STOPPING SEMIMARTINGALES ON FOCK SPACE

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Abstract

We define the value, at any non-commutative finite stop time, of some vector semimartingales in Fock space. We apply it to stop a large class of operator processes, including all processes given by Hudson-Parthasarathy quantum stochastic integrals.

1. Introduction and notations

The theory of non-commutative stop times, as an extension of the classical theory of stop times on a probability space, has been initiated in [Hud] and mainly developed in [P-S] (revisited in [Me1]) and [B-W]. In [P-S] the value at any stop time τ of some Weyl processes in the Fock space is computed. It gives rise to a factorization of the Fock space into a part "before τ" tensor a part "after τ". In this way they generalize the strong Markov property of the quantum Brownian motion proved in [Hud].

In [Me1] it is emphasised that the work of [P-S] gives the value at a quantum stop time of processes of vectors which are constituted of the tensor product of a complete martingale and a process in the future.

The aim of this article is to extend this latter result to processes of vectors which are constituted of the tensor product of a regular vector semimartingale and a process in the future. As a byproduct we obtain a method of stopping a large class of operator-valued processes, which contains all the non-commutative stochastic integrals. We recover in this latter case a definition given in [P-S]. This work differs from Barnett-Wilde's work in the sense that they deal with the general context of finite von Neumann algebras which does not contain the Fock space case. Here our study makes strong use of the Fock space structure.

Let us now examine the context in which we work. Let Φ be the boson Fock space over $L^2(\mathbb{R}^+)$. Let $\Phi_{t]}$, *resp.* $\Phi_{[t}$, be the Fock space over $L^2([0,t])$, *resp.*

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$L^2([t,+\infty[)$, for $t \in \mathbb{R}^+$. One then has the continuous tensor product structure: $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$ for all $t \in \mathbb{R}^+$. Recall that the Fock space Φ is isomorphic to the Guichardet space $L^2(\mathcal{P})$ ([Gui]). Indeed, let $\mathcal{P}$ be the set of finite subsets of $\mathbb{R}^+$ that is, $\mathcal{P} = \cup_n \mathcal{P}_n$ where $\mathcal{P}_n$ is the set of subsets of $\mathbb{R}^+$ with cardinal n and $\mathcal{P}_0 = \{\varphi\}$. Each $\mathcal{P}_n$ is equipped with the restriction of the corresponding Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{P}_0$ is equipped with the unit mass. Thus $\mathcal{P}$ is equipped with a σ-finite measure, denoted $d\sigma$, and Φ is isomorphic to $L^2(\mathcal{P})$. Consequently for $f \in \Phi$ we have

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma.$$

For all element $u \in L^2(\mathbb{R}^+)$ define the associated coherent vector $\varepsilon(u) \in \Phi$ by $[\varepsilon(u)](\sigma) = \prod_{s\in\sigma} u(s)$, for $\sigma \in \mathcal{P}$ (with the usual convention that the empty product equals 1). For all $s \le t$ define $u_{t]} = u\mathbb{1}_{[0,t]}, u_{[s,t]} = u\mathbb{1}_{[s,t]}, u_{[t} = u\mathbb{1}_{[t,+\infty[}$; define $\sigma_{t]} = \{r \in \sigma; r \le t\}$ and $\sigma_{[t} = \{r \in \sigma; r \ge t\}$ (note that it makes no difference whether we take strict inequalities in the latter definitions or not, as the set of $\sigma \in \mathcal{P}$ with $t \in \sigma$ is of null measure for every fixed t).

The tensor product structure $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$ can be seen to correspond to the following :

(i) $f \in \Phi_{t]}$ if and only if $f(\sigma) = 0$ unless $\sigma \subset [0,t]$

(ii) $h \in \Phi_{[t}$ if and only if $h(\sigma) = 0$ unless $\sigma \subset [t,+\infty[$

(iii) $g = f \otimes h$ if and only if $g(\sigma) = f(\sigma_{t]})h(\sigma_{[t})$.

In this way we observe that $\varepsilon(u) = \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t})$ for all u and all t.

Following [H-P] an *adapted operator process* in Φ is a family $(H_t)_{t\ge 0}$ of operators on Φ, defined on the dense subspace $\mathcal{E} = \text{span}\,\{\varepsilon(u);\, u \in L^2(\mathbb{R}^+)\}$, such that $t \to H_t\,\varepsilon(u)$ is strongly measurable and

$$\begin{cases} H_t\,\varepsilon(u_{t]}) \in \Phi_{t]} \\ H_t\,\varepsilon(u) = [H_t\,\varepsilon(u_{t]})] \otimes \varepsilon(u_{[t}) \end{cases}$$

for all $u \in L^2(\mathbb{R}^+)$, $t \in \mathbb{R}^+$. In this case we say that H_t is *adapted at time t*.

2. Calculus on the Guichardet space

The results of this section can be found in great details in [A-L]. For all $t \in \mathbb{R}^+$, define the operator E_t on Φ by

$$[E_t f](\sigma) = f(\sigma)\mathbb{1}_{\sigma\subset[0,t]},\ \sigma \in \mathcal{P}.$$

The operator E_t is actually the orthogonal projection from Φ onto $\Phi_{t]}$. The operator E_0, also denoted $\mathbb{E}$, is given by

$$\mathbb{E}[f](\sigma) = f(\emptyset)\mathbb{1}_{\sigma=\emptyset},\ \sigma \in \mathcal{P}.$$

For all $f \in \Phi$, all $t \in \mathbb{R}^+$, define

$$[D_t f](\sigma) = f(\sigma \cup \{t\})\mathbb{1}_{\sigma\subset[0,t]},\ \sigma \in \mathcal{P}.$$

It can be easily seen ([A-L]), from the $\not\int$-Lemma ([L-P]) that

$$\int_0^\infty \int_{\mathcal{P}} \left|[D_t f](\sigma)\right|^2 d\sigma\, dt < \infty.$$

