HARMONIC THETA FUNCTIONS AND HECKE OPERATORS

by Anatoli ANDRIANOV

ABSTRACT. — The introduction contains an historical survey of the transformation theory of theta functions and theta series of integral quadratic forms under Hecke operators. In the main part the explicit formulas are deduced which express images under Hecke operators of general vector-valued theta functions with (pluri)harmonic coefficients of integral nonsingular quadratic forms in an even number of variables in the form of linear combination of similar theta functions. The trigonometric sums appearing as coefficients of the linear combinations, the interaction sums, are studied. Similar formulas are derived for the action of Hecke operators on corresponding theta-series with rational translations.

Contents

Introduction

One of characteristic features of the Diophantine theory of quadratic forms is the presence of certain multiplicative properties of solutions of quadratic diaphantine equations. For example, the Jacobi formula

$$r_4(a) = 8\sum_{d|a} d = 8\sigma(a)$$

for the number of representations of an odd a by the sum of 4 integral squares expresses it through the function $\sigma(a)$, the sum of positive divisors of a, satisfying the multiplicative relations

$$\sigma(a)\sigma(b) = \sum_{d|a,b} d\sigma(ab/d^2) \quad (a,b = 1,2,\cdots)$$

which, in particular, allows one to reduce calculation of $r_4(a)$ for arbitrary odd a to the cases of prime divisors of a. Many particular results of such kind were obtained until E. Hecke has proposed a general approach to the problem of quadratic multiplicativity in 1937.

The Hecke approach is based on consideration of the numbers $r(\mathbf{q},a)$ of integral solutions of the equation

$$\mathbf{q}(x_1,\ldots,x_m) = a \tag{0.1}$$

with an integral positive definite quadratic form $\mathbf{q}$ in an even number of variables as Fourier coefficients of the generating function

$$\begin{aligned}\Theta(z,\mathbf{q}) &= \sum_{a=0}^{\infty} r(\mathbf{q},a)\exp(2\pi iaz) \qquad (0.2)\\ &= \sum_{x_1,\ldots,x_m=-\infty}^{\infty} \exp(2\pi i\mathbf{q}(x_1,\ldots,x_m)z), \quad (z = x+iy, y > 0),\end{aligned}$$

the theta series (of genus 1) of the form $\mathbf{q}$. The theta series is well known to belong to a finite dimensional space of (holomorphic) modular forms of weight $m/2$ for a subgroup of finite index of the modular group $SL_2(\mathbb{Z})$ (see, for example, Hecke [21], Schoeneberg [25]). Hecke [20] has introduced certain multiplicative families of linear operators, called now the Hecke operators, on the spaces of modular forms. Multiplicative properties of the operators allows one to reveal certain multiplicative properties of Fourier coefficients of modular forms and, in particular, imply that the Fourier coefficients of common eigenfunctions of the Hecke operators are proportional to the corresponding eigenvalues and therefore inherit the multiplicative properties of the operators.

A few years later H. Petersson [24] has proved that the spaces of modular forms are spanned by the common eigenfunction of the Hecke operators. This implied that the Fourier coefficients of an arbitrary modular form and in particular those of the theta series, *i.e.* the functions $a \to r(\mathbf{q},a)$, are linear combinations with constant coefficients of a finite number of multiplicative functions, whose values can be interpreted as eigenvalues of Hecke operators.

If the number of variables of an integral positive definite quadratic form $\mathbf{q}$ is odd, then the theta series (0.2) is a modular form of the half-integer weight $m/2$, and in order to apply a similar approach one have first to develop appropriate theory of Hecke operators on modular forms of noninteger weights. The first timid attempt to do it was undertaken by K. Wohlfahrt [31] in 1957. But only in the beginning of the seventieth G. Shimura [28] had succeeded to study multiplicative properties of Fourier coefficients of modular forms of half-integer weights and relate them to modular forms of integer weights. This allows one to reveal multiplicative properties of the numbers $r(\mathbf{q},a)$ for integral positive definite $\mathbf{q}$ in an odd number of variables.

Let us associate to each quadratic form $\mathbf{q}$ in m variables its matrix, *i.e.* the matrix Q of order m satisfying

$$ {}^tQ = Q \quad \text{and} \quad \mathbf{q}(\mathbf{x}) = \frac{1}{2}\,{}^t\mathbf{x}Q\mathbf{x}, \quad \left(\mathbf{x} = {}^t(x_1,\ldots,x_m)\right). \tag{0.3}$$

If $\mathbf{q}'$ is another quadratic form in n variables with the matrix Q', then *representations* of the form $\mathbf{q}'$ by the form $\mathbf{q}$ are solutions in $m \times n$-matrices of the matrix equation

$$ {}^tXQX = Q' \tag{0.4}$$

which is a natural generalization of the equation (0.1) and turns into it when $n = 1$, $\mathbf{q}' = ay^2$, and $Q' = 2a$. In a complete analogy to the case $n = 1$ the study of integral solutions of (0.4) for an integral positive definite $\mathbf{q}$ leads to consideration of the theta series of $\mathbf{q}$ of genus n,

$$ \Theta(Z;\mathbf{q}) = \sum_{N\in\mathbb{Z}_n^m} \exp\left(\pi i\,\mathrm{tr}({}^tNQNZ)\right), \tag{0.5}$$

where Z belongs to *the Siegel upper half-plane of genus n*,

$$ \mathbb{H}_n = \{Z = X + iY \in \mathbb{C}_n^n;\ {}^tZ = Z, Y > 0\} \tag{0.6}$$

and tr means the trace, which is a (holomorphic) modular form of weight $m/2$ for a subgroup of finite index of the Siegel modular group $\mathrm{Sp}_n(\mathbb{Z})$ (see Andrianov, Maloletkin [12]). The numbers $r(Q,Q')$ of integral solutions of (0.4) appear as the Fourier coefficients of the theta series (0.5) and their multiplicative properties can be approached by generalizing the Hecke ideas. The theory of Hecke operators on spaces of modular forms in several variables was originated by M. Sugawara in [29] and [30] soon after the original Hecke works and then developed by H. Maass [23] and many others. The modern state of the theory is exposed in the books of Andrianov [5] (the case of integer weight) and Andrianov, Zhuravlev [13] (the cases both of integer and half-integer weights). As applications certain multiplicative properties of integral representations of quadratic forms by quadratic forms were revealed (see Andrianov [2]).

Note that the values of multiplicative functions appearing in the applications of Hecke operators to quadratic Diophantine problems are expressible in terms of Fourier coefficients of modular forms belonging to the corresponding spaces. The spaces of modular forms are not necessary spanned by the theta series, and so the Fourier coefficients of general modular forms are not necessary related to some quadratic Diophantine equations. On the other hand it would be just natural to expect that the answer to such an arithmetical question as the nature of multiplicativity of solutions of quadratic Diophantine problems can be given in terms of quadratic forms themselves without participation of Fourier coefficients of general modular forms. This leds to the question whether the theta series span invariant under Hecke operators subspaces of the spaces of modular forms. In 1977 E. Freitag [14] had obtained the general positive answer to the question of invariance of spaces spanned by the theta series of the form (0.5) of integral positive definite quadratic forms in an even number of variables under Hecke operators. It was noted later that the Freitag's approach based on the theory of singular modular forms and use of the Siegel operator leave some open cases (see Freitag [15]), which are still open. Besides, no explicit formulas for images of the theta series under Hecke operators, which would be important for applications, were obtained at that time.

The first general explicit formulas expressing the images of the theta series (0.5) of integral positive definite quadratic forms in an even number of variables under regular Hecke operators were obtained by A.N. Andrianov in [1] by a new method based on his theory of factorization in parabolic extensions of abstract Hecke rings of symplectic groups. Soon after E. Freitag [16] has noticed that his method can also produce explicit formulas at least for the theta series of level 1. On the other hand, a few years later V.G. Zhuravlev has extended in [32] the Andrianov's method to the theta series of integral positive definite quadratic forms in an odd number of variables.

In general, if the quadratic form $\mathbf{q}$ is not positive definite, the series (0.5) diverges. In order to go around this difficulty, one can along with C.L. Siegel [26], [27] introduce additional variables into the definition of the theta series. Let Q be a real symmetric nonsingular matrix of order m. Let us define *the majorant space* $\mathcal{H}(Q)$ *of* Q by

$$\mathcal{H}(Q) = \{H = {}^tH \in \mathbb{R}^m_m; H > 0, HQ^{-1}H = Q\}. \tag{0.7}$$

This is a homogeneous space of the real orthogonal group

$$O(Q) = \{U \in \mathbb{R}^m_m;\ {}^tUQU = Q\} \tag{0.8}$$

acting on $\mathcal{H}(Q)$ by

$$O(Q) \supset U : H \longrightarrow {}^tUHU, \quad (H \in \mathcal{H}(Q)).$$

If $Z = X + iY \in \mathbb{H}_n$, where $n \geq 1$, and $H \in \mathcal{H}(Q)$, let us set now

$$\Theta(Z; H, Q) = \sum_{N \in \mathbb{Z}^m_n} \exp(\pi i \operatorname{tr}(X\, {}^tNQN + iY\, {}^tNHN). \tag{0.9}$$

The series converges absolutely and uniformly on compacts in $\mathbb{H}_n \times \mathcal{H}(Q)$ and therefore defines a real-analytic function, which is called the *theta series of genus n of the pair Q,H*.

If Q is positive definite, it is readily verified that the space $\mathcal{H}(Q)$ reduces to the single point $H = Q$, and the series (0.9) turns into the theta series (0.5) of the quadratic form $\mathbf{q}$ with the matrix Q. The series (0.9) were used by C.L. Siegel to study certain mean numbers of integral representations by integral indefinite quadratic forms.

Hecke operators can be extended to the theta series (0.9) when Q is the matrix of an integral quadratic form, but, in the best case, they can reveal some information only on multiplicative properties of certain numbers of integral representations of quadratic forms by quadratic forms and say nothing on multiplicative properties of the representations themselves. At the same time there are strong evidences that such properties exist. First of all, for some quadratic forms in 2, 4 and 8 variables it is confirmed by the classical composition theory of quadratic forms (see, for example, Gauss [19], Linnik [22]). Furthermore, it would be just natural to expect that in general the multiplicative properties of the numbers of integral representations is nothing else but a reflection of some multiplicative properties of the representations themselves. A possible approach to the problem of multiplicative properties of individual integral representations of quadratic forms by quadratic forms can be based on consideration of the action of Hecke operators on certain series obtained from the theta series by introducing some additional "linear" variables. Let Q be again a real symmetric nonsingular matrix of order m and H belongs to the majorant space $\mathcal{H}(Q)$ of Q. Let $n \geq 1$ and V_1, V_2 be complex $m \times n$-matrices. Then the series

$$\Theta(V,Z;H,Q) = \sum_{N \in \mathbb{Z}_n^m} \exp\left(\pi i \operatorname{tr}(XQ[N - V_2] + iYH[N - V_2] + 2\,{}^tV_1QN - {}^tV_1QV_2)\right), \qquad (0.10)$$

where $V = (V_1, V_2) \in \mathbb{C}_{2n}^m, Z = X + iY \in \mathbb{H}_n$, and we set

$$S[M] = {}^tMSM, \qquad (0.11)$$

if the product matrices on the right makes sense, converges absolutely and uniformly on compacts in $\mathbb{C}_{2n}^m \times \mathbb{H}_n \times \mathcal{H}(Q)$ and so defines there a real-analytic function. The function is called *the theta function of genus n of the pair Q,H*. When the matrix Q is even, *i.e.* it is the matrix (0.3) of an integral quadratic form, and m is even, explicit formulas expressing the images of the theta functions (0.10) under Hecke operators as linear combinations of similar theta function were derived by A.N. Andrianov [3] by extending the method of his work [1]. The originally complicated formulas for the coefficients of the linear combinations were replaced by simple trigonometrical sums in the work [4]. Applications of the explicit formulas to multiplicative properties of integral representations of quadratic forms by quadratic forms were obtained by A.N. Andrianov in [7], [8], [9], and [10].

