FOCK SPACE IS QUASI-LEFT CONTINUOUS

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Abstract

We develop the theory of quantum stop times on the boson Fock space $\Phi = \Gamma(L^2(I\!R^+))$. We define and study, for any quantum stop times S and T, the spaces Φ_T, Φ_{T-}, $(S \leq T)$ and $(S < T)$ which are the non-commutative analogues of the usual σ-fields $\mathcal{F}_T$ and $\mathcal{F}_{T-}$ and events $(S \leq T)$, $(S < T)$ in classical probability theory. They are shown to satisfy the same basic properties as in the commutative case. We apply these properties to define quantum predictable stop times on Φ and to prove that the Fock space is quasi-left continuous that is, $\Phi_T = \Phi_{T-}$ for all predictable stop time T .This result proves that the well-known property of quasi-left continuity for the filtration of all normal martingale satisfying the chaotic representation property (that is, every probabilistic interpretation of the Fock space) is actually a particular case of a more general and non-probabilistic property of the Fock space Φ.

1. Introduction

In classical probability theory a stop time T is a random variable on $(\Omega, \mathcal{F}, P)$ which is characterized by the increasing family of events $(T \leq t)$, $t \in I\!R^+$, which are adapted to a given filtration. In the quantum probability context Hudson ([Hud]) has extended these properties to the case of quantum random variables and defined a quantum stop time to be a non-negative self-adjoint operator $T = \int_0^\infty \lambda \, dE(\lambda)$ whose spectral measure E is adapted to a given filtration of von Neumann algebras. In the context of the boson Fock space $\Phi = \Gamma(L^2(I\!R^+))$ a (quantum) stop time is thus a spectral measure T on $I\!R^+ \cup \{+\infty\}$, with values in the set of orthogonal projections on Φ, such that for all $t \in I\!R^+$ the operator $T([0,t])$ is a t-adapted operator in the sense of Hudson-Parthasarathy's stochastic calculus ([H-P]).

This theory has been developed in many directions. In the context of finite von Neumann algebras (which does not contain the Fock space case), Barnett and Wilde ([B-W]) develop a theory of quantum predictable processes which appear to be powerful in term of quantum stochastic integration (in the same way as in the classical probability theory). In the Fock space context the theory of stop times is intensively studied by Parthasarathy and Sinha in [P-S]. For all quantum stop time T they define the "Fock space before T", denoted Φ_T, and of "Fock space after T", denoted Φ^T. They show that the usual continuous tensor product structure of the Fock space extends to (quantum) stop times : $\Phi \simeq \Phi_T \otimes \Phi^T$. The notion of quantum stop time is also successfully added to Bhat-Parthasarathy's ([B-P]) theory of quantum Markov processes in order to define quantum strong Markov processes and to solve non-commutative Dirichlet problems ([A-P]).

In the classical one the main ingredients of stop time theory and stochastic integration is the notion of predictable stop time. In order to use this notion one needs to be able to identify when two stop times S and T are such that $S < T$ and one needs to define the σ-field Φ_{T-}. In the quantum context the definition of $S \leq T$ and Φ_T has been given by Parthasarathy and Sinha, but there is no clear definition for $S < T$ and Φ_{T-}. This is one the aims of this article.

We revisit Parthasarathy-Sinha's definition of the space Φ_T and prove that it admits an equivalent definition which mimicks the usual definition of $\mathcal{F}_T$, the σ-field of events anterior to T in the classical theory. Actually we prove that Φ_T is exactely the set of $f \in \Phi$ such that $T([0,t])f$ belongs to Φ_t for all t. Pursuing the analogy we define the spaces $(S \leq T)$ and $(S < T)$ for any two quantum stop times S and T. Of course we get all the expected Most of the usual properties remain validin this non-commutative context. This leads to a natural definition of the notion of two stop times S and T such that $S < T$ by saying that this happens when $(S < T)$ is the whole Φ. This definition then corresponds to the intuitive idea : one has $S < T$ if and only if $S \leq T$ and S and T "never coincide". Finally, we define the space Φ_{T-} which is the quantum analogue of the σ-field $\mathcal{F}_{T-}$ of events strictly anterior to T in the classical theory. We can then mimick some of the usual properties of the two σ-fields $\mathcal{F}_T$ and $\mathcal{F}_{T-}$ and extend them to our non-commutative context.

Now, consider a normal martingale $(x_t)_{t\geq 0}$ (that is, $<x,x>_t = t$ for all t, where $<x,x>_.$ denote the angle bracket process) on its canonical space $(\Omega, \mathcal{F}, P)$. If x has the *predictable representation property* (that is, all martingales in $L^2(\Omega)$ admit a representation as a stochastic integral with respect to x) then it is easy too check that the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(x_t)_{t\geq 0}$ is quasi-left continuous that is, $\mathcal{F}_{T-} = \mathcal{F}_T$ for all predictable stop time T. But the boson Fock space Φ is isomorphic to a subspace of $L^2(\Omega, \mathcal{F}, P)$ called the *chaotic space* of $(x_t)_{t\geq 0}$ (that is, the space of random variables f in $L^2(\Omega, \mathcal{F}, P)$ which admit a chaotic representation with respect to $(x_t)_{t\geq 0}$). As our definitions on the Fock space correspond to the usual one when one considers such a probabilistic interpretation of Φ, we have $\Phi_{T-} = \Phi_T$ for those predictable stop times on Φ which correspond to a commutative stop time in a probabilistic interpretation of Φ. In this article, after defining quantum predictable stop times (which is easy since we hold a correct definition for $S < T$), we show that the property $\Phi_{T-} = \Phi_T$ holds true for all quantum predictable stop time T. Thus the Fock space is intrinsicaly quasi-left continuous. This property has nothing to do with any probabilistic interpretation of Φ.

2. Calculus on Fock space

Let Φ denote the Boson Fock space over $L^2(\mathbb{R}^+)$. In this article we identify Φ with the Guichardet symmetric space over $\mathbb{R}^+$. Indeed, let Γ denote the finite power set of $I\!R^+$ that is, the set of finite subsets of $I\!R^+$. Thus Γ has a disjoint partition : $\Gamma = \cup_{n \in I\!N} \Gamma^{(n)}$ where $\Gamma^{(n)}$ is the set of n elements subsets of $I\!R^+$. For $n \geq 1$ the set $\Gamma^{(n)}$ can be identified with the set $\{(t_1, \ldots, t_n) \in I\!R^n; 0 < t_1 < \ldots < t_n\}$ and then the Lebesgue measure on $I\!R^n$ induces a measure on $\Gamma^{(n)}$. By declaring the measure of $\Gamma^{(0)} = \{\emptyset\}$ to be unity, we obtain a σ-finite measure on Γ called the *symmetric measure* on $I\!R^+$ ([Gui]). The element of volume of this measure is denoted $d\sigma$. The space $L^2(\Gamma)$ is then isomorphic to the Fock space ([Par], p. 132).