Thus, for almost all $t, D_t f$ defines an element of Φ and

$$\int_0^\infty \|D_t f\|^2 dt < \infty.$$

From these definitions one can easily check the following properties :

(i) for all $f \in \Phi$, $\quad D_u E_t f = \begin{cases} 0 & \text{for a.a. } u > t \\ D_u f & \text{for a.a. } u < t \end{cases}$

(ii) for all $u \in L^2(\mathbb{R}^+)$, $D_t\, \varepsilon(u) = u(t)\varepsilon(u_{t]})$ for a.a. t

(iii) if $g = f \otimes h$ in the structure $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$, then for almost all $u \geq t$

$$D_u\, g = f \otimes D_u h.$$

For a strongly measurable family $g_. = (g_t)_{t\geq 0}$ of elements of Φ such that $g_t \in \Phi_{t]}$ for all t and $\int_0^\infty \|g_t\|^2 dt$ is finite, define

$$[I(g_.)](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ g_{\vee\sigma}(\sigma-) & \text{otherwise} \end{cases}$$

where $\vee\sigma = \max\, \{s \in \sigma\}$ and $\sigma- = \sigma \backslash \{\vee\sigma\}$. It can be easily seen ([A-L]) that $I(g_.)$ defines an element of $L^2(\mathcal{P})$, thus an element of Φ. From now on $I(g_.)$ is denoted $\int_0^\infty g_t\, dx_t$. This notation is justified by the following. Let $f \in \Phi$, we can compute $\int_0^\infty D_t f\, dx_t$ and observe that

$$[\int_0^\infty D_t f\, dx_t](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ f(\sigma) & \text{otherwise} \end{cases}$$

Thus, we have the following *Fock space predictable representation.*

Theorem 1 – *For all $f \in \Phi$ one has the representation*

$$f = \mathbb{E}[f] + \int_0^\infty D_t f\, dx_t$$

and

$$\langle f,\, g\rangle = \overline{\mathbb{E}[g]}\, \mathbb{E}[f] + \int_0^\infty \langle D_t\, g,\, D_t f\rangle\, dt$$

for all $g \in \Phi$. ■

This result is an analogue of the probabilistic predictable representation property of Brownian motion, compensated Poisson process, Azéma's martingales... (for which the notation x_t stands), but here it is purely intrinsic to the Fock space structure and has nothing to do with probabilistic interpretations of Φ.

A *vector process* is a strongly measurable family $(z_t)_{t\geq 0}$ of elements of Φ. A vector process z is *adapted* if $z_t \in \Phi_{t]}$ for all t. An adapted process $(m_t)_{t\geq 0}$ is a *martingale* if $E_s\, m_t = m_s$ for all $s \leq t$. A martingale $(m_t)_{t\geq 0}$ is *complete* if there exists a $m \in \Phi$ such that $m_t = E_t\, m$ for all t (or equivalently, if m_t converges in Φ to a vector m when t tends to $+\infty$).

If $(g_t)_{t\geq 0}$ is an adapted vector process such that $\int_a^b \|g_t\|^2 dt < \infty$ for every $0 \leq a < b \leq +\infty$, then define

$$\int_a^b g_t\, dx_t = \int_0^\infty g_t\, 1\!\!1_{[a,b]}(t)\, dx_t.$$

Lemma 2 – *Let $(m_t)_{t\geq 0}$ be a martingale in Φ. Then there exists a unique adapted process $(\xi_t)_{t\geq 0}$ in Φ such that for all $t \in \mathbb{R}^+$ one has $\int_0^t \|\xi_u\|^2 du < \infty$ and*

$$m_t = m_0 + \int_0^t \xi_u\, dx_u.$$

The process $(\xi_t)_{t\geq 0}$ is given by $\xi_t = D_t\, m_{t+} = D_t\, m_{t+h}$ for almost all $t, h > 0$.

Proof

From Theorem 1 one has $m_t = m_0 + \int_0^\infty D_u m_t\, dx_u$. But we have seen that for all $f \in \Phi$, almost all $u > t$ one has $D_u E_t f = 0$. Thus one has actually $m_t = m_0 + \int_0^t D_u m_t\, dx_u$. Furthermore, as $D_u E_t = D_u$ for almost all $u < t$, we have for all $t < t'$, almost all $u < t, D_u m_{t'} = D_u E_t m_{t'} = D_u m_t$. Consequently the vector $D_u m_{u+h}$ does not depend on $h > 0$, one can choose ξ_u to be $D_u m_{u+h}$ for any $h > 0$. ■

Lemma 3 – *Let H be a bounded operator on Φ, adapted at time t. Let $(g_s)_{s\geq t}$ be an adapted vector process in Φ such that $\int_t^\infty \|g_s\|^2 ds < \infty$. Then one has*

$$H \int_t^\infty g_s\, dx_s = \int_t^\infty H g_s\, dx_s.$$

Proof

First of all notice that the boundedness and adaptedness of H implies that $(Hg_t)_{t\geq 0}$ is strongly measurable and $\int_t^\infty Hg_s\, dx_s$ is well-defined. For all $u \in L^2(\mathbb{R}^+)$, one has

$$\begin{aligned}
&\langle \varepsilon(u),\, H \int_t^\infty g_s\, dx_s \rangle = \\
&\quad = \langle H^*\varepsilon(u),\, \int_t^\infty g_s\, dx_s \rangle = \int_t^\infty \langle D_s H^*\varepsilon(u), g_s\rangle\, ds \\
&\quad = \int_t^\infty \langle D_s(H^*\varepsilon(u_{t]}) \otimes \varepsilon(u_{[t})),\, g_s\rangle\, ds = \int_t^\infty \langle H^*\varepsilon(u_{t]}) \otimes D_s\varepsilon(u_{[t}), g_s\rangle\, ds \\
&\quad = \int_t^\infty \langle u(s)H^*\varepsilon(u_{t]}) \otimes \varepsilon(u_{[t,s]}),\, g_s\rangle\, ds = \int_t^\infty \langle u(s)H^*\varepsilon(u_{s]}),\, g_s\rangle\, ds \\
&\quad = \int_t^\infty \langle u(s)\varepsilon(u_{s]}),\, Hg_s\rangle\, ds = \int_t^\infty \langle D_s\varepsilon(u), H\, g_s\rangle\, ds = \langle \varepsilon(u),\, \int_t^\infty Hg_s\, dx_s\rangle.
\end{aligned}$$

One concludes by density of the space $\mathcal{E}$ in Φ. ■

3. Non-commutative stop times on Fock space

Let us recall the main definitions of [P-S].

