GEOMETRY OF CENTERS OF ANALYTIC DISCS AND DOMAINS OF DEPENDENCE

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Abstract. In this paper we introduce a new technique of *spinning analytic discs*, which allows us to control the direction of CR extension of CR functions from rigid CR manifolds.

1. Introduction

In this paper we study the geometry of centers of analytic discs for rigid CR manifolds, and domains of dependence for the CR extension of CR functions from such manifolds.

Let M be a real $(2n+k)$-dimensional manifold which has a locally closed embedding in $\mathbb{C}^{k+n}$. Then M is said to be a *generically embedded CR manifold of type* (n, k), provided that $\dim_{\mathbb{C}}(T_pM \cap JT_pM) \equiv n$, where J is the complex structure operator on $\mathbb{R}^{2(k+n)} \simeq \mathbb{C}^{k+n}$. Note that M has real codimension k. Associated with such a manifold, there is the system of tangential CR equations, $\overline{\partial}_M f = 0$, which define the *CR functions* f on M. For a detailed discussion of these concepts, the reader can consult [HN1], [HN2].

There is a sizable literature devoted to the study of the local holomorphic, or CR, extension of CR functions f, which are defined and CR in a small neighborhood of a point $p \in M$. For example one can consult Baouendi and Rothschild [BR1], [BR2], [BR3], [BR4], Baouendi, Rothschild and Trépreau [BRTp], Baouendi, Rothschild and Trèves [BRTv], Boggess and Pitts [BPi], Hanges and Trèves [HaT], Trépreau [Tr1], [Tr2], [Tr3], and Tumanov [Tu1], [Tu2], as well as the many references found there to earlier work. Insofar as the local extension of CR functions is concerned, a main issue has been to attempt to gain some control over the "directions", "size" and "shape" of the place where the CR functions extend.

In this paper we take a slightly different point of view: Given a direction in which it might be possible to expect CR extension, we investigate just when it is possible, and determine a "domain of dependence" for the extension; i.e., a region on M of a specific size and shape, where the given Cauchy data f suffices to accomplish the extension in the given direction. Thus we are taking here a semi-global, rather than a local approach to the CR extension problem. Actually in this paper we consider manifolds M which have a global presentation as $M\colon y = h(w)$, where $(z, w) \in \mathbb{C}^k \times \mathbb{C}^n$, $z = x + iy$, $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_k)$, $h = (h_1, \dots, h_k)$ and $w = (w_1, \dots, w_n)$. Such manifolds have been called rigid in [BRTv].

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Our approach is to use analytic discs, with boundaries on M, and to study the influence of their centers. An *analytic disc* is a holomorphic mapping g of the unit disc of the complex plane (in some smoothness class on the closure) into $\mathbb{C}^{k+n}$. By the *center* of an analytic disc we mean $g(0)$.

We now describe briefly our main results: Let $q = g(0)$ be a center which lies in $\mathcal{N}_pM \cong \mathbb{R}^k$, the normal space to M at $p \in M$, and consider the line segment $\overline{pq}$ in $\mathcal{N}_pM$. Then we can find an arc γ_p in $\mathcal{N}_pM$, which starts at p, and whose tangent direction at p makes an arbitrarily small angle with $\overline{pq}$. Each point of γ_p lies on the image of some analytic disc with its boundary on M. As p' moves around on M, so as to sweep out some open neighborhood of p, these arcs $\gamma_{p'}$ sweep out a "rib" $\tilde{M}_p$ attached to M near p (see Theorem 6.4). This means that $\tilde{M}_p$ is a real $(2n+k+1)$-dimensional generically embedded CR manifold of type $(n+1, k-1)$, which has M as its (partial) boundary. The analytic discs used in the construction of the rib $\tilde{M}_p$ all have their boundaries contained in an open neighborhood $\mathcal{D}_p$ of p, which is "thin" in the w-directions, and "long" in the x-directions. Moreover, we show that any f which is CR on this domain of dependence $\mathcal{D}_p$, has a unique CR extension to the rib $\tilde{M}_p$ (see Theorem 7.3). Suppose next that we have l such centers $q_1, q_2, \ldots, q_l$ in $\mathcal{N}_pM$ with $\overline{pq_1}, \overline{pq_2}, \ldots, \overline{pq_l}$ linearly independent. Then the same CR function f has a unique CR extension to a "thin wedge", of real dimension $2n+k+l$, whose intersection with $\mathcal{N}_pM$ is a curved l-dimensional simplex spanned by the arcs $\gamma_p^{(1)}, \gamma_p^{(2)}, \ldots, \gamma_p^{(l)}$ (see Theorem 8.1). Finally assume that at $p \in M$, every open half-space in $\mathcal{N}_pM$ contains such a center q. Then our same CR function f has a unique holomorphic extension to a full open neighborhood of p in $\mathbb{C}^{k+n}$ (see Theorem 9.1).

We introduce here a new technique, that of *spinning analytic discs* (see Section 5). It is used in Section 6 in the construction of a typical rib, and allows control over the direction of the rib, while at the same time giving a domain of dependence.

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2. Preliminaries

Let M be a rigid CR manifold of type (n, k) which is generically embedded in $\mathbb{C}^{k+n}$. This means that CR-$\dim_{\mathbb{C}} M = n$ and $\operatorname{codim}_{\mathbb{R}} M = k$. Consider a point p on M, and let us assume that p is the origin, and the defining equations for M are

$$M : \begin{cases} y = h(w) \\ h(0) = 0, \quad dh(0) = 0, \end{cases} \tag{2.1}$$

where h is a smooth map $: \mathbb{C}^n \to \mathbb{R}^k$ defined near the origin.

Next we consider a point p on M near the origin. We note that h depends on w only. In a small neighborhood of the origin we can fix a mapping $\Omega\colon M \ni p \to \Omega_p \in AU(\mathbb{C}^{k+n})$, which depends smoothly on p, where $AU(\mathbb{C}^{k+n})$ is the space of affine unitary transformations, such that for each $p \in M$, $\Omega_p\colon \mathbb{C}^{k+n} \to \mathbb{C}^{k+n}$ arranges that

$$\Omega_p(p) = (0,0), \quad \Omega_{p*}(T_pM) = \{y = 0\}, \quad \Omega_{p*}(H_pM) = \{z = 0\}, \tag{2.2}$$

where $(z, w) \in \mathbb{C}^k \times \mathbb{C}^n$ and $z = x + iy$. The map Ω can be given explicitly in terms of h, but in this paper we need only some of its properties, not explicit formulas.

Using the standard inner product on $\mathbb{C}^k \times \mathbb{C}^n$, we have a natural choice of normal space at p, with

$$\mathcal{N}_p M = \Omega_{p*}^{-1}(\{x = 0,\ w = 0\}). \tag{2.3}$$

Then for each $p \in M$ we can express $\Omega_p M$ locally as a graph over its tangent space $\{y = 0\}$ at the origin:

$$\Omega_p M\colon \begin{cases} y = h^p(w) \\ h^p(0) = 0,\ dh^p(0) = 0, \end{cases} \tag{2.4}$$

where h^p is a smooth map $: \mathbb{C}^n \to \mathbb{R}^k$, defined near the origin.

Here we collect in a condensed form some essential background material. For more complete details see [HT].

Fix an α with $0 < \alpha < 1$; we use the standard Hölder spaces: let K be a compact subset of $\mathbb{R}^m$, then

$$C^\alpha(K) = \{u\colon K \longrightarrow \mathbb{R}\,;\quad |u|_\alpha \ \equiv\ \sup_{x\in K}|u(x)| + \sup_{x,y\in K}\frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty\}.$$

Then $C^\alpha(K)$ is a Banach algebra with respect to the $|\cdot|_\alpha$ norm.

