On the Kernel of the Casson Invariant

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Abstract

We prove: Any integral homology sphere with Casson invariant zero can be obtained from S^3 by surgery on a boundary link each component of which has a trivial Alexander polynomial.

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1 Introduction

We prove the following result (stated with standard definitions and notations recalled in Section 2):

Theorem 1.1 *If H_1 and H are two integral homology spheres which have the same Casson invariant, then H can be obtained from H_1 by surgery on a framed boundary link $(K_i, \varepsilon_i \in \{1,-1\})_{i=1,\dots,n}$ such that for any $i \in \{1,\dots,n\}$, the Alexander polynomial $\Delta(K_i)$ of K_i is 1.*

(Equivalently, H can be obtained from H_1 by a sequence of (± 1)-surgeries on knots with trivial Alexander polynomial.)

I thank Andrew Casson for telling me that he thought that this result should be true. This statement is his.

I also thank Lucien Guillou and Alexis Marin for their accurate remarks.

2 Background

In this section, we introduce all our notations and conventions and we point out all the standard facts that we will use throughout the paper.

Here, all the manifolds are compact and oriented. The homology is always with coefficients in $\mathbf{Z}$; and when it does not seem to cause confusion, the curves are denoted like their homology classes. An *integral homology sphere* or *homology sphere* is a 3-manifold with the same (integral) homology as the usual sphere S^3. In such a manifold, every knot K bounds a (compact, oriented) connected embedded surface, which is oriented according to the 'outward normal first convention'. Such a surface is called a *Seifert surface* of K. The *linking number* $lk_H(J,K)$ of two disjoint knots J and K in a homology sphere H is the algebraic intersection number of K and of a Seifert surface of J. It is symmetric. For a surface Σ, $< .,. >_\Sigma$ denotes the symplectic intersection form on $H_1(\Sigma)$.

Definition 2.1 The *Seifert form* V_Σ of a Seifert surface Σ of a knot in a homology sphere is the bilinear form defined on $H_1(\Sigma)$ by:

For any two curves x and y of Σ

$$V_\Sigma([x],[y]) = lk(x^+, y)$$

where the brackets stand for the homology classes and x^+ denotes the curve x pushed off Σ in the direction of the positive normal to Σ. We also denote by V_Σ the matrix of V_Σ with respect to some basis of $H_1(\Sigma)$, and by V_Σ^T its transposed.

The Seifert form V_Σ may be used to define the following knot invariant.

Definition 2.2 Let K be a knot in a homology sphere and let Σ be a Seifert surface of K. The *Alexander polynomial* $\Delta(K)$ of K is the determinant of $(t^{1/2}V_\Sigma - t^{-1/2}V_\Sigma^T)$. It is a well-defined invariant of K which belongs to $\mathbf{Z}[t,t^{-1}]$. (See [G] or [L, Appendix] for example.)

Definition 2.3 Let H be a homology sphere. A (± 1)-*framed link* of H is a link $L = (K_i)_{i\in\{1,\dots,n\}}$ each component of which is equipped by an integer $\varepsilon_i \in \{-1,+1\}$.

The manifold $\chi_H(\mathbf{L}) = \chi(H;\mathbf{L})$ obtained by *surgery* on this framed link $\mathbf{L} = (K_i,\varepsilon_i)_{i\in\{1,\dots,n\}}$ is defined as follows.

Let $T(K_i)$ denote a tubular neighborhood of K_i and let $\partial T(K_i)$ denote its boundary. Let $\ell_i \subset \partial T(K_i)$ denote the *preferred parallel* of K_i, that is the parallel which satisfies $lk(\ell_i, K_i) = 0$. Let $m_i \subset \partial T(K_i)$ denote the *oriented meridian* of K_i, that is the meridian such that $lk(m_i, K_i) = 1$. Let μ_i be the curve of $\partial T(K_i)$ such that

$$\mu_i = m_i + \varepsilon_i \ell_i \tag{2.4}$$

in $H_1(\partial T(K_i))$. Then

$$\chi_H(\mathbf{L}) = \overline{H \setminus T(L)} \cup_{\partial T(L)} \coprod_{i=1}^{n} D_i \times S^1$$

where $T(L)$ is the union of the $T(K_i)$, D_i is a 2-disk, and $\partial(D_i \times S^1)$ is glued with $\partial T(K_i)$ by a homeomorphism which maps $\partial(D_i \times \{1\})$ to μ_i.

We denote by $\hat{K}_i$ the *core* of the surgery performed on K_i that is the core $\{0\} \times S^1$ of the solid torus $D_i \times S^1$. $\hat{K}_i$ is oriented so that μ_i is its oriented meridian in $\chi_H(\mathbf{L})$ which inherits its orientation from H.

The link L is said to be a *boundary link* if its components bound pairwise disjoint Seifert surfaces.

We call ε-*surgery* a surgery on a knot with coefficient ε.

In this paper, we will often consider surgeries on (± 1)-framed boundary links. We first point out some standard facts about these links.

Until Remark 2.8, we retain the notation from Definition 2.3 and we further assume that $\mathbf{L}$ is a (± 1)-framed boundary link.