It turns out that it would be useful for some applications (see, for example, Hecke [21], Freitag [18]) to extend the transformation formalism of theta series and theta functions under Hecke operators to the functions defined by similar summations of exponents but equipped with certain coefficients. In general we arrive at the series of the form

$$\Theta_P(V,Z;H,Q) = \sum_{N\in\mathbb{Z}_n^m} P(N-V_2)e\{XQ[N-V_2]+iYH[N-V_2]+2\,{}^tV_1QN-{}^tV_1QV_2\}, \qquad (0.12)$$

where Q is an even nonsingular matrix of order m, $H \in \mathcal{H}(Q)$, $Z = X + iY \in \mathbb{H}_n$ and $V = (V_1,V_2) \in \mathbb{C}_{2n}^m$, and where along with the notation (0.11) we use also the notation

$$e\{A\} = \exp(\pi i\,\mathrm{tr}(A)) \qquad (0.13)$$

for a complex square matrix A. As to the appropriate coefficients, it is natural to consider, generally speaking, vector-valued functions P on $\mathbb{C}_n^m$, which provide a good convergence of the series (0.12) and lead to the sums with good automorphic properties under integral symplectic transformations. It was recently shown by A.N. Andrianov [11] that such conditions are satisfied by (vector-valued) *harmonic forms P relative to the pair Q,H* (the explicit definition see in § 1 below). The corresponding (vector-valued) theta function (0.12) is called *the harmonic theta function of genus n of the pair Q,H with the coefficient form P*. The principal objective of this paper is to derive explicit formulas expressing the images of general harmonic theta functions of integral nonsingular quadratic forms in an even number of variables under the Hecke operators in the form of linear combinations of analogous theta functions. It is accomplished in § 4 (see Theorem 4.1). Our method is actually an extension of the method of the papers Andrianov [1], [3], [4] and is essentially based on explicit factorization of certain polynomials over abstract Hecke rings in their parabolic extensions.

As coefficients in the explicit formulas for the action of Hecke operators appear certain finite trigonometrical sums, *the interaction sums*, linking in fact Hecke rings of symplectic and orthogonal groups. Properties of the interaction sums are considered in § 3.

When the linear variables $V = (V_1,V_2)$ of the harmonic theta function (0.12) take certain fixed rational values one gets *the harmonic theta series with the rational translations*, which are important when the numbers of solution of quadratic Diophantine problems satisfying congruential conditions are considered. In § 5 we apply the explicit formulas of § 4 in order to derive analogous formulas for the action of Hecke operators on the harmonic theta series with rational translations of integral nonsingular quadratic forms in an even number of variables. The similar formulas for the harmonic theta series of integral positive definite forms in an even number of variables were formerly obtained by A.N. Andrianov in [6] (the case of special scalar harmonic forms), by E. Freitag in [17] (the case of quadratic forms of level one and the zero translation), and by V.G. Zhuravlev in [33] (the case of the zero translation).

It would be just natural to expect that the existing technique will allow one to extend the principal results of this paper to theta functions and theta series of integral quadratic forms in an odd number of variables.

Completing the introduction I would like to express my grateful thanks to Professor Eberhard Freitag of Heidelberg University for numerous stimulating discussions of Hecke operators. I am also thankful to Professor Roland Gillard and Alexei Panchishkin of Institut Fourier in Grenoble, where the present paper was completed during the spring of 1996, for their kind invitation, hospitality and useful discussions. Finally I want to thank Madame Myriam Charles who has skillfully prepared the TEX -file of my manuscript.

Notation. — As usual, the letters $\mathbb{Z}$,$\mathbb{Q}$$\mathbb{R}$ and $\mathbb{C}$ are reserved for the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. $\mathbb{A}_n^m$ is the set of all $m\times n$-matrices with entries in a set $\mathbb{A}$,$\mathbb{A}^m = \mathbb{A}_1^m$, and $\mathbb{A}_n = \mathbb{A}_n^1$.

If $M \in \mathbb{A}_n^m$, then ${}^tM \in \mathbb{A}_m^n$ is the transposed matrix. We write

$$\mathbf{1}_n = \mathrm{diag}(\underbrace{1,\ldots,1}_{n})$$

for the unit matrix of order n. Besides, we use systematically the notation (0.11) and (0.13).

We shall say that a square matrix with entries in $\mathbb{Z}$ is *even* if it is symmetric and has only even entries on the main diagonal, *i.e.* if it is the matrix (0.3) of an integral quadratic form. *The level* on an even nonsingular matrix Q is the least natural number d such that dQ^{-1} is even.

1. Harmonic theta functions

In this section we shall remind the basic definitions and automorphic properties under symplectic transformations of the harmonic theta functions. For more details and proofs see Andrianov [11], § § 4-6.

We shall first remind that a function $P : \mathbb{C}_n^m \to \mathbb{C}$ is called *harmonic*, if it is harmonic in mn variables in the sense that

$$\Delta P = \sum_{i,j} \frac{\partial^2 P(T)}{(\partial t_{ij})^2} = 0,$$

it is called *pluriharmonic*, if the functions $T \to P(TA)$ are harmonic for all $A \in GL_n(\mathbb{C})$. Let

$$\rho : GL_n(\mathbb{C}) \longrightarrow GL_a(\mathbb{C})$$

be *a polynomial group representation*, *i.e.* a group homomorphism given component-wise by polynomial functions, then *a ρ-harmonic form* is a component-wise harmonic polynomial mapping

$$P : \mathbb{C}_n^m \longrightarrow \mathbb{C}^a$$

satisfying the condition

$$P(TA) = \rho({}^tA)P(T) \text{ for all } A \in GL_n(\mathbb{C}).$$

It is clear that each harmonic form is component-wise pluriharmonic. The definition of ρ-harmonic forms is related to the quadratic form $\mathbf{q}(\mathbf{x}) = x_1^2 + \cdots + x_m^2$ in the sense that the mapping $T \to P(\varepsilon T)$ is a ρ-harmonic form for each ε of the complex orthogonal group $O_m(\mathbb{C})$ of order m, if the mapping P is a ρ-harmonic form. Let us define now harmonic forms related to arbitrary real nonsingular quadratic forms. Let $\mathbf{q}(\mathbf{x}) = \mathbf{q}(x_1, \ldots, x_m)$ be such a form, Q the matrix of $\mathbf{q}$ (see (0.3)), and $H \in \mathcal{H}(Q)$ a matrix of the majorant space (0.6) of Q. Since the matrix H is positive definite and satisfies $Q^{-1}[H] = Q$, it follows that there exists a real invertible m-matrix S such that

$$Q = \begin{pmatrix} \mathbf{1}_k & 0 \\ 0 & -\mathbf{1}_\ell \end{pmatrix} [S], \text{ and } H = \mathbf{1}_m[S] = {}^tSS \tag{1.1}$$

(see (0.11)), where (k, ℓ) is *the signature of the form* $\mathbf{q}$. We set

$$S = \begin{pmatrix} S_+ \\ S_- \end{pmatrix}, \text{ where } S_+ \in \mathbb{R}_m^k, \quad S_- \in \mathbb{R}_m^\ell. \tag{1.2}$$

Further, let

$$\rho_+ : GL_n(\mathbb{C}) \longrightarrow GL_a(\mathbb{C}), \rho_- : GL_n(\mathbb{C}) \longrightarrow GL_b(\mathbb{C}) \tag{1.3}$$

be two polynomial representations and let

$$P_+ : \mathbb{C}_n^k \longrightarrow \mathbb{C}^a, P_- : \mathbb{C}_n^\ell \longrightarrow \mathbb{C}^b$$

be a ρ_+-harmonic form and a ρ_--harmonic form, respectively. Then the mapping

$$P_0 = P_+ \otimes P_- : \mathbb{C}_n^m = \mathbb{C}_n^{k+\ell} \longrightarrow \mathbb{C}^{ab}$$

defined by

$$(P_+ \otimes P_-) \begin{pmatrix} T_+ \\ T_- \end{pmatrix} = P_+(T_+) \otimes P_-(T_-)$$

where $T_- \in \mathbb{C}_n^k$, $T_- \in \mathbb{C}_n^\ell$, and *the tensor* (or *Knonecker*) *product of a* $c \times d$-matrix A *by a* $e \times f$*-matrix B* is the $ce \times df$-matrix

$$A \otimes B = \begin{pmatrix} Ab_{11} \cdots Ab_{1f} \\ \cdots\cdots\cdots \\ Ab_{e1} \cdots Ab_{ef} \end{pmatrix},$$

is a $\rho_+ \otimes \rho_-$-harmonic form with

$$(\rho_+ \otimes \rho_-)(A) = \rho_+(A) \otimes \rho_-(A) \in GL_{ab}(\mathbb{C}),$$

as it easily follows from well known properties of the tensor product. Finally, returning to the pair Q,H, we shall say that the polynomial mapping

$$P = P_S : \mathbb{C}_n^m \longrightarrow \mathbb{C}^{ab} \tag{1.4}$$

defined by

$$P(T) = (P_0 \mid S)(T) = P_+(S_+T) \otimes P_-(S_-T),$$

where S is a matrix of the form (1.2) satisfying (1.1), is *a $\rho_+ \otimes \rho_-$-harmonic form relative to the pair Q,H*. Such a form satisfies the relation

$$\begin{aligned} P(TA) &= P_+(S_+TA) \otimes P_-(S_-TA) \\ &= (\rho_+ \otimes \rho_-)({}^tA)P(T) \end{aligned} \tag{1.5}$$

for each $A \in GL_n(\mathbb{C})$. In addition, if a complex m-matrix ε satisfies $Q[\varepsilon] = Q$ and $H[\varepsilon] = H$, then one can easily see that the mapping

$$T \longrightarrow (P \mid \varepsilon)(T) = P(\varepsilon T) = P_0(S\varepsilon T)$$

is again a $\rho_+ \otimes \rho_-$-harmonic form relative to the pair Q,H.

With the above notation and assumptions, if P is a harmonic form relative to the pair Q,H, then the series (0.12), where $V_1, V_2 \in \mathbb{C}_n^m$, $V = (V_1, V_2)$, $Z = X + iY \in \mathbb{H}_n$, and $H \in \mathcal{H}(Q)$, converges absolutely and uniformly on compacts in $\mathbb{C}_{2n}^m \times \mathbb{H}_n \times \mathcal{H}(Q)$. The resulting $\mathbb{C}^{ab}$-valued (real-analytic) function is called *the harmonic theta function of the pair Q,H of genus n with the coefficient form P*.