A *stop time* τ on Φ is a spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$ with values in the space of orthogonal projections on Φ and such that, for all t, the operator $\tau([0,t])$ is adapted at time t.

In the following we adopt a probabilistic-like notation: for any Borel subset $A \subset \mathbb{R}^+ \cup \{+\infty\}$, the operator $\tau(A)$ is denoted $\mathbb{1}_{\tau \in A}$; in the same way $\tau(\{t\})$ is denoted $\mathbb{1}_{\tau=t}$, $\tau([0,t])$ is denoted by $\mathbb{1}_{\tau \leq t}$, etc...

A stop time τ is *finite* if $\mathbb{1}_{\tau=+\infty} = 0$. It is bounded by T if $\mathbb{1}_{\tau \leq T} = I$ for some $T \in \mathbb{R}^+$.

A point t in $\mathbb{R}^+$ is a *continuity point* for τ if $\mathbb{1}_{\tau=t} = 0$. Note that, unless $\tau \equiv 0$, the point 0 is always a continuity point for τ. It is also easy to check that the set of points $t \in \mathbb{R}^+$ which are not continuity points for τ is at most countable.

If τ and τ' are two stop times on Φ, one says that $\tau \leq \tau'$ if, for all $t \in \mathbb{R}^+$ one has $\mathbb{1}_{\tau \leq t} \geq \mathbb{1}_{\tau' \leq t}$ (in the usual sense of comparison of two projections).

A stop time τ is *discrete* if there exists a finite set $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ such that $\mathbb{1}_{\tau \in E} = I$.

A sequence of stop times $(\tau_n)_n$ is said to *converge* to a stop time τ if, for all continuity point t for τ, the operators $\mathbb{1}_{\tau_n \leq t}$ converge strongly to $\mathbb{1}_{\tau \leq t}$.

A *sequence of refining τ-partitions* is a sequence $(E_n)_n$ of partitions $E_n = \{0 \leq t_1^n < t_2^n < \cdots < t_{i_n}^n < +\infty\}$ of partitions of $\mathbb{R}^+$ such that

(i) all the t_j^i are continuity points for τ;

(ii) $E_n \subseteq E_{n+1}$ for all n;

(iii) the diameter, $\max\, \{t_{i+1}^n - t_i^n\,; i = 1, ..., i_n\}$, of E_n tends to 0 as n tends to $+\infty$;

(iv) $t_{i_n}^n$ tends to $+\infty$ when n tends to $+\infty$.

The following result is taken from [P-S], Proposition 3.3 and [Me1].

Proposition 4 – *Let τ be any stop time. Then there exists a sequence $(\tau_n)_n$ of discrete stop times such that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau$ and $(\tau_n)_n$ converges to τ.*

Proof

Let $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ be a partition of $\mathbb{R}^+$. Define a spectral measure τ_E by

$$\tau_E(\{t_i\}) = \begin{cases} \mathbb{1}_{\tau < t_1} & \text{if } i = 1 \\ \mathbb{1}_{\tau \in [t_{i-1}, t_i[} & \text{if } 1 < i \leq n-1, \end{cases}$$
$$\tau_E(\{t_n\}) = \mathbb{1}_{\tau \geq t_{n-1}}.$$

The spectral measure τ_E clearly defines a discrete stop time on Φ and $\tau_E \geq \tau$. Taking a sequence $(E_n)_n$ of refining τ-partitions of $\mathbb{R}^+$ gives the required sequence $(\tau_n)_n = (\tau_{E_n})_n$. ■

4. Stopping vector processes

Our aim is to define the value z_τ at a finite stop time τ of a large class of vector processes $(z_t)_{t \geq 0}$ in Φ.

An adapted vector process $(z_t)_{t\geq 0}$ in Φ is a *regular vector semimartingale* if $(z_t)_{t\geq 0}$ admits a decomposition (always unique) as $z_t = m_t + a_t$ where m_t is a martingale and $a_t = \int_0^t h_s\,ds$ with $h_t \in \Phi_{t]}$ and $\int_0^t \|h_s\|ds < \infty$ for all t. The integral $\int_0^t h_s\,ds$ is understood in the usual hilbertian sense that is, $\langle f, \int_0^t h_s\,ds\rangle = \int_0^t \langle f, h_s\rangle ds$ which defines a vector in Φ for

$$|\langle f, \int_0^t h_s\,ds\rangle| \leq \int_0^t |\langle f, h_s\rangle|ds \leq \|f\| \int_0^t \|h_s\|ds.$$

It is interesting to recall a characterisation of the regular semimartingales of vectors in Φ.

Theorem 5 – *An adapted vector process $(z_t)_{t\geq 0}$ in Φ is a regular vector semimartingale if and only if there exists a locally integrable function g on $\mathbb{R}^+$ such that, for all $s \leq t$, one has*

$$\|E_s z_t - z_s\| \leq \int_s^t g(u)du.$$

Proof

If $(z_t)_{t\geq 0}$ is a regular vector semimartingale the estimate is trivial. The converse is a simple consequence of Enchev's characterization of Hilbertian quasimartingales in [Enc] (see also [Me2]). ■

A vector process $(y_t)_{t\geq 0}$ in Φ is said to be *adapted to the future* if $y_t \in \Phi_{[t}$ for all t.

