Plane curves with a big fundamental group of the complement

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Abstract

Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of geometric genus g whose dual $C^* \subset \mathbb{P}^{2*}$ is an immersed curve of degree $d \geq 2g-1$. The main result states that the Poincaré group $\pi_1(\mathbb{P}^2 \setminus C)$ contains a free group with two generators. We construct universal families of immersed plane curves and their Picard bundles. This allows us to reduce the consideration to the case of Plücker curves. Such a curve C can be regarded as a plane section of the corresponding discriminant hypersurface (cf. [Zar, DoLib]). Applying Zariski–Lefschetz type arguments we deduce the result from 'the bigness' of the d-th braid group $B_{d,g}$ of the Riemann surface of C.

Introduction

The fundamental groups of the plane curve complements are of permanent interest (see e.g. [Di, DoLib, Lib, MoTe, No, O, Zar] and the literature therein). Here we look for the most coarse properties of these groups (cf. e.g. [MoTe]). Namely, we distinguish between *big* and *small* groups.

0.1. Definition. We say that a group G is *big* if it contains a non–abelian free subgroup. We call G *small* if it is *almost solvable*, i.e. it has a solvable subgroup of finite index.

Recall the Tits alternative [Ti]: *any subgroup G of a general linear group $GL(n, k)$ over a field k of characteristic zero is either big or small.* This alternative holds true, even in a stronger form, for some classes of discrete groups, such as hyperbolic groups in sense of Gromov and the mapping class groups (see sect.1 below for references).

In [MoTe] classes of plane Plücker curves were indicated with infinite almost solvable (i.e. small) non–abelian fundamental groups of the complement. An example is the branching divisor of a generic projection of the Veronese surface V_3 of order 3 onto $\mathbb{P}^2$ [MoTe].

The well known Deligne–Fulton Theorem asserts that the complement of a nodal plane curve has abelian fundamental group. Here we show (and this is the main purpose of the paper) that in certain cases the fundamental group of the complement of the dual of a nodal plane curve is big.

0.2. Theorem. *Let $C \subset \mathbb{P}^2$ be an irreducible immersed curve*[1] *of degree d and geometric genus g. Suppose that either $g \geq 1$, or $g = 0$ and $d \geq 4$. Let $C^* \subset \mathbb{P}^{2*}$ be the dual curve. Then*

a) The group $\pi_1(\mathbb{P}^{2*} \setminus C^*)$ is not almost nilpotent[2].

b) If $d \geq 2g - 1$ then the group $\pi_1(\mathbb{P}^{2*} \setminus C^*)$ is big.

Obviously, the statement does not hold for $g = 0$, $d < 3$. In the exceptional case when $g = 0$ and $d = 3$ we have that C is a nodal cubic, C^* is a three–cuspidal quartic and $\pi_1(\mathbb{P}^{2*} \setminus C^*)$ is the metacyclic group of order 12 [Zar, p.143–145]. We do not know whether the theorem is true for $g > 0$ without the assumption $d \geq 2g - 1$.

The paper is organized as follows. Section 1 is preliminary. We discuss there a relation between bigness of the fundamental group and C-hyperbolicity. It is illustrated by several examples, in particular, of quasi–projective quotients of bounded symmetric domains and of complements of certain reducible plane curves. Due to this relation (actually, to a theorem of Lin [Li]), part (a) of Theorem 0.2 is reduced to a theorem from [DeZa].

The proof of Theorem 0.2.b) will be done in Sect. 4. The results in Sect. 2 and 3 on nodal and Plücker approximations of immersed curves reduce the proof to the case of a nodal Plücker curve (see Theorem 2.1 below; see also [AC, Ha] for related results, especially concerning (a) and (b) of Theorem 2.1).

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1 Big groups and C-hyperbolicity

1.1. Generalities on big groups

By a theorem of von Neumann, a big group is non–amenable. The converse is not true, in general; the corresponding examples are due to A. Ol'shanskij, S. I. Adian and M. Gromov (see [OSh]). Note that the group in all these examples is not finitely presented. For a finitely presented group the equivalence of bigness and non–amenability

[1] i.e. all the analytic branches at the singular points of C are smooth.

[2] by *almost nilpotent* we mean a group which has a nilpotent subgroup of finite index.

is unknown[3]. Being non–amenable, a big group can not be almost nilpotent or even almost solvable. As follows from the Nielsen–Schreier Theorem, a subgroup of finite index of a big group is big, as well as a normal subgroup with a solvable quotient. Clearly, a group with a big quotient is big.

We remind several classical examples of big groups. First of all, for $g \ge 1$ the Siegel modular group $Sp_{2g}(\mathbb{Z})$ is big. In addition, it has no infinite normal solvable subgroup (see (1.3)-(1.4) below).

Another examples are: the Artin group $B_{d,g}$ of the d–string braids of a genus g compact Riemann surface R_g, and the mapping class group $\mathrm{Mod}_{g,n}$, i.e. the group of classes of isotopy of orientation preserving diffeomorphisms of a genus g Riemann surface with n punctures (see e.g. [Bi]). Namely, we have the following

1.2. Lemma.
(a) The braid group $B_{d,g}$ ($d \ge 1$) is big iff $(d, g) \ne (1, 0), (2, 0), (3, 0), (1, 1)$.
(b) The mapping class group $Mod_{g,n}$ is big iff $g \ge 1$, or $g = 0$ and $n \ge 4$.

Proof. (a) By definition, $B_{d,g} = \pi_1(S^d R_g \setminus \Delta_{d,g})$, where $S^d R_g$ denotes the d-th symmetric power of R_g and $\Delta_{d,g} \subset S^d R_g$ denotes the discriminant hypersurface consisting of the d-tuples of points with coincidences. The pure braid group $P_{d,g} := \pi_1((R_g)^d \setminus D_{d,g})$, where $D_{d,g} \subset (R_g)^d$ is the union of diagonal hypersurfaces, is the normal subgroup of $B_{d,g}$ of index $d!$ which corresponds to the Vieta covering $(R_g)^d \setminus D_{d,g} \to S^d R_g \setminus \Delta_{d,g}$. The fibration $(R_g)^{d+1} \setminus D_{d+1,g} \to (R_g)^d \setminus D_{d,g}$ with the fibre $R_g \setminus \{d \text{ points}\}$ yields the short exact sequence [Bi, sect. 1.3]

$$\mathbf{1} \to \pi_1(R_g \setminus \{d \text{ points}\}) \to P_{d+1,g} \to P_{d,g} \to \mathbf{1}\,.$$

For $d > 0$ the group $\pi_1(R_g \setminus \{d \text{ points}\})$ is a free group $\mathbf{F}_k$ with $k = 2g + d - 1$ generators. For $d = 0$ by a theorem of Magnus [CoZi, (2.5.1)], after removing any of the standard generators $a_1, b_1, \ldots, a_g, b_g$ of $\pi_1(R_g)$, the subgroup generated by the remaining ones is the free group $\mathbf{F}_{2g-1}$.

Hence, under the above restrictions the pure braid group $P_{d,g}$, and therefore also the braid group $B_{d,g}$, contains a subgroup isomorphic to a non-abelian free group. In the exceptional cases when $(d, g) = (1, 1)$ or $g = 0$, $1 \le d \le 3$ the same exact sequence shows that the corresponding group $B_{d,g}$ is not big. This proves (a).

(b) There is a natural surjection $j : \mathrm{Mod}_{g,n} \to \mathrm{Mod}_g := \mathrm{Mod}_{g,0}$, where the kernel $\mathrm{Ker}\, j$ is the braid group $B_{n,g}$ if $g \ge 2$ and its quotient by the center if $g = 1$, $n \ge 2$ or $g = 0$, $n \ge 3$ [Bi, Theorem 4.3]. Therefore, the group $\mathrm{Mod}_{g,n}$ is big as soon as the corresponding braid group $B_{n,g}$ is so.

[3]we are thankful to V. Sergiescu and V. Guba for this information.

For $g \geq 1$ the induced representation of Mod_g into the first homology group of R_g yields a surjection $\mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$ (actually, $\mathrm{Mod}_1 \cong GL(2, \mathbb{Z})$). This shows that Mod_g, $g \geq 1$, is a big group.

For $g = 0$ we have that $\mathrm{Mod}_{0,3} = B_{3,0}/(\mathrm{center})$ is a finite group, the groups $\mathrm{Mod}_{0,0}$ and $\mathrm{Mod}_{0,1}$ are trivial, whereas $\mathrm{Mod}_{0,2} = \mathbb{Z}/2\mathbb{Z}$ [Bi, Theorem 4.5]. This completes the proof. □

Remark. In fact, the Tits alternative holds in Mod_g, $g \geq 1$ [Iv, MC] (note that for $g \geq 2$ the latter group is not isomorphic to any arithmetic linear group [Iv]). Furthermore, for $g \geq 2$ any almost solvable subgroup of Mod_g is almost abelian [BiLuMC].

Recall the following notion. A complex space X is said to be *C-hyperbolic* if it has a *Carathéodory hyperbolic* covering $Y \to X$, i.e. such that the bounded holomorphic functions on Y separate points of Y. We say that X is *almost C–hyperbolic* if the bounded holomorphic functions on Y separate points only up to finite subsets. As follows from Lin's Theorem [Lin, Theorem B], *the fundamental group of an almost C-hyperbolic algebraic variety can not be almost nilpotent.* Concerning the part (a) of Theorem 0.2 note that for $g \geq 1$ the complement $\mathbb{P}^{2*} \setminus C^*$ is actually C–hyperbolic [DeZa]. This allowed us to derive Theorem 0.2.a) from Lin's Theorem (indeed, (a) follows from (b) in the case where $g = 0$ and $d \geq 4$). The following question arises naturally:

Question. *Let X be an almost C–hyperbolic algebraic variety. Is then necessarily $\pi_1(X)$ a big group?*

Note that by another theorem of Lin [Lin, Thm. B(b)], $\pi_1(X)$ cannot be an amenable group with a non–trivial center assuming that the universal covering space $\tilde{X}$ is Carathéodory hyperbolic.

An easy observation is that the answer is 'yes' for $\dim X = 1$. Indeed, an algebraic curve C is C–hyperbolic iff it is hyperbolic, or, in turn, iff its normalization C_{norm} has a non-abelian fundamental group. In the latter case the group $\pi_1(C_{\mathrm{norm}})$ is big (see the proof of Lemma 1.2). Note, however, that by a result of [LySu], any compact Riemann surface R of genus $g \geq 2$ admits a Galois covering $\tilde{R}$ with a metabelian (i.e. two-step solvable) Galois group such that $\tilde{R}$ carries a non–constant bounded holomorphic function. Modifying this example, one may even assume $\tilde{R}$ being Carathéodory hyperbolic[4].