$\int$-Lemma– *Let g be an integrable measurable function on $\Gamma \times \Gamma$. Then $G : \sigma \mapsto \sum_{\alpha \subset \sigma} g(\alpha, \sigma \backslash \alpha)$ defines an integrable measurable function on Γ satisfying*

$$\int_\Gamma G(\sigma)\, d\sigma = \int_\Gamma \int_\Gamma g(\alpha, \beta)\, d\alpha\, d\beta.$$

An elementary proof may be found in [L-P]. ■

In Φ we distinguish the class of coherent vectors $\mathcal{E} = \mathrm{span}\{\varepsilon(f); f \in L^2(I\!R^+)\}$ where $\varepsilon(f)$ is the usual coherent vector associated to f that is, in the Guichardet space notations, $[\varepsilon(f)](\sigma) = \prod_{s \in \sigma} f(s)$, $\sigma \in \Gamma$. The space $\mathcal{E}$ is dense in Φ.

For all $\sigma \in \Gamma$ and $t \in I\!R^+$ we use the following notations : $\sigma \cup t = \sigma \cup \{t\}$, $\sigma \backslash t = \sigma \backslash \{t\}$; for $\sigma \neq \emptyset$, $\vee\sigma = \max\{s \in \sigma\}$, $\sigma_- = \sigma \backslash \{\vee\sigma\}$, $\sigma_{t]} = \sigma \cap [0, t]$, $\sigma_{(t} = \sigma \cap]t, +\infty[$; $\Gamma_t = \{\sigma \in \Gamma; \sigma \subset [0, t]\}$, $\Gamma^t = \{\sigma \in \Gamma; \sigma \subset]t, +\infty[\}$. Recall that the Fock space Φ_t (*resp.* Φ^t) over $L^2([0, t])$ (*resp.* $L^2(]t, +\infty[)$) is then isomorphic to $L^2(\Gamma_t)$ (*resp.* $L^2(\Gamma^t)$).

It is easy to check from the $\int$-Lemma that for any fixed $t \in I\!R^+$ the mapping

$$\begin{aligned} J \ : \Phi_t \otimes \Phi^t &\longrightarrow \Phi \\ f \otimes g &\longmapsto \big(\sigma \mapsto f(\sigma_{t]})\, g(\sigma_{(t})\big) \end{aligned}$$

extends to a unitary isomorphism between $\Phi_t \otimes \Phi^t$ and Φ. From now we do not make any difference between Φ and $\Phi_t \otimes \Phi^t$ and we omit the mapping J. This structure $\Phi = \Phi_t \otimes \Phi^t$ is known as the *continuous tensor product structure of the Fock space* (cf [Par], Proposition 19.6 or [Me2] IV.2.6). Note that in this structure we get $\varepsilon(f) = \varepsilon(f_{t]}) \otimes \varepsilon(f_{[t})$ for all t, where $f_{t]} = f 1\!\!1_{[0,t]}$ and $f_{[t} = f 1\!\!1_{[t,+\infty[}$.

The definitions and results to come in this section can be found in great details in [A-L].

For every $t \in I\!R^+$ define the operator $I\!E_t$ on Φ by

$$[I\!E_t f](\sigma) = 1\!\!1_{\Gamma_t}(\sigma) f(\sigma)$$

for all $f \in \Phi$, all $\sigma \in \Gamma$. The operator $I\!E_t$ is clearly the orthogonal projection from Φ onto Φ_t.

For every $t \in I\!R^+$, $f \in \Phi$, $\sigma \in \Gamma$ define the quantity

$$[D_t f](\sigma) = 1\!\!1_{\Gamma_t}(\sigma) f(\sigma \cup t).$$

Lemma 2.1 – *For all $f \in \Phi$ one has*

$$\int_0^\infty \int_\Gamma \left|[D_t f](\sigma)\right|^2 d\sigma\, dt = \int_\Gamma |f(\sigma)|^2\, d\sigma - |f(\emptyset)|^2.$$

This lemma is an easy consequence of the $\int$-Lemma (see [A-L]). ■

This implies that for all $f \in \Phi$, for a.a. $t \in I\!R^+$ the mapping $\sigma \mapsto [D_t f](\sigma)$ is square integrable. Thus, for a.a. t, $D_t f$ is an element of Φ. In other words D_t may be considered as an almost everywhere defined operator that is, for all $f \in \Phi$, $D_t f$ is defined as an element of Φ only for a.a. $t \in I\!R^+$.

Notice that this lemma implies

$$\|f\|^2 = \left|I\!E_0[f]\right|^2 + \int_0^\infty \|D_t f\|^2\, dt. \tag{2.1}$$

A *process in* Φ is a family $f_\cdot = (f_t)_{t\geq 0}$ of vectors in Φ such that the mapping $t \mapsto f_t$ is strongly measurable.

A process $f_\cdot$ in Φ is *adapted* if for all t the vector f_t belongs to Φ_t.

An process $f_\cdot$ in Φ is *Ito integrable* if it is adapted and

$$\int_0^\infty \|f_t\|^2\, dt < \infty.$$

In this case we define the *Ito integral* $I(f_\cdot)$ of $f_\cdot$ by

$$[I(f_\cdot)](\sigma) = \begin{cases} f_{\vee\sigma}(\sigma-), & \sigma \neq \emptyset \\ 0, & \sigma = \emptyset. \end{cases}$$

One easily checks (same reference as for $D_t f$) from the $\int$-Lemma that $I(f_\cdot)$ defines an element of Φ. In the following we denote $I(f_\cdot)$ by $\int_0^\infty f_s\, dX_s$. This notation will be justified latter. For $0 \leq a \leq b \leq \infty$ we denote by $\int_a^b f_s\, dX_s$ the Ito integral $\int_0^\infty f_s \mathbb{1}_{(a,b)}(s)\, dX_s$, when it is defined.

Because of (2.1) one can compute $\int_0^\infty D_s f\, dX_s$ for any $f \in \Phi$. One then easily get the following result.

Theorem 2.2 (Fock-space predictable representation property) – *For all $f \in \mathcal{F}$, the process $(D_t f)_{t\geq 0}$ is Ito integrable and we have the representation*

$$f = I\!E_0[f] + \int_0^\infty D_s f\, dX_s \tag{2.2}$$

and the isometry formula

$$< f\,,\, g > = \overline{I\!E_0[f]}\, I\!E_0[g] + \int_0^\infty < D_s f\,,\, D_s g > ds. \tag{2.3}$$

for all $g \in \Phi$. ■

Let t be fixed in $I\!R^+$. Recall that an operator H on Φ, with domain containing $\mathcal{E}$, is said to be *t-adapted* in Hudson-Parthasarathy's sense ([H-P]) if, for all $f \in L^2(I\!R^+)$,

$$\begin{cases} H\varepsilon(f_{t]}) \in \Phi_t \\ H\varepsilon(f) = \left[H\varepsilon(f_{t]})\right] \otimes \varepsilon(f_{[t}). \end{cases}$$

Note that if H is a t-adapted operator and if its adjoint H^* is defined on $\mathcal{E}$, then H^* is also a t-adapted operator.