Remark 2.5 *Any Seifert surface of $(K_i \subset H)$ disjoint of $L \setminus K_i$ may also be considered as a Seifert surface of $(\hat{K}_i \subset \chi_H(\mathbf{L}))$.* Indeed an obvious small isotopy supported near $T(K_i)$ moves such a surface to a Seifert surface of the preferred parallel ℓ_i of K_i embedded in $\overline{H \setminus T(L)} = \overline{\chi_H(\mathbf{L}) \setminus T(\hat{L})}$. Now, since $\ell_i \subset \partial T(K_i) = \partial T(\hat{K}_i)$ is also the preferred parallel of $\hat{K}_i$, this surface may be considered as a Seifert surface of $\hat{K}_i$.

It is easy to see that $\chi_H(\mathbf{L})$ is a homology sphere. Note also that if K is a knot disjoint of L such that $lk_H(K, K_i) = 0$ for all i, and if J is a knot disjoint of $L \cup K$,

$$lk_H(J,K) = lk_{\chi(H;\mathbf{L})}(J,K)$$

(Indeed, in this case, K bounds a Seifert surface disjoint of L.) Therefore, from Remark 2.5, we get:

Remark 2.6 For any $i \in \{1, \ldots, n\}$, $(K_i \subset H)$ and $(\hat{K}_i \subset \chi_H(\mathbf{L}))$ bound Seifert surfaces which carry the same Seifert form. Hence,

$$\Delta(K_i \subset H) = \Delta(\hat{K}_i \subset \chi_H(\mathbf{L}))$$

Remark 2.7 Similarly, if K is a knot of H such that $L \cup K$ is a boundary link, it can also be viewed as a knot in $\chi_H(\mathbf{L})$ and we have:

$$\Delta(K \subset H) = \Delta(K \subset \chi_H(\mathbf{L}))$$

Remark 2.8 The surgery on the framed link

$$\hat{\mathbf{L}} = (\hat{K}_i, -\varepsilon_i)_{i \in \{1,\ldots,n\}} \subset \chi_H(\mathbf{L})$$

transforms $\chi_H(\mathbf{L})$ into H. (See Equation 2.4 and Remark 2.5.) It is called the *inverse* of the surgery on $\mathbf{L}$.

Lemma 2.9 *Let $\mathcal{S}$ be a sequence of (± 1)-surgeries on knots from a homology sphere H_1 to another one H_{n+1}, that is a sequence $(H_i, K_i, \varepsilon_i)_{i=1,\ldots,n}$ such that:*

- *H_i is a homology sphere,*
- *K_i is a knot of H_i which bounds a Seifert surface Σ_i in H_i*
- *$\varepsilon_i = \pm 1$*
- *$H_{i+1} = \chi(H_i; (K_i, \varepsilon_i))$, for $i = 1, \ldots, n$*

Such a sequence $\mathcal{S}$ is equivalent to a surgery on a (± 1)-framed boundary link $\mathbf{L} = (\tilde{K}_i = \partial\tilde{\Sigma}_i, \varepsilon_i)_{i=1,\ldots,n} \subset H_1$ such that:

- *The $\tilde{\Sigma}_i$ are pairwise disjoint Seifert surfaces in H_1,*
- *With the notation $\tilde{H}_i = \chi(H_1; \cup_{j=1}^{i-1}(\tilde{K}_j, \varepsilon_j))$, the pair (H_i, Σ_i) is homeomorphic to the pair $(\tilde{H}_i, \tilde{\Sigma}_i)$ for $i = 1, \ldots, n$, and $\tilde{H}_{n+1}$ is homeomorphic to H_{n+1} (by orientation-preserving homeomorphisms).*

  In particular, $\tilde{\Sigma}_i$ and Σ_i have the same genus; and $(K_i \subset H_i)$ and $(\tilde{K}_i \subset H_1)$ have the same Alexander polynomial.

Of course, two sequences of surgeries starting from a homology sphere H_1 are said to be *equivalent* if they transform H_1 into the same manifold.

PROOF OF LEMMA 2.9: Proceed by induction on n. There is nothing to say if $n = 1$. According to the induction hypothesis, there are $(n-1)$ disjoint Seifert surfaces $\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_{n-1}$ in H_1 such that (with the notations of the statement) $(H_i, \Sigma_i) \cong (\tilde{H}_i, \tilde{\Sigma}_i)$ for $i = 1, \ldots, n-1$, and,

$$H_n \cong \chi(H_1; \cup_{j=1}^{n-1}(\tilde{K}_j = \partial\tilde{\Sigma}_j, \varepsilon_j))$$

The surfaces $\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_{n-1}$ may be seen in H_n where they are still disjoint. (See Remark 2.5.) Perform an isotopy of H_n to move Σ_n to a surface $\tilde{\Sigma}_n$ which is disjoint of them. (This is possible because all these Seifert surfaces are nothing but regular neighborhoods of wedges of circles. See Figure 5.) Of course, this isotopy has changed neither the homeomorphism type of (H_n, Σ_n) nor the homeomorphism type of $\chi(H_n; \partial\Sigma_n, \varepsilon_n)$. Now, $\tilde{\Sigma}_n$ may be seen in H_1 and it is easy to see that the $\tilde{\Sigma}_i$ satisfy the required properties.

□

The following fact is well-known (see [G-M, Lemme 2.1, p.238]):

Fact 2.10 *Any two homology spheres can be obtained from each other by a sequence of* (± 1)*-surgeries on knots.*

Recall now the following theorem (see [G-M] or [A-M]):

Theorem 2.11 (Casson, 1985) *There exists a unique integral topological invariant* λ *of oriented homology spheres such that*

1. $\lambda(S^3) = 0$
2. *For any knot* K *in a homology sphere* H*, for any* $\varepsilon = \pm 1$*,*

$$\lambda(\chi_H(K, \varepsilon)) = \lambda(H) + \frac{\varepsilon}{2}\Delta(K)''(1) \tag{2.12}$$

Because of this Casson surgery formula the knot invariant $\Delta(K)''(1)/2$ is called the *Casson Invariant of* K and is denoted by $\lambda'(K)$.