We shall remind that the integral symplectic group (or *the Siegel modular group*) *of genus n* is defined by

$$\mathrm{Sp}_n(\mathbb{Z}) = \Gamma^n = \{M \in \mathbb{Z}_{2n}^{2n};\ {}^tMJ_nM = J_n\} \tag{1.6}$$

where

$$J_n = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}. \tag{1.7}$$

We formulate now transformation properties of the harmonic theta functions of integral nonsingular quadratic forms in an even number of variables under integral symplectic transformations which are specializations of general formulas obtained in Andrianov [11], Theorems 4.1, 4.2 and 4.3.

THEOREM 1.1. — *Let Q be the matrix of an integral nonsingular quadratic form in an even number m of variables, i.e. an even nonsingular matrix of an even order m, (k,ℓ) the signature of the form and H belongs to the majorant space (0.6) of Q. Let $P = P_S$ be a $\rho_+ \otimes \rho_-$-harmonic form (1.4) on $\mathbb{C}_n^m$ relative to the pair Q,H, where ρ_+ and ρ_- are* *polynomial representations* (1.3) *with* $n,a,b = 1,2,\cdots$. *Then, for each integral symplectic matrix* $M = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix}$ *of the group*

$$\Gamma_0^n(d) = \left\{ M = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix} \in \Gamma^n; C \equiv 0(\mathrm{mod}\, d) \right\}, \tag{1.8}$$

where d *is the level of* Q *(see Notation), the theta function* (0.12) *of the pair* Q,H *of genus* n *with the coefficient form* P *satisfies the following functional equation*

$$\Theta_P(V\, {}^tM,(AZ+B)(CZ+\mathcal{D})^{-1};H,Q) = \\ \mu_Q(M)\, j_{k,\ell}(M,Z)\rho_+(CZ+\mathcal{D}) \otimes \rho_-(C\overline{Z}+\mathcal{D})\Theta_P(V,Z;H,Q) \tag{1.9}$$

where

$$j_{k,\ell}\Big(\begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix},Z \Big) = (\det(CZ+\mathcal{D}))^{(k-\ell)/2} |\det(CZ+\mathcal{D})|^{\ell}, \tag{1.10}$$

and where

$$\mu_Q(M) = \mu_Q\Big(\begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix} \Big) = \chi_Q(\det \mathcal{D}) \tag{1.11}$$

with the real Dirichlet character χ_Q *modulo* d *satisfying*

$$\chi_Q(-1) = (-1)^{(k-\ell)/2}, \tag{1.12}$$

$$\chi_Q(p) = \left(\frac{(-1)^{m/2} \det Q}{p} \right) \quad \text{(the Legendre symbol)}$$

if p *is an odd prime, and*

$$\chi_Q(2) = 2^{-m/2} \sum_{\mathbf{r} \in \mathbb{Z}^m/2\mathbb{Z}^m} \exp(\pi i Q[\mathbf{r}]/2),$$

if d *is odd.*

2. Hecke operators

Here we shall recall definitions and basic properties of Hecke rings and Hecke operators. For details and proofs, see Andrianov [5], Chapter 3, § § 3.1, 3.3.

Let G be a multiplicative semigroup, and $\Gamma \subset G$ a subgroup. We let

$$\mathbb{L} = L_{\mathbb{A}}(\Gamma \setminus G), \tag{2.1}$$

where $\mathbb{A}$ is a commutative and associative ring with the identity, be the free (left) $\mathbb{A}$-module consisting of all formal finite linear combinations $t = \sum_i a_i(\Gamma g_i)$ with coefficients $a_i \in \mathbb{A}$

of symbols (Γg_i), $g_i \in G$, which are in one-to-one correspondence with the left cosets Γg of the set G with respect to the group Γ. Further, we let

$$\mathbb{D} = D_{\mathbb{A}}(\Gamma \setminus G) = \{t = \sum_i a_i(\Gamma g_i) \in \mathbb{L}; t\gamma = \sum_i a_i(\Gamma g_i\gamma) = t \text{ for all } \gamma \in \Gamma\} \quad (2.2)$$

denote the submodule of $\mathbb{L}$ consisting of all elements which are invariant relative to the natural (right) action of the group Γ on $\mathbb{L}$. If $[G]_\Gamma$ denotes the subset of G consisting of all elements $g \in G$ such that the double coset $\Gamma g\Gamma$ consists only of finitely many left cosets modulo Γ :

$$|\Gamma \setminus \Gamma g\Gamma| < \infty, \quad (2.3)$$

then one can easily see that $[G]_\Gamma$ is a subsemigroup of G and the elements

$$[g] = [g]_\Gamma = \sum_{g_i \in \Gamma\setminus\Gamma g\Gamma} (\Gamma g_i) \in \mathbb{L} \quad (g \in [G]_\Gamma) \quad (2.4)$$

corresponding to different double cosets $\Gamma g\Gamma \subset [G]_\Gamma$ belong to $\mathbb{D}$ and form a free basis of $\mathbb{D}$ over $\mathbb{A}$, in particular,

$$D_{\mathbb{A}}(\Gamma \setminus G) = D_{\mathbb{A}}(\Gamma \setminus [G]_\Gamma). \quad (2.5)$$

The product of elements of $\mathbb{D}$ given by the formula

$$\Big(\sum_i a_i(\Gamma g_i)\Big)\Big(\sum_j b_j(\Gamma h_j)\Big) = \sum_{i,j} a_i b_j(\Gamma g_i h_j),$$

does not depend on the choice of representatives $g_i \in \Gamma g_i$, $h_j \in \Gamma h_j$, belongs to $\mathbb{D}$, and makes $\mathbb{D}$ into an associative ring, called *the Hecke ring of the pair* (Γ, G) *over* $\mathbb{A}$. The Hecke rings were first introduced by G. Shimura in the late fifties in a slightly different but equivalent form. When considering Hecke rings one can replace, by (2.5), the semigroup G by $[G]_\Gamma$ and so assume at the beginning that

$$G = [G]_\Gamma. \quad (2.6)$$

The pair (Γ, G) satisfying (2.6) will be called ℓ *(eft)-finite*. Suppose now that the semigroup G acts on a (left) $\mathbb{A}$ module $\mathbb{V}$ by linear operators

$$g : v \longrightarrow v \mid g \quad (v \in \mathbb{V}, g \in G)$$

satisfying

$$v|g|g_1 = v \mid gg_1 \quad (v \in \mathbb{V}, g, g_1 \in G) \quad (2.7)$$

and we let

$$\mathbb{V}(\Gamma) = \{v \in \mathbb{V}; v \mid \gamma = v \text{ for all } \gamma \in \Gamma\} \quad (2.8)$$

be the submodule of Γ-invariant elements of $\mathbb{V}$. Then for every $v \in \mathbb{V}(\Gamma)$ and every $t = \sum_i a_i(\Gamma g_i) \subset \mathbb{D}$ the element

$$v \mid t = \sum_i a_i v \mid g_i \quad (2.9)$$

does not depend on the choice of representatives $g_i \in \Gamma g_i$, it also belongs to the submodule $\mathbb{V}(\Gamma)$ and we have

$$v|t|t_1 = v \mid tt_1 \tag{2.10}$$

for any $v \in \mathbb{V}(\Gamma)$, $t, t_1 \in \mathbb{D}$. Thus, the correspondence $t \to \mid t$ is a linear representation of the Hecke ring $\mathbb{D}$ in the module $\mathbb{V}(\Gamma)$. The operators $\mid t$ are called *Hecke operators.*

In the situation which interests us – the action of Hecke operators on the harmonic theta series of genus n of an integral quadratic form of level d considered in Theorem 1.1 – the role of G is played by the semigroup

$$\mathbf{S} = S_0^n(d) = \left\{ M = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix} \in \mathbb{Z}_{2n}^{2n}, \ {}^tMJ_nM = \mu(M)J_n, \right.$$
$$\left. \mu(M) > 0, g.c.d(d,\mu(M)) = 1, C \equiv 0(\mathrm{mod}\ d) \right\}, \tag{2.11}$$

where J_n is the skew-symmetric matrix (1.7), and the role of Γ is played by the group $\Gamma_0^n(d)$ defined by (1.8). The pair $(\Gamma_0^n(d), S_0^n(d))$ is ℓ-finite, by Lemma 3.3.1 of Andrianov [5], and the corresponding Hecke ring over $\mathbb{A} = \mathbb{C}$

$$L_0^n(d) = D_{\mathbb{C}}(\Gamma_0^n(d), S_0^n(d)) \tag{2.12}$$

will be called *the Hecke ring of the group* $\Gamma_0^n(d)$. To define the Hecke operators, we shall take as a representation space for the semigroup $S_0^n(d)$ the space $\mathcal{F} = \mathcal{F}(m,n,ab)$ of all $\mathbb{C}^{ab}$-valued real analytical functions $F = F(V,Z)$ on $\mathbb{C}_{2n}^m \times \mathbb{H}_n$ and define the action of $S = S_0^n(d)$ on $\mathcal{F}$ by the formula

$$S \ni M = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix} : F \to F \mid M = J(M,Z)^{-1} F(V\ {}^tM, M\langle Z\rangle), \tag{2.13}$$

where

$$M\langle Z\rangle = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix} \langle Z\rangle = (AZ + B)(CZ + \mathcal{D}^{-1}) \in \mathbb{H}_n,$$

and where in the notation of Theorem 1.1 we set

$$J(M,Z) = \chi_Q(\det \mathcal{D}) j_{k,\ell}(M,Z)\rho_+(CZ + \mathcal{D}) \otimes \rho_-(C\overline{Z} + \mathcal{D}). \tag{2.14}$$

If $M = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix}$ and $M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & \mathcal{D}_1 \end{pmatrix}$ belong to $\mathbf{S}$, and $M' = \begin{pmatrix} A' & B' \\ C' & \mathcal{D}' \end{pmatrix} = MM_1$, it easily follows by the direct computation that

$$(CM_1\langle Z\rangle + \mathcal{D})(C_1Z + \mathcal{D}_1) = C'Z + \mathcal{D}', \quad (Z \in \mathbb{H}_n), \tag{2.15}$$

and

$$\chi_Q(\det \mathcal{D}') = \chi_Q(\det(CB_1 + \mathcal{D}\mathcal{D}_1)) = \chi_Q(\det \mathcal{D}\mathcal{D}_1) = \chi_Q(\det \mathcal{D})\chi_Q(\det \mathcal{D}_1)$$

since $C \equiv 0(\mathrm{mod}\ d)$. These relations imply that the matrices $J(M,Z)$ satisfy the relations

$$J(M, M_1\langle Z\rangle)J(M_1, Z) = J(MM_1, Z)$$

for every $M, M_1 \in S_0^n(d)$ and $Z \in \mathbb{H}_n$, hence we conclude that the action (2.13) satisfies the condition (2.7),

$$F|M|M_1 = F \mid MM_1 \quad (F \in \mathcal{F}, M, M_1 \in \mathbf{S}). \tag{2.16}$$

This allows us to define the standard representation $T \to \mid T$ of the Hecke ring $L_0^n(d)$ on the subspace

$$\mathcal{F}(\Gamma_0^n(d)) = \{F \in \mathcal{F}; F \mid \gamma = F \text{ for all } \gamma \in \Gamma_0^n(d)\} \tag{2.17}$$

of all $\Gamma_0^n(d)$-invariant functions of $\mathcal{F}$. Let

$$T = \sum_{M_i \in \Gamma_0^n(d) \backslash S_0^n(d)} a_i(\Gamma_0^n(d) M_i) \in L_0^n(d) \tag{2.18}$$

be an element of the Hecke ring (2.12). According to Theorem 1.1, in the notation and assumptions of the theorem, the theta function $\Theta_P(V, Z; H, Q)$ of genus n of the pair Q, H with the coefficient form P, when considered as a function of V and Z,

$$\Theta_P(V, Z; H, Q) = \Theta(V, Z), \tag{2.19}$$

belongs to the space $\mathcal{F}(\Gamma_0^n(d))$, and so its image

$$\begin{aligned} \Theta_P(V, Z; H, Q) \mid T = (\Theta \mid T)(V, Z) &= \sum_i a_i(\Theta \mid M_i)(V, Z) \\ &= \sum_i a_i J(M_i, Z)^{-1} \Theta_P(V\, {}^tM_i, M_i\langle Z\rangle; H, Q) \end{aligned} \tag{2.20}$$

under the Hecke operator $\mid T$ corresponding to the element (2.18) does not depend on the particular choice of representatives $M_i \in \Gamma_0^n(d) M_i$, and belongs again to the space $\mathcal{F}(\Gamma_0^n(d))$.