Lemma 2.3 – *Let $t \in \mathbb{R}^+$. Let H be a bounded t-adapted operator. Then for any Ito integrable process $(f_u)_{u\geq t}$, the process $(Hf_u)_{u\geq t}$ is Ito integrable and*

$$H \int_t^\infty f_u \, dX_u = \int_t^\infty Hf_u \, dX_u .$$

Proof

The proof of this lemma is here sketched quickly as it is a consequence of [A-L]'s redefinition of adaptedness; anyway, all the elements have been given in this section for the reader to be able to follow the proof.

The Ito integrability of $(Hf_u)_{u\geq t}$ is clear from the adaptedness and the boundedness of H.

For every $g \in L^2(\mathbb{R}^+)$ one has

$$
\begin{aligned}
&< \varepsilon(g) \, , \, H \int_t^\infty f_u \, dX_u > \\
&= < H^* \varepsilon(g) \, , \, \int_t^\infty f_u \, dX_u > \\
&= < \left[H^*\varepsilon(g_{t]})\right] \otimes \varepsilon(g_{[t}) \, , \, \int_t^\infty f_u \, dX_u > \\
&= < \left[H^*\varepsilon(g_{t]})\right] \otimes 1 \, , \, \int_t^\infty f_u \, dX_u > \\
&\quad + < \left[H^*\varepsilon(g_{t]})\right] \otimes \int_t^\infty g(u)\varepsilon(g_{[t,u]}) \, dX_u \, , \, \int_t^\infty f_u \, dX_u > \\
&\quad \text{by (2.2), where } g_{[t,u]} = g \mathbb{1}_{[t,u]} \\
&= < \left[H^*\varepsilon(g_{t]})\right] \otimes \int_t^\infty g(u)\varepsilon(g_{[t,u]}) \, dX_u \, , \, \int_t^\infty f_u \, dX_u > \\
&\quad \text{for } \mathbb{E}_t \textstyle\int_t^\infty f_u \, dX_u = 0 \\
&= < \int_t^\infty g(u)\left[H^*\varepsilon(g_{t]})\right] \otimes \varepsilon(g_{[t,u]}) \, dX_u \, , \, \int_t^\infty f_u \, dX_u > \\
&= < \int_t^\infty g(u) H^*\varepsilon(g_{u]}) \, dX_u \, , \, \int_t^\infty f_u \, dX_u > \\
&= \int_t^\infty < g(u) H^*\varepsilon(g_{u]}) \, , \, f_u > du \\
&= \int_t^\infty < g(u)\varepsilon(g_{u]}) \, , \, Hf_u > du \\
&= < \int_t^\infty g(u)\varepsilon(g_{u]}) \, dX_u \, , \, \int_t^\infty Hf_u \, dX_u > \\
&= < \varepsilon(g) \, , \, \int_t^\infty Hf_u \, dX_u >.
\end{aligned}
$$

■

3. Probabilistic interpretations

This section contains only remarks which connect the operators $I\!E_t, D_t, I(.)$ to some well-known probabilistic operations.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (x_t)_{t\geq 0})$ be a *probabilistic interpretation* of the Fock space Φ. That is, $(x_t)_{t\geq 0}$ is a *normal martingale* (i.e. a martingale such that $(x_t^2 - t)_{t\geq 0}$ is also a martingale) which possesses the chaotic representation property ; for example the Brownian motion, the compensated Poisson process, some Azma martingales (cf [Eme]). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be its canonical filtered space. One knows ([Me2]) that, throught the chaotic representation of random variables, the Fock space Φ is isomorphic to $L^2(\Omega, \mathcal{F}, P)$. Let denote this isomorphism from Φ to $L^2(\Omega, \mathcal{F}, P)$. All the operators $I\!E_t, D_t, I(.)$ have an interpretation on $L^2(\Omega, \mathcal{F}, P)$ as well-known probabilistic operators. Note that one can assume that $(x_t)_{t\geq 0}$ only possesses the predictable representation property and then identify the Fock space Φ to the chaotic space of $(x_t)_{t\geq 0}$ that is, the subspace of those $f \in L^2(\Omega, \mathcal{F}, P)$ which admit a chaotic representation with respect to $(x_t)_{t\geq 0}$.

First of all, one has $I\!E_t = {}^{-1}E_t$ where E_t is the operator of conditionnal expectation $E[\,\cdot\,|\mathcal{F}_t]$; by the way the space $\Phi_t = \mathrm{Im} I\!E_t$ is isomorphic to $L^2(\Omega, \mathcal{F}_t, P)$.

As the normal martingale $(x_t)_{t\geq 0}$ has the chaotic representation property, it has in particular the predictable representation property. Thus, for every random variable f in $L^2(\Omega, \mathcal{F}, P)$ there exists a predictable process $(\xi_t(f))_{t\geq 0}$ in $L^2(\Omega, \mathcal{F}, P)$ such that $f = E[f] + \int_0^\infty \xi_t(f)\, dx_t$. One can see ξ_t as an a.e. defined operator on $L^2(\Omega, \mathcal{F}, P)$. From this point of view ξ_t is nothing but the probabilistic interpretation of the operator D_t; that is, $D_t = {}^{-1}\xi_t$.

The operator $I(.)$ corresponds to the usual Ito integral with respect to $(x_t)_{t\geq 0}$. Theorem 2.2, when interpreted in $L^2(\Omega, \mathcal{F}, P)$, only expresses the predictable representation property of $(x_t)_{t\geq 0}$ and the isometry formula for the Ito integral.

This means that all the operators introduced in section 2 can be interpreted as well-known operators coming from the stochastic calculus, when the Fock space is interpreted as the chaotic space of some normal martingales. In fact, one should think the other way round. Probabilistic operations such as Ito integration, predictable representation, etc... can be expressed in term of the chaotic expansion of the random variables ; in their definition they do not use any specific property of the normal martingale involved except the chaotic representation property and the Ito isometry formula (which is the same for all the probabilistic interpretation as the normal martingale property implies that the angle bracket $<x, x>_t$ is equal to t). Thus they can be translated into intrinsic operators on the Fock space.