3 Sketch of the proof

We will first prove the following proposition:

Proposition 3.1 *Let* H_1 *and* H *be two homology spheres. Then* H *can be obtained from* H_1 *by surgery on a* (± 1)*-framed boundary link* $\mathbf{L} = (K_i = \partial\Sigma_i, \varepsilon_i)_{i=1,\ldots,n}$ *such that:*

1. *The Seifert surfaces* Σ_i *of the* K_i *are pairwise disjoint, and,*
2. *For any* $i \in \{1, \ldots, n\}$*,*
    - *either* $\Delta(K_i) = 1$*,*
    - *or the genus of* Σ_i *is one and* $\lambda'(K_i) = -1$*.*

Then we will eliminate the knots with non trivial Alexander polynomial from this statement, when their contributions to the Casson invariant cancel each other. This will be possible because of the following lemma:

Lemma 3.2 *Let H be a homology sphere. Let*

$$\mathbf{L} = ((K_1 = \partial\Sigma_1, 1), (K_2 = \partial\Sigma_2, -1))$$

be a framed link in H such that Σ_1 and Σ_2 are two disjoint genus one Seifert surfaces, and $\lambda'(K_1) = \lambda'(K_2) = -1$. Then the surgery on $\mathbf{L}$ is equivalent to a sequence of (± 1)-surgeries on knots with Alexander polynomial 1.

Proof of the theorem assuming Proposition 3.1 and Lemma 3.2: We obtain H from H_1 by performing the surgeries prescribed by the framed link $\mathbf{L}$ given by Proposition 3.1 one by one in any order. By Remark 2.7, the Alexander polynomial of a component K_i of $\mathbf{L}$ is the same in H and in any manifold obtained from H by surgery on a sublink of $\mathbf{L}$. Thus, we can obtain H from H_1 by a sequence of surgeries on knots with Alexander polynomial 1 followed by a sequence of surgeries on genus one knots for which $\lambda' = -1$. Since $\lambda(H_1) = \lambda(H)$ and because of the Casson surgery formula, we can perform the surgeries of the latter sequence two by two, and view this latter sequence as a sequence of surgeries on two component boundary links satisfying the hypotheses of Lemma 3.2. Now, this lemma makes clear that the whole sequence of surgeries transforming H_1 into H can be replaced by a sequence of surgeries on knots with Alexander polynomial 1; and by Lemma 2.9, we can replace the obtained sequence of surgeries by a single surgery on a boundary link each component of which has Alexander polynomial 1.

□

4 Proof of Proposition 3.1

With the help of Lemma 2.9, the proposition is the consequence of the two following lemmas:

Lemma 4.1 *Any two integral homology spheres can be obtained one from the other by a sequence of (± 1)-surgeries on knots such that every knot of the sequence has a trivial Alexander polynomial or bounds a genus one Seifert surface.*

Lemma 4.2 *A (± 1)-surgery on a genus one knot is equivalent to a sequence of (± 1)-surgeries on genus one knots with Alexander polynomial 1 or Casson invariant (-1).*

In order to prove them, we recall some standard facts about changes of crossings and Seifert surfaces.

4.1 About changes of crossings and Seifert surfaces

Definition 4.3 A positive (respectively negative) *change of crossings* of a knot K in a homology sphere is the effect on K of a positive twist t (respectively of

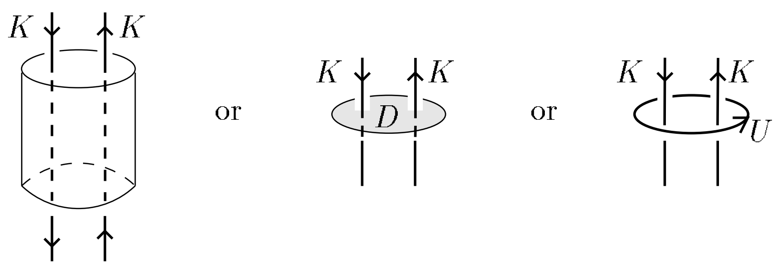

Figure 1: K and the cylinder

a negative twist t^{-1}) of a solid cylinder intersecting K as in Figure 1 around its axis. See Figure 2. (K is unchanged outside the cylinder.)

We call *disk of the change of crossings* the base D of the solid cylinder. The change of crossings is said to be *surrounded by the unknot* U which is the boundary of D. (See Figure 1.)

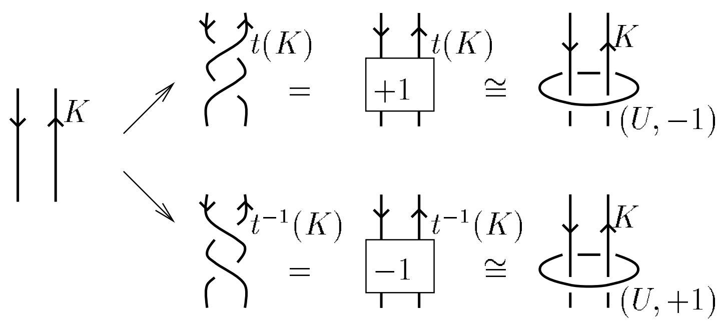

Figure 2: Effect of the two kinds of changes of crossings surrounded by U on K.

