ALGEBRAIC FAMILIES OF SMOOTH HYPERBOLIC SURFACES OF LOW DEGREE IN $\mathbb{P}^3_{\mathbb{C}}$

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Contents

0. INTRODUCTION

In 1970 S. Kobayashi conjectured that a generic hypersurface of $\mathbb{P}^n_{\mathbb{C}}$ of degree d large enough with respect to n is hyperbolic (it is usually expected that this holds for degrees $d \geq 2n-1$) and that its complement is hyperbolically embedded into $\mathbb{P}^n_{\mathbb{C}}$ (supposedly, for degrees $d \geq 2n+1$) [Ko70,pp. 131-132]. Here, generic is meant in the sense of Zariski topology in the parameter space $P_{n,d}$ of all degree d hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$.

By means of Brody's theorem [Br78], M. Zaidenberg [Za89] observed that for all $n, d \in \mathbb{N}$ the set $H_{n,d}$ of degree d hyperbolic hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$ is open in the classical Hausdorff topology of $P_{n,d}$. Therefore, if one had an algebraic characterisation of hyperbolicity, the above conjecture of Kobayashi would follow.

It has been suggested by S. Kobayashi [Ko76] and S. Lang [La86] that hyperbolicity is equivalent to the non existence of rational or elliptic curves and of nontrivial holomorphic images of a complex torus; these properties are of course of an algebraic nature.

According to J.-P. Demailly [De95], hyperbolicity could also be equivalent to the non existence of sequences of compact curves (C_ℓ) with the condition: genus(C_ℓ)/degree$(C_\ell) \to 0$.

Unfortunately an algebraic criterion for hyperbolicity seems to be still far beyond reach until now, although there has been some progress towards a solution of some of Kobayashi's conjectures, at least for the question of complements of curves in $\mathbb{P}^2_{\mathbb{C}}$ with a given number k of irreducible components.

For the compact case, that is, the first part of Kobayashi's above conjecture, the answer is known to be positive only when $n = 2$. For $n > 2$, a consequent number of examples have been constructed. R.Brody and M.Green [BG77] provided the first example of smooth hyperbolic surface in $\mathbb{P}^3_{\mathbb{C}}$ of even degree ≥ 50. A.M.Nadel [Na89] obtained examples of hyperbolic surface in $\mathbb{P}^3_{\mathbb{C}}$ of arbitrary degrees $6p+3 \geq 21$. Recently K.Masuda and J.Noguchi [MN93] proved the existence of such examples in arbitrary dimension $n > 0$.

In the present paper, we consider only the compact case, for which we follow closely methods introduced by A.M. Nadel [Na89]. Nadel's original method was based on a rather difficult and technical theorem of Y.T. Siu dealing with meromorphic connections [Si87]. We present here a much simpler approach depending only on elementary considerations about Brody curves. Also, it seems that some of Nadel's arguments related to genus calculations are not entirely correct (due to some minor computational mistakes). We take the opportunity to clarify these points here.

Our technique yields examples of smooth surfaces in $\mathbb{P}^3_{\mathbb{C}}$ which are hyperbolic for all degrees ≥ 14, and as a consequence we obtain that $H_{3,d}$ contains a non empty open set for all degrees $d \geq 14$; this brings us somewhat nearer than previously known from the expected range $d \geq 5$.

Furthermore, we prove that for arbitrary integers k_0, k_1, k_2 and k_3 two of them ≥ 2 and $d > 9 + \sum_{\ell=0}^{3} k_\ell$, then $H_{3,d}$ contains a quasi-projective subvariety of dimension

$$\sum_{\ell=0}^{3} \binom{k_\ell + 3}{k_\ell} - 1.$$

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1. SIMPLE PROOF OF A SPECIAL CASE OF SIU'S THEOREM

We prove here (after Y.T. Siu [Si87]) a simple vanishing theorem for the Wronskian of Brody curves under a suitable assumption on the existence of global meromorphic connections. Instead of using the Ahlfors lemma, our method only requires the following observation.

1.1 Lemma . — *Let L be a holomorphic line bundle on $\mathbb{C}$, equipped with a hermetien metric of negative curvature . If s is a bounded holomorphic section of L, then s vanishes identically.*

Proof . — We have by the extended Lelong-Poincaré equation applied to a non identically vanishing meromorphic section s of a holomorphic line bundle L:

$$id'd'' \ln |s|^2 = 2\pi \sum m_j [Z_j] - \mathrm{i}\Theta(L)$$

where the sets Z_j are the irreducible components of the sets of poles and zeros of s with respective multiplicities m_j and current of integrations $[Z_j]$ and where $\Theta(L)$ is the curvature of L. By applying this to our holomorphic section supposed not to vanish identically, because the set of poles is empty and the curvature is negative in our case, we obtain : $id'd'' \ln |s|^2 > 0$ which implies $\ln |s|$ to be a strictly subharmonic and bounded function on $\mathbb{C}$. This is a contradiction since all bounded subharmonic functions on $\mathbb{C}$ are constant.