The purpose of this paper is to show that under certain conditions on the elements T of $L_0^n(d)$ the images (2.20) of the theta function $\Theta_P(V, Z; H, Q)$ under the Hecke operators $\mid T$ are linear combination with constant coefficients of similar theta functions, but first we derive direct formulas for the images expressing them as infinite sums with explicitly given coefficients.

By (2.11), each matrix $M \in S_0^n(d)$ satisfies the relation ${}^tM J_n M = \mu(M) J_n$, where $\mu(M)$ is a positive integer coprime with d. The number $\mu(M)$ is called *the multiplier of M*. It is obvious that

$$\mu(MM_1) = \mu(M)\mu(M_1) \quad (M, M_1 \in S_0^n(d)), \tag{2.21}$$

and

$$\mu(M) = 1 \Longleftrightarrow M \in \Gamma_0^n(d). \tag{2.22}$$

It follows that the function μ is constant on the left and double cosets of M modulo the group $\Gamma_0^n(d)$, and so one can speak on *the multiplier of the corresponding cosets*,

$$\mu(\Gamma_0^n(d) M) = \mu\big(\Gamma_0^n(d) M \Gamma_0^n(d)\big) = \mu(M).$$

We say that a nonzero (formal) finite linear combination T of left or double cosets modulo $\Gamma_0^n(d)$ contained in $S_0^n(d)$ is *homogeneous of the multiplier* $\mu(T) = \mu$, if all of the cosets have multiplier μ. It is obvious that each of the finite linear combinations is a finite sum of homogeneous linear combinations of different multipliers, its *homogeneous components*, which are uniquely determined. In particular, this allows one to reduce the general Hecke operator $|\ T$ to the case when T is homogeneous.

Another reduction is related to a special choice of representatives in the left cosets $\Gamma_0^n(d)M$ contained in $S_0^n(d)$. By Lemma 3.3.4 of Andrianov [5] and the definition of $S_0^n(d)$, each of the left cosets contains a representative of the form

$$M = \begin{pmatrix} A & B \\ 0 & \mathcal{D} \end{pmatrix} \text{ with } A, B, \mathcal{D} \in \mathbb{Z}_n^n,\ {}^tA\mathcal{D} = \mu(M)1_n,\ {}^tB\mathcal{D} = {}^t\mathcal{D}B. \tag{2.23}$$

These representatives are very convenient for computations with Hecke operators and will be referred as *triangular*.

With the above definitions we can now prove the following proposition.

Proposition 2.1. — *In the notation and assumptions of Theorem 1.1, the image* (2.20) *of the theta function* $\Theta_P(V,Z;H,Q)$ *under the Hecke operator corresponding to an homogeneous element*

$$T = \sum_i a_i(\Gamma_0^n(d)M_i) \in L_0^n(d) \tag{2.24}$$

with the multiplier $\mu(T) = \mu$, *where all of the representatives* M_i *are supposed to be triangular,*

$$M_i = \begin{pmatrix} A_i & B_i \\ 0 & \mathcal{D}_i \end{pmatrix},\ {}^tA_i\mathcal{D}_i = \mu 1_n,\ {}^tB_i\mathcal{D}_i = {}^t\mathcal{D}_iB_i, \tag{2.25}$$

is equal to the series

$$\sum_{N\in C^n(Q/\mu)} I(N,Q,T)P(\mu^{-1}(N-\mu V_2))e\{X\cdot\mu^{-1}Q[N-\mu V_2] + \sqrt{-1}Y\mu^{-1}H[N-\mu V_2]$$

$$+2\cdot {}^t(\mu V_1)\mu^{-1}QN - {}^t(\mu V_1)\mu^{-1}Q\cdot(\mu V_2)\}, \tag{2.26}$$

where

$$C^n(Q/\mu) = \{N\in\mathbb{Z}_n^m; \mu^{-1}Q[N] \text{ is even}\}, \tag{2.27}$$

$$I(N,Q,T) = \sum_{i;N^t\mathcal{D}_i\equiv 0(\operatorname{mod}\mu)} a_i j_Q(\mathcal{D}_i)^{-1} e\{\mu^{-2}Q[N]\cdot {}^t\mathcal{D}_iB_i\}, \tag{2.28}$$

with

$$j_Q(\mathcal{D}) = \chi_Q(|\det\mathcal{D}|)|\det\mathcal{D}|^{m/2}, \quad (\mathcal{D}\in\mathbb{Z}_n^n) \tag{2.29}$$

and where the other notation are the same as in (0.12).

Proof. — To shorten notation we shall write

$$e\{XQ[N-V_2]+\sqrt{-1}YH[N-V_2]+2\,{}^tV_1QN-{}^tV_1QV_2\}=e(V,Z,HQ;N),\quad (2.30)$$

for the exponential factor in the general term of the series (0.12) for $\Theta_P(V,Z;H,Q)$. An easy direct computation based on definitions shows that for each matrix M of the form (2.23) one has the identity

$$e(V\cdot{}^tM,M\langle Z\rangle,H,Q;N)=e\{B\mathcal{D}^{-1}Q[N]\}e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q;NA)\qquad (2.31)$$

with the notation, the image (2.20) can be written in the form

$$\sum_{N\in\mathbb{Z}_n^m}\sum_i a_iJ(M_iZ)^{-1}P(N-V_2\cdot{}^t\mathcal{D}_i)e(V\cdot{}^tM_i,M_i\langle Z\rangle,H,Q;N).\qquad (2.32)$$

By (2.14), (1.10), (1.11) and (1.12), one can write

$$\begin{aligned}J(M_i,Z)&=\chi_Q(\det\mathcal{D}_i)(\det\mathcal{D}_i)^{\frac{k-\ell}{2}}|\det\mathcal{D}_i|^{\ell}\rho_+(\mathcal{D}_i)\otimes\rho_-(\mathcal{D}_i)\\&=j_Q(\mathcal{D}_i)\rho_+(\mathcal{D}_i)\otimes\rho_-(\mathcal{D}_i),\end{aligned}$$

whence, by (1.5) and (2.25),

$$J(M_i,Z)^{-1}P(N-V_2\cdot{}^t\mathcal{D}_i)=j_Q(\mathcal{D}_i)^{-1}P(\mu^{-1}NA_i-V_2).$$

Hence, using the relations (2.31) for $M=M_i$, we can rewrite the sum (2.32) in the form

$$\sum_{N\in\mathbb{Z}_n^m}\sum_i a_ij_Q(\mathcal{D}_i)^{-1}e\{B_i\mathcal{D}_i^{-1}Q[N]\}P(\mu^{-1}(NA_i-\mu V_2))e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q;NA_i).$$

After combining here all of the terms with a fixed matrix

$$N'=NA_i=\mu N\cdot{}^t\mathcal{D}_i^{-1},$$

using the obvious relations

$$e\{B_i\mathcal{D}_i^{-1}Q[N]\}=e\{\mathcal{D}_i^{-1}Q[\mu^{-1}N'\cdot{}^t\mathcal{D}_i]B_i\}=e\{\mu^{-2}Q[N']\,{}^t\mathcal{D}_iB_i\}$$

and then omitting the stroke, we get the sum

$$\sum_{N\in\mathbb{Z}_n^m}I(N,Q,T)P\big(\mu^{-1}(N-\mu V_2)\big)e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q,N).\qquad (2.33)$$

In order to conclude the proof of the proposition, it suffices to show that

$$I(N,Q,T)=0\text{ unless }N\in C^n(Q/\mu),\quad (N\in\mathbb{Z}_n^m).\qquad (2.34)$$

To see it, we recall that the image $F(V,Z)=(\Theta\mid T)(V,Z)$ given by the sum (2.33) belongs to the space $\mathcal{F}(\Gamma_0^n(d))$, and, in particular, satisfies

$$\left(F\mid\begin{pmatrix}\mathbf{1}_n&B\\0&\mathbf{1}_n\end{pmatrix}\right)(V,Z)=F\big((V_1+V_2\cdot{}^tB,V_2),Z+B\big)=F(V,Z)$$

for every $B = {}^tB \in \mathbb{Z}_n^n$. These relations for the function F written in the form (2.33) with $V = (V_1, V_2) = (0,0)$ and $P \equiv 1$ turn into the relations

$$\sum_{N\in\mathbb{Z}_n^m} I(N,Q,T)e\{\mu^{-1}Q[N]B\}e\{X\cdot\mu^{-1}Q[N]+\sqrt{-1}Y\mu^{-1}H[N]\}$$
$$= \sum_{N\in\mathbb{Z}_n^m} I(N,Q,T)e\{X\mu^{-1}Q[N]+\sqrt{-1}Y\mu^{-1}H[N]\}$$

which imply the formulas

$$I(N,Q,T)e\{\mu^{-1}Q[N]B\} = I(N,Q,T), \quad (N\in\mathbb{Z}_n^m, B = {}^tB\in\mathbb{Z}_n^n)$$

by uniqueness of the Fourier expansion of the analytical and periodical in $X = {}^tX \in \mathbb{R}_n^n$ function $F(0, X+\sqrt{-1}Y)$. If $N \notin C^n(Q/\mu)$, then there is an integral symmetric B such that $\operatorname{tr}(\mu^{-1}Q[N]B) \notin 2\mathbb{Z}$, and so $e\{\mu^{-1}Q[N]B\} \neq 1$. This proves (2.34) and the proposition. □

3. Interaction sums

We call the trigonometrical sums $I(N,Q,T)$, defined for homogeneous T by (2.28) and extended to the whole of $L_0^n(d)$ by linearity in T, *the interaction sums* because they connect certain arithmetical structures related to orthogonal and symplectic groups. In this section we consider some of their basic properties.

Over the whole of the section we assume that Q is a matrix of an integral nonsingular quadratic form in an even number m of variables, *i.e.* an even nonsingular matrix of even order m, (k,ℓ) is the signature of the form, d is the level of Q, and χ_Q is the corresponding Dirichlet character modulo d, defined in Theorem 1.