For any vector process $(w_t)_{t\geq 0}$ in Φ and for any discrete stop time τ one can obviously define, following the case of classical stop times, w_τ by

$$w_\tau = \sum_i \mathbb{1}_{\tau=t_i} w_{t_i}. \tag{1}$$

But when τ is any finite stop time we wish to pass to the limit on the expression (1) for a sequence $(\tau_n)_n$ of discrete stop times converging to τ (Proposition 4).

In [P-S] and [Me1] it is shown that this convergence can be obtained when $(w_t)_{t\geq 0}$ is of the form $(m_t \otimes y_t)_{t\geq 0}$ where $(m_t)_{t\geq 0}$ is a complete martingale and $(y_t)_{t\geq 0}$ is a vector process adapted to the future. We are going to extend this result to processes $(w_t)_{t\geq 0}$ of the form $(z_t \otimes y_t)_{t\geq 0}$ where $(z_t)_{t\geq 0}$ is a regular vector semimartingale and $(y_t)_{t\geq 0}$ is adapted to the future and bounded in norm. We first need some preliminary results.

Remark : If c is an element of $\Phi_{0]} \simeq \mathbb{C}\,\mathbb{1}$ we have

$$\sum_i \mathbb{1}_{\tau=t_i}\, c = c.$$

Thus, in the following we assume that all our martingales $(m_t)_{t\geq 0}$ are such that $m_0 = 0$. Consequently, by Lemma 2, every regular semimartingale is of the form $z_t = \int_0^t \xi_s\,dx_s + \int_0^t h_s\,ds, \quad t \geq 0.$

Proposition 6 – *Let*

$$z_t = \int_0^t \xi_s\, dx_s + \int_0^t h_s\, ds, \quad t \geq 0$$

be a regular vector semimartingale. Let τ be a bounded stop time with bound T. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Put $\tau_n = \tau_{E_n}$, for all $n \in \mathbb{N}$. Then the sequence $(z_{\tau_n})_n$ converges to a vector z_τ in Φ which is given by

$$z_\tau = \int_0^T \mathbb{1}_{\tau > s}\, \xi_s\, dx_s + \int_0^T \mathbb{1}_{\tau > s}\, h_s\, ds.$$

Proof

One has

$$\begin{aligned} z_{\tau_n} &= \sum_i \mathbb{1}_{\tau_n = t_i}\, z_{t_i} = \sum_i \mathbb{1}_{\tau_n = t_i} \Big[\int_0^{t_i} \xi_s\, dx_s + \int_0^{t_i} h_s\, ds\Big] \\ &= \sum_i \sum_{j<i} \mathbb{1}_{\tau_n = t_i} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\ &= \sum_j \sum_{i>j} \mathbb{1}_{\tau_n = t_i} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\ &= \sum_j \mathbb{1}_{\tau_n > t_j} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\ &= \sum_j \Big[\int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n > t_j}\, \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n > t_j}\, h_s\, ds\Big] \end{aligned}$$

(by boundedness and t_j-adaptedness of $\mathbb{1}_{\tau_n > t_j}$, and by Lemma 3)

$$\begin{aligned} &= \sum_j [\int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n > s}\, \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n > s}\, h_s\, ds] \\ &= \int_0^T \mathbb{1}_{\tau_n > s}\, \xi_s\, dx_s + \int_0^T \mathbb{1}_{\tau_n > s}\, h_s\, ds. \end{aligned}$$

Now, one has

$$\begin{aligned} &\Big|\Big| z_{\tau_n} - \Big[\int_0^T \mathbb{1}_{\tau > s}\, \xi_s\, dx_s + \int_0^T \mathbb{1}_{\tau > s}\, h_s\, ds\Big]\Big|\Big|^2 \\ &\qquad \leq 2 \int_0^T \|(\mathbb{1}_{\tau_n > s} - \mathbb{1}_{\tau > s})\xi_s\|^2 ds + 2\Big[\int_0^T \|(\mathbb{1}_{\tau_n > s} - \mathbb{1}_{\tau > s}) h_s\| ds\Big]^2. \end{aligned}$$

The quantities inside the integrals converge to 0 when n tends to $+\infty$ and are respectively dominated by $4\|\xi_s\|^2$ and $2\|h_s\|$ which are integrable on $[0, T]$. Thus, one concludes by the dominated convergence Theorem. ■

For any finite stop time τ and $n \in \mathbb{N}$ one can define the stop time $\tau \wedge n$ by

$$\mathbb{1}_{\tau\wedge n\leq t} = \begin{cases} 0 & \text{if } n > t \\ \mathbb{1}_{\tau\leq t} & \text{if } n \leq t. \end{cases}$$

It is clear that $\tau \wedge n$ is a bounded stop time with bound n and that $(\tau \wedge n)_n$ converges to τ. Thus, we easily deduce the following result.

Proposition 7 – *Let $z_t = \int_0^t \xi_s\, dx_s + \int_0^t h_s\, ds$, $t \in \mathbb{R}^+$, be a regular vector semimartingale. Let τ be a finite stop time such that*

$$\int_0^\infty \|\mathbb{1}_{\tau>s}\, \xi_s\|^2 ds < \infty \text{ and } \int_0^\infty \|\mathbb{1}_{\tau>s}\, h_s\| ds < \infty$$

then the sequence $(z_{\tau\wedge n})_n$ converges to a vector z_τ in Φ given by

$$z_\tau = \int_0^\infty \mathbb{1}_{\tau>s}\, \xi_s\, dx_s + \int_0^\infty \mathbb{1}_{\tau>s}\, h_s\, ds. \qquad \blacksquare$$

If τ is a finite stop time, the mapping $A \to \|\mathbb{1}_{\tau\in A} f\|^2$ defines a measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ for any $f \in \Phi$. We denote this measure by $\|\mathbb{1}_{\tau\in ds} f\|^2$.