More generally, we have the following fact. Its proof given below was communicated to us by D. Akhiezer[5].

[4] V. Lin, M. Zaidenberg; unpublished.

[5] and it is placed here with his kind permission.

1.3. Theorem. *Let $D \subset \mathbb{C}^n$ be a bounded symmetric domain, and let $\Gamma \subset \operatorname{Aut} D$ be a discrete subgroup. If the Bergman volume of a fundamental domain of Γ is finite, then Γ is a big group and it has no infinite solvable normal subgroup.*

Proof. According to a result of A. Borel and J.–L. Koszul [Bo, Kos], a homogeneous domain D is symmetric iff the identity component G of the automorphism group $\operatorname{Aut} D$ is semisimple. Recall that G has trivial center, and therefore it is a connected linear group [He, Ch. VIII.6]. Being semisimple G is not solvable. Moreover, since G is connected, it is not small. We have $D \cong G/K$, where $K \subset G$ is a maximal compact subgroup [ibid, Ch. VIII. 7]. The automorphism group $\operatorname{Aut} D$ has finitely many connected components, i.e. $[\operatorname{Aut} D : G] < \infty$ (indeed, being a compact Lie group the stabilizer $\operatorname{Stab}_z \subset \operatorname{Aut} D$ of a point $z \in D$ has a finite number of connected components, which is the same as those of $\operatorname{Aut} D$, because the quotient $D \simeq \operatorname{Aut} D/\operatorname{Stab}_z$ is connected). Hence, $\Gamma \cap G$ has finite index in Γ, and the Bergman volume of $(\Gamma \cap G) \setminus D$ is finite, too. Therefore, the invariant volume $\operatorname{Vol}((\Gamma \cap G) \setminus G)$ is finite, and so $\Gamma \cap G$ is a lattice of G.

Fix a faithful linear representation $G \hookrightarrow GL(n, \mathbb{C})$. Let $G_{\mathbb{C}}$ be the Zariski closure of G in $GL(n, \mathbb{C})$. By Borel's Density Theorem (see e.g. [Ra, 5.16]), the conditions "G is semisimple and $\operatorname{Vol}((\Gamma \cap G) \setminus G) < \infty$" imply that the subgroup $\Gamma \cap G$ is Zariski dense in $G_{\mathbb{C}}$. Hence, if Γ is almost solvable, $G_{\mathbb{C}}$ should be also almost solvable, which is not the case. By the Tits alternative, Γ must be big.

The last assertion follows from a theorem of V. Gorbatsevich [GoShVi, Proposition 3.7]. According to this theorem, the lattice $\Gamma \cap G$ in a connected Lie group G possesses no infinite solvable normal subgroup iff G is reductive and its semisimple part has a finite center. It is easily seen that in our case both conditions are fulfilled. □

1.4. *Remark.* In fact, it would be enough in the above theorem that Γ was a Zariski dense subgroup of a semisimple linear algebraic group G with a finite center, which acts holomorphically in D. This may be illustrated by the following example 1.5(a).

1.5. Examples.

(a) Let $D = \mathcal{Z}_g$ be the Siegel upper half–plane and $\Gamma = \operatorname{Sp}_{2g}(\mathbb{Z})$, $G = \operatorname{Sp}_{2g}(\mathbb{R})$, $g \geq 1$, are resp. the Siegel modular group and the simplectic group. Then $\Gamma \setminus D$ is a coarse moduli space of principally polarized abelian varieties of dimension g, which is a quasiprojective variety. Here Γ is Zariski dense in G. Actually, by a theorem of A. Borel and Harish–Chandra [BoHC, Thm. 7.8], the arithmetic subgroup $G_{\mathbb{Z}}$ of a semisimple real algebraic group $G_{\mathbb{R}}$ defined over $\mathbf{Q}$ is a lattice in $G_{\mathbb{R}}$, and so by Borel's Density Theorem, it is Zariski dense in $G_{\mathbb{C}}$. (By the way, these arguments show that $\Gamma = \operatorname{Sp}_{2g}(\mathbb{Z})$ is a big group without infinite normal solvable subgroups.)

(b) Let, furthermore, $D := T_{g,n} \subset\subset \mathbb{C}^{3g-3+n}$ be the Teichmüller space of the n–punctured genus g marked Riemann surfaces under the Bers realization, where $2-$

$2g - n < 0$. By Royden's Theorem, $\Gamma := \mathrm{Aut}\, D$ is the Teichmüller modular group, which coincides with the mapping class group $\mathrm{Mod}_{g,n}$. The quotient $\Gamma \setminus D$ is a coarse moduli space $\mathcal{M}_{g,n}$ of genus g n–punctured Riemann surfaces, which is a quasiprojective variety. By Lemma 1.2 above, except the case when $(g, n) = (0, 3)$ the group $\mathrm{Mod}_{g,n}$ is big.

(c) (see e.g. [Sh1, 2]). A smooth projective surface S is called *a Kodaira surface* if there is a smooth fibration $\pi : S \to B$ over a curve B, where both B and a generic fibre F of π are of genus ≥ 2 (usually π is supposed being a non-trivial deformation of F, but we don't need this assumption here). It is well known that the universal covering $\tilde{S}$ of S can be realized as a bounded pseudo–convex Bergman domain in $\mathbb{C}^2$. Thus, the projective surface $S = \Gamma \setminus D$ is C–hyperbolic; clearly, $\Gamma \simeq \pi_1(S)$ is a big group. More generally, the same is true when both B and F are quasiprojective hyperbolic curves.

Next we pass to the simplest examples of reducible plane projective curves with a big fundamental group of the complement.

1.6. Examples.

(a) Let $C \subset \mathbb{P}^2$ be a finite line arrangement. If these lines are in general position, then by the Deligne–Fulton Theorem, $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian. Otherwise, this group is big. Indeed, let C has a point A of multiplicity at least 3. The union L of lines in C passing through A contains at least three members of the associated linear pencil. The linear projection $\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{3 \text{ points}\}$ with center at A yields an epimorphism of the fundamental groups. Thus, $\pi_1(\mathbb{P}^2 \setminus C)$ dominates the free group $\mathbf{F}_2 = \pi_1(\mathbb{P}^1 \setminus \{3 \text{ points}\})$, and therefore, it is big.

(b) Consider further a configuration $C \subset \mathbb{P}^2$ of a plane conic together with two of its tangent lines (cf. [DeZa, 6.1(b)]). The Zariski–van Kampen method yields a presentation

$$G := \pi_1(\mathbb{P}^2 \setminus C) = \langle\, a, b \,|\, abab = baba \,\rangle\,.$$

The following proof of the bigness of G was communicated to us by V. Lin[6].

Remind that the Coxeter group $\mathbf{B}_k$ is the group generated by the orthogonal reflections in $\mathbb{R}^k$ with respect to the coordinate planes and the diagonals $x_i - x_j = 0$, $i, j = 1, \ldots, k$. The corresponding Artin–Brieskorn braid group is the fundamental group $\pi_1(G_k(\mathbf{B}_k))$ of the domain

$$G_k(\mathbf{B}_k) := \{z = (z_1, \ldots, z_k) \in \mathbb{C}^k \,|\, d_k(z) \cdot z_k \neq 0\}\,,$$

where $d_k(z)$ is the discriminant of the universal polynomial $p_k(t) = p_k(t, z) := t^k + z_1 t^{k-1} + \ldots + z_k$ of degree k. Put $G_k := \{z \in \mathbb{C}^k \,|\, d_k(z) \neq 0\}$, and let $E_k^1 \to G_k$ be the

[6] We are grateful to V. Lin for a kind permission to place here this proof.

standard k–sheeted covering over G_k, where

$$E_k^1 := \{(z,\,\lambda) = (z_1,\ldots,z_k,\,\lambda) \in \mathbb{C}^{k+1}\,|\,p_k(\lambda,\,z) = 0\}\,.$$

Define a mapping $\varphi\,:\,E_{k+1}^1 \to G_k(\mathbf{B}_k)\times\mathbb{C}$ as follows:

$$\varphi(z_1,\ldots,z_{k+1},\,\lambda) = (q_k,\,\lambda) = (\xi_1,\ldots,\xi_k,\,\lambda)\,,$$

where

$$q_k = q_k(t,\,\xi) = t^k + \xi_1 t^{k-1} + \ldots + \xi_k := p_{k+1}(t+\lambda,\,z)/t \in G_k(\mathbf{B}_k)\,.$$

Note that $t\,|\,p_{k+1}(t+\lambda,\,z)$, because $p_{k+1}(\lambda,\,z)\equiv 0$ for $(z,\,\lambda)\in E_{k+1}^1$. Since $p_{k+1}(t+\lambda)$ is a polynomial with simple roots, the same is true for $q_k(t)$. Moreover, $q_k(0)\neq 0$; thus, indeed, $q_k\in G_k(\mathbf{B}_k)$. It is easily seen that φ is a biregular isomorphism. Hence, the isomorphism

$$\pi_1(G_k(\mathbf{B}_k)) \cong \pi_1(E_{k+1}^1) \hookrightarrow \pi_1(G_{k+1})$$

represents the Artin–Brieskorn braid group $\pi_1(G_k(\mathbf{B}_k))$ as a subgroup of finite index (equal to $k+1$) of the standard Artin braid group[7] $B_{k+1} := \pi_1(G_{k+1})$. Therefore, the former group is big as soon as the latter one is so. Both of them are big starting with $k=2$ (for the Artin group B_{k+1} this can be checked in the same way as it was done in the proof of Lemma 1.2 for the braid groups $B_{k,\,g}$). It remains to note that $\mathbb{P}^2\setminus C \cong G_2(\mathbf{B}_2)$, and therefore $G=\pi_1(\mathbb{P}^2\setminus C)$ is isomorphic to the braid group $\pi_1(G_2(\mathbf{B}_2))$ which is big.

2 Nodal approximation of immersed curves

Due to Theorem 2.1 below, Theorem 0.2.b) can be reduced to the case where C is a generic nodal Plücker curve. We also believe that Theorem 2.1 has an independent interest.

