated. We obtain $V(I) = \emptyset$ or $V(I) = \{x_0\}$.

If we assume that $X' := \mathrm{Mor}(X,R)$ is noetherian (for this X has to be replaced by a Stein compact subset!), then there are $g_1, \ldots, g_k \in I$ such that

$$I = \sum_{i=1}^{k} X' g_i X'.$$

Let us denote by $\mathcal{O}_X(R)$ the *sheaf* of R-valued holomorphic functions on X. If $V(I) = \emptyset$, the morphism of sheaves

$$\begin{aligned} F: \qquad \mathcal{O}_X(R)^k \otimes_{\mathcal{O}_X} \mathcal{O}_X(R)^k &\longrightarrow \mathcal{O}_X(R) \\ (a_1, \ldots, a_k) \otimes (b_1, \ldots, b_k) &\longmapsto \sum_{i=1}^{k} a_i g_i b_i \end{aligned}$$

is an epimorphism. Assuming $H^1(X, \mathrm{Ker}(F)) = 0$, we are done since then $1 \in I$ which is impossible. Imposing some additional conditions on R, we would gain that $\mathrm{Ker}(F)$ is indeed a coherent $\mathcal{O}_X$-module, for example if $\dim_{\mathbb{C}} R < \infty$, so the above vanishing would hold.

Let us now assume that $V(I) = \{x_0\}$. We want to show that $\chi = \phi_{x_0}$. If $f \in I$, then $\chi(f) = 0$ and also $\phi_{x_0}(f) = 0$, so $I \subset \mathrm{Ker}(\phi_{x_0})$. For f arbitrary, we have $f - \chi(f) \in I$ and, consequently $f(x_0) = \chi(f)$ which gives $\chi = \phi_{x_0}$. The above argument shows that the Gelfand map is in fact bijective under (rather mild) assumptions on R. In order to recover also the structure sheaf of X, one would have to apply Forster's method ([F]).

In the affine case, we have to take $X = \mathrm{Spec}(A)$, where A is a $\mathbb{C}$-algebra of finite type. $\mathrm{Mor}(X,R)$ denotes here the set of algebraic maps of X into R which is a $\mathbb{C}$-algebra (non necessarily commutative) of finite dimension. Then our arguments apply also for this situation.

1.2. — We intend to replace the $\mathbb{C}$-algebra R by a complex Lie group G and consider the corresponding Gelfand map

$$X \longrightarrow \mathrm{Hom}_e(\mathrm{Mor}(X,G),G)$$
$$x \longmapsto \phi_x$$

X is again a Stein space, $\mathrm{Mor}(X,G)$ denotes the set of holomorphic maps from X to G and $\mathrm{Hom}_e(-,-)$ are the (continuous) group homomorphisms which are in addition G-left and right equivariant.

Injectivity. — It is verified if we can separate points of X by holomorphic maps with values in G. This is satisfied in all non trivial cases.

Surjectivity. — Given $\chi : \mathrm{Mor}(X,g) \to G$ as above. Then we put

$$I := \chi^{-1}(1_G),$$
$$V(I) := \left\{x \in X \mid f(x) = 1_G, \text{ for all } f \in I\right\}.$$

Obviously $V(I)$ is an analytic subset of X. If there are $x_0, x_1 \in V(I)$ such that $x_0 \neq x_1$, then we get a contradiction: take $f \in \mathrm{Mor}(X,G)$, then $\chi(f)^{-1} \cdot f \in I$, so finally $f(x_0) = f(x_1)$ and consequently

$$\mathrm{Card}(V(I)) \leq 1.$$

The rest of the argument in section (1.1) breaks down since we cannot expect always surjectivity here (as seen by the example $G = (\mathbb{C},+)$). There is a trivial necessary condition for χ to be of evaluation form ϕ_x. It is

$$\chi(f) \in \mathrm{Im}(f).$$

In the case where $f : X \to G$ is an embedding, a candidate for x_0 is obviously $\chi(f) = f(x_0)$.

If we *assume* now that $V(I)$ is exactly one point x_0, then we infer quickly that

$$\chi = \phi_{x_0}$$

since for any $f : X \to G$ we have $\chi(f)^{-1} \cdot f \in I$ and so $\chi(f)^{-1} \cdot f(x_0) = 1$.