LEMMA 3.1. — *The interaction sums $I(N,Q,T)$ with $N \in \mathbb{Z}_n^m$ and $T \in L_0^n(d)$ do not depend on the particular choice of triangular representatives in expansion of homogeneous component of T, and they satisfy the relations*

$$I(UNU', Q, T) = I(N, Q[U], T) \tag{3.1}$$

for every

$$U, U' \in \Lambda^m = GL_m(\mathbb{Z}). \tag{3.2}$$

Proof. — The first statement easily follows from definitions. As to the relations (3.1), one can assume that T is an homogeneous element of the form (2.24) with the representatives M_i satisfying (2.25). Then we have

$$I(UNU', Q, T) = \sum_{i;\, U\cdot N\cdot {}^t(\mathcal{D}_i\, {}^tU')\equiv 0 (\operatorname{mod} \mu)} a_i j_Q(\mathcal{D}_i\, {}^tU')e\{\mu^{-2}Q[U][N]\, {}^t(\mathcal{D}_i\cdot {}^tU')B_i\cdot {}^tU'\}$$

$$= I\Big(N, Q[U], T \begin{pmatrix} (U')^{-1} & 0 \\ 0 & {}^tU' \end{pmatrix}\Big) = I(N, Q[U], T),$$

since T is invariant under right multiplication by elements of $\Gamma_0^n(d)$. □

Next we consider relations between interaction sums for different genera n. For this we shall introduce the corresponding *Zharkovskaya mappings*

$$\psi^{n,r} = \psi_Q^{n,r} : L_0^n(d) \longrightarrow L_0^r(d) \tag{3.3}$$

for $n > r \geq 1$. If $T \in L_0^n(d)$ is an element of the form (2.24) with the representatives M_i of the form (2.25), then after replacing of M_i by $\begin{pmatrix} {}^tU_i^{-1} & 0 \\ 0 & U_i \end{pmatrix} M_i$ with suitable $U_i \in GL_n(\mathbb{Z})$, we may assume that all of the blocks $\mathcal{D}_i$ of M_i have the form $\mathcal{D}_i = \begin{pmatrix} \mathcal{D}'_i & * \\ 0 & \mathcal{D}''_i \end{pmatrix}$ with $\mathcal{D}'_i \in \mathbb{Z}_r^r$ (see, for example, Andrianov [5], Lemma 3.2.7), and so

$$M_i = \begin{pmatrix} \begin{pmatrix} A'_i & 0 \\ * & A''_i \end{pmatrix} & \begin{pmatrix} B'_i & * \\ * & * \end{pmatrix} \\ 0 & \begin{pmatrix} \mathcal{D}'_i & * \\ 0 & \mathcal{D}''_i \end{pmatrix} \end{pmatrix} \text{ with } A'_i, B'_i, \mathcal{D}'_i \in \mathbb{Z}_r^r. \tag{3.4}$$

It follows from (2.25), that

$${}^tA'_i\mathcal{D}'_i = \mu \mathbf{1}_r,\ {}^tB'_i\mathcal{D}'_i = {}^t\mathcal{D}_i B'_i, \tag{3.5}$$

so that

$$M'_i = \begin{pmatrix} A'_i & B'_i \\ 0 & \mathcal{D}'_i \end{pmatrix} \in S_0^r(d). \tag{3.6}$$

We set then

$$\psi^{n,r} T = \sum_i a_i j_Q(\mathcal{D}''_i)^{-1} (\Gamma_0^r(d) M'_i). \tag{3.7}$$

It is easy to check that the linear combination (3.7) belongs to the Hecke ring $L_0^r(d)$, and the mapping $T \to \psi^{n,r} T$ when extended by linearity to all of $L_0^n(d)$ is a $\mathbb{C}$-linear ring homomorphism.

Proposition 3.2. — *Let T be an element of $L_0^n(d)$ and $N \in \mathbb{Z}_r^m$ with $n > r \geq 1$, then*

$$I\big((N, 0), Q, T\big) = I(N, Q, \psi^{n,r} T) \tag{3.8}$$

where $\psi^{n,r}$ is the Zharkovskaya mapping (3.3)

Proof. — One can assume that T is homogeneous of the form (2.24) with the representatives M_i taken in the form (3.4). Then, by (2.28), we get

$$I\big((N,0),Q,T\big) = \sum_{i;(N,0)\begin{pmatrix} {}^t\mathcal{D}'_i & 0 \\ * & {}^t\mathcal{D}''_i \end{pmatrix}\equiv 0 (\mathrm{mod}\,\mu)} a_i j_Q(\mathcal{D}''_i)^{-1} j_Q(\mathcal{D}'_i)^{-1}\times$$

$$\times\, e\left\{\mu^{-2}\begin{pmatrix} Q[N] & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} {}^t\mathcal{D}'_i B'_i & * \\ * & * \end{pmatrix}\right\}$$

$$= \sum_{i;N\ {}^t\mathcal{D}'_i\equiv 0(\mathrm{mod}\,\mu)} a_i j_Q(\mathcal{D}''_i)^{-1} j_Q(\mathcal{D}'_i)^{-1} e\{\mu^{-2}Q[N]\cdot\ {}^t\mathcal{D}'_i B'_i\},$$

which proves (3.8). □

It is clear that the Zharkovskaya homomorphism maps homogeneous elements to homogeneous. For some applications it is important to know what homogeneous elements belong to its image. The following proposition answers the question

Proposition 3.3. — *Let $n > r \geq 1$ and T be a nonzero homogeneous element of $L_0^r(d)$ with the multiplier $\mu(T) = \mu$. Then the inclusion*

$$T \in \psi^{n,r}(L_0^n(d)) \tag{3.9}$$

is equivalent with the conditions

$$\left.\begin{array}{r} \text{either } r \geq m/2, \text{ or } r < m/2 \text{ and } \chi_Q(p) = 1 \\ \text{for each prime number } p \text{ appearing in } \mu \\ \text{to an odd power.} \end{array}\right\} \tag{3.10}$$

Proof. — If μ is a power of a prime number, the assertion follows from definitions and Proposition 2.13 of chapter 4 of the book Andrianov, Zhuravlev [13] (see also Andrianov [5], Proposition 4.2.19) with obvious changes. The general case follows then by [13], chapter 3, Theorem 3.12 (see also [5], Theorem 3.3.12). □

The following properties of the interaction sums are fundamental for applications to the action of Hecke operators on theta functions.

Theorem 3.4. — *Let Q be an even nonsingular matrix of an even order m, d the level of Q, and χ_Q the corresponding Dirichlet character modulo d. Let T be an homogeneous element of the Hecke ring $L_0^n(d)$, where $n \geq 1$, with $\mu(T) = \mu$ satisfying the condition*

$$T \in \psi^{m,n}(L_0^n(d)), \text{ if } n < m, \tag{3.11}$$

where $\psi^{m,n}$ is the Zharkovskaya homomorphism. Then, for each matrix $N \in \mathbb{Z}_n^m$, the interaction sum $I(N,Q,T)$ satisfies

$$I(N,Q,T) = \sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m;\mathcal{D}|N} I(\mathcal{D},Q,\psi^{n,m}T), \tag{3.12}$$

where

$$A(Q,\mu) = \{\mathcal{D} \in \mathbb{Z}^m_m; |\det \mathcal{D}| = \mu^{m/2}, \text{ the matrix } \mu^{-1}Q[\mathcal{D}] \text{ is even}\}$$

$$\Lambda^m = GL_m(\mathbb{Z}),$$

the condition $\mathcal{D} \mid N$ means that the matrix $\mathcal{D}^{-1}N$ is integral, and where

$$\psi^{n,m}T = T' \in L_0^m(d)$$

denotes an homogeneous element with $\mu(T') = \mu(T) = \mu$, satisfying the conditions

$$\begin{cases} T' = \psi^{n,m}T, & \text{if } n > m \\ T' = T, & \text{if } n = m \\ \psi^{m,n}T' = T, & \text{if } n < m. \end{cases}$$

We shall prove first three lemmas related to the action (2.20) of the Hecke operators $\mid T'$ for homogeneous $T' \in L_0^m(d)$ with $\mu(T') = \mu$ on the theta function

$$\Theta^m(V,Z) = \Theta(V,Z;H,Q),(V \in \mathbb{C}^m_{2m}, Z \in \mathbb{H}_m, H \in \mathcal{H}(Q)) \tag{3.13}$$

of genus m of Q,H with the coefficient form $P = 1$.

Lemma 3.5. — *Let Q be an even nonsingular matrix of an even order m with the level d, and let $T' \in L_0^m(d)$ be an homogeneous element with $\mu(T') = \mu$. Then the image of the theta function (3.13) under the Hecke operator $\mid T'$ can be written in the form*

$$(\Theta^m \mid T')(V,Z) = \sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m} c(\mathcal{D},Q,T')\,\Theta\left(\mu\mathcal{D}^{-1}V,Z;\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}]\right), \tag{3.14}$$

with some coefficients $c(\mathcal{D},Q,T)$ independent of V,Z,H.

Proof. — If μ is a power of a prime number p, the formula follows from Andrianov [3], Theorem 1. The general case follows, since by Andrianov, Zhuravlev [13], chapter 3, Theorem 3.12 or Andrianov [5], Theorem 3.3.12 each homogeneous T is a sum of products of homogeneous elements whose multipliers are powers of prime numbers. □

Lemma 3.6. — *The coefficients $c(\mathcal{D},Q,T)$ in (3.14) satisfy the relations*

$$\sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m;\mathcal{D}|N} c(\mathcal{D},Q,T') = I(N,Q,T') \tag{3.15}$$

for each matrix $N \in \mathbb{Z}^m_m$, where $I(N,Q,T')$ is the interaction sum (2.28).

Proof. — Using the abbreviation (2.30), one can rewrite the right hand side of (3.14) in the form

$$\sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m, N'\in\mathbb{Z}^m_m} c(\mathcal{D},Q,T')e(\mu\mathcal{D}^{-1}V,Z,\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}],N')$$

$$= \sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m, N'\in\mathbb{Z}^m_m} c(\mathcal{D},Q,T')e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q,\mathcal{D}N')$$

$$= \sum_{N\in\mathbb{Z}^m_m}\Bigg(\sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m,\mathcal{D}|N} c(\mathcal{D},Q,T')\Bigg)e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q,N).$$

On the other hand, by (2.33), the left hand side of (3.14) can be written in the form

$$\sum_{N\in\mathbb{Z}_m^m} I(N,Q,T')e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q,N).$$

Comparing the expansions we get (3.15), by the uniqueness of Fourier coefficients of the holomorphic and periodical in $V_1 \in \mathbb{C}_m^m$ function $(\Theta^m \mid T')((V_1,0),Z)$. □

LEMMA 3.6. — *The relations* (3.15) *with* $N = \mathcal{D}' \in A(Q,\mu)$ *turn into the equalities*

$$c(\mathcal{D}',Q,T') = I(\mathcal{D}',Q,T'), \quad (\mathcal{D}' \in A(Q,\mu)). \tag{3.16}$$

Proof. — In fact, if $\mathcal{D} \in A(Q,\mu)$ and $\mathcal{D} \mid \mathcal{D}'$, then $\mathcal{D}' = \mathcal{D}U$ with an integral matrix U of the determinant $\det U = \det\mathcal{D}/\det\mathcal{D}' = \pm1$, *i.e.* $U \in \Lambda^m$. □

Proof of Theorem 3.4. — The relations (3.15) and (3.16) prove the theorem in the case $n = m$.