We use the following notation and terminology. Let $\mathbb{P}^N$, $N=N(d)=\binom{d+2}{2}-1$, $d\geq 1$, be the Hilbert scheme of degree d plane curves. Denote $Imm_{d,g}$ the locus of points of $\mathbb{P}^N$ which correspond to reduced irreducible immersed curves of geometric genus g, $0\leq g\leq\binom{d-1}{2}$, and by $Nod_{d,\,g}$ resp. $PlNod_{d,\,g}$ the subset of points of $Imm_{d,\,g}$ which correspond to the nodal resp. to the Plücker nodal curves. Remind that an irreducible curve $C\subset\mathbb{P}^2$ is called *Plücker* if the only singular points of C and the dual curve C^* are ordinary nodes and cusps. Let $Pl\ddot{u}Nod_{d,\,g}\subset PlNod_{d,\,g}$ be the subset of curves which have no flex at a node.

[7] From now on we denote $B_m=\pi_1(G_m)$ the standard Artin braid group with n strings; don't confuse with the Coxeter group $\mathbf{B}_k$.

Denote $\mathcal{S}_d \to \mathbb{P}^N$ the universal family of curves over the Hilbert scheme $\mathbb{P}^N$, and let $\mathcal{S}_{d,g} \to Imm_{d,g}$ be its restriction to $Imm_{d,g}$. By *a family of curves* we mean a proper morphism $\varphi : X \to Y$ of relative dimension one of quasiprojective varieties. If X, Y are smooth and φ is a submersion, then the family φ is called *smooth*. We say that φ admits *a simultaneous normalization* if Y is smooth and there exists a smooth family of curves $\varphi' : X' \to Y$ and a morphism $f : X' \to X$ commuting with the projections onto Y such that for every point $y \in Y$ the restriction $f \,|\, X'_y : X'_y \to X_y$ onto the fibre over y is a normalization map.

2.1. Theorem.
a) $Imm_{d,g} \subset \mathbb{P}^N$ *is a smooth locally closed subvariety of pure dimension* $3d+g-1$.
b) The universal family of curves $\mathcal{S}_{d,g} \to Imm_{d,g}$ *admits a simultaneous normalization* $f : \mathcal{M}_{d,g} \to \mathcal{S}_{d,g}$.
c) $Nod_{d,g}$ *and, for* $n \geq 2g-1$, $PlNod_{d,g}$ *are Zariski open subsets of* $Imm_{d,g}$.

Remark. The first statement of (c) and the dimension count in (a) can be found in [Ha, Sect. 2], while the proofs are quite different. Note that, by Harris [Ha], the variety $Imm_{d,g}$ is irreducible; it is non–empty for any (d, g) with $0 \leq g \leq \binom{d-1}{2}$ [Se, sect.11, p.347; Ha; O].

In this section we prove (a), (b) and the first part of (c) of Theorem 2.1; the proof of (c) is completed in sect. 3. First we study $Imm_{d,g}$ locally, in a neighborhood of a given curve $C \in Imm_{d,g}$. This needs certain preparation, including a portion of plane curve singularities.

2.2. The Gorenstein–Rosenlicht invariant, the boundary braid and its algebraic length

Recall that the Gorenstein–Rosenlicht invariant δ_P of a singular analytic plane curve germ (A, P) can be expressed as $\delta_P = \frac{1}{2}(\mu + r - 1)$, where μ is the Milnor number and r is the number of local branches of A at P [Mi, sect. 10]. For a reduced curve F on a smooth surface W we set $\delta(F) = \sum_{P \in \mathrm{Sing} F} \delta_P$. If F is a complete irreducible curve, then by the genus formula and the adjunction formula [BPVV, II.11] we have

$$\pi_a(F) = g(F) + \delta(F) = 1/2\, F(K_W + F) + 1\,, \tag{1}$$

where π_a resp. g denotes arithmetic resp. geometric genus, K_W is the canonical divisor of W, and where for a non–compact surface W we put $FK_W = \deg(K_W \,|\, F)$.

Let $U \subset \mathbb{C}$ be the unit disc, $\Sigma = U \times \mathbb{C} \subset \mathbb{C}^2$ be the solid cylinder $\Sigma = \{(u, v) \in \mathbb{C}^2 \,|\, |u| < 1\}$, and $p : \mathbb{C}^2 \to \mathbb{C}$ be the first projection. Let $A \subset \Sigma$ be an analytic curve extendible transversally through the boundary $\partial\Sigma$, so that *the link* $\partial A = \bar{A} \cap \partial\Sigma$ is smooth. Suppose also that the projection $p : A \to U$ is proper, i.e. it is a (ramified) covering over the unit disc U of degree, say, m. The link ∂A carries a (closed) braid with m strings $b_A \in B_m$ defined uniquely up to conjugation, where B_m is the Artin braid group (see (1.6(b) above). To define the braid b_A, cut the cylinder $\partial U = S^1 \times \mathbb{C}$ along its generator $1 \times \mathbb{C}$ and then identify it with $[0, 1] \times \mathbb{R}^2 \subset \mathbb{R}^3$. Fix a numbering of the points of the fibre of ∂A over $1 \in \partial U$. Passing once along the circle $S^1 = \partial U$ counterclockwise, we obtain the braid b_A.

Let $\sigma_1, \dots, \sigma_{m-1}$ be the standard generators of B_m. Recall [Be, p.100] that there is a unique epimorphism $B_m \to \mathbb{Z}$ which associates to each braid $b = \sigma_{i_1}^{\alpha_1} \dots \sigma_{i_n}^{\alpha_n} \in B_m$ its *algebraic length* $l(\mathrm{b}) := \sum_{k=1}^n \alpha_k$.

2.3. Lemma. *Let $A \subset \Sigma$ as above be a nodal curve with δ nodes. Suppose that all the ramification points of the covering $p : A \to U$ are simple (i.e. with ramification indices 2) and no two of them are at the same fibre. If the branching divisor $D \subset U$ consists of $\delta + \tau$ points, then*

$$l(b_A) = 2\delta + \tau\,.$$

Proof. Choose small disjoint discs ω_i in U, $i = 1, \dots, \delta + \tau$, centered at the points of D. Fix a point at the boundary of the disc ω_i and join it by a path γ_i with the point $1 \in \partial U$, where γ_i, $i = 1, \dots, \delta + \tau$, are disjoint. The complement $U \setminus \bigcup_{i=1}^{\delta+\tau} (\bar{\omega}_i \cup \gamma_i)$ being simply connected, the braid b_A is the product of the local braids b_{A_i} which correspond to the curves $A_i := A \cap p^{-1}(\omega_i)$. It is easily seen that the local braid which corresponds to a node of A is conjugate in the braid group B_m with the square of a generator, and those at an irreducible ramification point is conjugate with a generator. Now the lemma easily follows. □

With each plane curve singularity $(A, \bar{0}) \subset (\mathbb{C}^2, \bar{0})$ we associate its *braid* $b_{A,\bar{0}}$ defined as follows. Fix a generic linear projection $p : (\mathbb{C}^2, \bar{0}) \to (\mathbb{C}, \bar{0})$, so that the direction of p is different from the tangent directions of the branches of A at $\bar{0}$, and proceed in the same way as above.

2.4. Lemma. *Suppose that $(A, \bar{0}) \subset (\mathbb{C}^2, \bar{0})$ is an immersed singularity (i.e. a singular point of a reduced curve having only smooth local branches $A_1, \dots, A_r$) with the Gorenstein–Rosenlicht invariant $\delta = \delta(A, \bar{0})$. Then*

a) $\delta = \frac{1}{2}\, l(b_{A,\bar{0}})\,.$

b) Let $\tilde{A}$ be a small nodal deformation of A defined in a fixed small ball B_ϵ centered at the origin. Denote by r resp. $\tilde{r}$ the number of irreducible components of A resp. of $\tilde{A}$ in B_ϵ. Then $\delta(\tilde{A}) \le \delta(A)$, and $\delta(\tilde{A}) = \delta(A)$ iff $r = \tilde{r}$. In the latter case the irreducible components $\tilde{A}_1, \dots, \tilde{A}_r$ of $\tilde{A}$ in B_ϵ approximate the corresponding irreducible components $A_1, \dots, A_r$ of $A \cap B_\epsilon$.

Proof. (a) We have $\delta = \sum_{1\le k<l\le r}(A_k \cdot A_l)_{\bar 0}$ [Mil, (10.20)]. Let $A'_i \subset B_\epsilon$ be a small generic deformation of the branch A_i, $i = 1,\dots,r$. Set $A' = \bigcup_{i=1}^r A'_i$. Then A' is a nodal curve with

$$\delta = \sum_{1\le k<l\le r} A'_k \cdot A'_l = \sum_{1\le k<l\le r} (A_k \cdot A_l)_{\bar 0}$$

nodes, and clearly, $b_{A,\bar 0} = b_{A',\bar 0}$. Since for all $i = 1,\dots,r$ the generic linear projection $p : A_i \to U_{\epsilon'}$ is non–ramified, the same is true for the branches A'_i, $i = 1,\dots,r$. Thus, $p : A' \to U_{\epsilon'}$ is ramified only at δ nodes, and therefore, in the notation of Lemma 2.3, $\tau = \tau(A') = 0$. By this lemma, we have $\delta = 1/2\, l(b_{A',\bar 0}) = 1/2\, l(b_{A,\bar 0})$. This proves (a).

(b) Once again here $b_{A,\bar 0} = b_{\tilde A,\bar 0}$. Due to (a) and to Lemma 2.3, we have

$$2\delta(A) = l(b_{A,\bar 0}) = l(b_{\tilde A,\bar 0}) = 2\delta(\tilde A) + \tau(\tilde A)\,,$$

and the inequality of (b) follows. The equality holds iff $\tau(\tilde A) = 0$, which means that the projection $p : \tilde A \to U_{\tilde\epsilon}$ is ramified only at nodes of $\tilde A$. Therefore, for any irreducible component $\tilde A_i$ of $\tilde A \cap B_\epsilon$ the composition of the normalization map $(\tilde A_i)_{\rm norm} \to \tilde A_i$ with the projection $p : \tilde A_i \to U_{\tilde\epsilon}$ is non–ramified and hence, one–sheeted. It follows that both of these mappings are biholomorphic, so that the irreducible components $\tilde A_i$ of $\tilde A \cap B_\epsilon$ are smooth. The degree of the branched covering $p : \tilde A \to U_{\tilde\epsilon}$ being equal to r, $\tilde A \cap B_\epsilon$ consists of r smooth irreducible components close to those of A. □

Let X be a smooth projective surface, $C \subset X$ be an irreducible immersed curve with a normalization $\varphi_0 : M_0 \cong C_{\rm norm} \to C$. By [No, (1.8)-(1.12)], there exists a smooth open complex surface V which contains M_0 as a closed subvariety, and a holomorphic immersion $\varphi : V \to X$ that extends φ_0; it is called *a tubular neighborhood of* φ_0. To obtain V one simply normalizes C together with a tubular neighborhood of C in X.