2. The case $G\ell_2(\mathbb{C})$

Let us consider a continuous group homomorphism $\chi\colon \mathrm{Mor}(X, G\ell_2(\mathbb{C})) \to G\ell_2(\mathbb{C})$ which is $G\ell_2(\mathbb{C})$-left and right equivariant. We try to extract from χ a point $x_0 \in \mathbb{C}$. For this we identify $\mathrm{Mor}(X, G\ell_2(\mathbb{C}))$ with $G\ell_2(\Gamma(X,\mathcal{O}_X))$ and define

(2.1) $$\mathfrak{a} := \left\{ f \in \Gamma(X,\mathcal{O}_X) \mid \chi \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Clearly $\mathfrak{a}$ is a closed subset of $\Gamma(X,\mathcal{O}_X)$ and it is stable under addition by using the group homomorphism property. But it is also stable by multiplication with complex numbers (see formula (2.2), using equivariancy. We want to show that $\mathfrak{a}$ is an ideal. For this we take first an element $g \in \Gamma(X,\mathcal{O}_X^*)$ and use

$$\begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} 1 & gf \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

So if $f \in \mathfrak{a}$, then evidently also gf. Now for a general element $g \in \Gamma(X,\mathcal{O}_X)$, we consider the expression for $t \in \mathbb{C}^*$

$$a(t,g) := (\exp(tg) - 1)/t.$$

By the above argument we know that $a(t,g)f \in \mathfrak{a}$ if $f \in \mathfrak{a}$. Since $\mathfrak{a}$ is closed and $a(t,g)$ is continuous for $t \to 0$, we infer that $gf \in \mathfrak{a}$. This shows that $\mathfrak{a} \in \Gamma(X,\mathcal{O}_X)$ is a (closed) ideal, clearly different from $\Gamma(X,\mathcal{O}_X)$.

Next we prove that $\mathfrak{a}$ is of codimension 1 in $\Gamma(X,\mathcal{O}_X)$ and so $\mathfrak{a}$ is maximal. Those ideals correspond exactly to points of X. In order to show this, it is sufficient to verify

$$\chi\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta(f) \\ 0 & 1 \end{pmatrix} \tag{2.3}$$

for $\beta : \Gamma(X,\mathcal{O}_X) \to \mathbb{C}$ continuous, $\mathbb{C}$-linear. First we have a general expression

$$\chi\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(f) & \beta(f) \\ \gamma(f) & \delta(f) \end{pmatrix}.$$

By applying formula (2.2) for $g = \lambda =$ constant, we see that

$$\chi\begin{pmatrix} 1 & \lambda f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(f) & \lambda\beta(f) \\ \lambda^{-1}\gamma(f) & \delta(f) \end{pmatrix}$$

so $\alpha(\lambda f) = \alpha(f)$, $\delta(\lambda f) = \delta(f)$, $\lambda\gamma(\lambda f) = \gamma(f)$. These identities still remain true if $\lambda \to 0$, so $\alpha(f) = \alpha(0) = 1$, $\delta(f) = \delta(0) = 1$, $\gamma(f) = 0$ and (2.3) is proven. We get therefore

$$V(\mathfrak{a}) = \{x_0\}, \qquad \chi\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & f(x_0) \\ 0 & 1 \end{pmatrix}.$$

Since χ commutes with permutation matrices, we find also

$$\chi\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f(x_0) & 1 \end{pmatrix}.$$

We have to consider now diagonal matrices in $G\ell_2(\Gamma(X,\mathcal{O}_X))$. We note, by Schur's lemma, that $\chi(gI)$ is again diagonal where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For a general $g \in \Gamma(X,\mathcal{O}_X^*)$ we write again

$$\chi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(g) & \beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix}.$$

By $G\ell_2(\mathbb{C})$-equivariancy, we obtain

$$\chi\begin{pmatrix} g\lambda & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda\alpha(g) & \beta(g) \\ \lambda\gamma(g) & \delta(g) \end{pmatrix},$$

$$\chi\begin{pmatrix} \lambda g & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda\alpha(g) & \lambda\beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix},$$

so $\lambda\gamma(g) = \gamma(g)$, $\beta(g) = \lambda\beta(g)$ for all $\lambda \in \mathbb{C}^*$. This implies $\gamma(g) = \beta(g) = 0$, so

$$\chi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(g) & 0 \\ 0 & \delta(g) \end{pmatrix}.$$