(Note the symbols that we will use to avoid drawing the results of twists of cylinders).

Of course, this definition of change of crossings is equivalent to the standard definition where a change of crossings transforms a knot K which intersects a 3-ball along two strands as in Figure 3 by making the two strands pass through each other.

Notice the following homeomorphism of pairs:

$$(\chi_H(U, \varepsilon \in \{-1, +1\}), K) \cong (H, t^{-\varepsilon}(K)) \tag{4.4}$$

Note also that U bounds a genus one surface in $H \setminus K$, namely the surface obtained by *tubing* the disk D, that is by replacing two disks neighborhoods of $K \cap D$ in D by a tube around a connected component of $K \setminus (K \cap D)$. See Figure 4.

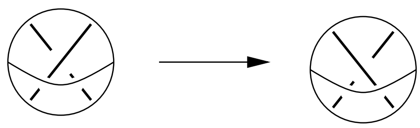

Figure 3: Effect of a change of crossings.

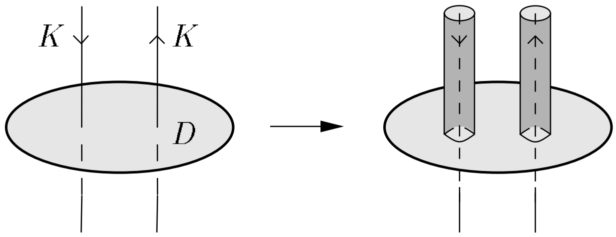

Figure 4: Tubing D.

Because of the well-known transitivity of the action of the group of homeomorphisms (up to isotopy) of a Seifert surface Σ on the symplectic bases of $H_1(\Sigma;\mathbf{Z})$, we have the following standard fact:

Fact 4.5 *Let Σ be a Seifert surface of K in H. For any symplectic basis $\mathcal{B} = (x_1, y_1, x_2, y_2, \ldots, x_g, y_g)$ of $(H_1(\Sigma), < .,. >_\Sigma)$, Σ is isotopic to a neighborhood of representatives of the x_i and the y_i of the form shown in Figure 5. The surface is represented as 2g one-handles $h_{x_1}, h_{y_1}, \ldots, h_{x_g}, h_{y_g}$ with respective cores $x_1, y_1, \ldots, x_g, y_g$ attached to a disk.*

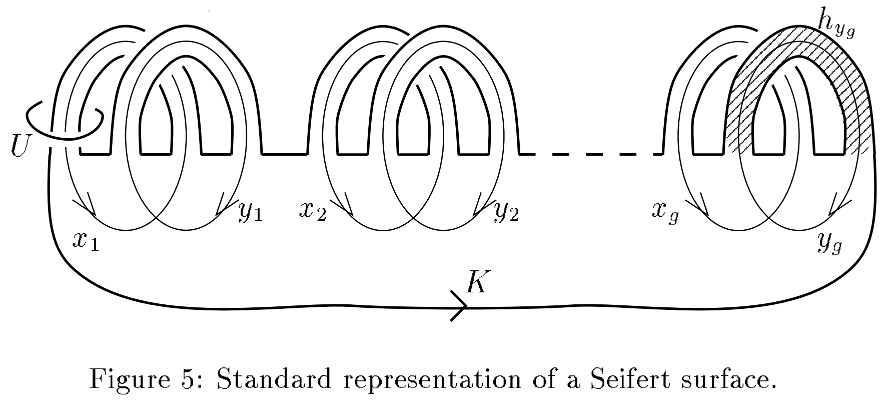

Figure 5: Standard representation of a Seifert surface.

4.2 Proof of Lemma 4.1

Proposition 4.6 *Any knot K in a homology sphere H can be tranformed into a knot with Alexander polynomial 1 by changes of crossings.*

PROOF: Let Σ_K be a Seifert surface of K. Let $\mathcal{B} = (x_1, y_1, x_2, y_2, \ldots, x_g, y_g)$ be a symplectic basis of $(H_1(\Sigma_K), <.,.>_{\Sigma_K})$ and view Σ_K as in Figure 5.

For any pair $\{z,t\}$ of elements of $\mathcal{B}$ and for any $\varepsilon = \pm 1$, we can pass the handles h_z and h_t through each other by four changes of crossings (see Figure 6) so that ε is added to $V_{\Sigma_K}(z,t)$ and $V_{\Sigma_K}(t,z)$, and the other coefficients of the matrix of V_{Σ_K} with respect to $\mathcal{B}$ are unchanged.

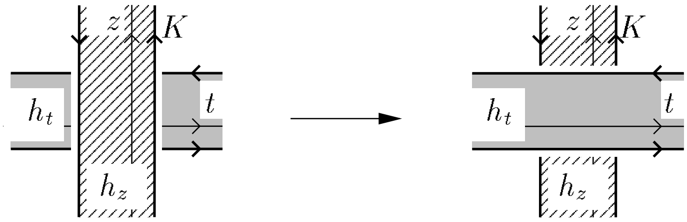

Figure 6: Passing the handles through each other.

For any element z of $\mathcal{B}$ and for any $\varepsilon = \pm 1$, we can twist the handle h_z by one change of crossings so that ε is added to $V_{\Sigma_K}(z,z)$, and the other coefficients of the matrix of V_{Σ_K} with respect to $\mathcal{B}$ are unchanged.