2.5. Lemma. *Let $C \subset X$, M_0 and V be as above, and let $N \to M_0$ be the normal bundle of M_0 in V. Then*

$$\deg N = M_0^2 = C^2 - 2\delta(C)\,. \qquad (2)$$

If $X = I\!\!P^2$ and $C \in Imm_{d,g}$, then

$$\deg N = 3d + 2(g-1)\,. \qquad (3)$$

Proof. By the adjunction formula, we have

$$2g - 2 = C^2 + CK_X - 2\delta(C) = M_0^2 + M_0 K_V \, . \tag{4}$$

Since $K_V = \varphi^* K_X$, by the projection formula we have $M_0 K_V = CK_X$, and so (2) follows. (3) is a corollary of (2) and the genus formula (1). □

From this lemma, using the well known criterion of ampleness and spannedness of a line bundle over a curve (see e.g. [Hart,IV.3.2] or [Na, 5.1.12]), the Kodaira Vanishing Theorem and the Riemann–Roch Formula we obtain

2.6. Corollary. *a) N is very ample iff $C^2 - 2\delta(C) \geq 2g + 1$. For $X = \mathbb{P}^2$ and $C \in Imm_{d,g}$ this is always the case, and furthermore, $h^1(M_0, \mathcal{O}(N)) = 0$ and $h^0(M_0, \mathcal{O}(N)) = 3d + g - 1$.*

b) For any pair of points $p_1, p_2 \in M_0$ the line bundle $N_{p_1,p_2} = N - [p_1] - [p_2]$ on M_0 is spanned[8] if $C^2 - 2\delta(C) \geq 2g+2$. In particular, this is so if $X = \mathbb{P}^2$ and $C \in Imm_{d,g}$, where $d \geq 2$.

The Kodaira Theorem on embedded deformations [Ko] implies such a

2.7. Corollary. *There exists a maximal smooth family $\pi_{loc} : \mathcal{M}_{loc} \to T_{loc}$ of embedded deformations of the curve $M_0 \cong \pi^{-1}(t_0)$ in V over a smooth base T_{loc} such that the Kodaira–Spencer map $T_{s_0} T_{loc} \to H^0(M_0, \mathcal{O}(N))$ is an isomorphism. In particular, if $X = \mathbb{P}^2$ and $C \in Imm_{d,g}$, then[9] $\dim T_{loc} = 3d + g - 1$.*

2.8. Definition. We say that a curve $C \in Imm_{d,g}$ is *strongly approximated* by curves $C' \subset Imm_{d,g}$ if C' approximate C in the Hausdorff topology, and for any singular point P of C of multiplicity $r(C, P)$ and for a fixed small neighborhood $B_{\epsilon,P}$ of P, the number $r(C', P)$ of irreducible components of $C' \cap B_{\epsilon,P}$ is equal to $r(C, P)$, and the irreducible components of $C' \cap B_{\epsilon,P}$ approximate those of $C \cap B_{\epsilon,P}$. Or, which is equivalent, if for a given tubular neighborhood $\varphi : V \to \mathbb{P}^2$ of a normalization $\varphi_0 : M_0 \to C$, the curves C' have normalizations $\varphi \,|\, M' : M' \to C'$, where $M' \subset V$ are obtained from M_0 by a small deformation.

We use below the following simple observation: a curve $C \in Nod_{d,g}$ is Plücker iff C has only ordinary flexes, no multitangent line, i.e. a line tangent to C in at least three points, and no bitangent line which is an inflexional tangent. One says that a curve $C \subset \mathbb{P}^2$ *has only ordinary singularities* iff all the local branches of C' at any

[8] i.e. the linear system $|N_{p_1,p_2}|$ has no base point.

[9] cf. [GH, sect.2.4; Ha].

of its singular point are smooth and pairwise transversal. Denote by $Ord_{d,g}$ the set of all such curves of degree d and genus g; clearly, $Ord_{d,g} \subset Imm_{d,g}$.

The next proposition should be known at least partially; in view of the lack of references, we give its proof.

2.9. Proposition. *The subspaces $Ord_{d,g}$, $Nod_{d,g}$, $PlNod_{d,g}$ and $PlüNod_{d,g}$ are dense in $Imm_{d,g}$ in the topology of strong approximation, and hence also in the Hausdorff topology of $\mathbb{P}^N$.*

Proof. Fix an arbitrary curve $C \in Imm_{d,g}$. We may assume that $d \geq 3$. First we show that C can be strongly approximated by curves $C' \in Ord_{d,g}$. For a curve $C' \in Imm_{d,g}$ denote by $\delta_1(C')$ the number of all non–ordered pairs (A'_i, A'_j) of local analytic branches of C' which meet normally at their common center $P \in C'$, so that $(A'_i, A'_j)_P = 1$. Clearly, $C' \in Ord_{d,g}$ iff $\delta(C') = \delta_1(C')$.

Suppose that C as above has a non–ordinary singular point P of multiplicity m. Consider the blow up $\sigma : X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at P, and let $\hat{C} \subset X$ be the proper transform of C. It is easily seen that $\delta(\hat{C}) = \delta(C) - \binom{m}{2}$. Since $\hat{C}^2 = C^2 - m^2$ we have

$$\hat{C}^2 - 2\delta(\hat{C}) = C^2 - 2\delta(C) - m = 3d + 2(g-1) - m \geq 2g + 2\,.$$

Let $\varphi : V \to X$ be a tubular neighborhood of a normalization $\varphi_0 : M_0 \to \hat{C}$ of $\hat{C}$. For a pair (A_i, A_j) of local branches of C at P with $(A_i, A_j)_P > 1$ let $\hat{P} \in X$ be the common center of their proper preimages $\hat{A}_i$, $\hat{A}_j$ in X, and let $P_i, P_j \in M_0$ be resp. the centers of the branches $\varphi^{-1}(\hat{A}_i)$, $\varphi^{-1}(\hat{A}_j)$ of the curve $M_0 \subset V$. By Lemma 2.5, for the normal bundle N of M_0 in V we have

$$\deg\,(N - [P_i]) = M_0^2 - 1 = \hat{C}^2 - 2\delta(\hat{C}) - 1 \geq 2g + 1\,.$$

Therefore, being spanned, the line bundle $N - [P_i]$ possesses a section which does not vanish at P_j, $j \neq i$. It follows that N has a section that vanishes at P_i, but not at P_j. This yields a deformation M'_0 of M_0 in V which passes through P_i, but not through P_j. Thus, for the curve $C' := \sigma\varphi(M'_0) \in Imm_{d,g}$ close enough to C we have $\delta_1(C') > \delta_1(C)$. By induction on $\delta_1(C')$, we get a strong approximation $C' \in Ord_{d,g}$ of C.

Suppose further that $C \in Ord_{d,g}$ is not nodal, i.e. it has a point P of multiplicity $m \geq 3$. Applying the same procedure as above to a triple of points $P_i, P_j, P_k \in M_0$ which lie over P, and using the inequality

$$\deg\,(N - [P_i] - [P_j]) \geq 2g\,,$$

by the spannedness of the line bundle $N - [P_i] - [P_j]$, we obtain a section of N which vanishes at the points P_i and P_j, but not at P_k. This leads to a curve $C' \in Ord_{d,g}$

which strongly approximates C and is simpler than C in the following sense: $m(C') < m(C)$, where

$$m(C) := \sum_{P_i \in \mathrm{sing}\,(C)} (\mathrm{mult}\,(P_i) - 1)\,.$$

Induction on $m(C')$ now shows that C can be strongly approximated by curves $C' \in Nod_{d,\,g}$.

Next we show that a curve $C \in Nod_{d,\,g}$ can be strongly approximated by curves $C' \in Nod_{d,\,g}$ with only ordinary flexes. We proceed by induction on the number $ofl(C')$ of ordinary flexes of C'. Since such a flex is a normal intersection point of the Hesse curve of C' with a smooth local branch of C', clearly, the bounded function $ofl(C')$ is lower semi–continuous on $Nod_{d,\,g}$ with respect to the Hausdorff topology.

Suppose that C has a non–ordinary flex at a local branch A of C centered at $P \in C$, so that $(A,\, L)_P \geq 4$, where L is the tangent line to A at P. In the notation as above, let $\sigma \,:\, X \to I\!\!P^2$ be the composition of three successive blow ups over P with centers at the proper preimages of A. Let $\hat{L} \subset X$ be the proper transform of L and $\hat{P}$ be the center of the proper transform $\hat{A} \subset X$ of A. We have $(\hat{A},\, \hat{L})_{\hat{P}} \geq 1$. If P is a smooth point of C then $\hat{C}^2 = C^2 - 3$ and $\delta(\hat{C}) = \delta(C)$. If P is a node of C then $\hat{C}^2 = C^2 - 6$ and $\delta(\hat{C}) = \delta(C) - 1$. In any case,

$$\deg N = \hat{C}^2 - 2\delta(\hat{C}) \geq C^2 - 2\delta(C) - 4 \geq 2g\,.$$

Therefore, the normal bundle N of M_0 in V is spanned, and hence it has a section which does not vanish at the point $\hat{P}$. The corresponding Kodaira-Spencer deformation yields a curve M_0' on X close enough to M_0 which does not pass through $\hat{P}$. It is easily seen that the projection $C' := \sigma\varphi(M_0') \subset I\!\!P^2$ is a nodal curve with an ordinary flex at P and such that $ofl(C') > ofl(C)$. After a finite number of steps we obtain a strong approximation $C' \in Nod_{d,\,g}$ of C with only ordinary flexes.

Suppose further that $C \in Nod_{d,\,g}$ has only ordinary flexes (note that this is an open condition). We will find a strong approximation C' of C without multiple tangents. Denote by $b(C')$ the total number of distinct intersection points with C of all the bitangent lines of C'. Clearly, the bounded function $b(C')$ is lower semi–continuous on $Nod_{d,\,g}$.