Since χ commmutes with permutation matrices, we have

$$\chi\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} \delta(h) & 0 \\ 0 & \alpha(h) \end{pmatrix}$$

so

$$\chi\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} \alpha(g)\delta(h) & 0 \\ 0 & \alpha(h)\delta(g) \end{pmatrix}.$$

By (2.2) we get the following identity

$$\begin{pmatrix} \delta(g^{-1}) & 0 \\ 0 & \alpha(g^{-1}) \end{pmatrix}\begin{pmatrix} 1 & f(x_0) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta(g) & 0 \\ 0 & \alpha(g) \end{pmatrix} = \begin{pmatrix} 1 & g(x_0)f(x_0) \\ 0 & 1 \end{pmatrix}$$

which gives immediately $\alpha(g) = g(x_0)\delta(g)$ and so

$$\chi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g(x_0)\delta(g) & 0 \\ 0 & \delta(g) \end{pmatrix}.$$

Moreover, we conclude from that

$$\chi\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} g(x_0) & 0 \\ 0 & g^{-1}(x_0) \end{pmatrix}$$

and also

$$\chi\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & g(x_0) \\ g^{-1}(x_0) & 0 \end{pmatrix}$$

again by permutations.

Gathering these facts now, we obtain by the classical Bruhat decomposition for $S\ell_2(\mathbb{C})$ (see for instance [H] 28.2) that χ is the evaluation map in x_0 on all matrices in $S\ell_2(\Gamma(X,\mathcal{O}_X))$ where at least *one* entry belongs to $\Gamma(X,\mathcal{O}_X^*)$. A candidate for δ of the theorem is the δ just found.

Concrete examples of such δ's may be constructed as follows: assume that X is simply connected and P is a linear differential operator on $\Gamma(X,\mathcal{O}_X)$ with zero constant term. Then define

$$\delta : \Gamma(X,\mathcal{O}_X^*) \to \mathbb{C}^*$$

by

$$f \longmapsto \exp(P(\log f))(x_1)$$

where $x_1 \in X$ is a fixed point. Clearly $\delta(fg) = \delta(f)\delta(g)$ and $\delta(c) = 1$ for c constant.

Let us now take x_0 and δ, both extracted from χ, and put

$$\chi' : \Gamma(X, G\ell_2(\mathcal{O}_X)) \longrightarrow G\ell_2(\mathbb{C})$$
$$A \longmapsto A(x_0) \cdot \delta(\det(A))\,.$$

We already know that χ and χ' coincide on matrices where at least one entry belongs to $\Gamma(X, \mathcal{O}_X^*)$. Our aim is first to show that χ and χ' coincide on the 1-component of $G := \Gamma(X, G\ell_2(\mathcal{O}_X))$. For this, we observe that for a Runge compact subset $K \subset X$, the restriction map

$$G^0 \longrightarrow G_K^0$$

has dense image, where $G_K := B(K, G\ell_2(\mathcal{O}_X))$ is the Banach Lie group of continuous 2×2-matrices on K which are holomorphic in the interior of K, and "^{0}" denotes always the 1-component. In fact G_K^0 is the completion of G^0 with respect to uniform convergence on K. By continuity of χ, there exists such a K (which can be assumed sufficiently large in order to contain x_0 as an interior point) and a commutative diagram

$$\begin{tikzcd} G^0 \arrow[rr] \arrow[dr, "\chi"'] & & G_K^0 \arrow[dl, "\chi_K"] \\ & G\ell_2(\mathbb{C}) & \end{tikzcd}$$

where χ_K is a continuous homomorphism of Banach Lie groups which is left and right $G\ell_2(\mathbb{C})$-equivariant. We apply our method above to χ_K (extended to G_K) and obtain $\chi_K(A) = A(x_0) \cdot \delta(\det(A))$ for matrices $A \in G_K^0$ admitting a Bruhat decomposition. But this is true here if A is sufficiently close to the unit matrix in G_K. Therefore $\chi_K = \phi_{x_0} \cdot \delta$ in a neighborhood of 1 in G_K and so $\chi_K = \phi_{x_0} \cdot \delta$ on G_K^0. This shows now, by the diagram above, that