If $n > m$ and $N \in \mathbb{Z}_n^m$, then it is well known that there is a matrix $U \in \Lambda^n$ such that $NU = (N',0)$ with $N' \in \mathbb{Z}_m^m$. Then, by (3.1) and (3.8) we have

$$I(N,Q,T) = I((N',0),Q,T) = I(N',Q,\psi^{n,m}T).$$

Since $T' = \psi^{n,m}T \in L_0^m(d)$, and T' is an homogeneous element with $\mu(T') = \mu(T)$, we can write

$$I(N',Q,T') = \sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m,\mathcal{D}\mid N'} I(\mathcal{D},Q,T'),$$

which proves the theorem in this case, because the conditions $\mathcal{D} \mid N'$ and $\mathcal{D} \mid N$ are clearly equivalent.

Finally, let $n < m$. Then, by (3.11), there is an homogeneous element $T' = \psi^{n,m}T \in L_0^m(d)$ with $\mu(T') = \mu(T)$ satisfying $\psi^{m,n}T' = T$. By the above, we have

$$I((N,0),Q,T') = \sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m;\mathcal{D}\mid(N,0)} I(\mathcal{D},Q,T')$$

for every $N \in \mathbb{Z}_n^m$. On the other hand, by (3.8), we conclude that

$$I((N,0),Q,T') = I(N,Q,\psi^{m,n}T') = I(N,Q,T).$$

The case follows, since $\mathcal{D} \mid (N,0)$ if and only if $\mathcal{D} \mid N$. □

Unfortunately, in order to prove the elementary formulated theorem 3.3 we had to use the nonelementary lemma 3.4 whose proof is based on a complicated factorization theory of standard Rankin polynomials in parabolic extensions of symplectic Hecke rings. It would be very interesting to find an elementary (and simple) proof of the theorem.

Finally, for future applications we shall formulate here important *composition relations* for the interaction sums. We shall prove the relations in the next section as an application of explicit formulas for the action of Hecke operators on theta functions.

PROPOSITION 3.7. — *Let Q be an even nonsingular matrix of even order m, of level d and character χ_Q. Let T,T' be two homogeneous elements of $L_0^n(d)$, where $n \geq 1$, such that $T \in \psi^{m,n}(L_0^m(d))$, if $n < m$. Then, for each matrix $N \in \mathbb{Z}_n^m$ the following relation holds*

$$I(N,Q,TT') = \sum_{\substack{(\mathcal{D},N') \in (A(Q,\mu)/\Lambda^m, \mathbb{Z}_n^m/\Lambda^n); \\ \mathcal{D}N' \in N\Lambda^n}} I(\mathcal{D},Q,\psi^{n,m}T)I(N',\mu^{-1}Q[\mathcal{D}],T') \tag{3.17}$$

where $\mu = \mu(T)$. In particular, if $n = m$ and $N \in A(Q,\mu\mu')$, where $\mu' = \mu(T')$, then

$$I(N,Q,TT') = \sum_{\substack{(\mathcal{D},N') \in (A(Q,\mu)/\Lambda^m, A(\mu^{-1}Q[\mathcal{D}],\mu')/\Lambda^m); \\ \mathcal{D}N' \in N\Lambda^m}} I(\mathcal{D},Q,T)I(N',\mu^{-1}Q[\mathcal{D}],T'). \tag{3.18}$$

4. Theta-theta formulas for Hecke operators

In this section we obtain explicit formulas expressing the images (2.20) of harmonic theta functions of integral nonsingular quadratic forms in an even number of variables under Hecke operators in the form of linear combinations with explicitly given coefficients of similar theta functions (the theta-theta formulas).

THEOREM 4.1. — *Let Q be an even nonsingular matrix of even order m,d the level of Q,χ_Q the Dirichlet character modulo d defined in Theorem 1.1, and H belongs to the majorant space (0.7) of Q. Let T be an homogeneous element with $\mu(T) = \mu$ of the Hecke ring $L_0^n(d)$, where $n \geq 1$, satisfying the condition*

$$T \in \psi^{m,n}(L_0^m(d)), \text{ if } n < m,$$

where $\psi^{m,n}$ is the Zharkovskaya homomorphism (3.3). Then the image (2.20) of the theta function $\Theta_P(V,Z;H,Q)$ of the pair Q,H of genus n with a harmonic coefficient form P under the Hecke operator $\mid T$ is a linear combination of theta functions of the form

$$\Theta_P(V,Z;H,Q) \mid T = \sum_{\mathcal{D} \in A(Q,\mu)/\Lambda^m} I(\mathcal{D},Q,\psi^{n,m}T)\Theta_{P|\mu^{-1}\mathcal{D}}(\mu\mathcal{D}^{-1}V,Z;\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}]), \tag{4.1}$$

where

$$(P \mid \mu^{-1}\mathcal{D})(U) = P(\mu^{-1}\mathcal{D}U) \quad (U \in \mathbb{C}_n^m) \tag{4.2}$$

and where the other notation are the same as in Theorem 3.4.

Proof. — By Proposition 2.1, we can write the left hand side of (4.1) in the form

$$\sum_{N\in C^n(Q/\mu)} I(N,Q,T)P(\mu^{-1}(N-\mu V_2))e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q;N),$$

where we use the abbreviation (2.30). Substituting here the expression (3.12) for the interaction sum $I(N,Q,T)$ we get the expression

$$\begin{aligned}
&\sum_{\substack{\mathcal{D}\in A(Q,\mu)/\Lambda^m\\ N\in\mathbb{Z}_n^m}} I(\mathcal{D},Q,\psi^{n,m}T)P(\mu^{-1}(\mathcal{D}N-\mu V_2))e(\mu V,Z,\mu^{-1}H,\mu^{-1}Q,\mathcal{D}N)\\
&=\sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m} I(\mathcal{D},Q,\psi^{n,m}T)\sum_{N\in\mathbb{Z}_n^m} P(\mu^{-1}\mathcal{D}(N-\mu\mathcal{D}^{-1}V_2))\times\\
&\qquad\times e(\mu\mathcal{D}^{-1}V,Z,\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}];N),
\end{aligned}$$

which proves the formula (4.1). □

As an application of the theorem we shall prove the proposition 3.7.

Proof of Proposition 3.7. — Applying the theorem in the case $P=1$, we get

$$\Theta(V,Z;H,Q)\mid T=\sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m} I(\mathcal{D},Q,\psi^{n,m}T)\Theta(\mu\mathcal{D}^{-1}V,Z;\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}]).$$

If we apply now the operator $\mid T'$ to the both sides of this relation, we get

$$\begin{aligned}
\Theta(V,Z;H,Q)|T|T' &= \Theta(V,Z;H,Q)\mid TT'\\
&=\sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m} I(\mathcal{D},Q,\psi^{n,m}T)(\Theta^{\mathcal{D}}\mid T')(V,Z),
\end{aligned}\tag{4.3}$$

where

$$\Theta^{\mathcal{D}}(V,Z)=\Theta(\mu\mathcal{D}^{-1}V,Z,\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}].$$

By Proposition 2.1, we have

$$\Theta(V,Z;H,Q)\mid TT'=\sum_{N\in C^n(Q/\mu\mu')} I(N,Q,TT')e(\mu\mu' V,Z,(\mu\mu')^{-1}H,(\mu\mu')^{-1}Q;N)$$

and

$$\begin{aligned}
(\Theta^{\mathcal{D}}\mid T')(V,Z) &= \sum_{N'\in C^n(\mu^{-1}Q[\mathcal{D}]/\mu')} I(N',\mu^{-1}Q[\mathcal{D}],T')\times\\
&\qquad\times e(\mu\mu'\mathcal{D}^{-1}V,Z,(\mu\mu')^{-1}H[\mathcal{D}],(\mu\mu')^{-1}Q[\mathcal{D}],N)\\
&=\sum_{N'\in C^n(\mu^{-1}Q[\mathcal{D}]/\mu')} I(N',\mu^{-1}Q[\mathcal{D}],T')\times\\
&\qquad\times e(\mu\mu' V,Z,(\mu\mu')^{-1}H,(\mu\mu')^{-1}Q;\mathcal{D}N')
\end{aligned}$$

(see (2.30)). Then the relation (4.3) turns into

$$\sum_{N\in\mathbb{Z}_n^m} I(N,Q,TT')e(\mu\mu' V,Z,(\mu\mu')^{-1}H,(\mu\mu')^{-1}Q;N) =$$
$$\sum_{\substack{\mathcal{D}\in A(Q,\mu)/\Lambda^m\\ N'\in\mathbb{Z}_n^m}} I(\mathcal{D},Q,\psi^{n,m}T)I(N',\mu^{-1}Q[\mathcal{D}],T')e(\mu\mu' V,Z,(\mu\mu')^{-1}H,(\mu\mu')^{-1}Q;\mathcal{D}N'),$$

where we have extended the summation sets for N and N' because of (2.34). If we set here $V_2 = 0$, compare corresponding Fourier coefficients of the holomorphic and periodical in V_1 function $\Theta((V_1,0),Z;H,Q) \mid TT'$, and use (3.1), we get the relations (3.17). The relations (3.18) follow from (3.17) because the conditions $\mathcal{D} \in A(Q,\mu)$, $N \in A(Q,\mu\mu')$, and $\mathcal{D} \mid N$ imply clearly that

$$N' = \mathcal{D}^{-1}N \in A(\mu^{-1}Q[\mathcal{D}],\mu'). \ \square$$

5. Harmonic theta series

In this section we assume again that Q is an even nonsingular matrix of an even order $m,(k,\ell)$ is the signature of the quadratic form with the matrix Q, and d is the level of Q. We assume also that H is a matrix of the majorant space (0.7) of Q and $P : \mathbb{C}_n^m \to \mathbb{C}^{ab}$ is a $\rho_+ \otimes \rho_-$-harmonic form relative to the pair Q,H with $n \geq 1$ in the sense of § 1.1.