Let C have a multitangent line L which is tangent to C at points $P,\, Q,\, R \in C$ and, perhaps, at some other points. Let $\sigma \,:\, X \to I\!\!P^2$ be the composition of the blow-ups of $I\!\!P^2$ at the points P, Q and R, and let $\hat{C}$ be the proper transform of C at X. Note that $d = \deg C = L \cdot C \geq 6$. As above, the blow up at a smooth point (resp. at a node) of C decreases the difference $C^2 - 2\delta(C)$ by 1 (resp. by 2). Thus, we have

$$\hat{C}^2 - 2\delta(\hat{C}) \geq C^2 - 2\delta(C) - 6 = 2g + 3d - 8\,.$$

Let $\varphi : V \to X$ be a tubular neighborhood of a normalization $\varphi_0 : M_0 \to \hat{C}$ of $\hat{C}$, and let N be the normal bundle of M_0 in V. The line bundle $N - [\hat{P}] - [\hat{Q}]$ on M_0 of degree $\geq 2g + 3d - 10 > 2g + 2$ is spanned (cf. Corollary 2.6). This yields a deformation $C' := \sigma\varphi(M_0') \subset Nod_{d,\,g}$ of C such that L is still tangent to C' at the points P and Q, and meets C' normally at R, so that $b(C') > b(C)$. Maximizing $b(C')$ we get a strong approximation $C' \in Nod_{d,\,g}$ of C with only ordinary flexes and without multiple tangents.

Suppose now that $C \in Nod_{d,\,g}$ has only ordinary flexes and no multiple tangent line, which is an open condition. To find a strong Plücker approximation C' of C, we will proceed by induction on the total number $inf(C')$ of distinct intersection points of C' with all of its inflexional tangent lines. We have to ensure that no inflexional tangent line of C' is a bitangent line.

Let a bitangent line L of C be an inflexional tangent of C at a point $P \in C$ and tangent to C at a point $Q \in C$. Then $d = \deg C = C \cdot L \geq 5$. Blowing up $I\!P^2$ at Q we get a surface $X = \sigma_Q(I\!P^2)$. In the notation as above, we have

$$\deg\,(N - 3[\hat{P}]) \geq 2g + 3d - 7 > 2g + 2\,.$$

Therefore, there exists a deformation $C' = \sigma\varphi(M_0') \in Nod_{d,\,g}$ of C such that L is still an inflexional tangent of C at P, but it meets C normally at Q. Thus, $inf(C') > inf(C)$. By induction, we obtain a strong approximation C' of C which belongs to $PlNod_{d,\,g}$.

Suppose finally that $C \in PlNod_{d,\,g} \setminus Pl\ddot{u}Nod_{d,\,g}$, so that, although all the flexes of C are ordinary, one of them, say $(A,\,P)$, is located at a node of C with the second branch, say, B. This time we proceed by induction on the number $sfl(C')$ of flexes of C' which are smooth points. Evidently, $sfl(C')$ is a bounded lower semi–continuous function on $Nod_{d,\,g}$.

Performing two successive blow ups, the first one at $P \in C$ and the second one at the center of the proper transform of the branch A, we obtain a surface X. Denote by $\hat{Q}$ the center of the proper preimage $\hat{B}$ of the branch B in X. We have

$$\deg N = \hat{C}^2 - 2\delta(\hat{C}) = C^2 - 2\delta(C) - 3 \geq 2g + 1\,,$$

so that the line bundle $N - [\hat{Q}]$ on V is spanned. Hence, we can find a section of N which vanishes at $\hat{Q}$ and does not vanish at $\hat{P}$. This yields a small deformation M_0' of M_0 on V which passes through $\hat{Q}$ but not through $\hat{P}$. The curve $C' := \sigma\varphi(M_0') \subset I\!P^2$ is close enough to C, still has a node at P which is not any more a flex, while L is still a tangent line of C' at P. Note that a small deformation of C yields a small deformation of the Hesse curve H_C of C, so that the flexes of C which are the (normal) intersection points of C and H_C are also perturbed a little. Thus, C' has a flex at a smooth point close to P. It follows that $sfl(C') > sfl(C)$. In a finite number of steps we obtain a desired strong approximation $C' \in Pl\ddot{u}Nod_{d,g}$ of C. This completes the proof. $\square$

The next lemma shows that the strong approximation of immersed curves coincides with the usual one.

2.10. Lemma. *Let $C \in Imm_{d,g}$, and let $\varphi : V \to \mathbb{P}^2$ be a tubular neighborhood of its normalization $\varphi_0 : M_0 \to C$. Then any curve $C' \in Imm_{d,g}$ close enough to C in the Hausdorff topology of $\mathbb{P}^N$ (or, which is the same, coefficientwise) is the image of a unique smooth curve $M \cong C'_{\text{norm}} \subset V$ under the holomorphic mapping $\varphi : V \to \mathbb{P}^2$.*

Proof. Let P be a singular point of C, and let $B_{\epsilon,P}$ be a fixed small neighborhood of P. Denote by $r(C, P)$ the multiplicity of C at P, and by $r(C', P)$ the number of irreducible components in $B_{\epsilon,P}$ of a curve C' close enough to C (cf. Definition 2.8). Once we show that $r(C, P) = r(C', P)$ for any singular point P of C, then the irreducible components of $C' \cap B_{\epsilon,P}$ approximate those of $C \cap B_{\epsilon,P}$, i.e. C' is a strong approximation of C, and the statement follows.

Actually, it is sufficient to prove the equality $r(C, P) = r(C', P)$ under the additional assumption that the approximating curve C' is nodal. Indeed, by Proposition 2.9, the curve $C' \subset Imm_{d,g}$ can be, in turn, strongly approximated by a curve $C'' \in Nod_{d,g}$. Since C'' approximates both C and C' in the Hausdorff topology, from the equalities $r(C'', P) = r(C, P)$ and $r(C'', P) = r(C', P)$ it follows that $r(C', P) = r(C, P)$.

Assuming further that C' is nodal, by (1) and Lemma 2.4(b), we obtain

$$\binom{n-1}{2} - g = \delta(C') = \sum_{P \in \operatorname{Sing} C'} \delta(C' \cap B_{\epsilon,P}) \le \sum_{P \in \operatorname{Sing} C} \delta(C, P) = \binom{n-1}{2} - g\,.$$

Henceforth, $\delta(C' \cap B_{\epsilon,P}) = \delta(C, P)$ for all $P \in \operatorname{Sing} C$. Applying Lemma 2.4($b$) once again, we get that $r(C', P) = r(C, P)$ for all $P \in \operatorname{Sing} C$, as desired. $\square$

2.11. Lemma. *(a) $Imm_{d,g}$ is a locally closed complex analytic submanifold of $\mathbb{P}^N$ of dimension $3d + g - 1$.*

(b) The universal family of curves $\mathcal{S}_{d,g} \to Imm_{d,g}$ over $Imm_{d,g}$ admits a complex analytic simultaneous normalization $f = f_{d,g} : \mathcal{M}_{d,g} \to \mathcal{S}_{d,g}$.

Proof. Fix a curve $C \in Imm_{d,g}$, and consider a tubular neighborhood $\varphi : V \to \mathbb{P}^2$ of a normalization φ_0 of C. By Corollary 2.7 and Lemma 2.10, the projection φ yields a local analytic chart U_C of dimension $3d + g - 1$ on $Imm_{d,g}$ centered at C which covers the whole intersection of $Imm_{d,g}$ with a sufficiently small ball in $\mathbb{P}^N$ around C. This proves (a).

To prove (b) denote by $\mathcal{S}_C$ the restriction of the family $\mathcal{S}_{d,g}$ onto the chart U_C. Note that the same projection φ yields an analytic simultaneous normalization $f_C : \mathcal{M}_C \to \mathcal{S}_C$ of $\mathcal{S}_C$. Any two such normalizations $f_C : \mathcal{M}_C \to \mathcal{S}_C$ and $f'_C : \mathcal{M}'_C \to \mathcal{S}_C$ over the same chart U_C which arise from two different tubular neighborhoods φ, φ', can be naturally biholomorphically identified via their projections. Hence, the equivalence class of these normalizations over the same chart U_C in $Imm_{d,g}$ can be regarded as an equivalence class of charts on a new complex manifold $\mathcal{M}_{d,g}$ of dimension $3d+g$. Indeed, suppose that two charts U_C and $U_{C'}$ on $Imm_{d,g}$ have a non–empty intersection $U_{C,C'} := U_C \cap U_{C'}$. Consider a fibrewise bimeromorphic mapping of smooth manifolds $f_{C,C'} := f_{C'}^{-1} \circ f_C : \mathcal{M}_C \,|\, U_{C,C'} \to \mathcal{M}_{C'} \,|\, U_{C,C'}$. It is biholomorphic at the complement of the 'multiple point locus' $D_{C,C'} := f_C^{-1}(\operatorname{sing} S_{C,C'})$, where $S_{C,C'} := S_C \,|\, U_{C,C'}$, and by Riemann's extension Theorem, it has a holomorphic extension through $D_{C,C'}$. Clearly, the projection $f_{d,g} : \mathcal{M}_{d,g} \to \mathcal{S}_{d,g}$ induced by the local mappings $f_C : \mathcal{M}_C \to \mathcal{S}_C$ is a holomorphic simultaneous normalization, which proves (b). $\square$

Next we show that all the above subvarieties of the Hilbert scheme $\mathbb{P}^N$ are algebraic. Although the following statement holds in much bigger generality[10] (cf. e.g. [BinFl, Theorem 2.2]), it will be enough for us this restricted version which has a rather easy proof.

2.12. Lemma. *Let $f : X \to Y$ be a family of curves over an irreducible base Y. Then there exists a Zariski open subset $U \subset Y$ such that the restriction $f \,|\, f^{-1}(U)$ of f over U admits a simultaneous normalization.*

Proof. Without loss of generality we may suppose Y being smooth. Let $\nu : X_{\text{norm}} \to X$ be a normalization. Consider the induced family of curves $f' := f \circ \nu$. Since the singular locus S of the normal variety X_{norm} has codimension at least 2, its image $f'(S) \subset Y$ has codimension at least 1. Restricting f and f' onto the complement of the Zariski closure $\overline{f'(S)}$ of the constructible subset $f'(S)$ in Y, we may suppose X_{norm} being smooth. By the Bertini–Sard Theorem [Hart, III.10.7], f' is an immersion over a Zariski open subset $U \subset Y$. Therefore, each fibre $(f')^{-1}(y)$, $y \in U$, is smooth, and the restriction $\nu \,|\, (f')^{-1}(y)$ yields a normalization of the curve $X_y := f^{-1}(y)$. Thus, we have obtained the desired simultaneous normalization of the original family f over U. $\square$

We use below the following notation. Given a family of curves $f : X \to Y$, for any $g \ge 0$ denote by $Curv_g(f)$ the subset of points $y \in Y$ such that the fibre X_y over y is a reduced irreducible curve of geometric genus g. For the universal family $f_d : \mathcal{S}_d \to \mathbb{P}^N$ of degree d curves in $\mathbb{P}^2$, set $Curv_{d,g} = Curv_g(f_d)$.

[10] We are grateful to H. Flenner who introduced to us this circle of ideas.