4. Stop times on Fock space

Following [Hud] and [P-S], we define a *stop time* T on Φ to be a spectral measure on $I\!R^+ \cup \{+\infty\}$ with values in the set of orthogonal projection operators on Φ and such that, for all $t \geq 0$ the operator $T([0,t])$ is a t-adapted operator in Hudson-Parthasarathy's sense.

In the following we adopt a probabilistic-like notation : for every Borel set $E \subset \mathbb{R}^+ \cup \{+\infty\}$ we write $\mathbb{1}_{T\in E}$ instead of $T(E)$, in the same way $T([0,t])$ is denoted $\mathbb{1}_{T\leq t}$, $T(\{t\})$ is denoted $\mathbb{1}_{T=t}$, etc...

A stop time T is *discrete* if there exists a finite set $E = \{0 \leq t_1 < \cdots < t_n \leq +\infty\}$ such that $\mathbb{1}_{T\in E} = I$.

A point $t \in \mathbb{R}^+$ is a *continuity point* for T if $\mathbb{1}_{T=t} = 0$. Note that any stop time T has an at most countable set of points which are not continuity points; also note that if T is not the null stop time, the point 0 in $\mathbb{R}^+$ is always a continuity point for T.

A sequence $(T_n)_n$ of stop times *converges* to a stop time T if for every continuity point t of T, the operators $\mathbb{1}_{T_n\leq t}$ converge strongly to $\mathbb{1}_{T\leq t}$.

For any two stop times S and T one says that $S \leq T$ if $\mathbb{1}_{S\leq t} \geq \mathbb{1}_{T\leq t}$ for all t (in the sense of the positivity of the operator $\mathbb{1}_{S\leq t} - \mathbb{1}_{T\leq t}$).

Let T be any stop time. By a sequence of *refining T-partitions* of $\mathbb{R}^+$ we mean a sequence $(E_n)_n$ of finite sets $E_n = \{0 \leq t_1^n < t_2^n < \cdots < t_{i_n}^n < +\infty\}$ such that

i) all the t_j^i are continuity points for T

ii) $E_n \subseteq E_{n+1}$ for all n

iii) the diameter, sup $\{t_{i+1}^n - t_i^n, i = 1, ..., i_n\}$, of E_n tends to 0 when n tends to $+\infty$.

iv) $t_{i_n}^n$ tends to $+\infty$ when n tends to $+\infty$.

The following result is a combination of [P-S] Proposition 3.3 and [Me1].

Proposition 4.1 – *For every stop time T there exists a sequence $(T_n)_n$ of discrete stop times such that $T_1 \geq T_2 \geq \cdots \geq T$ and $(T_n)_n$ converges to T.*

Proof

Let $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ be a partition of $\mathbb{R}^+$. Define a spectral measure T_E by

$$T_E(\{t_i\}) = \begin{cases} \mathbb{1}_{T\leq t_1} & \text{if } i = 1 \\ \mathbb{1}_{T\in]t_{i-1},t_i]} & \text{if } i > 1, \end{cases}$$

$$T_E(\{+\infty\}) = \mathbb{1}_{T>t_n}.$$

The spectral measure T_E clearly defines a discrete stop time on Φ and $T_E \geq T$. Taking a sequence $(E_n)_n$ of refining T-partitions of $\mathbb{R}^+$ gives the required sequence $(T_n)_n = (T_{E_n})_n$. Details are left to the reader. ∎

5. The space Φ_T

For a $t \in \mathbb{R}^+$ the space Φ_t is interpreted as the "Fock space before t". Indeed, it is generated by the coherent vectors $\varepsilon(g)$ where g has its support included in $[0,t]$. One can also think of the case of any probabilistic interpretation

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (x_t)_{t\geq 0})$ of Φ (cf section 3), in which case Φ_t is isomorphic to $L^2(\Omega, \mathcal{F}_t, P)$. In the same way as in the classical theory of stop times we wish to define, for a quantum stop time T, the projection $\mathbb{E}_T$ and the associated "Fock space before time T", $\Phi_T = \mathrm{Im}\,\mathbb{E}_T$. Mimicking the classical case, one can define $\mathbb{E}_T$ in the case where T is a discrete stop time :

$$\mathbb{E}_T = \sum_i \mathbb{1}_{T=t_i}\,\mathbb{E}_{t_i}.$$

Note that because of the t_i-adaptedness of $\mathbb{1}_{T=t_i}$, the projections $\mathbb{1}_{T=t_i}$ and $\mathbb{E}_{t_i}$ do commute.

For a general stop time T we wish to pass to the limit on a sequence $(T_n)_n$ of discrete stop times converging to T (Proposition 4.1). The following result is inspired from [Me1].

Proposition 5.1 – *Let T be any stop time on Φ. Let $(E_n)_n$ be refining sequence of T-partitions of $\mathbb{R}^+$. Let $T_n = T_{E_n}$, $n \in \mathbb{N}$. Then the projections $\mathbb{E}_{T_n}$ converge strongly to a projection $\mathbb{E}_T$ which satisfies*

$$\mathbb{E}_T f = \mathbb{E}_0[f] + \int_0^\infty \mathbb{1}_{T>s}\,D_s f\,dX_s.$$

Therefore the limit $\mathbb{E}_T$ is independent of the chosen sequence $(E_n)_n$.

Proof

If $c \in \mathbb{C}$ one gets $\mathbb{E}_{T_n} c = c$. Thus one can assume that $\mathbb{E}_0[f] = 0$. By definition one has

$$\begin{aligned}
\mathbb{E}_{T_n} f &= \sum_i \mathbb{1}_{T\in]t_i,t_{i+1}]}\mathbb{E}_{t_{i+1}} f \\
&= \sum_i \mathbb{1}_{T\in]t_i,t_{i+1}]} \int_0^{t_{i+1}} D_s f\,dX_s \\
&= \sum_i \sum_{j\leq i} \mathbb{1}_{T\in]t_i,t_{i+1}]} \int_{t_j}^{t_{j+1}} D_s f\,dX_s \\
&= \sum_j \sum_{i\geq j} \mathbb{1}_{T\in]t_i,t_{i+1}]} \int_{t_j}^{t_{j+1}} D_s f\,dX_s \\
&= \sum_j \mathbb{1}_{T>t_j} \int_{t_j}^{t_{j+1}} D_s f\,dX_s \\
&= \sum_j \int_{t_j}^{t_{j+1}} \mathbb{1}_{T>t_j} D_s f\,dX_s \quad \text{(Lemma 2.3)} \\
&= \sum_j \int_{t_j}^{t_{j+1}} \mathbb{1}_{T_n>s}\,D_s f\,dX_s \\
&= \int_0^\infty \mathbb{1}_{T_n>s}\,D_s f\,dX_s.
\end{aligned}$$