The purpose of this section is to derive explicit transformation formulas under Hecke operators for the series of the form

$$\Theta_P(Z;H,Q \mid R) = \Theta_P((0,R),Z;H,Q) =$$
$$\sum_{N\in\mathbb{Z}_n^m} P(N-R)e\{XQ[N-R] + \sqrt{-1}YH[N-R]\} \tag{5.1}$$

(see (0.13)), where $R \in \mathbb{Q}_n^m$ is a rational $m\times n$-matrix. These series appears when one considers numerical characteristics of integral representations of quadratic forms by quadratic forms satisfying congruential conditions. The series (5.1) will be referred as *the theta series of the pair Q,H of genus n with the coefficient form P and the translation R.*

We shall consider the theta series (5.1) as an element of the space $\mathcal{G} = \mathcal{G}(n,ab)$ of all real analytic function $F : \mathbb{H}_n \to \mathbb{C}^{ab}$. If $F \in \mathcal{G}$ and $M = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix}$ is a real $2n \times 2n$-matrix satisfying the condition

$$^tMJ_nM = \mu(M)J_n \text{ with } \mu(M) > 0, \quad \left(J_n = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}\right) \tag{5.2}$$

we set

$$F \circ M = (F \circ M)(Z) = J_0(M,Z)^{-1} F((AZ + B)(CZ + \mathcal{D})^{-1}), \quad (Z \in \mathbb{H}_n) \tag{5.3}$$

where

$$J_0\left(\begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix}, Z \right) =$$
$$(\det(CZ + \mathcal{D}))^{(k-\ell)/2} |\det(CZ + \mathcal{D})|^{\ell} \rho_+(CZ + \mathcal{D}) \otimes \rho_-(CZ + \mathcal{D}) \tag{5.4}$$

(compare with (2.13),(2.14)). It follows from (2.15) that

$$J_0(M, M_1\langle Z\rangle) J_0(M_1, Z) = J_0(MM_1, Z),$$

whence

$$F \circ M \circ M_1 = F \circ (MM_1)$$

for every real $2n \times 2n$-matrices M, M_1 satisfying (5.2). In order to define the action of Hecke operators on the theta series (5.1), we shall follow the general pattern set forth in § 2 starting from the action (5.3) on $\mathcal{G}$ of the semigroups

$$S = S^n(h) = \left\{ M \in S_0^n(h); M = \begin{pmatrix} \mu(M)\mathbf{1}_n & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} (\bmod h) \right\} \tag{5.5}$$

(see (2.11)) and the corresponding *principal congruence subgroups*

$$\Gamma = \Gamma^n(h) = \{ M \in \Gamma^n; M \equiv \mathbf{1}_{2n} (\bmod h) \} \tag{5.6}$$

of the modular group (1.6) with such levels h that the function $F(Z) = \Theta_P(Z; H, Q \mid R)$ belongs to the subspace

$$\mathcal{G}(\Gamma^n(h)) = \{ F \in \mathcal{G}; F \circ M = F \text{ for all } M \in \Gamma^n(h) \} \tag{5.7}$$

of all $\Gamma^n(h)$-invariant elements of G. To find these levels, we first prove a lemma.

LEMMA 5.1. — *For given Q and R, let h be a natural number satisfying the conditions :*

$$hR \in \mathbb{Z}_n^m, \text{ and the matrix } hQ[R] \text{ is even.} \tag{5.8}$$

Then

$$\Theta_P((RC, R\mathcal{D}), Z; H, Q) = \Theta_P((0, R), Z; H, Q),$$

for each pair $C, \mathcal{D}$ of integral n-matrices satisfying $(C, \mathcal{D}) \equiv (0, \mathbf{1}_n) (\bmod h)$ and $\mathcal{D} \cdot {}^tC = C \cdot {}^t\mathcal{D}$.

Proof. — By the definition (see (0.12)), we have

$$\Theta_P((RC, R\mathcal{D}), Z; H, Q) =$$
$$= \sum_{N \in \mathbb{Z}_n^m} P(N - R\mathcal{D}) e\{X \cdot Q[N - R\mathcal{D}] + iY \cdot H[N - R\mathcal{D}] + 2 \cdot {}^t(RC)QN - {}^t(RC)QR\mathcal{D}\}.$$

(Since h divides C and the matrix hR is integral, it follows that the matrix $2\ {}^t(RC)QN$ has even trace. The trace of ${}^t(RC)QR\mathcal{D} = {}^tCQ[R]\mathcal{D}$ is also even, because it is equal to the trace of $Q[R]\mathcal{D} \cdot {}^tC = hQ[R]h^{-1}C \cdot {}^t\mathcal{D}$, the matrix $hQ[R]$ is even, and the matrix $h^{-1}C \cdot {}^t\mathcal{D}$ is integral and symmetric. It follows that the last series is equal to

$$\sum_{N \in \mathbb{Z}_n^m} P(N - R\mathcal{D})e\{X \cdot Q[N - R\mathcal{D}] + iY \cdot H[N - R\mathcal{D}]\}.$$

Since $N - R\mathcal{D} = N + R(\mathbf{1}_n - \mathcal{D}) - R$ and the matrix $R(\mathbf{1}_n - \mathcal{D}) = hR(h^{-1}(\mathbf{1}_n - \mathcal{D}))$ is integral, it follows that the last sum coincides with $\Theta_P((0,R),Z; H,Q)$. □

We can prove now the following :

Proposition 5.2. — *Let Q be a nonsingular even matrix of an even order m, let $R \in \mathbb{Q}_n^m$, and let h be a positive integer satisfying the following three conditions :*

$$hR \in \mathbb{Z}_n^m, \text{ the matrix } hQ[R] \text{ is even}, h \text{ is divisible by } d, \tag{5.9}$$

where d is the level of Q. Then the theta series (5.1) *satisfies*

$$\Theta_P((AZ + B)(CZ + \mathcal{D})^{-1}, H,Q \mid R) = J_0(M,Z)\Theta_P(Z; H,Q \mid R)$$

for each matrix $M = \begin{pmatrix} A & B \\ C & \mathcal{D} \end{pmatrix} \in \Gamma^n(h)$, where J_0 is the automorphy factor (5.4).

Proof. — Since h is divisible by d, we have $M \in \Gamma^n(d)$, and so

$$\Theta_P(M\langle Z\rangle; H,Q \mid R) = \Theta_P\big(((0,R)\ {}^tM^{-1})\ {}^tM, M\langle Z\rangle; H,Q\big)$$
$$= J_0(M,Z)\Theta_P((0,R)\ {}^tM^{-1}, Z; H,Q),$$

where $M\langle Z\rangle = (AZ + B)(CZ + \mathcal{D})^{-1}$, by Theorem 1.1. Since $\begin{pmatrix} A_1 & B_1 \\ C_1 & \mathcal{D}_1 \end{pmatrix} = {}^tM^{-1} \in \Gamma^n(h)$, and h satisfies (5.8), by Lemma 5.1 we can write

$$\Theta_P((0,R) \cdot {}^tM^{-1}, Z; H,Q) = \Theta_P((RC_1, R\mathcal{D}_1), Z; H,Q) =$$
$$\Theta_P((0,R),Z; H,Q) = \Theta_P(Z; H,Q \mid R). \ \square$$

The proposition shows that the theta series (5.1) belongs to the space (5.7) whenever h satisfies the conditions (5.9). These h are, in general, greater then the level d of Q. However, one can take $h = d$ in a number of important cases. For example, if

$$L \in \mathbb{Z}_n^m \text{ and } QL \equiv 0 (\text{mod } d),$$

then

$$\Theta_P(Z; H,Q \mid d^{-1}L) \in \mathcal{G}(\Gamma^n(d)). \tag{5.10}$$

Actually, it is sufficient to check that the matrix $dQ[d^{-1}L] = d^{-1}Q[L]$ is even. But the conditions on L imply that $L = dQ^{-1}L_1$ with an integral matrix L_1, and so the matrix $d^{-1}Q[L] = (dQ^{-1})[L_1]$ is even, since dQ^{-1} is even, by the definition of level.

We shall fix a positive integer h satisfying the conditions (5.9). By Andrianov [5] Lemma 3.3.1, the pair $\Gamma^n(h), S^n(h)$ is $\mathcal{E}$-finite, and so one can consider the Hecke ring

$$L^n(h) = D_{\mathbb{C}}\left(\Gamma^n(h), S^n(h)\right)$$

of the pair over $\mathbb{C}$, which is called *the Hecke ring of the group* $\Gamma^n(h)$ (over $\mathbb{C}$), and the linear representation

$$L^n(h) \ni T = \sum_i a_i(\Gamma^n(h)M_i) : F \to F \circ T = \sum_i a_i F \circ M_i$$

of the ring on the space $\mathcal{G}(\Gamma^n(h))$ given by Hecke operators. We shall derive explicit formulas for the action of the Hecke operators $\circ T$ with $T \in L^n(h)$ on the theta series (5.1) from the formulas (4.1) for certain operators $|\ T'$ with $T' \in L_0^n(d)$. In order to translate the Hecke operators from the language of the group $\Gamma_0^n(d)$ into the language of the group $\Gamma^n(h)$ and back, we briefly describe relations between the corresponding Hecke rings. For details and proofs, see Andrianov [5], § 3.3, especially, Theorem 3.3.3, Lemma 3.3.4, and Lemma 3.3.5, or Andrianov, Zhuravlev [13], Chapter 3, § 3, Theorem 3.3, Lemma 3.4, and Lemma 3.5.

Since d divides h, it follows that

$$\begin{aligned} S_0^n(d,h) &= \{M \in S_0^n(d); g.c.d(\mu(M),h) = 1\} \\ &= \Gamma_0^n(d)S^n(h) = S^n(h)\Gamma_0^n(d). \end{aligned} \tag{5.11}$$

The pair $\Gamma_0^n(d)$, $S_0^n(d,h)$ is clearly $\mathcal{E}$-finite, and the corresponding Hecke ring

$$L_0^n(d,h) = D_{\mathbb{C}}\left(\Gamma_0^n(d), S_0^n(d,h)\right)$$

is naturally a subring of $L_0^n(d)$,

$$L_0^n(d,h) \subset L_0^n(d),$$

because $S_0^n(d,h) \subset S_0^n(d)$. Let

$$T' = \sum_i a_i(\Gamma_0^n(d)M_i) \in L_0^n(d,h).$$

Without loss of generality we may assume, by (5.11), that all of the representative M_i of the left cosets $\Gamma_0^n(d)M_i$ belong to $S^n(h)$. Then we obviously have

$$T = \eta(T') = \sum_i a_i(\Gamma^n(h)M_i) \in L^n(h),$$

and the map η is a homomorphic embedding of the ring $L_0^n(d,h)$ into $L^n(h)$. In fact, η is a ring isomorphism, and the inverse isomorphism

$$\zeta : L^n(h) \longrightarrow L_0^n(d,h)$$

is determined by

$$\zeta : L^n(h) \supset \sum_j b_j(\Gamma^n(h)N_j) \longrightarrow \sum_j b_j(\Gamma_0^n(d)N_j).$$

Using the isomorphism of Hecke rings, we can transfer the definition of interaction sums considered in § 3 to the elements T of $L^n(h)$ by

$$I(N,Q,T) = I(N,Q,\zeta(T)), \quad \bigl(N \in \mathbb{Z}^m_n, T \in L^n(h)\bigr) \tag{5.12}$$

and we can carry over the above Zharkovskaya homomorphism (3.3) to the rings $L^n(h)$: for $1 \le r \le n$ we define the homomorphism

$$\psi^{n,v} = \psi^{n,r}_Q : L^n(h) \longrightarrow L^r(h) \tag{5.13}$$

starting from the condition of commutativity of the diagram

$$\begin{array}{ccc} L^n_0(d,h) & \xrightarrow{\eta} & L^n(h) \\ \downarrow\psi^{n,r} & & \downarrow\psi^{n,r} \\ L^r_0(d,h) & \xrightarrow{\eta} & L^r(h). \end{array} \tag{5.14}$$

(Note that, clearly, $\psi^{n,r}(L^n_0(d,h)) \subset L^r_0(d,h)$).

The principal result of this section can be now formulated in the following form.