We say that an abstract reduced irreducible curve C is *of immersed type* if its normalization map $\nu : C_{\text{norm}} \to C$ has a nowhere vanishing differential. Let $Imm_g(f)$ be the subset of points $y \in Curv_g(f)$ which correspond to the curves of immersed type, so that, in particular, $Imm_g(f_d) = Imm_{d,g}$.

2.13. Corollary. *(a) Given a family of curves $f : X \to Y$, the base Y can be represented as a disjoint union of smooth irreducible quasi–projective subvarieties $Y_i \subset Y$, $i = 1, \dots, n = n(f)$, such that for each $i = 1, \dots, n$ the restriction of f onto Y_i admits a simultaneous normalization.*

(b) For any $g \geq 0$ the subsets $Curv_g(f) \subset Y$ and $Imm_g(f) \subset Y$ are constructible. In particular, $Curv_{d,g}$ and $Imm_{d,g}$ are constructible subsets of the Hilbert scheme $\mathbb{P}^N$.

Proof. (a) Assuming for simplicity that Y is irreducible we start with $Y_1 := U$, where $U \subset Y$ is as in Lemma 2.12 above. Next we apply Lemma 2.12 to the restriction of f onto each of the irreducible components of the regular part of the Zariski closed subvariety $Y^{(1)} := Y \setminus Y_1$ of Y. Following this way, in a finite number of steps we obtain the desired partition of Y. □

(b) Since $f \,|\, Y_i$ admits a simultaneous normalization, for any $i = 1, \dots, n$ the number and the geometric genera of the irreducible components of a fibre $X_y = f^{-1}(y)$ do not depend on $y \in Y_i$. Thus, $Curv_g(f)$ is a union of some of the Y_i, and hence it is constructible.

Set $X_i = f^{-1}(Y_i)$ and $f_i = f \,|\, X_i$, where $Y_i \subset Curv_g(f)$ is a stratum of the above stratification. Let

$$\begin{array}{ccc} X_i' & \xrightarrow{\ \nu_i\ } & X_i \\ & {\scriptstyle p_i}\searrow \qquad \swarrow & \\ & Y_i & \end{array}$$

be a simultaneous normalization. Denote by $T_{Y_i}X_i' = \operatorname{Ker} dp_i$ the relative tangent bundle of p_i; p_i being a smooth family of curves, $T_{Y_i}X_i'$ is a smooth line bundle on X_i'. Let $D_i \subset X_i'$ be the locus of points where the restriction $d\nu_i \,|\, T_{Y_i}X_i'$ vanishes. Since D_i is Zariski closed its image $p_i(D_i) \subset Y_i$ is a constructible subset of Y_i. Clearly, the complement $Y_i \setminus p_i(D_i)$ coincides with $Imm_g(f) \cap Y_i$. Thus, the latter subset is constructible for all $i = 1, \dots, n$. Hence, $Imm_g(f)$ is constructible, too. □

2.14. *Starting the proof of Theorem 2.1.* (a) directly follows from Lemma 2.11(a) and Corollary 2.13(b). From Corollary 2.13(b) it also follows that the total space $\mathcal{S}_{d,g}$ of the universal family of curves $\mathcal{S}_{d,g} \to Imm_{d,g}$ is a quasi–projective variety. The holomorphic mapping $f = f_{d,g} : \mathcal{M}_{d,g} \to \mathcal{S}_{d,g}$ which realizes an analytic simultaneous normalization is finite and proper (see Lemma 2.11(b)). Therefore, by the Grauert–Remmert Theorem [Ha, B 3.2], $\mathcal{M}_{d,g}$ possesses a structure of a quasi–projective variety, so that f is a finite morphism of quasi–projective varieties. Thus, f yields an algebraic simultaneous normalization of the universal family of curves over $Imm_{d,g}$. This proves (b).

To prove the first part of (c) denote $T = Imm_{d,g}$, $\mathcal{S}_T = \mathcal{S}_{d,g}$, $\mathcal{M}_T = \mathcal{M}_{d,g}$ and $\mathbb{P}^2_T = \mathbb{P}^2 \times T$. There is a natural embedding $i : \mathcal{S}_T \hookrightarrow \mathbb{P}^2_T$. Consider the composition $\varphi := i \circ f : \mathcal{M}_T \to \mathbb{P}^2_T$ and its relative square $\varphi^{(2)} := \varphi^2_T : \mathcal{M}^2_T \to (\mathbb{P}^2_T)^2$, where $\mathcal{M}^2_T := \mathcal{M}_T \times_T \mathcal{M}_T$ and $(\mathbb{P}^2_T)^2 := \mathbb{P}^2_T \times_T \mathbb{P}^2_T$. Let $\mathcal{D}_T \subset \mathcal{M}^2_T$ resp. $D_T \subset (\mathbb{P}^2_T)^2$ be the diagonals. Clearly, $E := (\varphi^{(2)})^{-1}(D_T) \setminus \mathcal{D}_T$ is a closed subvariety of $\mathcal{M}^2_T$, and the restriction $\pi^{(2)} \,|\, E : E \to T$ of the projection $\pi^{(2)} : \mathcal{M}^2_T \to T$ has finite fibres. Its fibre over a point $t \in T = Imm_{d,g}$ corresponds to the multiple point divisor on the normalization M_t of the immersed curve $S_t \subset \mathbb{P}^2$. The restriction $\varphi^{(2)} \,|\, E : E \to D_T$ is a finite morphism. The image $\tilde{E} := \varphi^{(2)}(E)$ is proper over T. Moreover, the fibre $\tau^{-1}(t) \subset \tilde{E}$ over a point $t \in T$ under the restriction to $\tilde{E}$ of the projection $\tau : D_T \to T$ corresponds to the set of singular points of the curve S_t. Therefore, it consists of $\delta = \binom{d-1}{2} - g$ points iff S_t is a nodal curve. By Proposition 2.9, any irreducible component of T contains points which correspond to nodal curves. Thus, the finite morphism $\tau : \tilde{E} \to T$ has degree δ over every such component, and so, the complement $Imm_{d,g} \setminus Nod_{d,g} \subset T = Imm_{d,g}$ coincides with the ramification divisor R_τ of τ. Hence, $Nod_{d,g} \subset Imm_{d,g}$ is, indeed, a Zariski open subvariety. □

Remark. If S_t is a nodal curve with $\delta = \binom{d-1}{2} - g$ nodes, then the fibre $p^{-1}(t)$ of the above projection $p := \pi^{(2)} \,|\, E : E \to T$ consists of 2δ points. The latter holds true if S_t has only ordinary singularities. Hence, the subset $Imm_{d,g} \setminus Ord_{d,g}$ is contained in the ramification divisor $R_p \subset T$ of p.

3 Plücker conditions

It is known [Au] that in general, the subset of the rational Plücker curves is not Zariski open in the space $\mathcal{R}_d$ of all the rational plane curves of a given degree d, although it always contains a Zariski open subset of $\mathcal{R}_d$. Nevertheless, we will show that $PlNod_{d,g}$ is a Zariski open subset of $Imm_{d,g}$ for $d \ge 2g - 1$, which proves Theorem 2.1(c).

3.1. Lemma. *Let $C \subset \mathbb{P}^2$ be an irreducible nodal curve of degree d with the normalization M, and let g^2_d be the linear system on M of all line cuts of C. Then C is a Plücker curve iff g^2_d contains no divisor D of the form*

(i) $D = 4p_1 + \ldots$; *or* (ii) $D = 3p_1 + 2p_2 + \ldots$; *or* (iii) $D = 2p_1 + 2p_2 + 2p_3 + \ldots$,

where $p_i \in M$ are not necessarily distinct.

Proof. The system g^2_d contains no divisor of type (i) iff all the flexes of C are ordinary, i.e. all the singular branches of C^* are ordinary cusps. Under this condition, at most two of the local branches of C^* meet at a point iff g^2_d does not contain any divisor of type (iii). Furthermore, two branches of C^* meet at a point and one of them is singular iff g^2_d contains a divisor D as in (ii). Since C being nodal has no tacnode, C^* has no one, too. Therefore, C is a Plücker curve iff g^2_d does not contain any divisor D as in (i)–(iii). □

Recall the following notion (see e.g. [ACGH]).

3.2. Picard bundles

Let M be a smooth projective curve of genus g. The d–th symmetric power S^dM (which is a smooth manifold) might be regarded as the space of degree d effective divisors on M. Let $J_d(M) = \mathrm{Pic}^d(M)$ be the component of the Picard group $\mathrm{Pic}(M)$ which parametrizes the degree d line bundles on M, and let $\phi_d : S^dM \to J_d(M)$ be the morphism sending a degree d effective divisor on M into its linear equivalence class. Chosing a base point $p_0 \in M$ we may identify $J_d(M)$ with the Jacobian variety $J_0(M)$ and ϕ_d with the d-th Abel–Jacobi mapping. By a theorem of Mattuck [Ma] (see also [ACGH, Ch.IV]) for $d \geq 2g-1$ the morphism ϕ_d is a submersion and moreover, it defines a projective bundle (i.e. a projectivization of an algebraic vector bundle) with the standard fibre $\mathbb{P}^{d-g}$. This bundle is called *the d-th Picard bundle of M.*

Given a smooth family $\pi : \mathcal{M} \to T$ of complete genus g curves and given $d \geq 2g-1$, there is the associated Picard bundle $\Phi_d : S^d\mathcal{M} \to \mathcal{J}_d(\mathcal{M})$ of relative smooth schemes over T. Consider also the associated grassmanian bundle $Grass_{2,\,d-g}(\mathcal{M}) \to \mathcal{J}_d(\mathcal{M})$ which parametrizes the two–dimensional linear series g^2_d of degree d on the fibres $M_t = \pi^{-1}(t)$, $t \in T$.

Let $\pi : \mathcal{M} \to T$, $T = Imm_{d,g}$, be the family constructed in 2.14 above. Then for each $t \in T$ there is the linear series $g^2_d = g^2_d(t)$ on M_t of the line cuts of the plane curve $C_t = f(M_t) \subset \mathbb{P}^2$. This defines a regular section $\sigma : T \to Grass_{2,\,d-g}(\mathcal{M})$.