Thus the proposition holds true for discrete stop times. Let us check the convergence. Indeed, one has

$$\begin{aligned}
&\left|\left|\mathbb{E}_{T_n} f - \int_0^\infty \mathbb{1}_{T>s}\, D_s f\, dX_s\right|\right|^2 \\
&\quad = \int_0^\infty \left|\left|\mathbb{1}_{T_n>s}\, D_s f - \mathbb{1}_{T>s}\, D_s f\right|\right|^2 ds \\
&\quad = \int_0^\infty \left|\left|\mathbb{1}_{T\geq t_{i(s)}} D_s f - \mathbb{1}_{T>s}\, D_s f\right|\right|^2 ds \quad \text{where } t_{i(s)} \leq s < t_{i(s)+1} \\
&\quad = \int_0^\infty \left|\left|\mathbb{1}_{T\in]t_{i(s)},s]} D_s f\right|\right|^2 ds.
\end{aligned}$$

By the dominated convergence Theorem this term converges to 0 when the diameter of the partition $\{t_i; i = 1, \dots n\}$ tends to 0. ■

It is easy to check that $\mathbb{E}_T$ is a projection on Φ. We call Φ_T its range. We are going to see that the space Φ_T can be characterized in another way which is closer to the definition of the σ-field $\mathcal{F}_T$ in classical probability theory.

Lemma 5.2 – *For every stop time T on Φ, every $f \in \Phi$, every $t \in \mathbb{R}^+$*

i) $\mathbb{1}_{T\leq t}\, \mathbb{E}_T f$ *belongs to* Φ_t,

ii) $\mathbb{1}_{T<t}\, \mathbb{E}_T f$ *belongs to* Φ_t.

Proof

i) One has

$$\begin{aligned}
\mathbb{1}_{T\leq t}\, \mathbb{E}_{T_n} f &= \mathbb{1}_{T\leq t} \sum_i \mathbb{1}_{T\in]t_i,t_{i+1}]} \mathbb{E}_{t_{i+1}} f \\
&= \sum_{i\leq i_0-1} \mathbb{1}_{T\leq t}\, \mathbb{1}_{T\in]t_i,t_{i+1}]} \mathbb{E}_{t_{i+1}} f + \mathbb{1}_{T\in]t_{i_0},t]} \mathbb{E}_{t_{i_0+1}} f \\
&\quad \text{where } t_{i_0} \leq t < t_{i_0+1} \\
&= \mathbb{E}_t \sum_{i\leq i_0-1} \mathbb{1}_{T\leq t}\, \mathbb{1}_{T\in]t_i,t_{i+1}]} \mathbb{E}_{t_{i+1}} f + \mathbb{1}_{T\in]t_{i_0},t]} \mathbb{E}_{t_{i_0+1}} f \\
&\quad + \mathbb{E}_t \sum_{i\geq i_0+1} \mathbb{1}_{T\leq t}\, \mathbb{1}_{T\in]t_i,t_{i+1}]} \mathbb{E}_{t_{i+1}} f \\
&\quad \text{as the last term actually vanishes} \\
&= \mathbb{E}_t\, \mathbb{1}_{T\leq t} \sum_i \mathbb{1}_{T\in]t_i,t_{i+1}]} \mathbb{E}_{t_{i+1}} f + \mathbb{1}_{T\in]t_{i_0},t]} \mathbb{E}_{t_{i_0+1}} f \\
&\quad - \mathbb{E}_t\, \mathbb{1}_{T\leq t}\, \mathbb{1}_{T\in]t_{i_0},t_{i_0+1}]} \mathbb{E}_{t_{i_0+1}} f \\
&= \mathbb{E}_t\, \mathbb{1}_{T\leq t}\, \mathbb{E}_{T_n} f + \mathbb{1}_{T\in]t_{i_0},t]} (\mathbb{E}_{t_{i_0+1}} f - \mathbb{E}_t f).
\end{aligned}$$

The left hand side of this identity converges to $\mathbb{1}_{T\leq t}\, \mathbb{E}_T f$, the first term of the right hand side converges to $\mathbb{E}_t\, \mathbb{1}_{T\leq t}\, \mathbb{E}_T f$ and the second term converges to 0 by continuity of $t \mapsto \mathbb{E}_t\, f$. This proves (i).

The proof of ii) is the same, one just has to check that the operator $1\!\!1_{T<t}$ commutes with $I\!\!E_u$ for all $u \geq t$, even for those t which are not continuity points for T. Let $u \geq t$, $f \in \Phi$. One has

$$\begin{aligned}
&\|1\!\!1_{T<t}\, I\!\!E_u f - I\!\!E_u 1\!\!1_{T<t}\, f\|^2 \\
&\quad\leq 2\|1\!\!1_{T<t}\, I\!\!E_u f - 1\!\!1_{T\leq t-\frac{1}{n}}\, I\!\!E_u f\|^2 + 2\|I\!\!E_u 1\!\!1_{T\leq t-\frac{1}{n}}\, f - I\!\!E_u 1\!\!1_{T<t}\, f\|^2 \\
&\quad\leq 2\|1\!\!1_{T\in]t-\frac{1}{n},t[}\, I\!\!E_u f\|^2 + 2\|1\!\!1_{T\in]t-\frac{1}{n},t[}\, f\|^2.
\end{aligned}$$

This quantity converges to 0 when n tends to $+\infty$. ∎

Theorem 5.3 – *For every stop time T one has*

$$\Phi_T = \{f \in \Phi;\ 1\!\!1_{T\leq t} f \in \Phi_t \textit{ for all } t\} = \{f \in \Phi;\ 1\!\!1_{T<t} f \in \Phi_t \textit{ for all } t\}.$$

Proof

Let $E_\leq$ be the set $\{f \in \Phi; 1\!\!1_{T\leq t} f \in \Phi_t \text{ for all } t\}$ and let $E_<$ be the set $\{f \in \Phi; 1\!\!1_{T<t} f \in \Phi_t \text{ for all } t\}$. If $f \in \Phi_T$ then $f = I\!\!E_T f$ and by Lemma 5.2 $1\!\!1_{T\leq t} f$ and $1\!\!1_{T<t} f$ are elements of Φ_t for all t. Thus $\Phi_T \subseteq E_\leq$ and $\Phi_T \subseteq E_<$.