$$\chi | G^0 = \chi' | G^0.$$

In general χ and χ' may not coincide on G itself. But the difference will be a homomorphism

$$\alpha : G/G^0 \longrightarrow \mathbb{C}^*.$$

This is seen as follows: for $g \in G$ and any $h \in G^0$, we have $ghg^{-1} \in G^0$, so

$$\chi(ghg^{-1}) = \chi'(ghg^{-1}).$$

This means that $\chi'(g)^{-1} \cdot \chi(g)$ commutes with all $\chi(h) = \chi'(h)$. By Schur, $\chi'(g)^{-1} \cdot \chi(g)$ must be a multiple of the unit matrix and thus we obtain our α.

We want to show finally that δ is trivial if χ satisfies in addition the condition mentioned in the theorem. Take $g \in \Gamma(X, \mathcal{O}_X^*)$ and consider the invertible matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & g \end{pmatrix}, \qquad A' = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}.$$

Then

$$\chi(A') = \begin{pmatrix} 1 & g(x_0) \\ 0 & 1 \end{pmatrix},$$

$$\chi(A) = \chi\begin{pmatrix} 1 & g^{-1} \\ 0 & 1 \end{pmatrix} \cdot \chi\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

$$= \begin{pmatrix} \delta(g) & \delta(g) \\ 0 & g(x_0)\delta(g) \end{pmatrix}.$$

So if χ commutes with interchanging entries of matrices, we have necessarily $\delta \equiv 1$.

3. The general case $G\ell_n(\mathbb{C})$, $n \geq 3$

We can apply exactly the same method with some minor technical modifications. We fix a continuous group homomorphism

$$\chi : \mathrm{Mor}(X, G\ell_n(\mathbb{C})) \longrightarrow G\ell_n(\mathbb{C})$$

which is $G\ell_n(\mathbb{C})$-left and right invariant. We want to single out the ideal which defines the point x_0. For this we fix two indices $i,j \in \{1,\dots,n\}$, $i \neq j$ and denote by E_{ij} the $n \times n$-matrix with 1 at (i,j) and 0 elsewhere. We define

$$\mathfrak{a}_{ij} := \{f \in \Gamma(X, \mathcal{O}_X) \mid \chi(I + fE_{ij}) = I\}$$

where I is the $(n \times n)$-identity matrix. Analogously to the case $n = 2$, one shows that it is a closed ideal in $\Gamma(X, \mathcal{O}_X)$. By applying permutation matrices, one concludes that $\mathfrak{a}_{ij}$ is in fact independent of i and j and we write just $\mathfrak{a}$ for it.

Next we show that $\mathfrak{a}$ is of codimension 1 or more precisely that there is some continuous $\mathbb{C}$-linear $\beta : \Gamma(X, \mathcal{O}_X) \longrightarrow \mathbb{C}$ with

$$\chi(I + fE_{1n}) = I + \beta(f) \cdot E_{1n}$$

for any $f \in \Gamma(X, \mathcal{O}_X)$. This implies that $\mathfrak{a}$ is maximal and so we have found the point $x_0 \in X$. For $g \in \Gamma(X, \mathcal{O}_X)$ and $k \in \{1,\dots,n\}$, we define

$$D_k(g) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ & & & g & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

where g is on the position (k,k). We write also

$$\chi(I + fE_{1n}) = (\chi_{ij}(f))_{i,j=1,\dots,n}.$$

Using the identity (for $\lambda \in \mathbb{C}^*$) :

$$D_n(\lambda^{-1}) \cdot (I + fE_{1n}) \cdot D_n(\lambda) = I + \lambda f E_{1n}$$

we get

$$(\chi_{ij}(f)) \cdot D_n(\lambda) = D_n(\lambda) \cdot (\chi_{ij}(\lambda f))$$

and therefore

$$\begin{aligned} \chi_{ij}(f) &= \chi_{ij}(\lambda f), \qquad i,j \leq n-1 \\ \chi_{nj}(f) &= \lambda \chi_{nj}(\lambda f), \qquad j \leq n-1 \\ \lambda \chi_{nn}(f) &= \lambda \chi_{nn}(\lambda f) \end{aligned}$$

so finally (for $\lambda \to 0$):

$$\begin{aligned} \chi_{ij}(f) &= 0, \qquad i,j \leq n-1 \\ \chi_{nj}(f) &= 0, \qquad j \leq n-1 \\ \chi_{nn}(f) &= 0\,. \end{aligned}$$