We use D to denote the unit disc $\{\zeta \in \mathbb{C};\ \ |\zeta| < 1\}$, $S^1 = \partial D$, and $\mathcal{O}(D)$ to denote the space of holomorphic functions in D. When $K = S^1$ we indicate the norm by $|\cdot|_\alpha$, and when $K = \overline{D}$ we indicate it by $|\cdot|_\alpha^{\overline{D}}$. For $u \in \mathcal{O}(D) \cap C^\alpha(\overline{D})$ these two norms are equivalent:

$$|u|_\alpha \le |u|_\alpha^{\overline{D}} \le C_\alpha |u|_\alpha.$$

The operator

$$(Tx)(e^{it}) = \frac{1}{2\pi} p.v. \int_0^{2\pi} x(e^{i\theta})\, \Im\left(\frac{e^{i\theta}+e^{it}}{e^{i\theta}-e^{it}}\right) d\theta, \tag{2.5}$$

is known as the Hilbert transform on S^1. It is a bounded linear operator T : $C^\alpha(S^1) \longrightarrow C^\alpha(S^1)$. The significance of T is as follows. Let $y = Tx$ with $x \in C^\alpha(S^1)$ and let U be the unique harmonic function in D having boundary values x. Then $U \in C^\alpha(\overline{D})$. Construct the unique conjugate harmonic function V such that $V(0) = 0$. Then also $V \in C^\alpha(\overline{D})$. It follows that $f = U + iV \in \mathcal{O}(D) \cap C^\alpha(\overline{D})$, $\Im f(0) = 0$, and on S^1 has the boundary values $x + iy$.

We denote by $E_\zeta(x)$, $\zeta \in D$, the value of the Poisson integral at ζ, i.e.,

$$E_\zeta(x) = \frac{1}{2\pi}\int_0^{2\pi} x(e^{i\theta})\frac{1-|\zeta|^2}{|e^{i\theta}-\zeta|^2} d\theta. \tag{2.6}$$

For $\zeta \in S^1$ we have that $E_\zeta(x) = x(\zeta)$, provided that $x(\cdot)$ is continuous at ζ. In particular, if $\zeta = 0$, we obtain the mean value operator

$$E_0(x) = E(x) = \frac{1}{2\pi}\int_0^{2\pi} x(e^{i\theta})\, d\theta. \tag{2.7}$$

Later we need a modified operator T_ζ, namely $T_\zeta = T - E_\zeta T$.

3. Lifted analytic discs

A map $g\colon \overline{D} \longrightarrow \mathbb{C}^{k+n}$, with each component belonging to $\mathcal{O}(D)$ and to some differentiability class on $\overline{D}$, such as $C^\alpha(\overline{D})$, will be called an *analytic disc* in $\mathbb{C}^{k+n}$. The restriction of g to S^1 will be called the *boundary of the disc*, and the point $g(0)$ will be called the *center of the disc.*

We shall be interested in analytic discs whose boundaries lie on M, where M is as in (2.1). By a *parameter disc* we mean a pair $(c,\, w(\cdot))$, where c is a column vector of constants in $\mathbb{R}^k$ and $w(\cdot)$ belongs to $[\mathcal{O}(D)\cap C^\alpha(\overline{D})]^n$. We measure the size of a parameter disc in a natural way by $|c| + |w(\cdot)|_\alpha^{\overline{D}}$, where $|c|$ denotes the Euclidean norm. In the rigid case, to lift a parameter disc to M, i.e., the boundary of the lifted disc $g(\zeta) = (z(\zeta), w(\zeta))$ is on M, is a simple process. Namely we have

$$
\begin{aligned}
\Im z(e^{i\theta}) &= y(e^{i\theta}) = h(w(e^{i\theta})),\\
\Re z(e^{i\theta}) &= x(e^{i\theta}) = c - [Th(w(\cdot))](e^{i\theta})\\
&= c - \frac{1}{2\pi} p.v. \int_0^{2\pi} h(w(e^{it}))\, \Im\left(\frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}}\right) dt.
\end{aligned}
\tag{3.1}
$$

In this standard lifting we have

$$
\Re z(0) = x(0) = c. \tag{3.2}
$$

To get a parameter disc from a lifted disc $g(\zeta) = (z(\zeta), w(\zeta))$ is very easy. Simply we take $(\Re z(0),\, w(\cdot))$. So we see that lifted discs are in one-to-one correspondence with parameter discs.

The center of a lifted analytic disc $g(\zeta) = (z(\zeta), w(\zeta))$ is the point $(z(0), w(0))$. If additionally we assume that $c = 0$ and $w(0) = 0$, then the center of the lifted disc is of the form $(iy(0), 0)$, i.e., it lies on the space $i\mathbb{R}^k \times \{0\}$. In an equivalent way, the center $(iy(0), 0)$ is given by the formula

$$
y(0) = \frac{1}{2\pi} \int_0^{2\pi} h(w(e^{i\theta}))\, d\theta. \tag{3.3}
$$

We denote by $\mathcal{B}$ the Banach space

$$
\begin{aligned}
\mathcal{B} &= \{w(\cdot) \in [\mathcal{O}(D)\cap C^\alpha(\overline{D})]^n \quad | \quad w(0) = 0\},\\
\mathcal{B}_\delta &= \{w(\cdot) \in \mathcal{B} \quad | \quad |w|_\alpha < \delta\}.
\end{aligned}
\tag{3.4}
$$

4. Swinging analytic discs

Now we take a real parameter $0 \le \lambda \le 1$ and consider the parameter disc $\zeta \to w(\zeta) - w(\lambda)$. We lift this disc in a modified way (see [BPo]), so as to have

$\Re z(\lambda) = c = 0$:

(4.1)

$$\Im z_\lambda(e^{i\theta}) = y_\lambda(e^{i\theta}) = h(w(e^{i\theta}) - w(\lambda))$$

$$\Re z_\lambda(e^{i\theta}) = x_\lambda(e^{i\theta}) = -[T_\lambda h(w(\cdot) - w(\lambda))](e^{i\theta})$$

$$= -\frac{1}{2\pi} p.v. \int_0^{2\pi} h(w(e^{it}) - w(\lambda)) \Im \left[\frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} - \frac{e^{it} + \lambda}{e^{it} - \lambda} \right] dt,$$

where T_λ is the operator defined at the very end of §2.

We note that the image in $\mathbb{C}^n$ of $w(\cdot) - w(\lambda)$ is simply a rigid translation of the image of $w(\cdot)$. We denote the lifted disc given in (4.1) by

$$g_\lambda(\zeta) = (z_\lambda(\zeta), w(\zeta) - w(\lambda)).$$

We have that $x_\lambda(\lambda) = 0$. If $\lambda = 0$ and $w(0) = 0$, then we obtain the analytic disc which was discussed just before (3.3). If $\lambda = 1$, then the boundary of the disc $g_1(\zeta)$ passes through the origin, namely $g_1(1) = 0$. Note that this corresponds to a swinging of the original lifted disc.

For each such $w(\zeta)$ we consider the mapping $\lambda \to g_\lambda(\lambda)$, $0 \le \lambda \le 1$. Since $w(\lambda) - w(\lambda) = 0$ and $x_\lambda(\lambda) = 0$, the mapping takes the form $\lambda \to (iy_\lambda(\lambda), 0)$ and can be considered as a mapping from $[0,\, 1] \times \mathcal{B}$ into $\mathbb{R}^k$ (here and in what follows, we abuse notation; e.g., by $\varphi\colon \mathcal{B} \to \mathbb{R}^k$ the reader should understand it is meant that φ may be only defined in a suitably small neighborhood of the origin in $\mathcal{B}$.) For each λ we denote this mapping by F_λ, $F_\lambda\colon \mathcal{B} \to \mathbb{R}^k$; it is given by the formula

$$F_\lambda(w) = y_\lambda(\lambda) = \frac{1-\lambda^2}{2\pi} \int_0^{2\pi} \frac{h(w(e^{i\theta}) - w(\lambda))}{|e^{i\theta} - \lambda|^2} d\theta, \tag{4.2}$$

obtained by plugging $y_\lambda(\cdot) = h(w(\cdot) - w(\lambda))$ into (2.6).

For each fixed $w \in \mathcal{B}$, the mapping $\lambda \longrightarrow (iF_\lambda(w),\, 0)$ gives a curve in the normal space to M at $p = 0$. This mapping will be used in construction of "ribs" of one dimension higher than $\dim_{\mathbb{R}} M$.