The following result is an improvement of [P-S]'s results, together with a shortening of their proofs as we are dealing with a slightly simpler case.

Proposition 8 – *Let $(m_t)_{t\geq 0}$ be a complete martingale. Let $(y_t)_{t\geq 0}$ be a vector process adapted to the future. Put $w_t = m_t \otimes y_t$, $t \in \mathbb{R}^+$. Let τ be a finite stop time such that*

$$\int_0^\infty \|y_s\|^2 \|\mathbb{1}_{\tau\in ds}\, m\|^2 < \infty$$

where $m = \lim_{t\to+\infty} m_t$. Let $(E_n)_n$ be a sequence of refining τ-partitions. Put $\tau_n = \tau_{E_n}$, $n \in \mathbb{N}$. Then the sequence (w_{τ_n}) converges in Φ to a vector w_τ which is independent of the chosen sequence $(E_n)_n$.

Proof

Consider $p \leq q \in \mathbb{N}$. As $E_q \supset E_p$ we can assume that E_p is of the form $\{0 \leq t_1 < \cdots < t_n\}$ and E_q is of the form $\{0 \leq \cdots < t_i = t_i^0 < t_i^1 < \cdots < t_i^{n_i} = t_{i+1} < \cdots\}$. So it is sufficient to prove that the expression

$$\|\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}\, w_{t_{i+1}} - \sum_{i,j} \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\, w_{t_i^{j+1}}\|^2$$

converges to 0 when the diameter δ of E_p tends to 0.

As the space $\mathcal{E}$ is dense in Φ there exists a $\tilde{m} \in \mathcal{E}$ such that $\|m - \tilde{m}\|$ is small. Suppose $\tilde{m}$ is of the form

$$\tilde{m} = \sum_{k=1}^K \lambda_k\, \varepsilon(u^k).$$

In [P-S], Proposition 4.9, it is proved that there exists a sequence $(y^n)_n$ of vector processes adapted to the future such that, for all n, the mapping $t \to y_t^n$ is strongly continuous, $\int_0^\infty \|y_s^n\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2 < \infty$ and

$$\int_0^\infty \|y_s - y_s^n\|^2\, \|\mathbb{1}_{\tau\in ds} m\|^2 \to 0, \quad n \to +\infty.$$

To simplify the notation choose a $(\tilde{y}_t)_{t\geq 0}$ strongly continuous vector process adapted to the future such that

$$\int_0^\infty \|\tilde{y}_s\|^2 \|\mathbb{1}_{\tau\in ds}\, m\|^2 < \infty$$

and such that $\int_0^\infty \|y_s - \tilde{y}_s\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2$ is small.

Finally, notice that an operator H which is adapted at time t always satisfies $HE_u = E_u H$ for all $u \geq t$. One has

$$\begin{aligned}
\|\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} w_{t_{i+1}} - \sum_{i,j} \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} w_{t_i^{j+1}}\|^2 \\
&= \|\sum_{i,j} \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(w_{t_{i+1}} - w_{t_i^{j+1}})\|^2 \\
&= \sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m_{t_{i+1}} \otimes y_{t_{i+1}} - m_{t_i^{j+1}} \otimes y_{t_i^{j+1}})\|^2 \\
&\leq 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m_{t_{i+1}} \otimes y_{t_{i+1}} - m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}})\|^2 \\
&+ 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m_{t_i^{j+1}} \otimes y_{t_i^{j+1}} - m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2 \\
&+ 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}} - m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2. \quad (2)
\end{aligned}$$

We now concentrate on the last term of the right hand side of (2). For any fixed $b > 0$, it is equal to

$$\begin{aligned}
3 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}} - m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2 \\
+ 6 \sum_{i,j;\, t_i^{j+1}>b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\, m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}}\|^2 \\
+ 6 \sum_{i,j;\, t_i^{j+1}>b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\, m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}}\|^2. \quad (3)
\end{aligned}$$

We now concentrate on the first term of (3). It is dominated by

$$\begin{aligned}
9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m_{t_{i+1}} - \tilde{m}_{t_{i+1}}) \otimes \tilde{y}_{t_{i+1}}\|^2 \\
+ 9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m_{t_i^{j+1}} - \tilde{m}_{t_i^{j+1}}) \otimes \tilde{y}_{t_i^{j+1}}\|^2 \\
+ 9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(\tilde{m}_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}} - \tilde{m}_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2 \\
\leq 9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} E_{t_{i+1}}(m - \tilde{m})\|^2 \|\tilde{y}_{t_{i+1}}\|^2
\end{aligned}$$

$$+ 9 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} E_{t_i^{j+1}}(m - \tilde{m})\|^2 \|\tilde{y}_{t_i^{j+1}}\|^2$$

$$+ 9K \sum_{k=1}^{K} \lambda_k^2 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \big(\varepsilon(u^k_{t_{i+1}]}) \otimes \tilde{y}_{t_{i+1}} - \varepsilon(u^k_{t_i^{j+1}]}) \otimes \tilde{y}_{t_i^{j+1}}\big)\|^2$$

$$\leq 18 \max_{s \leq b} \|\tilde{y}_s\|^2 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} (m - \tilde{m})\|^2$$

$$+ 9K \sum_{k=1}^{K} \lambda_k^2 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \varepsilon(u^k_{t_i^{j+1}]})\|^2 \|\varepsilon(u^k_{[t_i^{j+1}, t_{i+1}]}) \otimes \tilde{y}_{t_{i+1}} - 1 \otimes \tilde{y}_{t_i^{j+1}}\|^2$$