Theorem 5.3. — *Let Q be an even nonsingular matrix of an even order m and level d, and let $H \in \mathcal{H}(Q)$. Let $\Theta_P(Z;H,Q \mid R)$ be a theta series* (5.1) *of genus $n \ge 1$ of the pair Q,H with a harmonic form P and a rational translation R. Finally, let h be a positive integer satisfying the conditions* (5.9) *relative Q and R, and*

$$T = \sum_i a_i(\Gamma^n(h)M_i)$$

an element of the Hecke ring $L^n(h)$, where matrices $M_i \in S^n(h)$ have a fixed multiplier $\mu(M_i) = \mu$, and such that

$$T \in \psi^{m,n}(L^m(h)), \text{ if } m > n,$$

where $\psi^{m,n}$ is the Zharkovskaya map (5.13)*. Then the image of the theta series under the Hecke operator $\circ T$ defined by*

$$\Theta_P(Z;H,Q \mid R) \circ T = \sum_i a_i J_0(M_i,Z)^{-1}\Theta_P(M_i\langle Z\rangle;H,Q \mid R)$$

(see (5.3) *and* (5.4)*) is equal to the sum*

$$\sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m} I(\mathcal{D},Q,\psi^{n,m}T)\Theta_{P|\mu^{-1}\mathcal{D}}(Z;\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}] \mid \mu\mathcal{D}^{-1}R) \tag{5.15}$$

where $\Lambda^m = GL_m(\mathbb{Z})$,

$$A(Q,\mu) = \{\mathcal{D} \in \mathbb{Z}^m_m; |\det\mathcal{D}| = \mu^{m/2}, \text{ the matrix } \mu^{-1}Q[\mathcal{D}] \text{ is even}\},$$

$I(\ )$ are the interaction sums (5.12)*, $\psi^{n,m}$ with $n \ge m$ means the Zharkovskaya map* (5.13)*, and $\psi^{n,m}T$ with $n < m$ stands for a linear combination of the left cosets $(\Gamma^m(h)M'_j)$ with*

$\mu(M'_j) = \mu$ belonging to the inverse image $(\psi^{m,n})^{-1}L^n(h)$, and where $(P \mid \mu^{-1}\mathcal{D})(U) = P(\mu^{-1}\mathcal{D}U)$.

Proof. — By the formula (4.1) for

$$T' = \zeta(T) = \sum_i a_i(\Gamma_0^n(d)M_i) \in L_0^n(d,h) \subset L_0^n(d)$$

and $V = (0,R)$, we get the formula

$$\begin{aligned}&\sum_i a_i J(M_i,Z)^{-1}\Theta_P((0,R)\cdot {}^tM_i,M_i\langle Z\rangle; H,Q)\\ &= \sum_{\mathcal{D}\in A(Q,\mu)/\Lambda^m} I(\mathcal{D},Q,\psi^{n,m}T')\Theta_{P|\mu^{-1}\mathcal{D}}(\mu\mathcal{D}^{-1}(0,R),Z,\mu^{-1}H[\mathcal{D}],\mu^{-1}Q[\mathcal{D}]).\end{aligned} \tag{5.16}$$

Since all of M_i belong to $S^n(h)$, it follows from (2.14), (5.4), and Lemma 5.1 that $J(M_i,Z) = J_0(M_i,Z)$, and

$$\Theta_P((0,R)\,{}^tM_i,M_i\langle Z\rangle; H,Q) = \Theta_P(M_i\langle Z\rangle; H,Q \mid R).$$

Hence the left hand side of (5.16) is equal to $\Theta_P(Z; H,Q \mid L)\circ T$. As to the right hand side, we have $I(\mathcal{D},Q,\psi^{n,m}T') = I(\mathcal{D},Q,\psi^{n,m}T)$, by (5.12), since the diagram (5.14) is commutative, and so it coincides with the sum (5.15). □

Remark 5.4. — Since det Q and μ are coprime, it easily follows that $\mu\mathcal{D}^{-1}$ belongs to $A(Q,\mu)$ for each $\mathcal{D} \in A(Q,\mu)$. In particular, the matrix $\mu\mathcal{D}^{-1}$ is integral.

Bibliography

[1] Andrianov a.n. — *The multiplicative arithmetic of Siegel modular forms*, Uspekhi Mat. Nauk **34:1** (1979), 67–135 ; English transl. in Russian Math. Surveys **34:1** (1979), 75–148.

[2] Andrianov a.n. — *Integral representations of quadratic forms by quadratic forms : multiplicative properties*, Proc. Intern. Congress of Mathematicians, (1983), Warszawa, **vol. 1**, 465–474, DWN-Polish Sci. Publ., Warszawa, Nort-Holland, Amsterdam, New York, Oxford, 1984.

[3] Andrianov a.n. — *The action of Hecke operators on nonhomogeneous theta series*, Matem. Sbornik **131:3** (1986), 275–292 ; English transl. in Math. USSR Sbornik **59:2** (1988), 269–285.

[4] Andrianov a.n. — *The calculation of coefficients in the formulas for the transformations of inhomogeneous theta series under the action of Hecke operators*, Zapiski Nauchn. Sem. LOMI **160** (1987), 9–15 ; English transl. in J. Soviet Math. **52:3** (1990), 2999–3004.

[5] Andrianov a.n. — *Quadratic forms and Hecke operators*, Grundlehren Math. Wiss. Band **286**, Springer-Verlag Berlin, Heidelberg, 1987.

[6] Andrianov a.n. — *Spherical theta series*, Matem. Sbornik **134:3** (1987), 291–305 ; English transl. in Math. USSR Sbornik **62:2** (1989), 289–304.

[7] Andrianov a.n. — *Multiplicative properties of solutions of quadratic Diophantine problems*, Algebra i Analis **2:1** (1990), 3–46 ; English transl. in Leningrad Math. J. **2:1** (1991), 1–39.

[8] Andrianov a.n. — *Composition of solutions of quadratic Diophantine equations*, Uspekhi Mat. Nauk **46:2** (1991), 3–40 ; English transl. in Russian Math. Surveys **46:2** (1991), 1–44.

[9] Andrianov a.n. — *Queen's lectures on arithmetical composition of quadratic forms*, Queen's papers in Pure and Applied Mathematics, Queen's University, **vol. 92**, 1992.

[10] Andrianov a.n. — *Factorizations of integral representations of binary quadratic forms*, Algebra i Analis **5:1** (1993), 3–29 ; English transl. in St. Petersburg Math. J. **5:1** (1994), 1–27.

[11] Andrianov a.n. — *Symmetries of harmonic theta functions of integral quadratic forms*, Uspekhi Mat. Nauk **50:4** (1995), 3–44 ; English transl. in Russian Math. Surveys **50:4** (1995).

[12] Andrianov a.n., Maloletkin g.n. — *The behaviour of theta series of genus n of indefinite quadratic forms under modular substitutions*, Trudy Mat. Inst. Steklov **148** (1978), 5–15 ; English transl. in Proc. Steklov Inst. Math. **4** (1980), 1–12.

[13] Andrianov a.n., Zhuravlev v.g. — *Modular forms and Hecke operators*, Nauka, Moskow, 1990,Transl. of Math. Monographs, **145**, AMS, Providens, Rhode Island, 1995.

[14] Freitag e. — *Die Invarianz gewisser von Thetareihen erzeugter Vektorräume unter Heckeoperatoren*, Math. Z. **156** (1977), 141–155.

[15] Freitag e. — *Berichtigung zu der Arbeit "Die Invarianz gewisser von Thetareihen erzeugter Vektorräume unter Heckeoperatoren"*, Math. Z. **168** (1979), 289–290.

[16] Freitag e. — *Eine Bemerkung zu Andrianovs expliziten Formelu für die Wirkung der Heckeoperatoren auf Thetareihen*, In E.B. Christofel, Basel Aachen : Birkhäuser (1981), 336–351.

[17] Freitag e. — *Die Wirkung von Heckeoperatoren auf Thetareihen mit harmonischen Koeffizienten*, Math. Ann. **258** (1982), 419–440.

[18] Freitag e. — *Singular Modular forms and Theta relations*, Lecture notes in Math. **1487**, Springer, Berlin Heidelberg New York, 1986.

[19] Gauss c.f. — *Disquisitiones arithmeticae*, Fleischer, Lipsiae, Leipzig (= Werke, Bd. I), 1801.

[20] Hecke e. — *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktenwicklung*, I, II. Math. Ann. **114** (1937), 1–28, 316–351 (=Math. Werke, 644–707).

[21] HECKE E. — *Analytische Arithmetik der positiven quadratischen Formen*, Danske Vidensk. Selsk. Math. Fys. Meddel XVII, Copenhagen **12** (1940), (=Math. Werke, 789–918).

[22] LINNIK JU.V. — *Quaternions and Caley numbers ; some applications of quaternion arithmetic (in Russian)*, Uspekhi Mat. Nauk **4:5** (1949), 49–98.

[23] MAASS H. — *Die Primzahlen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **124** (1951), 87–122.

[24] PETERSSON H. — *Konstruktion der sämtlichen Lösungen einer Riemannschen Funktionalgleichung durch Dirichletreichen mit Eulerscher Produktentwicklung*, I, II, III, Math. Ann. **116** (1939), 401–412, **117**, (1940–1941), 39–64, 277–300.

[25] SCHOENEBERG B. — *Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen*, Math. Ann. **116** (1939), 511–523.

[26] SIEGEL C.L. — *On the theory of indefinite quadratic forms*, Ann. of Math. **45** (1944), 577–622 (=Ges. Abh, Springer, 1966 Bd 2, 421–466).

[27] SIEGEL C.L. — *Lectures on quadratic forms, Notes by K.G. Ramanathan*, Tata Inst. of Fund. Res., Bombay, 1955–1956 (Reissued 1967).

[28] SHIMURA G. — *On modular forms of half integral weight*, Ann. of Math. **97:3** (1973), 440–481.

[29] SUGAWARA M. — *On the transformation theory of Siegel's modular group of the n-th degree*, Proc. Imp. Acad. Japan **13** (1937), 335–336.

[30] SUGAWARA M. — *An invariant property of Siegel's modular functions*, Proc. Imp. Acad. Japan **14** (1938), 1–3.

[31] WOHLFAHRT K. — *Über Operatoren Heckescher Art bei Modulformen reeller Dimension*, Math. Nach. **16** (1957), 233–256.

[32] ZHURAVLEV V.G. — *Representations of Hecke rings on theta series of uneven quadratic forms*, Vsesoysnaya Konferencia "Teoria Chisel i ee prilozenia", Tbilisi (in Russian) (1985), 81–82.

[33] ZHURAVLEV V.G. — *Generalized Eichler-Brandt matrices, Hecke operators, and vector-valued theta-series*, Algebra i Analis **5:3** (1993), English transl. in St. Petersburg Math. J. **5:3** (1994), 545–576.

– ◊ –

Université de Grenoble I
Institut Fourier
UMR 5582 du CNRS et de l'UJF
UFR de Mathématiques
B.P. 74
38402 ST MARTIN D'HÈRES Cedex (France)
&
St. Petersburg Branch of the
Steklov Mathematical Institute
Russian Academy of Sciences,
Fontanka 27,
191011 ST PETERSBURG (Russia)
e-mail : anandr@pdmi.ras.ru

(20 mai 1996)