5. Spinning analytic discs

Consider the holomorphic mapping $\psi = \psi(\nu, z)$ of the unit disc D onto itself given by

$$\psi(\nu, z) = \left(\frac{\sqrt{\frac{z+i}{iz+1}} - 1}{\sqrt{\frac{z+i}{iz+1}} + 1} \right)^{2\nu}. \tag{5.1}$$

We note that the mapping $\psi(1, z)$ is a conformal mapping of the unit disc D onto $D_1 = D - [0, 1)$, and $\psi(1, 1) = 0$. Moreover, $\psi(1, z)$ can be holomorphically extended to a neighborhood of $\overline{D} \backslash \{\pm i\}$. Obviously also we have $\psi(\nu, z) = [\psi(1, z)]^\nu$, therefore the image $\psi(\nu, D)$ can be considered as a ν-fold covering of the unit disc. We denote by S_ν the boundary of the image $\psi(\nu, D)$ (in the corresponding Riemann surface) oriented counterclockwise. S_ν consists of the segment from 0 to 1, counted twice with opposite orientations, and the boundary of the unit disc counted ν-times with orientation counterclockwise.

We prove the following lemma.

Lemma 5.1. *Let $H = H(\zeta) \in C^1(V) \cap C^0(\overline{D})$ with $H(0) = 0$, where V is a neighborhood of 0 in $\mathbb{C}$. Then*

$$\int_0^{2\pi} \frac{H(\psi(\nu, e^{i\theta}))}{|e^{i\theta} - 1|^2}\, d\theta = \frac{1}{8\nu} \int_{S_\nu} H(\xi) \left[\xi^{-\frac{1}{2\nu}} - \xi^{\frac{1}{2\nu}}\right] \frac{1}{\xi}\, d\xi. \tag{5.2}$$

Proof. First we note that the integral on the left hand side of (5.2) converges since $H(0) = 0$, $\psi(\nu, 1) = 0$, and $\frac{d\psi}{dz}(\nu, 1) = 0$.

The inverse of the map $\psi(\nu, z)$, considered from the Riemann surface onto the unit disc, is

$$\begin{aligned} z(\xi) = \psi^{-1}(\nu, \xi) &= -i \frac{1 + i\left(\frac{1+\xi^{\frac{1}{2\nu}}}{1-\xi^{\frac{1}{2\nu}}}\right)^2}{1 - i\left(\frac{1+\xi^{\frac{1}{2\nu}}}{1-\xi^{\frac{1}{2\nu}}}\right)^2} \\ &= \frac{(1-i) + 2(1+i)\xi^{\frac{1}{2\nu}} + (1-i)\xi^{\frac{1}{\nu}}}{(1-i) - 2(1+i)\xi^{\frac{1}{2\nu}} + (1-i)\xi^{\frac{1}{\nu}}} \\ &= \frac{1 + 2i\xi^{\frac{1}{2\nu}} + \xi^{\frac{1}{\nu}}}{1 - 2i\xi^{\frac{1}{2\nu}} + \xi^{\frac{1}{\nu}}} \end{aligned} \tag{5.3}$$

The integral on the left-hand side of (5.2) can be rewritten as

$$\begin{aligned} \int_0^{2\pi} \frac{H(\psi(\nu, e^{i\theta}))}{|e^{i\theta} - 1|^2}\, d\theta &= \frac{1}{i} \int_0^{2\pi} \frac{H(\psi(\nu, e^{i\theta}))}{|e^{i\theta} - 1|^2} \frac{de^{i\theta}}{e^{i\theta}} \\ &= \frac{1}{i} \int_{S^1} \frac{H(\psi(\nu, z))}{|z-1|^2 z}\, dz \\ &= i \int_{S^1} \frac{H(\psi(\nu, z))}{(z-1)^2}\, dz. \end{aligned}$$

After the change of variables $\xi = \psi(\nu, z)$, the last integral becomes

$$i \int_{S_\nu} \frac{H(\xi)}{(z(\xi) - 1)^2} \frac{dz(\xi)}{d\xi}\, d\xi. \tag{5.4}$$

Before we calculate the above integral, we would like to simplify the expression

$$\frac{1}{(z(\xi)-1)^2}\frac{dz(\xi)}{d\xi}$$

From the form of $z(\xi)=\psi^{-1}(\nu,\xi)$ in (5.3), we have

$$z(\xi)-1=\frac{4i\xi^{\frac{1}{2\nu}}}{1-2i\xi^{\frac{1}{2\nu}}+\xi^{\frac{1}{\nu}}},$$

and hence

$$\frac{dz}{d\xi}=\frac{2i}{\nu}\xi^{\frac{1}{\nu}-1}\frac{\xi^{-\frac{1}{2\nu}}-\xi^{\frac{1}{2\nu}}}{[1-2i\xi^{\frac{1}{2\nu}}+\xi^{\frac{1}{\nu}}]^2}.$$

Consequently, we obtain

$$\frac{1}{(z(\xi)-1)^2}\frac{dz(\xi)}{d\xi}=\frac{\frac{2i}{\nu}\xi^{\frac{1}{\nu}-1}[\xi^{-\frac{1}{2\nu}}-\xi^{\frac{1}{2\nu}}]}{-4^2\xi^{\frac{1}{\nu}}}=\frac{1}{8i\nu\xi}\left[\xi^{-\frac{1}{2\nu}}-\xi^{\frac{1}{2\nu}}\right].$$

Now coming back to the integral (5.4), we get

$$i\int_{S_\nu}\frac{H(\xi)}{(z(\xi)-1)^2}\frac{dz(\xi)}{d\xi}\,d\xi=\frac{1}{8\nu}\int_{S_\nu}H(\xi)\left[\xi^{-\frac{1}{2\nu}}-\xi^{\frac{1}{2\nu}}\right]\frac{1}{\xi}\,d\xi,$$

as we wanted to prove. □

Lemma 5.2. *Let $H=H(\zeta)\in C^1(V)\cap C^0(\overline{D})$ and $|H(\zeta)|\le a|\zeta|$ for some constant a, where V is a neighborhood of 0 in $\mathbb{C}$. Then*

$$\left|\int_0^{2\pi}\frac{H(\psi(\nu,e^{i\theta}))}{|e^{i\theta}-1|^2}\,d\theta-\frac{1}{2\pi}\int_0^{2\pi}H(e^{i\theta})\,d\theta\right|\le\frac{20a}{\nu^2}. \tag{5.5}$$

In particular,

$$\lim_{\nu\to\infty}\int_0^{2\pi}\frac{H(\psi(\nu,e^{i\theta}))}{|e^{i\theta}-1|^2}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}H(e^{i\theta})\,d\theta. \tag{5.6}$$

Proof. Using the equation from Lemma 5.1 the integral on the right-hand side of (5.2) can be considered as the sum of three integrals: one over the segment from 0 to 1, the second over the same segment in the opposite direction (with a different branch of the root), and the third is over the unit circle (counted ν-times).

We parametrize the curve of the first integral by $\xi=t$, where $0\le t\le 1$. So the first integral becomes

$$\frac{1}{8\nu}\int_0^1 H(t)\left[t^{-\frac{1}{2\nu}}-t^{\frac{1}{2\nu}}\right]\frac{1}{t}\,dt.$$

We note that this integral converges because $H(0) = 0$ and H is C^1 in a neighborhood of 0. Moreover, if $|H(\zeta)| \le a|\zeta|$, we have

$$\begin{aligned}\frac{1}{8\nu}\left|\int_0^1 H(t)[t^{-\frac{1}{2\nu}} - t^{\frac{1}{2\nu}}]\frac{1}{t}dt\right| &\le \frac{a}{8\nu}\int_0^1 [t^{-\frac{1}{2\nu}} - t^{\frac{1}{2\nu}}]dt \\ &= \frac{a}{8\nu^2}\frac{1}{(1-\frac{1}{2\nu})(1+\frac{1}{2\nu})} \\ &\le \frac{a}{4\nu^2}.\end{aligned} \tag{5.7}$$

For the second integral, we parametrize the curve by $\xi = te^{2i\nu\pi}$, where t is running from 1 to 0. The second integral becomes

$$\frac{1}{8\nu}\int_1^0 H(t)\left[(te^{2i\nu\pi})^{-\frac{1}{2\nu}} - (te^{2i\nu\pi})^{\frac{1}{2\nu}}\right]\frac{1}{t}\,dt = \frac{1}{8\nu}\int_0^1 H(t)\left[t^{-\frac{1}{2\nu}} - t^{\frac{1}{2\nu}}\right]\frac{1}{t}\,dt,$$

so we see that the first two integrals are the same.