Let $f \in E_\leq$. One has

$$\begin{aligned}
I\!\!E_{T_n} f &= \sum_i 1\!\!1_{T\in]t_i,t_{i+1}]} I\!\!E_{t_{i+1}} f \\
&= \sum_i (1\!\!1_{T\leq t_{i+1}} - 1\!\!1_{T\leq t_i}) I\!\!E_{t_{i+1}} f \\
&= \sum_i I\!\!E_{t_{i+1}} (1\!\!1_{T\leq t_{i+1}} - 1\!\!1_{T\leq t_i}) f
\end{aligned}$$

But, as f belongs to $E_\leq$, $(1\!\!1_{T\leq t_{i+1}} - 1\!\!1_{T\leq t_i})f$ is an element of $\Phi_{t_{i+1}}$. Thus $I\!\!E_{T_n} f = \sum_i (1\!\!1_{T\leq t_{i+1}} - 1\!\!1_{T\leq t_i}) f = f$. Passing to the limit we get $I\!\!E_T f = f$, thus $f \in \Phi_T$ and we have proved that $E_\leq \subseteq \Phi_T$.

The proof of $E_< \subseteq \Phi_T$ is the same as for $E_\leq \subseteq \Phi_T$ by noticing that as all the t_i are continuity points for T one has $1\!\!1_{T\in]t_i,t_{i+1}]} = 1\!\!1_{T\in[t_i,t_{i+1}[} = 1\!\!1_{T<t_{i+1}} - 1\!\!1_{T<t_i}$.
∎

With this theorem one recovers a non-commutative extention of the usual characterization of the σ-field $\mathcal{F}_T$ in the theory of classical stop times. By the way we are able to recover most of the usual properties of $\mathcal{F}_T$ in our context.

Proposition 5.4 –

i) *If S, T are two stop times on Φ such that $S \leq T$ then $\Phi_S \subset \Phi_T$.*

ii) *If $(T_n)_n$ is a decreasing sequence of stop times converging to T then $\Phi_T = \cap_n \Phi_{T_n}$.*

Proof

i) Let f be an element of Φ_S. One has $1\!\!1_{T\leq t} f = 1\!\!1_{T\leq t} 1\!\!1_{S\leq t} f$ as $S \leq T$. But by Theorem 5.3 $1\!\!1_{S\leq t} f$ belongs to Φ_t. Therefore, by t-adaptedness and boundedness of $1\!\!1_{T\leq t}$ one gets $1\!\!1_{T\leq t} f \in \Phi_t$. This proves i).

ii) By i), Φ_T is a subspace of Φ_{T_n} for all n, thus a subspace of $\cap_n \Phi_{T_n}$. Conversly, let $f \in \cap_n \Phi_{T_n}$. Let t be a continuity point of T. One has $\mathbb{1}_{T\le t} f = \lim_n \mathbb{1}_{T_n \le t} f$. As each $\mathbb{1}_{T_n\le t} f$ is an element of Φ_t then so is $\mathbb{1}_{T\le t} f$. If t is not a continuity point of T, let $(\varepsilon_n)_n$ a sequence of strictly positive real numbers such that $\varepsilon_1 > \varepsilon_2 > ...$, such that ε_n tends to 0 when n tends to $+\infty$ and such that, for all n, $t+\varepsilon_n$ is a continuity point of T. For all m and n we have $\mathbb{1}_{T_m\le t+\varepsilon_n} f \in \Phi_{t+\varepsilon_n}$. As $\mathbb{1}_{T\le t+\varepsilon_n} f = \lim_m \mathbb{1}_{T_m\le t+\varepsilon_n} f$ one gets that $\mathbb{1}_{T\le t+\varepsilon_n} f$ also belongs to $\Phi_{t+\varepsilon_n}$. Furthermore $\mathbb{1}_{T\le t} f$ is equal to $\lim_n \mathbb{1}_{T\le t+\varepsilon_n} f$. Noticing that for all $n' \ge n$ one has $\mathbb{1}_{T\le t+\varepsilon_{n'}} f \in \Phi_{t+\varepsilon_n}$, one deduces that $\mathbb{1}_{T\le t} f$ belongs to $\Phi_{t+\varepsilon_n}$ for all n, thus it belongs to $\Phi_t = \cap_n \Phi_{t+\varepsilon_n}$. ■

6. The spaces $(S \le T)$ and $(S < T)$

For any stop time T on Φ one denotes by $(T \le t)$ (*resp.* $(T < t)$, etc...) the range of the projection $\mathbb{1}_{T\le t}$ (*resp.* $\mathbb{1}_{T<t}$, etc...).

For any two stop times S, T on Φ denote by $(S < T)$ the closed subspace

$$(S < T) = \bigvee_{r\in\mathbb{Q}^+} (S \le r) \cap (T \le r)^\perp = \bigvee_{r\in\mathbb{Q}^+} (S \le r) \cap (T > r),$$

where $\mathbb{Q}^+$ denotes the set of positive rational numbers. Denote by $(S \le T)$ the space $(T < S)^\perp$ that is,

$$(S \le T) = \bigcap_{r\in\mathbb{Q}^+} (S \le r) \vee (T \le r)^\perp = \bigcap_{r\in\mathbb{Q}^+} (S \le r) \vee (T > r).$$

Proposition 6.1 – *For any stop times S and T on Φ one has the following properties.*

i) *The space $(S < T)$ is a closed subspace of $(S \le T)$.*

ii) *One has $S \le T$ if and only if the space $(S \le T)$ is equal to the whole Φ.*

iii) *If T is the constant time $t \in \mathbb{R}^+$ then $(S \le T) = (S \le t)$ and $(S < T) = (S < t)$.*

iv) *Let J be the set of closed subspace A of Φ such that $(T \le r) \cap A = (S \le r) \cap (T \le r) \cap A$ for all $r \in \mathbb{Q}^+$. Then J admits one and only one maximal element; this element is the space $(S \le T)$.*

v) *Let J' be the set of closed subspace A of $(S \le T)$ such that for all closed non-trivial subspace A' of A there exists a $r \in \mathbb{Q}^+$ with $(T \le r) \cap A' \ne (S \le r) \cap A'$. Then J' admits one and only one maximal element; this element is the space $(S < T)$.*

Proof

i) If f belongs to $\cup_{r\in\mathbb{Q}^+} (S \le r) \cap (T \le r)^\perp$ then there exists a $q \in \mathbb{Q}^+$ such that $f \in (S \le q) \cap (T \le q)^\perp$. As $(S \le q)$ is a subspace of $(S \le r)$ for all $r \ge q$ and as $(T \le q)^\perp$ is a subspace of $(T \le r)^\perp$ for all $r \le q$, we get that $(S \le q) \cap (T \le q)^\perp$ is a subspace of $(S \le r) \vee (T \le r)^\perp$ for all $r \in \mathbb{Q}^+$. Thus

$(S \le q) \cap (T \le q)^\perp$ is a subspace of $\cap_{r\in\mathbb{Q}^+}(S \le r) \vee (T \le r)^\perp = (S \le T)$. So we have $\cup_{r\in\mathbb{Q}^+}(S \le r) \cap (T \le r)^\perp \subset (S \le T)$ and hence $(S < T) \subset (S \le T)$.