$$\leq 18 \max_{s \leq b} \|\tilde{y}_s\|^2 \|m - \tilde{m}\|^2$$

$$+ 18K \sum_{k=1}^{K} \lambda_k^2 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \varepsilon(u^k_{t_i^{j+1}]})\|^2 \|\varepsilon(u^k_{[t_i^{j+1}, t_{i+1}]}) - 1\|^2 \|\tilde{y}_{t_{i+1}}\|^2$$

$$+ 18K \sum_{k=1}^{K} \lambda_k^2 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \varepsilon(u^k_{t_i^{j+1}]})\|^2 \|\tilde{y}_{t_{i+1}} - \tilde{y}_{t_i^{j+1}}\|^2$$

$$\leq 18 \max_{s \leq b} \|\tilde{y}_s\|^2 \|m - \tilde{m}\|^2$$

$$+ 18K \sum_{k=1}^{K} \lambda_k^2 \max_{s \leq b} \|\tilde{y}_s\|^2 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \varepsilon(u^k_{t_i^{j+1}]})\|^2 \times$$

$$\times \Big(\int_{t_i^{j+1}}^{t_{i+1}} |u^k(s)|^2 \|\varepsilon(u^k_{[t_i^{j+1}, s]})\|^2 ds \Big)$$

$$+ 18\, K \sum_{k=1}^{K} \lambda_k^2 \sup_{\substack{s,t \leq b \\ |s-t| \leq \delta}} \|\tilde{y}_s - \tilde{y}_t\|^2 \sum_{i,j;\, t_i^{j+1} \leq b} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \varepsilon(u^k_{t_i^{j+1}]})\|^2$$

$$\leq 18 \max_{s \leq b} \|\tilde{y}_s\|^2 \|m - \tilde{m}\|^2$$

$$+ 18K \sum_{k=1}^{K} \lambda_k^2 \max_{s \leq b} \|\tilde{y}_s\|^2 \sum_{i,j} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \varepsilon(u^k)\|^2 \sup_{\substack{s,t \leq b \\ |s-t| \leq \delta}} \int_s^t |u^k(v)|^2 dv$$

$$+ 18K \sum_{k=1}^{K} \lambda_k^2 \sup_{\substack{s,t \leq b \\ |s-t| \leq \delta}} \|\tilde{y}_s - \tilde{y}_t\|^2 \sum_{i,j} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} \varepsilon(u^k)\|^2.$$

Inserting this in (3) and then in (2) we get

$$\|w_{\tau_p} - w_{\tau_q}\|^2$$

$$\leq 3 \sum_{i,j} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} m\|^2 \|y_{t_{i+1}} - \tilde{y}_{t_{i+1}}\|^2 + 3 \sum_{i,j} \|\mathbb{1}_{\tau \in [t_i^j, t_i^{j+1}]} m\|^2 \|y_{t_i^{j+1}} - \tilde{y}_{t_i^{j+1}}\|^2$$

$$+ 18 \max_{s\leq b} \|\tilde{y}_s\|^2 \|m-\tilde{m}\|^2$$

$$+ 18\, K \sum_{k=1}^{K} \lambda_k^2 \|\varepsilon(u^k)\|^2 \max_{s\leq b} \|\tilde{y}_s\|^2 \sup_{\substack{s,t\leq b\\ |s-t|\leq\delta}} \int_s^t |u^k(v)|^2 dv$$

$$+ 18\, K \sum_{k=1}^{K} \lambda_k^2 \|\varepsilon(u^k)\|^2 \sup_{\substack{s,t\leq b\\ |s-t|\leq\delta}} \|\tilde{y}_s - \tilde{y}_t\|^2$$

$$+ 12 \sum_{i,j;t_i^{j+1}>b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} m\|^2 \|\tilde{y}_{t_{i+1}}\|^2.$$

When δ tends to 0 the fourth and the fifth terms converge to 0 and the expression above converges to

$$6\int_0^\infty \|y_s-\tilde{y}_s\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2 + 18 \max_{s\leq b} \|\tilde{y}_s\|^2 \|m-\tilde{m}\|^2 + 12 \int_b^\infty \|\tilde{y}_s\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2.$$

This latter expression converges to $6\int_b^\infty \|y_s\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2$ when $\tilde{y}$ tends to y and $\tilde{m}$ tends to m. This finally tends to 0 when b tends to $+\infty$. We have thus proved the convergence of $(w_{\tau_n})_n$ to a limit $w_\tau \in \Phi$.

If $(E_n)_n$ and $(F_n)_n$ are two sequences of refining τ-partitions, denote by $E_n \vee F_n$ the τ-partition made of $E_n \cup F_n$. We then have

$$\|w_{\tau_{E_n}} - w_{\tau_{F_n}}\|^2 \leq 2\|w_{\tau_{E_n}} - w_{\tau_{E_n\vee F_n}}\|^2 + 2\|w_{\tau_{E_n\vee F_n}} - w_{\tau_{F_n}}\|^2.$$

From the estimate obtained above we see that, as $E_n \vee F_n \subset E_n$ and $E_n \vee F_n \subset F_n$ that $\|w_{\tau_{E_n}} - w_{\tau_{E_n\vee F_n}}\|^2$ (respectively $\|w_{\tau_{E_n\vee F_n}} - w_{\tau_{F_n}}\|^2$) is dominated by an expression which depends only on the diameter of E_n (respectively F_n) and converges to 0 with it. Thus, the limit w_τ does not depend on the choice of the sequence $(E_n)_n$. ■

Remark : The vector w_τ obtained from this proposition is denoted

$$\int \mathbb{1}_{\tau\in ds}(E_s m)\otimes y_s.$$

Let $w_t = m_t \otimes y_t$ and $w'_t = m'_t \otimes y'_t$ with integrability condition:

$$\int_0^\infty \|y_s\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2 < \infty \quad \text{and} \quad \int_0^\infty \|y'_s\|^2 \|\mathbb{1}_{\tau\in ds} m'\|^2 < \infty.$$