To compute the third integral, we parametrize the remaining part of the boundary by

$$\xi = e^{i\theta} \quad \text{where} \quad 0 \le \theta \le 2\nu\pi.$$

So the third integral becomes

(5.8)

$$\begin{aligned}&\frac{i}{8\nu}\int_0^{2\nu\pi} H(e^{i\theta})\left[e^{-\frac{i\theta}{2\nu}} - e^{\frac{i\theta}{2\nu}}\right]d\theta = \frac{1}{4\nu}\int_0^{2\nu\pi} H(e^{i\theta})\sin\frac{\theta}{2\nu}\,d\theta \\ &= \frac{1}{4\nu}\sum_{j=0}^{\nu-1}\int_{2j\pi}^{2(j+1)\pi} H(e^{i\theta})\sin\frac{\theta}{2\nu}\,d\theta = \frac{1}{4\nu}\sum_{j=0}^{\nu-1}\int_0^{2\pi} H(e^{i\theta})\sin\frac{\theta+2j\pi}{2\nu}\,d\theta \\ &= \frac{1}{4\nu}\sum_{j=0}^{\nu-1}\cos\frac{j\pi}{\nu}\int_0^{2\pi} H(e^{i\theta})\sin\frac{\theta}{2\nu}\,d\theta + \frac{1}{4\nu}\sum_{j=0}^{\nu-1}\sin\frac{j\pi}{\nu}\int_0^{2\pi} H(e^{i\theta})\cos\frac{\theta}{2\nu}\,d\theta\end{aligned}$$

The first sum on the very right-hand side of (5.8) converges to 0. To see this, we have a sequence of estimates:

$$\begin{aligned}\left|\frac{1}{4\nu}\sum_{j=0}^{\nu-1}\cos\frac{j\pi}{\nu}\int_0^{2\pi} H(e^{i\theta})\sin\frac{\theta}{2\nu}\,d\theta\right| &= \left|\frac{1}{4\nu}\int_0^{2\pi} H(e^{i\theta})\sin\frac{\theta}{2\nu}\,d\theta\right| \\ &\le \frac{\pi}{4\nu^2}\int_0^{2\pi}|H(e^{i\theta})|\,d\theta \\ &\le \frac{\pi^2 a}{2\nu^2}.\end{aligned} \tag{5.9}$$

Now we find the limit of the second sum on the very right-hand side of (5.8) as $\nu \to \infty$. First we estimate

$$\begin{aligned}
&\left| \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \int_0^{2\pi} H(e^{i\theta}) \cos \frac{\theta}{2\nu}\, d\theta - \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \int_0^{2\pi} H(e^{i\theta})\, d\theta \right| \\
&= \left| \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \int_0^{2\pi} H(e^{i\theta}) \left[\cos \frac{\theta}{2\nu} - 1 \right] d\theta \right| \\
&= \left| \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \int_0^{2\pi} H(e^{i\theta}) \left[-2 \sin^2 \frac{\theta}{4\nu} \right] d\theta \right| \\
&\le \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \int_0^{2\pi} |H(e^{i\theta})| \frac{2\theta^2}{16\nu^2}\, d\theta \qquad (5.10) \\
&\le \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \int_0^{2\pi} |H(e^{i\theta})| \frac{\pi^2}{2\nu^2}\, d\theta \\
&\le \frac{\pi^2}{8\nu^3} \sum_{j=0}^{\nu-1} \int_0^{2\pi} |H(e^{i\theta})|\, d\theta \\
&\le \frac{\pi^2}{8\nu^2} \int_0^{2\pi} |H(e^{i\theta})|\, d\theta \\
&\le \frac{\pi^3 a}{4\nu^2}.
\end{aligned}$$

From the above estimates we have that

$$\lim_{\nu\to\infty} \left\{ \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \int_0^{2\pi} H(e^{i\theta}) \cos \frac{\theta}{2\nu}\, d\theta \right\} = \int_0^{2\pi} H(e^{i\theta})\, d\theta \lim_{\nu\to\infty} \left\{ \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} \right\}.$$

But the last limit can be easily calculated:

$$\begin{aligned}
\lim_{\nu\to\infty} \frac{1}{4\nu} \sum_{j=0}^{\nu-1} \sin \frac{j\pi}{\nu} &= \lim_{\nu\to\infty} \frac{1}{4\nu} \sum_{j=0}^{\nu} \sin \frac{j\pi}{\nu} \\
&= \lim_{\nu\to\infty} \frac{1}{4\nu} \Im \sum_{j=0}^{\nu} e^{\frac{ij\pi}{\nu}} \\
&= \Im \lim_{\nu\to\infty} \frac{1}{4\nu} \frac{1 - e^{\frac{i(\nu+1)\pi}{\nu}}}{1 - e^{\frac{i\pi}{\nu}}} \\
&= \lim_{\nu\to\infty} \frac{1}{4\nu} \frac{\sin \frac{\pi}{\nu}}{1 - \cos \frac{\pi}{\nu}} \\
&= \frac{1}{2\pi}.
\end{aligned}$$

Now we calculate how fast the last sequence converges:

$$\left|\frac{1}{4\nu}\frac{\sin\frac{\pi}{\nu}}{1-\cos\frac{\pi}{\nu}}-\frac{1}{2\pi}\right| = \left|\frac{1}{4\nu}\frac{\frac{\pi}{\nu}-\frac{1}{3!}\frac{\pi^3}{\nu^3}+\dots}{\frac{1}{2!}\frac{\pi^2}{\nu^2}-\frac{1}{4!}\frac{\pi^4}{\nu^4}+\dots}-\frac{1}{2\pi}\right| \le \frac{1}{\nu^2}. \tag{5.11}$$

Combining (5.7) and (5.9) - (5.11), we obtain

$$\left|\int_0^{2\pi}\frac{H(\psi(\nu,e^{i\theta}))}{|e^{i\theta}-1|^2}\,d\theta-\frac{1}{2\pi}\int_0^{2\pi}H(e^{i\theta})\,d\theta\right| \le \frac{a}{\nu^2}\left[\frac{2}{4}+\frac{\pi^2}{2}+\frac{\pi^3}{4}+1\right] \le \frac{20a}{\nu^2},$$

and consequently (5.5). □

6. Construction of a rib

In this section we assume that a manifold M is given as in (2.1) with the function h of class C^3. For each point $p \in M$ we make a change of coordinates by using Ω_p as in (2.2) and then consider the manifold $\Omega_p M$ as in (2.4).

We recall that $\psi(1,z)$ defined in (5.1) can be holomorphically extended to a neighborhood of $\overline{D}\setminus\{\pm i\}$ in $\mathbb{C}$. At $\pm i$ the function $\psi(1,z)$ is of class $C^{\frac{1}{2}}$. Moreover, the half circle, which corresponds to $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$ is mapped onto the segment $[0,\,1]$ twice. The same is true for $\psi(\nu,z) = [\psi(1,z)]^\nu$. From the form of $\psi(\nu,z)$ we see that

$$\begin{aligned} &\psi(1,1)=0, \qquad \frac{d\psi(1,1)}{dz}=0, \qquad \frac{d^2\psi(1,1)}{dz^2}\neq 0,\\ &\frac{d\psi(\nu,z)}{dz}=\frac{d}{dz}[\psi(1,z)]^\nu=\nu[\psi(1,z)]^{\nu-1}\frac{d\psi(1,z)}{dz}. \end{aligned} \tag{6.1}$$

Now we consider the function

$$F_\lambda(\nu,p,w)=\frac{1-\lambda^2}{2\pi}\int_0^{2\pi}\frac{h^p(w(\psi(\nu,e^{i\theta}))-w(\psi(\nu,\lambda)))}{|e^{i\theta}-\lambda|^2}d\theta. \tag{6.2}$$