Note that $(V_{\Sigma_K} - V_{\Sigma_K}^T)$ represents the intersection form on Σ_K.

These three remarks make clear that K may be transformed by (a finite number of) changes of crossings into a knot K^0 which bounds a Seifert surface Σ_0 homeomorphic to Σ_K and such that:

- $V_{\Sigma_0}(x_i, y_i) = 1$ for any $i \in \{1, \ldots, g\}$
- $V_{\Sigma_0}(z,t) = 0$ for any other (z,t) of $\mathcal{B}^2$

Thus, $\Delta(K^0) = 1$ and we are done.

□

Thus, K is obtained from a knot K^0 with $\Delta(K^0) = 1$ by a finite number of changes of crossings. The disks of the changes of crossings which transform K^0 into K may be assumed to be pairwise disjoint. Indeed, such a disk D is only a regular neighborhood of an arc joining the two points of $K^0 \cap D$. Thus, the proposition may be rewritten as follows:

Statement 4.7 *For any knot K in a homology sphere H, there exist a knot K^0 of H, a collection $(D_i)_{i=1,\ldots,n}$ of pairwise disjoint disks and a collection $(\varepsilon_i)_{i=1,\ldots,n}$ of elements of $\{\pm 1\}$ such that:*

1. $\Delta(K^0) = 1$,

2. *Every* D_i *intersects* K^0 *transversally at exactly two points in* $\overset{o}{D_i}$ *with opposite signs.*

3. *The pair* (H, K) *is homeomorphic to* $(\chi_H((U_i = \partial D_i, \varepsilon_i)_{i=1,\dots,n}), K^0)$

In particular, for $\varepsilon = \pm 1$, if we set

$$\mathbf{L} = ((K^0, \varepsilon), (U_i, \varepsilon_i)_{i=1,\dots,n})$$

$$\chi_H(K, \varepsilon) = \chi_H(\mathbf{L})$$

where U_i bounds the genus one surface obtained by tubing D_i in $H \setminus (K^0 \cup (\cup_{j \neq i} U_i))$ and therefore in any manifold obtained from H by surgery on a sublink of $\mathbf{L}$.

Thus, we have proved that any (± 1)-surgery on a knot K in H is equivalent to a sequence of (± 1)-surgeries such that:

- The first surgery is performed on a knot K^0 with Alexander polynomial one
- The following ones are performed on genus one knots.

Of course, by Fact 2.10, this implies Lemma 4.1. □

4.3 Proof of Lemma 4.2

Let K be a knot bounding a genus one Seifert surface Σ in a homology sphere. In any symplectic basis of $H_1(\Sigma)$, the matrix of V_Σ has the form

$$V_\Sigma = \begin{pmatrix} a & c \\ c-1 & b \end{pmatrix} \tag{4.8}$$

where $a, b, c \in \mathbf{Z}$. In particular, $\Delta(K) = 1 + det(V_\Sigma)(t^{1/2} - t^{-1/2})^2$; thus

$$\lambda'(K) = det(V_\Sigma) \tag{4.9}$$

and the Alexander polynomial of genus one knots is determined by the Casson invariant, namely:

$$\Delta(K) = 1 + \lambda'(K)(t^{1/2} - t^{-1/2})^2 \tag{4.10}$$

Lemma 4.11 *Let* V *be a Seifert form on* $\mathbf{Z}^2$*, (i.e.* V *is an integral bilinear form on* $\mathbf{Z}^2$ *such that* $det(V - V^T) = 1$*). If* $det(V) \neq 0$*, then there exists* $x \in \mathbf{Z}^2 \setminus \{0\}$ *such that:*

$$|V(x,x)| \leq |det(V)|$$

If $|det(V)| > 1$*, then there exists* $x \in \mathbf{Z}^2 \setminus \{0\}$ *such that:*

$$|V(x,x)| < |det(V)|$$

PROOF: Let $a = min_{x \in \mathbf{Z}^2 \setminus \{0\}} |V(x,x)|$. The statement is clear if $a < 2$. Assume

$$a \geq 2$$

and after possibly changing V into $-V$, assume that there exists x such that:

$$V(x,x) = a$$

Let Y be the set of y of $\mathbf{Z}^2$ such that $V(x,y) = V(y,x) + 1$. (For such an y, (x,y) is a basis of $\mathbf{Z}^2$.) Y is non-empty.

If there exists $y \in Y$ such that $V(y,y) \leq 0$, then set $b = V(y,y)$, observe $b \leq -a$, set also $c = V(x,y)$, and observe $c(c-1) \geq 0$. Thus,

$$det(V) = ab - c(c-1) \leq ab \leq -2a < -a$$

and we are done. Hence, we assume that $V(y,y) \geq 0$ for all y of Y. Set

$$b = min_{y \in Y} V(y,y) \ (\geq a)$$

and choose $y \in Y$ such that

$$b = V(y,y)$$

Set

$$c = V(x,y)$$

Since $(y \pm x) \in Y$, $V(y \pm x, y \pm x) \geq b$. Hence,

$$b + a \pm 2(c - \frac{1}{2}) \geq b$$

This implies

$$|c - \frac{1}{2}| \leq \frac{a}{2}$$

Therefore

$$det(V) = ab - c^2 + c = ab - (c - \frac{1}{2})^2 + \frac{1}{4} \geq ab - \frac{a^2}{4} + \frac{1}{4}$$

where $b - a/4 \geq \frac{3}{4}a \geq \frac{3}{2}$. Hence

$$det(V) > a$$

□

Lemma 4.12 *Let K be a knot bounding a genus one Seifert surface Σ in a homology sphere. Assume that $\lambda'(K) = \pm 1$. Then there exists a symplectic basis of $H_1(\Sigma)$ where the matrix of V_Σ has the form:*

$$\begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon\lambda'(K) \end{pmatrix}$$

with $\varepsilon = \pm 1$; and if $\lambda'(K) = -1$, then ε may be arbitrarily chosen in $\{-1, +1\}$.