Then it follows from the above that

$$\langle w_\tau, w'_\tau\rangle = \int_0^\infty \langle y_s, y'_s\rangle \langle \mathbb{E}_s \mathbb{1}_{\tau\in ds} m, m'\rangle.$$

Theorem 9 – *Let $z_t = \int_0^t \xi_s dx_s + \int_0^t h_s\, ds$, $t\geq 0$ be a regular vector semimartingale. Let $(y_t)_{t\geq 0}$ be a vector process, adapted to the future and bounded in norm. Let τ be a finite stop time such that*

$$\int_0^\infty \|\mathbb{1}_{\tau>s}\, \xi_s\|^2 ds < \infty \quad \textit{and} \quad \int_0^\infty \|\mathbb{1}_{\tau>s}\, h_s\| ds < \infty.$$

Let $w_t = z_t \otimes y_t$, $t \geq 0$. Let $(E_n)_n$ be a sequence of refining τ-partitions of $I\!R^+$. Put $\tau_n = \tau_{E_n}, n \in I\!N$. Then the sequence $(w_{\tau_n})_n$ converges to a vector w_τ which is given by

$$w_\tau = \int 1\!\!1_{\tau\in ds}[E_s z_\tau] \otimes y_s.$$

Proof

One has

$$\begin{aligned}
z_{\tau_n} &= \sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}]}\left[\int_0^{t_{i+1}} \xi_s\,dx_s + \int_0^{t_{i+1}} h_s\,ds\right] \otimes y_{t_{i+1}} \\
&= \sum_i \sum_{j\leq i} 1\!\!1_{\tau\in[t_i,t_{i+1}]}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{j+1}} h_s\,ds\right] \otimes y_{t_{i+1}} \\
&= \sum_i \sum_{j\leq i} 1\!\!1_{\tau\in[t_i,t_{i+1}]}1\!\!1_{\tau\geq t_j}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{j+1}} h_s\,ds\right] \otimes y_{t_{i+1}} \\
&= \sum_i \sum_{j\leq i} 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}1\!\!1_{\tau\geq t_j}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{i+1}} h_s\,ds\right] \otimes y_{t_{i+1}} \\
&= \sum_i \sum_{j} 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}1\!\!1_{\tau\geq t_j}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{j+1}} h_s\,ds\right] \otimes y_{t_{i+1}}
\end{aligned}$$

(as for $j \geq i+1$ the terms inside the sum vanish)

$$\begin{aligned}
&= \sum_i \sum_j 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\left[\int_{t_j}^{t_{j+1}} 1\!\!1_{\tau\geq t_j}\,\xi_s\,dx_s + \int_{t_j}^{t_{j+1}} 1\!\!1_{\tau\geq t_j} h_s\,ds\right] \otimes y_{t_{i+1}} \\
&= \sum_i \sum_j 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\left[\int_{t_j}^{t_{j+1}} 1\!\!1_{\tau_n>s}\,\xi_s\,dx_s + \int_{t_j}^{t_{j+1}} 1\!\!1_{\tau_n\geq s} h_s\,ds\right] \otimes y_{t_{i+1}} \\
&= \sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\left[\int_0^{\infty} 1\!\!1_{\tau_n>s}\,\xi_s\,dx_s + \int_0^{\infty} 1\!\!1_{\tau_n>s} h_s\,ds\right] \otimes y_{t_{i+1}} \\
&= \sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}(z_{\tau_n}) \otimes y_{t_{i+1}} \\
&= \sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}(z_{\tau}) \otimes y_{t_{i+1}} \\
&+ \sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}(z_\tau - z_{\tau_n}) \otimes y_{t_{i+1}}.
\end{aligned}$$

The first term of the right hand side converges to

$$\int 1\!\!1_{\tau\in ds}\,E_s(z_\tau) \otimes y_s$$

by Proposition 8 (as the condition $\int_0^\infty \|y_s\|^2 \|1\!\!1_{\tau\in ds} z_\tau\|^2 < \infty$ is trivial since $s \to$

$\|y_s\|$ is bounded). The second term has the square of its norm dominated by

$$\sum_i \|\mathbb{1}_{\tau\in[t_i,t_{i+1}]} E_{t_{i+1}}(z_\tau - z_{\tau_n})\|^2 \sup_s \|y_s\|^2 \leq \sup_s \|y_s\|^2 \|z_\tau - z_{\tau_n}\|^2$$

which converges to 0 by Proposition 7. ■

5. Stopping operator processes

We now consider a process $(X_t)_{t\geq 0}$ of adapted operators on Φ. As for vectors in the previous section, we want to define the value X_τ of $(X_t)_{t\geq 0}$ at a finite stop time τ. In the case of discrete stop times, three non-equivalent definitions appear:

left-stopping : $\tau \circ X = \sum_i \mathbb{1}_{\tau=t_i} X_{t_i}$

right-stopping : $X \circ \tau = \sum_i X_{t_i} \, \mathbb{1}_{\tau=t_i}$

two-sided-stopping : $\tau \circ X \circ \tau = \sum_i \mathbb{1}_{\tau=t_i} X_{t_i} \, \mathbb{1}_{\tau=t_i}$

As previously we wish to pass to the limit of discrete stop times converging to a finite stop time, for a large class of processes of operators $(X_t)_{t\geq 0}$.