Proposition 6.1. *Let $w = w(\zeta) \in [\mathcal{O}(D)\cap C^0(\overline{D})]^n$ with $w(0)=0$ and $|w|_\infty$ sufficiently small. Then the function $(p,\lambda) \longrightarrow F_\lambda(\nu,p,w)$ is C^1 on $M\times[0,\,1]$ and C^2 on $M\times[0,1)$. Moreover, there is a constant C, which depends on $|h|_3$ only, such that for $\nu=1,2,\dots$ we have*

$$F_\lambda(\nu,p,w)=\frac{1-\lambda^2}{2\pi}\left[\int_0^{2\pi}\frac{h^p(w(\psi(\nu,e^{i\theta})))}{|e^{i\theta}-1|^2}d\theta+R_\lambda(\nu,p,w)\right], \tag{6.3}$$

where

$$|R_\lambda(\nu,p,w)|\le C|w|_\infty^2(1-\lambda)\log\frac{1}{1-\lambda} \qquad (\frac{1}{4}\le\lambda<1). \tag{6.4}$$

Proof. The property that the mapping $(p,\lambda) \longrightarrow F_\lambda(\nu,p,w)$ is of class C^2 on $M\times[0,\,1)$ immediately follows from the formula (6.2). If λ approaches 1, then C^1 dependence follows from (6.3) and from the estimate (6.4).

Now we prove the formula (6.3) with the estimate (6.4). Throughout the proof, $C_1, C_2, \ldots$ denote constants which depend on $|h|_3$. To simplify the notation, we drop the dependence on p.

We consider

$$\varphi(\nu,\lambda) = \int_0^{2\pi} \frac{h(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda)))}{|e^{i\theta}-\lambda|^2} d\theta, \tag{6.5}$$

and use a $'$ to denote $\dfrac{\partial}{\partial\lambda}$.

For $0<\lambda<1$, the derivative $\varphi'(\nu,\lambda)$ is

$$\begin{aligned}\varphi'(\nu,\lambda) = &-\int_0^{2\pi} \frac{h_w(w(\psi(\nu,e^{i\theta})) - w((\psi(\nu,\lambda)))\, w'(\psi(\nu,\lambda))\,\psi'(\nu,\lambda)}{|e^{i\theta}-\lambda|^2} d\theta + \\ &- \int_0^{2\pi} \frac{h_{\overline{w}}(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda)))\,\overline{w'}(\psi(\nu,\lambda))\,\overline{\psi}'(\nu,\lambda)}{|e^{i\theta}-\lambda|^2} d\theta + \\ &+ \int_0^{2\pi} \frac{h(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda)))\,(2\lambda - e^{i\theta} - e^{-i\theta})}{|e^{i\theta}-\lambda|^4} d\theta.\end{aligned} \tag{6.6}$$

To estimate $|\varphi'(\nu,\lambda)|$ as λ approaches 1, we need estimates of all expressions in the integrand.

In what follows λ will be taken sufficiently close to 1.

We start with an estimate for $|w'(\psi(\nu,\lambda))\,\psi'(\nu,\lambda)|$. We have

$$|w'(\psi(\nu,\lambda))\psi'(\nu,\lambda)| = |[w(\psi(\nu,\lambda))]'| \le C_1 |w|_\infty, \tag{6.7}$$

by the Cauchy-Schwarz estimate which may be used because $\psi(\nu,z)$ has a holomorphic extension to a neighborhood of $\overline{D}\setminus\{\pm i\}$ in $\mathbb{C}$.

The next estimate is of $|h_w(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda)))|$. After applying the Taylor formula, we use the form of derivatives for ψ with respect to z (6.1) and again the Cauchy-Schwarz estimate:

$$\begin{aligned}&|h_w(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda)))| \le \\ &\le |h_{ww}(0)\,(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda))) + \\ &+ h_{w\overline{w}}(0)\,(\overline{w}(\psi(\nu,e^{i\theta})) - \overline{w}(\psi(\nu,\lambda)))| + O(|w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda))|^2) \\ &\le C_2 |w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda))| \le C_3 |w|_\infty\, |e^{i\theta}-\lambda|.\end{aligned}$$

So we have

$$|h_w(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda)))| \le C_3 |w|_\infty\, |e^{i\theta}-\lambda|. \tag{6.8}$$

Next, we estimate $|h(w(\psi(\nu, e^{i\theta})) - w(\psi(\nu, \lambda)))|$. Using the same arguments as in the proof of (6.8), we have

$$|h(w(\psi(\nu, e^{i\theta})) - w(\psi(\nu, \lambda)))| =$$

$$= |\frac{1}{2} h_{ww}(0)\,(w(\psi(\nu, e^{i\theta})) - w(\psi(\nu, \lambda)))^2 +$$

$$+ h_{w\overline{w}}(0)\,(w(\psi(\nu, e^{i\theta})) - w(\psi(\nu, \lambda)))(\overline{w}(\psi(\nu, e^{i\theta})) - \overline{w}(\psi(\nu, \lambda))) +$$

$$+ \frac{1}{2} h_{\overline{w}\,\overline{w}}(0)\,(\overline{w}(\psi(\nu, e^{i\theta})) - \overline{w}(\psi(\nu, \lambda)))^2\,| + O(|w(\psi(\nu, e^{i\theta})) - w(\psi(\nu, \lambda))|^3)$$

$$\leq C_4 |w(\psi(\nu, e^{i\theta})) - w(\psi(\nu, \lambda))|^2$$

$$\leq C_5 |w|^2_\infty |e^{i\theta} - \lambda|^2.$$

So we have

$$|h(w(\psi(\nu, e^{i\theta})) - w(\psi(\nu, \lambda)))| \leq C_5 |w|^2_\infty\, |e^{i\theta} - \lambda|^2. \tag{6.9}$$

And finally we need an estimate of $(2\lambda - e^{i\theta} - e^{-i\theta})/|e^{i\theta} - \lambda|^4$. We have

$$\frac{|2\lambda - e^{i\theta} - e^{-i\theta}|}{|e^{i\theta} - \lambda|^4} \leq \frac{2}{|e^{i\theta} - \lambda|^3}. \tag{6.10}$$

Now we can estimate $\varphi'(\nu, \lambda)$ from (6.6). Combining (6.7) - (6.10), we obtain

$$|\varphi'(\nu, \lambda)| \leq$$

$$\leq C_6 \int_0^{2\pi} \frac{|w|_\infty\, |w|_\infty\, |e^{i\theta} - \lambda|}{|e^{i\theta} - \lambda|^2} d\theta + C_7 \int_0^{2\pi} \frac{|w|^2_\infty\, |e^{i\theta} - \lambda|^2}{|e^{i\theta} - \lambda|^3} d\theta =$$

$$= (C_6 + C_7)|w|^2_\infty \int_0^{2\pi} \frac{1}{|e^{i\theta} - \lambda|} d\theta.$$

To complete the proof of the proposition, we should consider the last integral just above. Using the elementary estimates

$$|e^{i\theta} - \lambda|^2 = 1 - 2\lambda \cos\theta + \lambda^2 \geq 1 - 2\lambda(1 - \frac{\theta^2}{8}) + \lambda^2 \geq (1 - \lambda)^2 + \frac{\theta^2}{16}$$

for $-\pi \leq \theta \leq \pi$ and $\lambda > \frac{1}{4}$, we have

$$\frac{1}{|e^{i\theta} - \lambda|} \leq \frac{1}{[(1 - \lambda)^2 + (\frac{\theta}{4})^2]^{1/2}}.$$

Consequently, we obtain

$$\int_0^{2\pi} \frac{1}{|e^{i\theta}-\lambda|}d\theta = 2\int_0^{\pi} \frac{1}{|e^{i\theta}-\lambda|}d\theta \leq$$
$$\leq 2\int_0^{\pi} \frac{1}{[(1-\lambda)^2+(\frac{\theta}{4})^2]^{1/2}}d\theta \leq -C_8\log(1-\lambda).$$

So we obtained

$$|\varphi'(\nu,\lambda)| \leq -C_9|w|_\infty^2 \log(1-\lambda) \qquad \text{for} \quad \frac{1}{4}\leq \lambda < 1.$$

Consequently,

$$|\varphi(\nu,\lambda)-\varphi(\nu,1)| \leq \int_\lambda^1 |\varphi'(\nu,t)|\,dt$$
$$\leq -C_9|w|_\infty^2 \int_\lambda^1 \log(1-t)\,dt \leq C_{10}|w|_\infty^2\,(\lambda-1)\log(1-\lambda).$$

The proposition is proved. □

For each ν we define a mapping $G(\nu,\lambda,p,w)$ by

$$(6.11) \qquad \begin{aligned} &G(\nu,\cdot,\cdot,\cdot)\colon [0,\,1]\times M\times\mathcal{B}\longrightarrow \mathcal{N}(M)\\ &G(\nu,\lambda,p,w)=\Omega_p^{-1}(0+iF_\lambda(\nu,p,w),\,0).\end{aligned}$$

We should make it clear that actually the mapping G is defined for p from a neighborhood of the origin. We hope it does not lead to a confusion and keeps the notation simpler.