PROOF: Because of the form of the Seifert matrix (4.8), since $\lambda'(K)$ is odd, there can be no primitive element x of $H_1(\Sigma)$ such that $V_\Sigma(x,x) = 0$. Thus, according to Lemma 4.11, there exists a nonzero x such that $V_\Sigma(x,x) = \pm 1$. Choose such an x and set $\varepsilon = V_\Sigma(x,x)$. Now, choose y such that

$$< ., y >_\Sigma = \varepsilon V_\Sigma(., x)$$

Clearly, (x,y) is a symplectic basis of $H_1(\Sigma)$ in which V_Σ has the desired form; and if $\lambda'(K) = -1$, then ε could have been arbitrarily chosen. $\square$

Lemma 4.13 *Let K be a knot bounding a genus one Seifert surface Σ in a homology sphere H. Let*

$$V_\Sigma = \begin{pmatrix} a & c \\ c-1 & b \end{pmatrix}$$

be the matrix of V_Σ in a symplectic basis (x,y) of $H_1(\Sigma)$. Then for $\eta = \pm 1$ and $\varepsilon = \pm 1$, the surgery on (K,η) is equivalent to a sequence of two surgeries on two genus one framed knots $(\tilde{K},\eta) \subset H$ and $(U,\varepsilon) \subset H_1 = \chi_H(\tilde{K},\eta)$ such that:

1. *$\tilde{K}$ bounds a Seifert surface $\tilde{\Sigma}$ whose Seifert matrix is*

$$V_{\tilde{\Sigma}} = \begin{pmatrix} a & c \\ c-1 & b+\varepsilon \end{pmatrix}$$

   in some symplectic basis of $H_1(\tilde{\Sigma})$.

   In particular,

$$\lambda'(\tilde{K} \subset H) = \lambda'(K) + \varepsilon a$$

2. $$\lambda'(U \subset H_1) = -\eta a$$

PROOF: View Σ as in Figure 5. Let U be an unknot surrounding h_y.

Let $\tilde{\Sigma}$ be such that $(\chi_H(U,-\varepsilon),\Sigma) \cong (H,\tilde{\Sigma})$ as in Subsection 4.1. $V_{\tilde{\Sigma}}$ has the desired form. Let $\tilde{K} = \partial\tilde{\Sigma}$. Since $(H,\Sigma) \cong (\chi_H(U,\varepsilon),\tilde{\Sigma})$, we have

$$\chi_H(K,\eta) = \chi_H((\tilde{K},\eta),(U,\varepsilon))$$

Now, the Casson surgery formula (2.12) or a direct computation (as in the proof of Sublemma 5.2 below) yields:

$$\lambda'(U \subset H_1) = -\eta a$$

and since U bounds a genus one surface in H_1, we are done. $\square$

We can now prove:

Lemma 4.14 *A (± 1)-surgery on a genus one knot is equivalent to a sequence of (± 1)-surgeries on genus one knots with Casson invariant 0, 1 or -1.*

PROOF: Let K be a knot bounding a genus one Seifert surface Σ in a homology sphere H such that $|\lambda'(K)| > 1$. Let $\eta = \pm 1$. We are about to prove that an η-surgery on K is equivalent to a (finite) sequence of (± 1)-surgeries on genus one knots with $|\lambda'| < |\lambda'(K)|$. This clearly proves the lemma.

1. If there exists x such that $0 < |V_\Sigma(x,x)| < |\lambda'(K)|$, then choose a symplectic basis (x,y) of $H_1(\Sigma)$ starting with such an x and apply Lemma 4.13 with this basis, with $a = V_\Sigma(x,x)$, and with an ε such that

$$|\lambda'(\tilde{K})| < |\lambda'(K)|$$

2. Otherwise, according to Lemma 4.11, there is a primitive x such that $V_\Sigma(x,x) = 0$. Then there is a symplectic basis (x,y) of $H_1(\Sigma)$ in which:

$$V_\Sigma = \begin{pmatrix} 0 & c \\ c-1 & b \end{pmatrix}$$

Several applications of Lemma 4.13 transform the surgery on (K,η) into a sequence of (± 1)-surgeries on genus one knots such that

   - the first knot K' has a Seifert matrix of the form

   $$V = \begin{pmatrix} 0 & c \\ c-1 & 1 \end{pmatrix}$$

   and satisfies $\lambda'(K') = \lambda'(K)$,
   - all the other knots have Casson invariant 0.

Now, K' satisfies the hypotheses of the first case of the proof, and we are done.

□

To conclude the proof of Lemma 4.2, it suffices to prove:

Lemma 4.15 *Let K be a genus one knot in a homology sphere H such that $\lambda'(K) = +1$. Let $\eta = \pm 1$. The surgery on (K,η) is equivalent to a sequence of (± 1)-surgeries on genus one knots with Casson invariant* 0 *or* (-1).