3.3. *Finishing up the proof of Theorem 2.1(c).* Let $\pi : \mathcal{M} := \mathcal{M}_T \to T$ be the family as in 2.14, and let $\Phi_d : S^d\mathcal{M} \to \mathcal{J}_d(\mathcal{M})$ be the associated Picard bundle. Denote $\mathcal{D}^{(i)}$ resp. $\mathcal{D}^{(ii)}$, $\mathcal{D}^{(iii)}$ the subvariety of $S^d\mathcal{M}$ which consists of the degree d effective divisors on the fibres M_t of π of the form (i) resp. (ii), (iii) of Lemma 3.1. Set $\mathcal{D} = \mathcal{D}^{(i)} \cup \mathcal{D}^{(ii)} \cup \mathcal{D}^{(iii)}$. Note that $\mathcal{D}$ is a closed subvariety of $S^d\mathcal{M}$ of $\mathrm{codim}_{S^d\mathcal{M}}\mathcal{D} \geq 3$ (and moreover, $\mathrm{codim}_{S^dM_t}\mathcal{D}_t \geq 3$ for each $t \in T$). Indeed, to be in $\mathcal{D}_t$ a divisor on M_t must satisfy a system of three independent equations.

Let $\mathcal{Z} \subset Grass_{2,\,d-g}(\mathcal{M}) \times S^d\mathcal{M}$ be the incidence relation. Its fibre $\mathcal{Z}_t$ over a point $t \in T$ consists of all pairs (L, v), where L is a two–plane in $\mathbb{P}^{d-g}_j := \phi_d^{-1}(j)$, $j \in J_d(M_t)$, and $v \in \mathbb{P}^{d-g}_j$ is a point of L. Let $pr_1 : \mathcal{Z} \to Grass_{2,\,d-g}(\mathcal{M})$, $pr_2 : \mathcal{Z} \to$

$S^d\mathcal{M}$ be the canonical projections, and let $\sigma : T \to Grass_{2,\,d-g}(\mathcal{M})$ be the regular section as in (3.2) above. Put $\mathcal{Z}_{\mathcal{D}} := pr_2^{-1}(\mathcal{D}) \subset \mathcal{Z}$, $\hat{\mathcal{D}} := pr_1(\mathcal{Z}_{\mathcal{D}}) \subset Grass_{2,\,d-g}(\mathcal{M})$ and $T' := \sigma^{-1}(\hat{\mathcal{D}}) \subset T$. Since the projection pr_1 is proper, $\hat{\mathcal{D}} \subset Grass_{2,\,d-g}(\mathcal{M})$, and therefore also $T' \subset T$ are closed subvarieties of the corresponding varieties. Clearly, $t \in T'$ iff the linear series $g_d^2(t) = \sigma(t)$ on M_t contains a divisor from $\mathcal{D}_t$.

Recall that $Nod_{d,\,g} = T \setminus R_\tau$, where R_τ is the ramification divisor as in (2.14). By Lemma 3.1, we have that $PlNod_{d,\,g} = T \setminus (R_\tau \cup T')$. By Proposition 2.9, any irreducible component I of $T = Imm_{d,\,g}$ contains a Plücker curve. Thus, $T' \cap I$ is a proper subvariety of I; in particular, $\mathrm{codim}_T T' \geq 1$. Hence, $PlNod_{d,\,g}$ is, indeed, a Zariski open subset of $T = Imm_{d,\,g}$. This completes the proof of Theorem 2.1. □

Theorem 2.1 implies

3.4. Corollary. *Any irreducible plane curve C^* of genus g and degree $n = 2(d+g-1)$, where $d \geq 2g-1$, whose dual C is an immersed curve, is a specialization of generic maximal cuspidal Plücker curves C'^* of the same degree and genus*[11]*. Hence, there is an epimorphism $\pi_1(\mathbb{P}^{2*} \setminus C^*) \to \pi_1(\mathbb{P}^{2*} \setminus C'^*)$. In particular, the former group is big (resp. non–amenable, non–almost solvable, non–almost nilpotent) if the latter one is so.*

Proof. By the class formula [Na, 1.5.4], the dual of an irreducible immersed plane curve of degree d and genus g has degree $d^* = 2(g+d-1)$. By Theorem 2.1(a) and (b), there is the diagram

$$\begin{array}{ccc} & \mathcal{M}_T & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f^*} \\ \mathbb{P}^2_T & \longleftrightarrow & \mathbb{P}^{2^*}_T \end{array}$$

where the morphism f^* yields a simultaneous normalization of the dual family, so that for each $t \in T = Imm_{d,\,g}$ the image $f^*(M_t) = S_t^*$ is the dual curve of the curve $S_t = f(M_t)$ (see e.g. [Na, 1.5.1]). By (c), the subset $PlNod_{d,\,g} \subset T$ is Zariski open. The dual S_t^*, where $t \in PlNod_{d,\,g}$, is a maximal cuspidal curve of degree d and genus g. Vice versa, any such curve is the dual S_t^* of a nodal Plücker curve S_t, $t \in PlNod_{d,\,g}$. This yields the first assertion. The second one follows from a well known theorem of Zariski (see [Zar, p.131, Thm.5] or [Di, 4.3.2]). As for the third one, see (1.1) above. □

[11] i.e. C'^* has the maximal number of cusps allowed by Plücker's formulas.

4 Proof of Theorem 0.2.b)

The following lemma is a particular case of the Varchenko Equisingularity Theorem [Va, Theorem 5.3].

4.1. Lemma. *Let $p : E \to B$ be a surjective morphism, where E, B are smooth connected quasi–projective varieties. Then there exist a proper subvariety $A \subset B$ such that the restriction $p \,|\, (E \setminus H)$, where $H = p^{-1}(A)$, determines a smooth locally trivial fibre bundle $p : E \setminus H \to B \setminus A$.*

Let Δ be a hypersurface in a complex manifold E, $e \in \mathrm{reg}\,\Delta$ be a smooth point of Δ, and ω be a small disc in E centered at e and transversal to Δ. By *a vanishing loop* of Δ at e we mean a loop δ in $E \setminus \Delta$ consisting of a path α which joins a base point $e_0 \in E \setminus \Delta$ with a point $e' \in \omega \setminus \Delta$ and a loop β in $\omega \setminus \Delta$ with the base point e' (i.e. e is in the interior of β in ω).

The next simple lemma is well known; for the sake of completeness we give its proof.

4.2. Lemma. *Let, as before, Δ be a hypersurface in a complex manifold E, and let $\gamma_0, \gamma_1 : S^1 \to E \setminus \Delta$ be two loops with the base point $e_0 \in E \setminus \Delta$ joined in E by a smooth homotopy $\gamma : S^1 \times [0, 1] \to E$ transversal to Δ, such that the image $S = Im\,\gamma$ meets Δ at the points $e_1, \ldots, e_k \in \mathrm{reg}\,\Delta$. Then γ_0 is homotopic in $E \setminus \Delta$ to a product $\gamma_1 \delta_{i_1} \ldots \delta_{i_k}$, where $(i_1, \ldots, i_k)$ is a permutation of $(1, \ldots, k)$ and δ_i is a vanishing loop of Δ at the point e_i, $i = 1, \ldots, k$.*

Proof. Slightly modifying the original homotopy and changing the numeration of the intersection points $e_1, \ldots, e_n \in \mathrm{reg}\,\Delta$ we may assume that $e_i \in \gamma_{t_i} \cap \Delta$, $i = 1, \ldots, k$, correspond to different values $0 < t_1 < \ldots < t_n < 1$ of the parameter of homotopy $t \in [0,1]$. If $s_i \in [0, 1]$, $0 < s_1 < t_1 < \ldots < t_n < s_{n+1} < 1$, and $\bar\gamma_i = \gamma_{s_i} : S^1 \to E \setminus \Delta$, $i = 1, \ldots, n+1$, then clearly $\bar\gamma_{i+1}^{-1} \cdot \bar\gamma_i \approx \delta_i$, i.e. $\bar\gamma_i \approx \bar\gamma_{i+1} \cdot \delta_i$ in $E \setminus \Delta$, where δ_i is a vanishing loop of Δ at the point e_i, and $\bar\gamma_1 \approx \gamma_0$, $\bar\gamma_{n+1} \approx \gamma_1$. Thus, $\gamma_0 \approx \gamma_1 \delta_n \cdot \ldots \cdot \delta_1$ in $E \setminus \Delta$, and the lemma follows. □

In the proof of Theorem 0.2.b) below we use the following

4.3. Proposition. *Let a morphism $p : E \to B$ of smooth quasiprojective varieties be a smooth fibration over B with a connected generic fibre F of positive dimension. Let $\Delta \subset E$ be a Zariski closed hypersurface which contains no entire fibre of p, i.e. $p^{-1}(b) \not\subset \Delta$ for each $b \in B$. Then we have the following exact sequence:*

$$\pi_1(F \setminus \Delta) \stackrel{i_*}{\to} \pi_1(E \setminus \Delta) \stackrel{p_*}{\to} \pi_1(B) \to \mathbf{1}\,.$$

Proof. By Lemma 4.1, there exist hypersurfaces $A \subset B$ and $D = H \cup \Delta \subset E$, where $H := f^{-1}(A)$, such that $p\,|\,(E \setminus D) \, : \, E \setminus D \to B \setminus A$ is a smooth fibration. In particular, $p\,|\,(E \setminus D)$ induces an epimorphism of the fundamental groups. Since the same is also true for the embedding $i \, : \, B \setminus A \hookrightarrow B$, and since $p_* = i_* \circ (p\,|\,E \setminus D)_*$, the exactness at the third term follows. It remains to prove that the homomorphism

$$i_* \, : \, \pi_1(F \setminus \Delta) \to \operatorname{Ker} p_* \subset \pi_1(E \setminus \Delta)$$

is surjective.

Fix a generic fibre $F \not\subset D$ and base points $e_0 \in F \setminus D$ and $b_0 = p(e_0) \in B \setminus A$. Let a class $[\gamma_0] \in \operatorname{Ker} p_*$ be represented by a loop $\gamma_0 \, : \, S^1 \to E \setminus \Delta$ with the base point e_0. We will show that γ_0 is homotopic in $E \setminus \Delta$ to a loop $\gamma_0' \, : \, S^1 \to F \setminus \Delta$ with the same base point.

The loop $\bar{\gamma}_0 := p \circ \gamma_0 \, : \, S^1 \to B$ with the base point $b_0 \in B$ is contractible. Let $\bar{\gamma} \, : \, S^1 \times [0;\, 1] \to B$ be a contraction to the constant loop $\bar{\gamma}_1 \equiv b_0$. Since $p \, : \, E \to B$ is a fibration, there exists a covering homotopy $\gamma \, : \, S^1 \times [0;\, 1] \to E$. Thus, we have $\bar{\gamma} = p \circ \gamma$ and $\gamma_1 \, : \, S^1 \to F$.