We write $\chi(I + fE_{1n}) = I + \sum_{i=1}^{n-1} \chi_{in}(f)E_{in}$. In order to show $\chi_{in} = 0$, $i > 1$, we use the formula

$$(I + fE_{1n}) \cdot D_2(\lambda) = D_2(\lambda) \cdot (I + fE_{1n})\,,$$

since $n \geq 3$. Applying χ, we get

$$\left(I + \sum_{i=1}^{n-1} \chi_{in}(f)E_{in}\right) \cdot D_2(\lambda) = D_2(\lambda) \cdot \left(I + \sum_{i=1}^{n-1} \chi_{in}(f)E_{in}\right)$$

so $\chi_{2n}(f)E_{2n} = \lambda \chi_{2n}(f)E_{2n}$ and for $\lambda \to 0$ we obtain $\chi_{2n}(f) = 0$. Similarly we argue with $D_3(\lambda)$ and at the end we can take $\beta := \chi_{1n}$.

We pass to the case of diagonal matrices. It is sufficient to describe the values of χ on matrices $D_1(g)$ for $g \in \Gamma(X, \mathcal{O}_X^*)$ (since χ commutes with permutation matrices). We set

$$\chi(D_1(g)) = (h_{ij}(g))_{i,j=1,\ldots,n}$$

and exploit the following identities for $\lambda \in \mathbb{C}^*$:

$$\begin{aligned} \chi(D_1(g\lambda)) &= \chi(D_1(g)) \cdot D_1(\lambda) \\ &= \chi(D_1(\lambda g)) \\ &= D_1(\lambda) \cdot \chi(D_1(g))\,. \end{aligned}$$

This gives

$$h_{ij}(g\lambda) = \begin{cases} \lambda h_{i1}(g), & j = 1 \\ h_{ij}(g), & j > 1 \end{cases}$$

and

$$h_{ij}(\lambda g) = \begin{cases} \lambda h_{1j}(g), & i = 1 \\ h_{ij}(g), & i > 1. \end{cases}$$

So putting together $h_{i1}(g\lambda) = \lambda h_{i1}(g) = h_{i1}(g)$ if $i > 1$. In the same way we get for $j > 1$: $h_{1j}(\lambda g) = \lambda h_{1j}(g) = h_{1j}(g)$. For $\lambda \to 0$ we obtain

$$
\begin{aligned}
h_{1j}(g) &= 0, \quad j > 1, \\
h_{i1}(g) &= 0, \quad i > 1,
\end{aligned}
$$

which means that

$$
\chi(D_1(g)) = \left(\begin{array}{c|c} h_{11}(g) & 0 \\ \hline 0 & * \end{array}\right).
$$

We may now multiply on both sides with $D_2(\lambda)$ and so forth. At the end we arrive at a diagonal form

$$
\chi(D_1(g)) = \begin{pmatrix} \alpha_1(g) & & \\ & \ddots & 0 \\ 0 & & \alpha_n(g) \end{pmatrix}.
$$

Now let us consider the following equality

$$
D_1(g) \cdot (I + E_{12}) \cdot D_1(g^{-1}) = I + gE_{12}\,.
$$

Applying χ yields

$$
\alpha_1(g)\alpha_2(g)^{-1} = g(x_0)
$$

so

$$
\alpha_1(g) = g(x_0)\alpha_2(g)\,.
$$

By using permutation matrices, we see that $\alpha_2 = \cdots = \alpha_n = \delta$. So we can write

$$
\chi(D_1(g)) = \delta(g) \cdot D_1(g(x_0))\,.
$$

For the rest of the argument, one follows the lines of section 2 and uses now the Bruhat decomposition for $G\ell_n(\mathbb{C})$ in order to reduce to upper/lower/diagonal matrices (up to permutation matrices). In fact, this decomposition is effectively only used for holomorphic matrices on a Runge compact set which are close to the unit matrix. Exactly as in the case $n = 2$, we get the desired structure theorem for χ.

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Université de Grenoble I
Institut Fourier
UMR 5582
UFR de Mathématiques
B.P. 74
38402 ST MARTIN D'HÈRES Cedex (France)

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