Let τ be a finite stop time. A regular vector semimartingale $z_t = \int_0^t \xi_s dx_s + \int_0^t h_s \, ds$ is said to be τ-integrable if

$$\int_0^\infty \|\mathbb{1}_{\tau>s}\xi_s\|^2 ds < \infty \text{ and } \int_0^\infty \|\mathbb{1}_{\tau>s} h_s\| ds < \infty.$$

Proposition 10 – *Let $(X_t)_{t\geq 0}$ be an adapted operator process on Φ. Let $u \in L^2(\mathbb{R}^+)$ be such that $(X_t \, \varepsilon(u_{t]}))_{t\geq 0}$ is a regular vector semimartingale. Let τ be a finite stop time such that $(X_t \, \varepsilon(u_{t]}))_{t\geq 0}$ is τ-integrable. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Then the sequence $(X_{\tau_{E_n}} \varepsilon(u))_n$ converges to a vector $X_\tau \, \varepsilon(u)$.*

Proof

As $(X_t)_{t\geq 0}$ is an adapted process of operators, one has

$$X_t \, \varepsilon(u) = X_t \, \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t}).$$

The vector process $(\varepsilon(u_{[t}))_{t\geq 0}$ is clearly adapted to the future and bounded in norm. By hypothesis the process $(X_t \, \varepsilon(u_{t]}))_{t\geq 0}$ is a τ-integrable regular vector semimartingale . Thus we can apply Theorem 9 to the process $(w_t)_{t\geq 0} = (X_t \, \varepsilon(u))_{t\geq 0}$. ■

Theorem 11 – *Let $(X_t)_{t\geq 0}$ be an adapted operator process on Φ. Suppose that for all $u \in L^2(\mathbb{R}^+)$ the process $(X_t \, \varepsilon(u_{t]}))_{t\geq 0}$ is a regular vector semimartingale. Let τ be a finite stop time such that, for all $u \in L^2(\mathbb{R}^+)$, the process $(X_t \, \varepsilon(u_{t]}))_{t\geq 0}$ is τ-integrable.*

Then the left stopping $\tau \circ X$ converges strongly on $\mathcal{E}$.

Proof

By Proposition 10 we have that the quantity

$$\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} (X_{t_{i+1}} \varepsilon(u_{t_{i+1}]})) \otimes \varepsilon(u_{[t_{i+1}})$$

admits a limit when the diameter δ of the τ-partition $\{t_i;\ i = 1, ..., n\}$ tends to 0. But this quantity is also equal to

$$\sum_i \mathbb{1}_{\tau \in [t_i, t_{i+1}]}(X_{t_{i+1}}\varepsilon(u)) = \left[\sum_i \mathbb{1}_{\tau \in [t_i, t_{i+1}]} X_{t_{i+1}}\right] \varepsilon(u).$$

This proves that the Riemann sums associated to the left stopping of X converge. ■

One can wonder what is this class of operator processes such that $(X_t\varepsilon(u_{t]}))_{t\geq 0}$ is a regular vector semimartingale, and what are the stop time τ such that the process $(X_t\varepsilon(u_{t]}))_{t\geq 0}$ is τ-integrable.

We now recall the definitions of the non-commutative stochastic integrals ([H-P]) of adapted processes of operators with respect to the creation $(A_t^+)_{t\geq 0}$, annihilation $(A_t)_{t\geq 0}$, conservation $(\Lambda_t)_{t\geq 0}$ and time $(tI)_{t\geq 0}$ processes, and also their extension as defined in [A-M].

Let H, K, L, M be adapted processes of operators defined on a domain $\mathcal{D}$ containing $\mathcal{E}$. Assume that the following integrals are meaningful (from the point of view of domains) and finite for all $f \in \mathcal{D}$, $t \in \mathbb{R}^+$:

$$\int_0^t \|H_s D_s f\|^2 ds,\ \int_0^t \|K_s E_s f\|^2 ds,\ \int_0^t \|L_s D_s f\| ds,\ \int_0^t \|M_s E_s f\| ds. \tag{4}$$

According to [A-M], we say that an adapted operator process $(T_t)_{t\geq 0}$ defined on $\mathcal{D}$ has the integral representation

$$T_t = \int_0^t H_s d\Lambda_s + \int_0^t K_s dA_s^+ + \int_0^t L_s dA_s + \int_0^t M_s ds$$

on *the domain* $\mathcal{D}$ if for all $f \in \mathcal{D}$ one has that $\int_0^t \|T_s D_s f\|^2 ds$ is well-defined meaningful and finite, and

$$T_t E_t f = \int_0^t T_s D_s f dx_s + \int_0^t H_s D_s f dx_s + \int_0^t K_s E_s f dx_s + \int_0^t L_s D_s f ds + \int_0^t M_s E_s f\, ds. \tag{5}$$

Theorem 12 ([A-M], Theorem 1, and [Me3] p. 123) – *On the domain $\mathcal{D} = \mathcal{E}$, this definition is equivalent to Hudson-Parthasarathy's definition of non-commutative stochastic integrals.* ■

A consequence of Theorem 12 is that if $(X_t)_{t\geq 0}$ is any process of the form (in [H-P]'s sense)

$$X_t = \int_0^t H_s d\Lambda_s + \int_0^t K_s dA_s^+ + \int_0^t L_s dA_s + \int_0^t M_s\, ds$$

then for all $f \in \mathcal{E}$, the process $(X_t E_t f)_{t\geq 0}$ is a regular vector semimartingale.

Now, if τ is a finite stop time such that the integral

$$X_\tau = \int_0^\infty \mathbb{1}_{\tau > s} H_s\, d\Lambda_s + \int_0^\infty \mathbb{1}_{\tau > s} K_s dA_s^+ + \int_0^\infty \mathbb{1}_{\tau > s} L_s dA_s + \int_0^\infty \mathbb{1}_{\tau > s} M_s ds \tag{6}$$

is well-defined in [H-P]'s sense we have by Theorem 12 and (4) that $(X_t\varepsilon(u_{t]}))_{t\geq 0}$ is a τ-integrable regular semimartingale of vectors. The left stopping $\tau \circ X$ given by Theorem 11 is then the operator X_τ given by (6). By this way we have seen that *the set of processes of operators concerned by Theorem 11 at least contains all the Hudson-Parthasarathy stochastic integrals.*

Note that, since $(\tau \circ X)^* = X^* \circ \tau$ for discrete stop times, we get obvious extensions of the results of this section in the case of right-stopping.

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