ii) If one has $S \le T$ we have $(S \le t) \subset (T \le t)$ for all $t \in \mathbb{R}^+$. Consequently $(T \le t) = (T \le t) \cap (S \le t)$ for all t. The space $(S \le T)$ can then be written as

$$\begin{aligned}(S \le T) &= \bigcap_{r\in\mathbb{Q}^+} (S \le r) \vee [(T \le r) \cap (S \le r)]^\perp \\ &= \bigcap_{r\in\mathbb{Q}^+} (S \le r) \vee (T \le r)^\perp \vee (S \le r)^\perp \\ &= \bigcap_{r\in\mathbb{Q}^+} \Phi \vee (T \le r)^\perp \\ &= \bigcap_{r\in\mathbb{Q}^+} \Phi = \Phi.\end{aligned}$$

Conversly, if $(S \le T)$ is equal to the whole Φ then we have $(S \le r) \vee (T \le r)^\perp = \Phi$ for all $r \in \mathbb{Q}^+$. Thus $(S \le r)^\perp \cap (T \le r) = \{0\}$ for all $r \in \mathbb{Q}^+$ that is, $(S \le r) \cap (T \le r) = (T \le r)$ for all $r \in \mathbb{Q}^+$. This means that $(T \le r)$ is a subspace of $(S \le r)$ for all $r \in \mathbb{Q}^+$. One concludes easily that $S \le T$.

iii) If T is the constant time $t \in \mathbb{R}^+$ we have

$$\begin{aligned}(S \le T) &= \bigcap_{r\in\mathbb{Q}^+} (S \le r) \vee (T \le r)^\perp \\ &= \bigcap_{r\in\mathbb{Q}^+} (S \le r) \vee (t \le r)^\perp \\ &= \bigcap_{\substack{r\in\mathbb{Q}^+\\ r<t}} (S \le r) \vee \{0\}^\perp \bigcap \bigcap_{\substack{r\in\mathbb{Q}^+\\ r\ge t}} (S \le r) \vee \Phi^\perp \\ &= \bigcap_{\substack{r\in\mathbb{Q}^+\\ r<t}} \Phi \bigcap \bigcap_{\substack{r\in\mathbb{Q}^+\\ r\ge t}} (S \le r) \\ &= (S \le t).\end{aligned}$$

The case of $(S < T)$ is treated in the same way.

iv) One has

$$\begin{aligned}(T \le r) \cap (S \le T) &= (T \le r) \bigcap \bigcap_{s\in\mathbb{Q}^+} (S \le s) \vee (T > s) \\ &= \bigcap_{\substack{s\in\mathbb{Q}^+\\ s<r}} (S \le s) \cap (T \le r) \vee (T > s) \cap (T \le r) \\ &\quad \bigcap \bigcap_{\substack{s\in\mathbb{Q}^+\\ s\ge r}} (S \le s) \cap (T \le r) \vee (T > s) \cap (T \le r) \\ &= \bigcap_{\substack{s\in\mathbb{Q}^+\\ s<r}} (S \le s) \cap (T \le r) \vee (T \in]s, r])\end{aligned}$$

$$\bigcap \bigcap_{\substack{s\in\mathbb{Q}^+\\ s\geq r}} (S\leq s)\cap(T\leq r)$$
$$= \bigcap_{\substack{s\in\mathbb{Q}^+\\ s< r}} (S\leq s)\cap(T\leq r)\vee(T\in]s,r])\cap(S\leq r)\cap(T\leq r).$$

Thus $(T\leq r)\cap(S\leq T)$ is a subspace of $(S\leq r)$ and consequently $(S\leq r)\cap(T\leq r)\cap(S\leq T)$ is equal to $(T\leq r)\cap(S\leq T)$. We have proved that the space $(S\leq T)$ is an element of J.

Now, let A be an element of J. We have $A\cap(S\leq T)=\cap_{r\in\mathbb{Q}^+}(S\leq r)\cap A\vee(T>r)\cap A$. But as A is in J we have $(T>r)\cap A=[(S>r)\vee(T>r)]\cap A$. This implies

$$A\cap(S\leq T)=\bigcap_{r\in\mathbb{Q}^+}(S\leq r)\cap A\vee(S>r)\cap A\vee(T>r)\cap A=\bigcap_{r\in\mathbb{Q}^+}A=A.$$

That is, A is a subspace of $(S\leq T)$. This ends the proof of iv).

v) We first want to prove that the space $(S<T)$ is an element of J'. From i) the property that $(S<T)$ is a closed subspace of $(S\leq T)$ is satisfied. Now, let A' be a subspace of $(S<T)$. If we have $(S\leq r)\cap A'=(T\leq r)\cap A'$ for all $r\in\mathbb{Q}^+$ then this means $(S\leq r)\cap(T>r)\cap A'=\{0\}$ for all $r\in\mathbb{Q}^+$. Consequently the space $\vee_{r\in\mathbb{Q}^+}(S\leq r)\cap(T>r)\cap A'$ is trivial. But on the other hand this space is equal to $(S<T)\cap A'=A'$. This means that A' is trivial. We have proved that the space $(S<T)$ is an element of J'.

Finally, let A be an element of J'. Let $A'=(S<T)^{\perp}\cap A=\cap_{r\in\mathbb{Q}^+}[(S>r)\vee(T\leq r)]\cap A$. For all $r\in\mathbb{Q}^+$ we have $(S\leq r)\cap(T>r)\cap A'=\{0\}$, that is $(S\leq r)\cap A'=(T\leq r)\cap A'$. But as A' is a subspace of $A\in J'$ this must imply that $A'=\{0\}$. Consequently $(S<T)\cap A$ is equal to A, that is $A\subset(S<T)$. ■

We say that two stop times S and T on Φ *coincide on a closed subspace* A of Φ if $(S\leq r)\cap A=(T\leq r)\cap A$ for all $r\in\mathbb{Q}^+$. We say that S and T *never coincide* if there exist no non-trivial closed subspace A of Φ such that S and T coincide on A.

We say that $S<T$ if the space $(S<T)$ is equal to the whole Φ. The following corollary says that this definition corresponds to the intuitive notion of the property $S<T$.

Corollary 6.2 – If S and T are two stop times on Φ then the following assertions are equivalent :

i) $S<T$

ii) $S\leq T$ and S and T never coincide.

Proof

If $S<T$ then $(S<T)=\Phi$ and thus $(S\leq T)=\Phi$. This means that one has $S\leq T$. Furthermore, by Proposition 6.1 v) we have that for all non-trivial closed

subspace A of Φ there exists a $r \in \mathbb{Q}^+$ such that $(T \leq r) \cap A \neq (S \leq r) \cap A$. That is, S and T never coincide.