Corollary 6.2. *(a) If $|w|_\infty$ is sufficiently small, then the mapping $[0,\,1]\times M\ni(\lambda,p)\to G(\nu,\lambda,p,w)$ is of class C^1.*

(b) If $|w|_\infty$ is sufficiently small, then $M\ni p\to G(\nu,\lambda,p,w)$ has maximal rank.

(c) If $|w|_\infty$ is sufficiently small, $\int_0^{2\pi} h(w(e^{i\theta}))\,d\theta\neq 0$, and ν is sufficiently large, then the mapping $[0,\,1]\times M\ni(\lambda,p)\to G(\nu,\lambda,p,w)$ has maximal rank for λ close to 1.

Proof of Corollary 6.2. The property that the map $(\lambda,p)\to G(\nu,\lambda,p,w)$ is of class C^1 follows immediately from Proposition 6.1. If $w\equiv 0$, then $G(\nu,\lambda,p,w)\equiv p$, consequently the rank of the map $p\to G(\nu,\lambda,p,w)$ is maximal if $|w|_\infty$ is sufficiently small.

The derivative of $G(\nu,\lambda,p,w)$ with respect to λ at $\lambda=1$ is, up to a nonzero constant, $\int_0^{2\pi}\frac{h^p(w(\psi(\nu,e^{i\theta})))}{|e^{i\theta}-1|^2}\,d\theta$. Using Lemma 5.2, this integral converges to the integral $\int_0^{2\pi} h^p(w(e^{i\theta}))\,d\theta$, which is nonzero if p is sufficiently close to the origin. This gives (c). □

The following corollary is an immediate consequence of Corollary 6.2.

Corollary 6.3. *If $|w|_\infty$ is sufficiently small, ν is sufficiently large, and if the integral $\int_0^{2\pi} h(w(e^{i\theta}))d\theta \neq 0$, then the image of the map $[0,1]\times M \ni (\lambda, p) \to G(\nu,\lambda,p,w) \in \mathcal{N}(M)$ is a C^1 submanifold M_ν^w of $\mathbb{C}^{k+n}$ of real dimension equal to* $\dim_{\mathbb{R}} M + 1$.

Theorem 6.4 (existence of a rib). *For any $\nu \in \mathbb{N}$ sufficiently large and $w \in \mathcal{B}$ there exists a C^1 manifold M_ν^w with the following properties:*

(i) $\dim_{\mathbb{R}} M_\nu^w = \dim_{\mathbb{R}} M + 1$;

(ii) $U \subset \partial M_\nu^w$, where U is a neighborhood of 0 in M;

(iii) The distance between the two following vectors which lie in $\mathcal{N}_p M$, $p \in M$, namely the vector

$$\Omega_p^{-1}\left(i\frac{1}{2\pi}\int_0^{2\pi} h^p(w(e^{i\theta}))\,d\theta,\ 0\right)$$

and the vector tangent to $M_\nu^w \cap \mathcal{N}_p(M)$

$$\Omega_p^{-1}\left(i\int_0^{2\pi}\frac{h^p(w(\psi(\nu,e^{i\theta})))}{|e^{i\theta}-1|^2}\,d\theta,\ 0\right)$$

is less than $\mathrm{const}\frac{1}{\nu^2}|w|_\infty^2$. *In particular, if $\nu \to \infty$, then the angle between these two vectors can be made arbitrarily small.*

Proof of Theorem 6.4. Actually the manifold M_ν^w is constructed in Corollary 6.3, and the properties (i) and (ii) are obvious from the construction and from Corollary 6.2.

The estimate of the distance between the tangent vectors immediately follows from Lemma 5.2. Namely we have $|h^p(w(\zeta))| \le \mathrm{const}|w|_\infty^2|\zeta|$ and we apply (5.5) for $a = \mathrm{const}|w|_\infty^2$. We obtain

$$\left|\int_0^{2\pi}\frac{h^p(w(\psi(\nu,e^{i\theta})))}{|e^{i\theta}-1|^2}d\theta - \frac{1}{2\pi}\int_0^{2\pi} h^p(w(e^{i\theta}))d\theta\right| \le \frac{20\mathrm{const}}{\nu^2}|w|_\infty^2.$$

7. Extension of CR functions to a rib

For each $w \in \mathcal{B}$ and $\nu = 1, 2, \ldots$, Theorem 6.4 gives a rib with edge M. From the construction, the rib consists of some points of a family of analytic discs $g_\lambda(\nu,p,\zeta) = (z_\lambda(\nu,p,\zeta), w(\psi(\nu,\zeta)) - w(\psi(\nu,\lambda)))$, namely the points are given in terms of F_λ. Of course z_λ and g_λ depend on the analytic disc $w(\cdot)$, but in order not to complicate the notation, we do not include w there. In §6 we discussed the behavior of $\Im z_\lambda(\nu,p,\zeta)$ at some points. For $\zeta = e^{i\theta}$ we have $\Im z_\lambda(\nu,p,e^{i\theta}) = h^p(w(\psi(\nu,e^{i\theta})) - w(\psi(\nu,\lambda)))$ and the $|\cdot|_\infty$ norm of this function can be controlled by const $|w|_\infty^2$. However, to have a control over the size of the analytic discs g_λ, we need an estimate of $\Re z_\lambda(\nu,p,\cdot)$. We prove the following lemma.

Lemma 7.1. *There is a constant C such that for any α, $0<\alpha<1$, we have the estimate*

$$|h^p(w(\psi(\nu,\cdot)))|_{\frac{\alpha}{2}} \le C|w|_\alpha^2\,\nu^\alpha \tag{7.1}$$

Proof. We have the following sequence of inequalities ($C_1, C_2, \ldots$ are constants):

$$|h^p(w(\psi(\nu,\cdot)))|_{\frac{\alpha}{2}} \le C_1|w|_\infty^2 + \sup_{\zeta,\eta\in S^1} \frac{|h^p(w(\psi(\nu,\zeta))) - h^p(w(\psi(\nu,\eta)))|}{|\zeta-\eta|^{\frac{\alpha}{2}}}$$

$$\le C_1|w|_\infty^2 + C_2|h_w^p(w(\psi(\nu,\cdot)))|_\infty \sup_{\zeta,\eta\in S^1} \frac{|w(\psi(\nu,\zeta)) - w(\psi(\nu,\eta))|}{|\zeta-\eta|^{\frac{\alpha}{2}}}$$

$$\le C_1|w|_\infty^2 + C_3|w|_\infty \sup_{\zeta,\eta\in S^1} \frac{|w(\psi(\nu,\zeta)) - w(\psi(\nu,\eta))|}{|\zeta-\eta|^{\frac{\alpha}{2}}}$$

$$\le C_1|w|_\infty^2 + C_3|w|_\infty \sup_{\zeta,\eta\in S^1} \frac{|w(\psi(\nu,\zeta)) - w(\psi(\nu,\eta))|}{|\psi(\nu,\zeta) - \psi(\nu,\eta)|^\alpha}\frac{|\psi(\nu,\zeta) - \psi(\nu,\eta)|^\alpha}{|\zeta-\eta|^{\frac{\alpha}{2}}}$$