PROOF: Let Σ be a genus one Seifert surface of K. Recall from Lemma 4.12 that there is a symplectic basis of $H_1(\Sigma)$ in which:

$$V_\Sigma = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

where

$$a = \pm 1$$

Thus, if $\eta = a$, we may apply Lemma 4.13 with $\varepsilon = -a$ to transform the surgery on (K, η) into a sequence of two (± 1)-surgeries on genus one knots with Casson invariant 0 and (-1).

Now, if $\eta = -a$, H is obtained from $\chi_H(K, \eta)$ by surgery on $(\hat{K}, -\eta) \subset \chi_H(K, \eta)$ (see Remark 2.8). By Remark 2.6, K and $\hat{K}$ bound Seifert surfaces with identical Seifert forms, and $\lambda'(\hat{K}) = \lambda'(K) = 1$. Thus, $(\hat{K}, -\eta)$ satisfies the hypotheses of the previous case. Therefore, H can be obtained from $\chi_H(K, \eta)$ by a sequence of (± 1)-surgeries on genus one knots with Casson invariant 0 and (-1). Hence, $\chi_H(K, \eta)$ can be obtained from H by the inverse sequence which is also a sequence of (± 1)-surgeries on genus one knots with Casson invariant 0 and (-1). □

5 Proof of Lemma 3.2

Lemma 3.2

Let H be a homology sphere. Let

$$\mathbf{L} = ((K_1 = \partial\Sigma_1, 1), (K_2 = \partial\Sigma_2, -1))$$

be a framed link in H such that Σ_1 and Σ_2 are two disjoint genus one Seifert surfaces, $\lambda'(K_1) = \lambda'(K_2) = -1$. Then the surgery on $\mathbf{L}$ is equivalent to a sequence of two (± 1)-surgeries on knots with Alexander polynomial 1.

PROOF: To prove this, we isotope K_1 to a knot $\tilde{K}_1$ in $\chi_H(K_2, -1)$ so that $\tilde{K}_1$ satisfies:

1. $\tilde{K}_1$ is disjoint from $\hat{K}_2$ in $\chi_H(K_2, -1)$
2. $lk_H(\tilde{K}_1, K_2) = 0$
3. $\Delta(\tilde{K}_1 \subset H) = 1$
4. $\Delta(K_2 \subset \chi_H(\tilde{K}_1, 1)) = 1$

These properties of $\tilde{K}_1$ ensure that

$$\chi_H((K_2, -1), (\tilde{K}_1, 1)) = \chi_H(\mathbf{L})$$

and that performing first the surgery on $(\tilde{K}_1, 1)$ and next on $((K_2, -1) \subset \chi_H(\tilde{K}_1, 1))$ yields the desired sequence.

Now, let us describe our isotopy and prove that it satisfies the required properties.

Let (x, y) be a symplectic basis of Σ_1 and let (z, t) be a symplectic basis of Σ_2 such that with respect to these bases:

$$V_{\Sigma_1} = V_{\Sigma_2} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

(See Lemma 4.12.) View Σ_1 as the union of a disk D_1 with two handles h_x and h_y and Σ_2 as the union of a disk D_2 with two handles h_z and h_t as usually. Denote $H_2 = \chi_H(K_2, -1)$. See Figure 7.

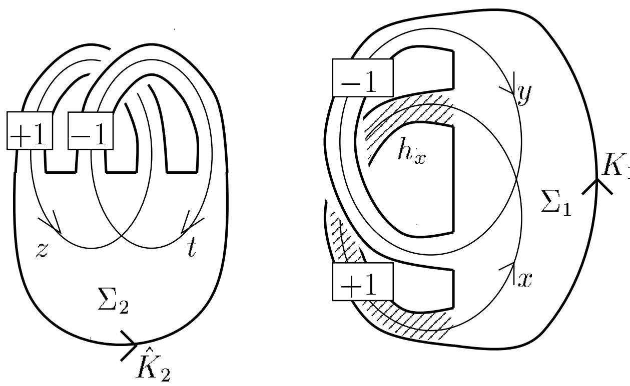

Figure 7: Σ_1 and Σ_2 in H_2 before the isotopy

Move Σ_1 to a surface ($\tilde{\Sigma}_1 \cong \Sigma_1$) by an isotopy of H_2 so that in H_2:

- $\tilde{\Sigma}_1$ intersects Σ_2 only in the interior of Σ_2 and exactly along one arc of $\tilde{\Sigma}_1$ separating h_x as in Figure 8. The core x of h_x intersects Σ_2 exactly at one point transversally and $lk_{H_2}(x, \hat{K}_2) = -1$.
- $lk_{H_2}(y, z) = 0$, $lk_{H_2}(y, t) = 1$