Conversely, if $S \leq T$ and if S and T never coincide we get that the whole space Φ is an element of the set J' of Proposition 6.1 v). As the space $(S < T)$ is the only maximal element of J' we must have $(S < T) = \Phi$. ■

7. The space Φ_{T-}

Pursuing the analogy between the definition of the space Φ_T from Theorem 5.3 and the classical definition of $\mathcal{F}_T$, one defines, for all quantum stop time T on Φ, the space Φ_{T-} to be the closure in Φ of the linear space generated by $\{\mathbb{1}_{T>t} f; f \in \Phi_t, t \in \mathbb{R}^+\}$.

Proposition 7.1 –

i) *For all stop time T one has $\Phi_{T-} \subset \Phi_T$.*

ii) *If S, T are two stop times in Φ such that $S \leq T$ then $\Phi_{S-} \subset \Phi_{T-}$.*

iii) *If $(T_n)_n$ is an increasing sequence of stop times converging to T then $\Phi_{T-} = \vee_n \Phi_{T_n -}$.*

Proof

i) For every $t \in \mathbb{R}^+$, every $f \in \Phi_t$ we have $\mathbb{1}_{T\leq s} \mathbb{1}_{T>t} f = 0$ if $s < t$, $\mathbb{1}_{T \in]t,s]} f$ if $s > t$. Consequently in any case $\mathbb{1}_{T\leq s} \mathbb{1}_{T>t} f$ is an element of Φ_s. This means that all the vectors which generate Φ_{T-} are elements of Φ_T, thus $\Phi_{T-} \subset \Phi_T$.

ii) Let $t \in \mathbb{R}^+$ and $f \in \Phi_t$. By adaptedness $\mathbb{1}_{S>t} f$ is an element of Φ_t and thus $\mathbb{1}_{T>t} \mathbb{1}_{S>t} f$ is an element of Φ_{T-}. But as $S \leq T$ the vector $\mathbb{1}_{T>t} \mathbb{1}_{S>t} f$ is also equal to $\mathbb{1}_{S>t} f$. This means that the generators of Φ_{S-} are all elements of Φ_{T-}, thus $\Phi_{S-} \subset \Phi_{T-}$.

iii) Because of property ii) all the spaces $\Phi_{T_n -}$ are subspaces of Φ_{T-}, thus so is $\vee_n \Phi_{T_n -}$. Conversly if f belongs to Φ_{T-} and if t is a continuity point of T we have $\mathbb{1}_{T>t} f = \lim_n \mathbb{1}_{T_n > t} f$, thus $\mathbb{1}_{T>t} f$ is an element of $\vee_n \Phi_{T_n -}$. If t is not a continuity point of T one choses a sequence $(\varepsilon_n)_n$ as in the proof of Proposition 5.4 and one concludes from the identity $\mathbb{1}_{T>t} f = \lim_m \lim_n \mathbb{1}_{T_n > t+\varepsilon_m} f$. ■

Proposition 7.2 – *If S and T are two stop times such that $S < T$ then one has $\Phi_S \subset \Phi_{T-}$*

Proof

As $(S < T) = \Phi$ we have in particular

$$\Phi_S = \bigvee_{r \in \mathbb{Q}^+} \big((S \leq r) \cap (T > r) \cap \Phi_S \big).$$

Let f be an element of $(S \leq r) \cap \Phi_S$. In particular f is an element of Φ_S thus $f = \mathbb{E}_S f$. Furthermore f is an element of $(S \leq r)$ thus $f = \mathbb{1}_{S\leq r} f$. All together this gives $f = \mathbb{1}_{S\leq r} \mathbb{E}_S f$. So, by Theorem 5.3, f is an element of Φ_r. If furthermore f belongs to $(T > r)$ we have $f = \mathbb{1}_{T>r} f$ and thus $f \in \Phi_{T-}$. This shows that for all $r \in \mathbb{Q}^+$ we have $(T > r) \cap (S \leq r) \cap \Phi_S \subset \Phi_{T-}$ and thus $\Phi_S \subset \Phi_{T-}$. ■

Proposition 7.3 –

i) *If* $(T_n)_n$ *is a decreasing sequence of stop times converging to a stop time* T *and such that* $T_n > T$ *for all* n *then* $\Phi_T = \cap_n \Phi_{T_n-}$.

ii) *If* $(T_n)_n$ *is an increasing sequence of stop times converging to a stop time* T *and such that* $T_n < T$ *for all* n *then* $\Phi_{T-} = \vee_n \Phi_{T_n}$.

Proof

i) By Proposition 7.2, Φ_T is included in Φ_{T_n-} for all n thus Φ_T is included in $\cap_n \Phi_{T_n-}$. Conversly, $\cap_n \Phi_{T_n-}$ is included in $\cap_n \Phi_{T_n}$ which is equal to Φ_T by Proposition 5.4. This proves i).

ii) By Proposition 7.2, Φ_{T_n} is included in Φ_{T-} for all n thus so is $\vee_n \Phi_{T_n}$. Finally, by Proposition 7.1, $\Phi_{T-} = \vee_n \Phi_{T_n-} \subset \vee_n \Phi_{T_n}$. ■

A stop time T is said to be *predictable* if there exists an increasing sequence $(T_n)_n$ of stop times converging to T and such that $T_n < T$ for all n.

Theorem 7.4 – *The Fock space is "quasi-left continuous" that is, for all predictable stop time* T *one has* $\Phi_T = \Phi_{T-}$.

Proof

We know that $\Phi_{T-} \subset \Phi_T$ and that $\Phi_{T-} = \vee_n \Phi_{T_n}$. It is thus sufficient to prove that $\Phi_T \subset \vee_n \Phi_{T_n}$. Let $f \in \Phi$. By Proposition 5.1 we have

$$\mathbb{E}_T f = \mathbb{E}_0[f] + \int_0^\infty \mathbb{1}_{T>s} D_s f \, dX_s$$

$$\text{and} \quad \mathbb{E}_{T_n} f = \mathbb{E}_0[f] + \int_0^\infty \mathbb{1}_{T_n>s} D_s f \, dX_s.$$

Thus one has

$$\begin{aligned} ||\mathbb{E}_T f - \mathbb{E}_{T_n} f||^2 &= \int_0^\infty ||\mathbb{1}_{T>s} D_s f - \mathbb{1}_{T_n>s} D_s f||^2 \, ds \\ &= \int_0^\infty ||\mathbb{1}_{T\le s} D_s f - \mathbb{1}_{T_n\le s} D_s f||^2 \, ds. \end{aligned}$$

For every s which is a continuity point of T the term $||\mathbb{1}_{T\le s} D_s f - \mathbb{1}_{T_n\le s} D_s f||^2$ converges to 0 as n tends to $+\infty$. As the Lebesgue measure of the set of continuity points of T is null, we conclude by the dominated convergence Theorem. ■

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