$$\le C_1|w|_\infty^2 + C_3|w|_\infty|w|_\alpha \sup_{\zeta,\eta\in S^1} \left[\frac{|\psi(\nu,\zeta) - \psi(\nu,\eta)|}{|\zeta-\eta|^{\frac{1}{2}}}\right]^\alpha$$

$$= C_1|w|_\infty^2 + C_3|w|_\infty|w|_\alpha \sup_{\zeta,\eta\in S^1} \left[\frac{|[\psi(1,\zeta)]^\nu - [\psi(1,\eta)]^\nu|}{|\zeta-\eta|^{\frac{1}{2}}}\right]^\alpha$$

$$\le C_1|w|_\infty^2 + C_3|w|_\infty|w|_\alpha \sup_{\zeta,\eta\in S^1} \left[\frac{|[\psi(1,\zeta)] - [\psi(1,\eta)]|}{|\zeta-\eta|^{\frac{1}{2}}} \sum_{j=0}^{\nu-1} |\psi(1,\zeta)|^j|\psi(1,\eta)|^{\nu-1-j}\right]^\alpha$$

$$\le C_1|w|_\infty^2 + C_4|w|_\infty|w|_\alpha\,\nu^\alpha$$

$$\le C_5|w|_\alpha^2\,\nu^\alpha$$

The lemma is proved. □

Since $\Re z_\lambda(\nu,p,\cdot) = -Th^p(w(\psi(\nu,\cdot)) - w(\psi(\nu,\lambda)))$, therefore we have the following

Corollary 7.2. *There is a constant C such that*

$$|\Re z_\lambda(\nu,p,\cdot)|_{\frac{\alpha}{2}} \le C\|T\|_{\frac{\alpha}{2}}\,|w|_\alpha^2\,\nu^\alpha \tag{7.2}$$

Theorem 7.3 (Extension of CR functions to a rib). *Let M be given as in (2.1) and is of class C^3. Then any CR function defined on M (globally, not locally) can be CR extended to the rib M_ν^w constructed in Theorem 6.4.*

Proof. It is enough to show that any CR function defined globally on M can be uniformly approximated by entire functions. The manifold M is given as in (2.1). In the proof we assume that $\|\Re w\| \le 3\delta$, $\|\Im w\| \le 3\delta$, where $\delta > 0$ is sufficiently small and $\|\cdot\|$ means the max norm of coordinates. More precisely, for any CR function defined on M and for any compact set $K \subset \{(z,w) \in M \mid \|\Re w\| \le \delta,\ \|\Im w\| \le$

$\delta\} \subset M \subset \mathbb{C}^{k+n}$ there is a sequence of entire functions $F_\mu\colon \mathbb{C}^{k+n} \to \mathbb{C}$ such that F_μ converges to f uniformly on K. Then the sequence will converge on the analytic discs constructed in §6 and will give the extension of f to the rib constructed in Theorem 6.4. The construction of such sequence F_μ is the same as in the paper of Baouendi-Trèves [BT] and also in [DG1]. In these two mentioned papers, the approximation is local, however when applied to the manifold M considered in this paper, the same formulas work semiglobally. Actually the formula in [DG1] for the approximation is invariant and because of that can be considered also as a global formula under some additional assumptions, which actually are satisfied in the present paper. We are not going to repeat all the calculations given in [BT] and [DG1], only we show the crucial steps and give precise references where to find complete computations.

We consider a family of totally real submanifolds N^η of M that are given by

(7.3)
$$N^\eta = \{(s+ih(t+i\eta), t+i\eta) \mid \quad s=(s_1,\dots,s_k)\in\mathbb{R}^k,\ t=(t_1,\dots,t_n)\in\mathbb{R}^n,\ \|t\|\le 3\delta\}, \qquad \|\eta\|\le 3\delta,$$

where $\delta > 0$ is sufficiently small. Here $\|\xi\| = \max|\xi_j|$ in the corresponding space. Let φ be a smooth real-valued function with compact support defined on M. More precisely the properties of the function φ will be given later. Let f be any CR function defined on M. We consider the following sequence of entire functions

$$\begin{aligned} &F_\mu(\cdot,\cdot;\eta)\colon \mathbb{C}^{k+n} \longrightarrow \mathbb{C} \\ &F_\mu(z,w;\eta) = \left(\frac{\mu}{\sqrt{\pi}}\right)^{k+n} \int_{N^\eta} \varphi(p)\exp\{-\mu^2[(z,w)-p]^2\} f(p)\,dp, \end{aligned} \tag{7.4}$$

where $p=(p_1,\dots,p_{k+n})$, $dp = dp_1\wedge\dots\wedge dp_{k+n}$, and $[\xi]^2$ denotes $\xi_1^2+\dots+\xi_{k+n}^2$ for $\xi\in\mathbb{C}^{k+n}$. Using the parametrization of N^η from (7.3), the integral in (7.4) can be written as:

$$\begin{aligned} (7.5)\quad F_\mu(z,w,\eta) &= \\ &= \left(\frac{\mu}{\sqrt{\pi}}\right)^{k+n} \int_{\mathbb{R}^{k+n}} \varphi(s,t)\exp\{-\mu^2[(z,w)-(s+ih(t+i\eta),t+i\eta)]^2\}\times \\ &\qquad\times f(s+ih(t+i\eta),t+i\eta)\Delta(s,t)\,ds\,dt, \end{aligned}$$

where $\Delta(s,t)$ is the Jacobian of the parametrization of N^η, i.e., $\Delta(s,t) = \det\left(\frac{\partial(s+ih(t+i\eta),t+i\eta)}{\partial(s,t)}\right)$. Here the function φ is smooth, depends on (s,t) only, has compact support which is contained in $\{(s,t) \mid \|t\| \le 3\delta\}$, its values are in the interval $[0, 1]$, and the function is 1 on the set $\{(s,t) \mid \|s\| \le R+1,\ \|t\|\le 2\delta\}$. The number R is large enough such that, for the compact set chosen at the beginning of the proof, we have $K\cap N^\eta$, $\|\eta\|\le 2\delta$, is contained in $\{(s+ih(t+i\eta),t+i\eta) \mid \|s\| \le R,\ \|t\|\le\delta\}$.

Using the arguments from [BT, p. 396], we have

$$\lim_{\mu\to\infty} F_\mu(x+ih(u+i\eta),\, u+i\eta;\, \eta) = f(x+ih(u+i\eta),\, u+i\eta) \tag{7.6}$$

for $\|x\| \le R$, $\|u\| \le \delta$, and is uniform on this set. To see that, it is enough to make the substitution $(s,t) \to (x - \frac{s}{\mu},\ u - \frac{t}{\mu})$ in the integral (7.5), and, at the limit, the integral becomes

$$\left(\frac{1}{\sqrt{\pi}}\right)^{k+n} \int_{\mathbb{R}^{k+n}} \exp\left\{ - \left[\frac{\partial(x+ih(u+i\eta), u+i\eta)}{\partial(x,u)}(s,t) \right]^2 \right\} ds\, dt\ \times$$

$$\times\ \Delta(x,u) f(x+ih(u+i\eta), u+i\eta) =$$

$$= f(x+ih(u+i\eta), u+i\eta),$$

where $\Delta(x,u) = \det\left[\frac{\partial(x+ih(u+i\eta),u+i\eta)}{\partial(x,u)}\right]$, i.e., it is the determinant of the matrix from the exponent of the exponential function.