Fix a stratification of $D = \Delta \cup H$ which satisfies the Whitney condition A and contains the regular part $\operatorname{reg} D$ of D as an open stratum. By Thom's Transversality Theorem, the homotopy γ can be chosen being transversal to the strata of this stratification, and therefore such that its image meets the divisor D only in a finite number of its regular points. Let it meet Δ at the points $e_1, \ldots, e_k \in \operatorname{reg}(\Delta \setminus H)$. We may also assume that the loop $\gamma_1 \, : \, S^1 \to F$ does not meet D; in particular, $[\gamma_1] \in \pi_1(F \setminus \Delta;\, e_0)$. By Lemma 4.2, γ_0 is homotopic in $E \setminus \Delta$ to the product $\gamma_1 \delta_{i_1} \ldots \delta_{i_k}$, where δ_i is a vanishing loop of Δ at the point e_i, $i = 1, \ldots, k$.

Note that all the transversal discs to Δ in E centered at e_i are homotopic (via the family of such discs). Hence, all the simple positive local vanishing loops of Δ at e_i are freely homotopic in $E \setminus \Delta$. Therefore, performing further deformation of the vanishing loops δ_i, $i = 1, \ldots, k$, and taking into account our assumptions that $\dim F > 0$ and Δ does not contain entirely a fibre of p, we may suppose that

(i) for each $i = 1, \ldots, k$ the fibre of p through the point e_i is transversal to Δ;

(ii) the loops δ_i, $i = 1, \ldots, k$, do not meet H, and the corresponding local loops β_i, $i = 1, \ldots, k$, are contained in the fibres of p.

Since $p^{-1}(A) \subset D$, we have that for each $i = 1, \ldots, k$ the projection $\bar{\delta}_i := p \circ \delta_i$ of the loop δ_i does not meet the hypersurface $A \subset B$. By the construction, the loops $\bar{\delta}_i$, $i = 1, \ldots, k$, are contractible in $B \setminus A$. Applying the covering homotopy theorem to the smooth fibration $p \, : \, E \setminus D \to B \setminus A$ we may conclude that for each $i = 1, \ldots, k$, the loop δ_i is homotopic in $E \setminus D \subset E \setminus \Delta$ to a loop $\delta_i' \, : \, S^1 \to F \setminus D \subset F \setminus \Delta$. Hence, γ_0 is homotopic in $E \setminus \Delta$ to the product $\gamma_0' := \gamma_1 \delta_{i_1}' \ldots \delta_{i_k}'$, $\gamma_0' \, : \, S^1 \to F \setminus \Delta$, and we are done. □

4.4. Duality, discriminants and the Zariski embedding

The following construction was used, for instance, in [Zar, pp.307, 326] and in [DoLib, sect.1, 3]. Let M be an irreducible smooth projective variety, and let $L \subset H^0(M, \mathcal{L})$ be a linear system of effective divisors on M, where $\mathcal{L}$ is a linear bundle on M. It defines a rational mapping $\Phi_L : M \to \mathbb{P}(L^*)$. If $K \subset L$ is a linear subsystem, then the mapping $\Phi_K : M \to \mathbb{P}(K^*)$ is composed of the mapping Φ_L followed by the linear projection $\pi_{L,K} : \mathbb{P}(L^*) \to \mathbb{P}(K^*)$ which is dual to the tautological embedding $\rho_{K,L} : \mathbb{P}(K) \hookrightarrow \mathbb{P}(L)$.

Set $C_L := \Phi_L(M) \subset \mathbb{P}(L^*)$ and $C_K := \Phi_K(M) \subset \mathbb{P}(K^*)$, so that $C_K = \pi_{L,K}(C_L)$. The dual variety $\Delta_L \subset \mathbb{P}(L)$ of $C_L \subset \mathbb{P}(L^*)$ is usually a hypersurface, which is called *the discriminant hypersurface of the linear system* L. The embedding $\rho_{K,L}$ yields the embedding of the discriminants $\Delta_K = \mathbb{P}(K) \cap \Delta_L \hookrightarrow \Delta_L$.

In particular, starting with a degree d irreducible plane curve $C \subset \mathbb{P}^2$ with a normalization $M \to C$, denote by $K = g^2_d$ the linear system on M of line cuts of C and by $L = |g^2_d|$ the corresponding complete linear system. Since g^2_d and therefore, also L are base point free, they define morphisms $\Phi_K : M \to C \subset \mathbb{P}^2 = \mathbb{P}(K^*)$ resp. $\Phi_L : M \to C_L := \Phi_L(M) \hookrightarrow \mathbb{P}(L^*)$, and $C \subset \mathbb{P}^2$ is a projection of the curve $C_L \subset \mathbb{P}(L^*)$. Set $\mathbb{P}^2_C := \mathbb{P}(K) \hookrightarrow \mathbb{P}(L)$. The discriminant $\Delta_L = C_L^*$ is, indeed, a projective hypersurface, and the dual curve $C^* \subset \mathbb{P}^2_C$ is an irreducible component of the plane cut $\Delta_K = \mathbb{P}(K) \cap \Delta_L$. The other irreducible components of Δ_K are special tangent lines of C^* dual to the cusps of C (by *a cusp* we mean here a singular point of a local irreducible analytic branch of C). We call these tangent lines *artifacts* [DeZa]. Thus, the plane cut $\mathbb{P}(K) \cap \Delta_L$ of the discriminant hypersurface Δ_L is irreducible iff C is an immersed curve.

The embedding $\mathbb{P}^{2*} \cong \mathbb{P}^2_C \hookrightarrow \mathbb{P}(L)$ which represents C^* as a plane cut of the discriminant hypersurface Δ_L is called *the Zariski embedding* (see [Zar, pp.307, 326; DeZa]).

By definition, the dual variety $C_L^* = \Delta_L$ consists of the points $x \in \mathbb{P}(L)$ such that the dual hyperplane $x^* \subset \mathbb{P}_L$ cuts out of C_L a non-reduced divisor on the normalization M of C_L. If $x \in C_L$ is a cusp, then, clearly, the dual hyperplane x^* is an irreducible component of Δ_L. Thus, the discriminant Δ_L is irreducible iff C_L was an immersed curve. In particular, this is the case if $C = \pi_{L,K}(C_L)$ is an immersed curve. Vice versa, if C_L is an immersed curve, then the same is true for its generic projection onto the plane. Or, what is the same, if the discriminant Δ_L is irreducible, then its generic plane section is irreducible, too.

The projectivization $\mathbb{P}(L)$ of the complete linear system L of degree d divisors on M coincides with a fibre of the Abel–Jacobi map $\phi_d : S^d M \to J_d(M)$ (see (3.2)), so that $\mathbb{P}^2_C$ is a plane in this fibre. We still call the morphism $\mathbb{P}^2_C \hookrightarrow S^d M$ *the Zariski embedding.* The hypersurface $\Delta_d \subset S^d M$ which consists of the non-reduced degree d effective divisors on M is also called *the discriminant hypersurface*. It is the image of the diagonal hypersurfaces of the direct product M^d via the Vieta map $M^d \to S^dM$. Thus, $\Delta_L = I\!P(L) \cap \Delta_d$, where $I\!P(L)$ has been identified with a fibre $F_j := \phi_d^{-1}(j)$, $j \in J_d(M)$, of ϕ_d.

4.5. *Proof of Theorem 0.2.b).* By Corollary 3.4, we may suppose that C is a generic nodal Plücker curve of degree d and geometric genus g, where $d \geq 2g-1$. By Mattuck's Theorem (see (3.2)), the d-th Picard bundle $\phi_d : S^dM \to J_d(M)$, where M is a normalization of C, is a projective bundle with a generic fibre $F \cong I\!P^{d-g}$. By (4.4), the dual curve C^* can be identified with the plane cut of the discriminant hypersurface $\Delta_d \subset S^dM$ by the plane $I\!P^2_C$ via its Zariski embedding $I\!P^2_C \hookrightarrow F_0 := I\!P(L) \subset S^dM$, where $L = |g^2_d|$ and g^2_d is the linear system on M of line cuts of C.

If the group $\pi_1(I\!P^2_C \setminus \Delta_d)$ is big for a generic plane $I\!P^2_C \subset F_0 \cong I\!P^{d-g}$, then by Zariski's Lefschetz type Theorem [Zar, p.279; Di, 4.1.17], it is big for any such plane, so that $I\!P^2_C \subset F_0$ might be assumed being generic. Indeed, a section S of the discriminant hypersurface $\Delta_L = F_0 \cap \Delta_d$ by a generic plane $I\!P \subset F_0$ is an irreducible curve with the same normalization M and with the dual $S^* \subset I\!P^2$ an immersed curve of degree d and genus g. Thus, we may start with $C = S^*$ and obtain $C^* = S = I\!P^2_C \cap \Delta_d$. Note that such a generic linear system $K = g^2_d \subset L$, where $I\!P = I\!P(K)$, defines a morphism $M \to I\!P^2$ such that its image coincides with $C = S^*$. Since by Theorem 2.1(c), $PlNod_{d,g}$ is a Zariski open subset of $Imm_{d,g}$, the curve $C = S^*$ obtained in this way is a nodal Plücker one.

Applying once again the Zariski Lefschetz type Theorem we get an isomorphism

$$\pi_1(F_0 \setminus \Delta_d) \cong \pi_1(I\!P^2_C \setminus \Delta_d) \cong \pi_1(I\!P^2 \setminus C^*)\,.$$

By Proposition 4.3, we have the exact sequence

$$\pi_1(F_0 \setminus \Delta_d) \to \pi_1(S^dM \setminus \Delta_d) \to \pi_1(J_d(M)) \cong Z\!\!Z^{2g} \to \mathbf{1}\,.$$

It follows that $\pi_1(I\!P^2 \setminus C^*)$ is a big group if $\pi_1(S^dM \setminus \Delta_d)$ is big (cf. (1.1)). But $\pi_1(S^dM \setminus \Delta_d)$ is the braid group $B_{d,g}$ of M with d strings which is big (see Lemma 1.2(a)). This completes the proof. $\square$

Remark. A presentation of the group $\pi_1(I\!P^2 \setminus C)$ for a generic maximal cuspidal curve $C \subset I\!P^2$ of genus 0 or 1 was found by Zariski [Zar, p. 307]; see also [Ka] for $g \leq \frac{d-1}{2}$, where $d = \deg C^*$ [12].

[12] It is stated in [DoLib] that for $d \geq 2g-1$ the d–th Abel–Jacobi mapping $\phi_d : S^dM \to J(M)$ restricted to the complement of the discriminant hypersurface $\Delta_d \subset S^dM$ is a Serre fibration, so that the long exact homotopy sequence is available. But the indication given in [DoLib] does not seem to be sufficient for the proof. The result of [Ka] is based on this statement.

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