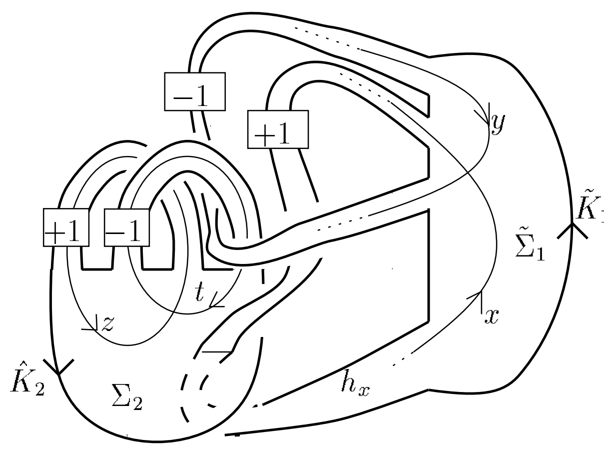

Figure 8: $\tilde{\Sigma}_1$ and Σ_2 in H_2 after the isotopy

Since $\tilde{\Sigma}_1$ does not intersect $\hat{K}_2$, it can be seen in H and we have:

Sublemma 5.1 *With respect to the basis (x,y) of $H_1(\tilde{\Sigma}_1)$,*

$$V_{\tilde{\Sigma}_1\subset H} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

PROOF: Of course,

$$V_{\tilde{\Sigma}_1\subset H_2} = V_{\Sigma_1\subset H_2} = V_{\Sigma_1\subset H}$$

Thus, since $lk_{H_2}(y\subset \tilde{\Sigma}_1, \hat{K}_2)=0$, $V_{\tilde{\Sigma}_1\subset H}(y,.) = V_{\Sigma_1\subset H}(y,.)$; and we are left with the computation of $lk_H(x^+,x)$ for $x\subset \tilde{\Sigma}_1$: Let m_x and m_2 denote the meridians of x and K_2 in $(H\setminus(x\cup K_2) = H_2\setminus(x\cup\hat{K}_2))$, and let ℓ_2 be the preferred parallel of K_2 in H. The following equalities take place in $H_1(H\setminus(x\cup K_2))$. The oriented meridian μ_2 of $\hat{K}_2$ satisfies:

$$\mu_2 = m_2 - \ell_2$$

Since $lk_H(K_2,x)=-1$,

$$\ell_2 = -m_x$$

Since $lk_{H_2}(x^+,x)=1$ and $lk_{H_2}(\hat{K}_2,x)=-1$,

$$x^+ = -\mu_2 + m_x = -m_2 + \ell_2 + m_x = -m_2$$

Thus, $lk_H(x^+,x)=0$ and the sublemma is proved. □

Let $\tilde{K}_1 = \partial\tilde{\Sigma}_1$. By the sublemma,

$$\Delta(\tilde{K}_1) = 1$$

Let $H_1 = \chi_H(\tilde{K}_1,1)$. In $(H\setminus\tilde{K}_1)$, K_2 bounds a genus 2 Seifert surface $\tilde{\Sigma}_2$ obtained by tubing Σ_2 as in Figure 9: The neighborhood of $\Sigma_2\cap\tilde{\Sigma}_1$ in Σ_2 is first isotoped towards D_1. Next, Σ_2 is tubed along a part of $\tilde{K}_1$ which is parallel to $y\subset\tilde{\Sigma}_1$. Denote by m the meridian of the tube which is also a meridian of $\tilde{K}_1$.

Then (z,t,m,y) is a symplectic basis of $H_1(\tilde{\Sigma}_2)$, and we have the following sublemma:

Sublemma 5.2 *With respect to (z,t,m,y), the Seifert matrix of $\tilde{\Sigma}_2$ in H_1 is:*

$$V_{\tilde{\Sigma}_2\subset H_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

PROOF: The computation of the coefficients involving z, t or y is easy after the two following remarks:

1. Since
$$lk_H(z,\tilde{K}_1) = lk_H(t,\tilde{K}_1) = lk_H(y,\tilde{K}_1)$$
$$= lk_H(z,K_2) = lk_H(t,K_2) = lk_H(y,K_2) = 0$$
the linking numbers involving z, t, y (or their parallels) are the same in H, in H_1 and in H_2.

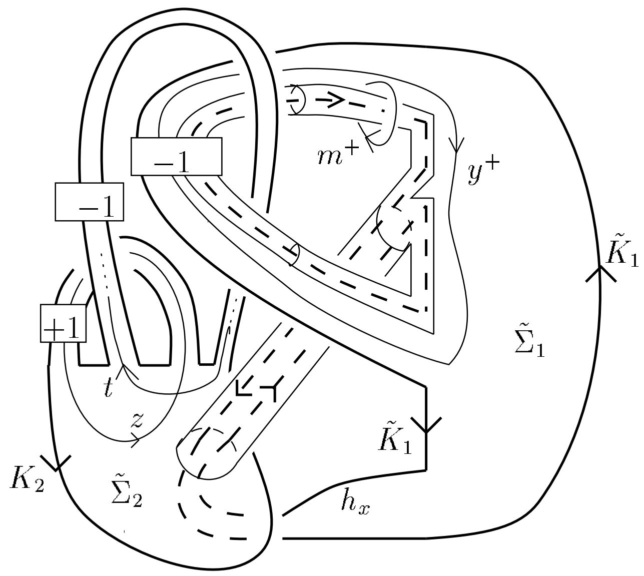

Figure 9: $\tilde{\Sigma}_2$

2. Let ℓ_1 denote the preferred parallel of $\tilde{K}_1$ on $\tilde{\Sigma}_1$. Then m is homologous to $-\ell_1$ inside the tubular neighborhood of $\hat{\tilde{K}}_1$ in H_1.

Thus, we are left with the computation of

$$lk_{H_1}(m^+, m) = lk_{H_1}(m^+, -\ell_1) = -1$$

□

Now, $\Delta(K_2) = 1$ and we are done. □

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