In particular, the limit (7.6) holds for $\eta = 0$. Using again the arguments of [BT, p. 397] or [DG1], we have

$$\lim_{\mu\to\infty} [F_\mu(x+ih(u+i\eta),\ u+i\eta;\ \eta) - F_\mu(x+ih(u),\ u;\ 0)] = 0 \tag{7.7}$$

uniformly with respect to $\|x\| \le R$, $\|u\| \le \delta$ and $\|\eta\| \le \delta$. To give a rough argument to prove (7.7) (for detailed calculations the reader is refered to [BT, p. 395 - 397]), the difference under the limit in (7.7) is given by an integral of the form

$$\left(\frac{\mu}{\sqrt{\pi}}\right)^{k+n} \int\limits_{[0,\,\eta]} \int\limits_{\mathbb{R}^{k+n}} \exp\{-\mu^2[(x+ih(u+i\xi), u+i\xi) - (s+ih(t),t)]^2\}\, A(s,t,\xi)\, ds\, dt\, d\xi, \tag{7.8}$$

where $A(s,t,\xi) \equiv 0$ for $\|s\| \le R+1$, $\|t\| \le 2\delta$, $\|\xi\| \le \delta$, and $[0,\,\eta]$ means the straight segment joining 0 with η. Moreover, the function A has compact support. Consequently, in the integral it is enough to integrate over $\|s\| \ge R+1$, $2\delta \le \|t\| \le 3\delta$. If $\|x\| \le R$, $\|u\| \le \delta$, $\|\eta\| \le \delta$, we have

$$\Re[(x+ih(u+i\xi),\ u+i\xi) - (s+ih(t),\ t)]^2 \ge \frac{1}{2},$$

if δ is small enough (it can be chosen once at the beginning, depending on h). Since the function $A(s,t,\xi)$ has compact support, the sequence in (7.8) converges to 0 faster than const $e^{-\frac{1}{2}\mu^2}$. The theorem is proved. □

8. Extension of CR functions to a wedge

In this section we consider a CR submanifold in $\mathbb{C}^{k+n}$ of codimension k given like in (2.1). Let $\mathcal{B}$ and $\mathcal{B}_\delta$ be as in (3.4). We consider the data $(0, w(\cdot))$, $w \in \mathcal{B}$, and solve the Bishop equation (3.1), and get $(z(\cdot), w(\cdot))$ with $x(0) = 0$. We denote the mean operator by

$$E(w) = \frac{1}{2\pi}\int_0^{2\pi} h(w(e^{i\theta}))\, d\theta, \quad E = (E_1, \dots, E_k), \tag{8.1}$$

which is a vector-valued function defined on $\mathcal{B}$. Actually $(iE(w),\ 0)$ is a vector in $\mathcal{N}_0 M$. We use the notation $E(p,w)$ if h is replaced by h^p.

We prove the following theorem.

Theorem 8.1 (Extension of CR functions to a wedge). *Assume that M is as in (2.1) and of class C^3, and suppose that there are analytic discs $w^1, \dots, w^l$ with $|\cdot|_\alpha$ norms sufficiently small and such that the vectors $E(w^1), \dots, E(w^l)$ are $\mathbb{R}$-linearly independent. Then any CR function defined globally on M can be CR extended (in a neighborhood of the origin) to a (curved) wedge spanned by $E(w^1), \dots, E(w^l)$.*

More precisely, any CR function defined on M can be CR extended to a wedge spanned by the ribs $M^j_\nu = M^{w^j}_\nu$, $j = 1, \dots, l$, $\nu = 1, 2, \dots,$ which were constructed in §6. If $\nu \to \infty$, then the angle between each rib M^j_ν and $E(w^j)$ can be made arbitrarily small. In particular, if the vectors $E(w^1), \dots, E(w^l)$ span an open cone (in the normal space $\mathcal{N}_0 M$), then the extension is holomorphic.

Proof. The vectors $E(w^1), \dots, E(w^l)$ represent the centers of the lifted discs. From Lemma 5.2, we have

$$\lim_{\nu\to\infty} \Omega_p^{-1} \left[i \int_0^{2\pi} \frac{h^p(\psi(\nu, e^{i\theta}))}{|e^{i\theta} - 1|^2}\, d\theta,\ 0 \right] = \Omega_p^{-1}(E(p, w^j)), \qquad j = 1, \dots, l,$$

uniformly with respect to p. For any $\varepsilon > 0$ we can find such ν that

$$\left| \Omega_p^{-1} \left[i \int_0^{2\pi} \frac{h^p(\psi(\nu, e^{i\theta}))}{|e^{i\theta} - 1|^2}\, d\theta,\ 0 \right] - \Omega_p^{-1}(E(p, w^j)) \right| < \varepsilon, \qquad j = 1, \dots, l.$$

This actually easily follows from (5.5). We fix such a ν.

For this fixed ν we consider the mapping $G(\lambda, p, w^j) = G(\nu, \lambda, p, w^j)$ defined in (6.11). The image of this mapping is a C^1 manifold M_j with edge M and of real dimension $\dim_{\mathbb{R}} M + 1$. From §7 we know that CR functions on M can be CR extended to M_j. For a fixed p, $\mathcal{N}_p M \cap M_j$ is a curve. Without any loss of generality, and to simplify the arguments, we can assume that $p = 0$. This curve can be parametrized by

$$[0,\, 1] \ni t \longrightarrow \mathfrak{m}^j(t) = (\mathfrak{m}^j_1(t), \dots, \mathfrak{m}^j_k(t)), \qquad \mathfrak{m}^j(0) = 0, \qquad j = 1, \dots, l.$$

The curve is of class C^2 on $(0,\, 1]$ and C^1 on $[0,\, 1]$. For each $j = 1, \dots, l$ we denote by $\mathfrak{a}_j$ a vector in $\mathcal{N}_0(M)$ that is tangent to M_j at the origin.

To get an extension to a wedge (we follow the argument of [A, p. 114]), we divide the unit circle S^1 into l equal arcs A_j, $j = 1, \dots, l$. Let φ_j be a fixed C^∞ function defined on S^1 such that $\varphi_j \in C_0^\infty(A_j)$ and $\varphi_j \equiv 0$ on $S^1 \setminus A_j$, and $\tilde{\varphi}_j(0) = 1$, where $\tilde{\varphi}_j(0)$ is the harmonic extension of φ_j onto the disc.

We consider the following mapping

$$S^1 \ni e^{i\theta} \longrightarrow \mathfrak{m}^1(t_1\varphi_1(e^{i\theta})) + \ldots + \mathfrak{m}^l(t_l\varphi_l(e^{i\theta})) \in \bigcup_j M_j.$$

We note that this mapping is of class C^2 and can be extended to an analytic disc with boundary on $\cup_j M_j$. Since the value at 0 of the real parts of the analytic disc can be assigned arbitrarily (the manifold is rigid), we are interested at the value at 0 of the imaginary parts. The values of the centers of the imaginary parts for $t_j \geq 0$

fill out the wedge spanned by the ribs M_j. To see this, it is enough to consider the first approximation of the ribs, namely by the "straight" ribs $t \to \mathfrak{a}_\alpha t$. Then the imaginary parts of the centers of lifted discs are

$$\sum_{j=1}^{l} t_j \mathfrak{a}_j \int_0^{2\pi} \varphi_j(e^{i\theta}) d\theta = \sum_{j=1}^{l} t_j \mathfrak{a}_j,$$

which gives the simplex spanned by the vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_l$.

The theorem is proved. □

9. Extension to a full neighborhood

In this section we prove the following theorem.

Theorem 9.1 (Extension of CR functions to a full neighborhood). *Let M be as in (2.1) and of class C^1, and assume that for any vector $\mathfrak{n} = (n_1, \ldots, n_k)$ there is an analytic disc $w \in \mathcal{B}_\delta$ (δ is sufficiently small) such that $\mathfrak{n} \cdot E(0,w) = n_1 E_1(0,w) + \ldots + n_k E_k(0,w) > 0$ in the normal space $\mathcal{N}_0 M$. Then any CR function defined globally on M can be holomorphically extended to a full neighborhood of $p = 0$ in $\mathbb{C}^{k+n}$.*

Proof. Let $S = \{E(0;w) \mid w \in \mathcal{B}_\delta\} \subset \mathcal{N}_0 M$ and let $\mathrm{ch}(S)$ be the convex hull of S in $\mathcal{N}_0(M)$. We note that the interior $\mathrm{Int}(\mathrm{ch}(S))$ contains $p = 0$ because otherwise we get a contradiction with the assumptions of the theorem. Consequently, there are analytic discs $w^1, \ldots, w^l$, $l \geq k+1$, such that the smallest convex set spanned by the vectors

$$E(w^1) = E(0, w^1), \ldots, E(w^l) = E(0, w^l)$$

contains the origin in its interior, which is taken in $\mathcal{N}_0 M$.

To complete the proof of the theorem, we apply Theorem 8.1 to the ribs which are determined by the analytic discs $w^1, \ldots, w^l$, and ν sufficiently large.

The theorem is proved. □

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