Let X be a $n-$dimensional compact complex manifold equipped with a hermitian metric ω. We now introduce meromorphic connections as suggested by Y.T. Siu in [Si87].

1.2 Definition . — *We say that a connection ∇ on T_X is meromorphic if for any two holomorphic vector fields U, V defined locally on X, the derivative $\nabla_U V$ of V in the direction U is a locally meromorphic vector field, equivalently if the Christoffel symbol $\Gamma^k_{i,j}$ of ∇ are meromorphic functions on the coordinate patch of definition.*

1.3 Notation. — *Let f be an entire curve in X and ∇ a connection on X we note f' the velocity vector of f and $f^{(k)} := \nabla_{f'}(f^{(k-1)})$.*

1.4 Definition . — *We say that an entire curve $f\colon \mathbb{C} \longrightarrow X$ is a Brody curve if:* $\sup_{z\in\mathbb{C}} |f'(z)|_\omega \leq 1$. *The manifold X is said to be Brody hyperbolic if there does not exist any Brody curve in X.*

We have now the following characterization of compact hyperbolic manifold due to R. Brody [Br78].

1.5 Corollary (Brody Criterion of Hyperbolicity) . — *A compact complex manifold X is hyperbolic if and only if X is Brody hyperbolic (and therefore, it is hyperbolic if and only if there does not exist any non constant entire curve in X).*

1.6 Lemma . — *Suppose that ∇ is a meromorphic connection on X, and t is a holomorphic section of a line bundle L on X such that $t\nabla$ is holomorphic. Let f be a Brody curve in X. Then $(t \circ f)^{p-1} \otimes f^{(p)}$ is bounded on $\mathbb{C}$ for every integer $p > 1$.*

Proof . — It is sufficient to prove the lemma on $f^{-1}(V)$ for every open relatively compact subset V of a trivialisation set U where both TX and L are trivial, because X is compact, and so it can be covered by a finite number of such sets V. Let θ_X and θ_L be respective trivialisations. Denote by δ the distance between $f^{-1}(V) \subset \mathbb{C}$ and $\mathbb{C} \setminus f^{-1}(U)$. We claim that δ is positive, otherwise there exist two sequences $(x_n)_n \subset f^{-1}(V)$ and $(y_n)_n \subset \mathbb{C} \setminus f^{-1}(U)$ such that $|x_n - y_n|$ tends to 0 at infinity, but f is a Brody curve so $|f(x_n) - f(y_n)|_\omega \leq |x_n - y_n|$ which is not allowed because V is relatively compact in U. Let $z_0 \in f^{-1}(V)$ and consider a disc $\Delta_0 \subset f^{-1}(V)$ with center z_0 and radius $r = \delta/2$. Then on Δ_0 Cauchy's integral formula implies

$$\frac{d\sigma}{dz}(z) = \frac{1}{2\pi} \int_{\Delta_0} \frac{\sigma(s)}{(s-z)^2} ds$$

where $\sigma = \theta_X \circ f'$ is the expression of f' in the trivialization θ_X. Thus $\|\frac{d\sigma}{dz}(z_0)\|$ is bounded by $(1/\delta)\|\sigma\|$, where $\| \ \ \|$ is the euclidian norm in $\mathbb{C}^n$. Recall

$$\frac{\nabla f'}{dz} \simeq_\theta \frac{d\sigma}{dz} + (A(f(z)) \cdot \sigma) \cdot \sigma$$

on our disc, where A is the connection form of ∇ with respect to θ_X. Now by hypothesis f is a Brody curve, thus f' is bounded, on the other hand $\theta_L(t)A$ is holomorphic on U hence bounded on V, thus we can conclude that $(\theta_L(t) \circ f) \otimes f''$ is bounded on $f^{-1}(V)$. The proof is complete for $p = 2$. By iterating the argument to higher order derivatives of $(\theta_L(t) \circ f) \otimes f''$ we can conclude for every $p > 2$.

1.7 Theorem (Siu's Theorem). — *Suppose that X is provided with a meromorphic connection ∇, a holomorphic line bundle L and a non identically vanishing global holomorphic section t of L such that $t\nabla$ is holomorphic. Assume that* $F := -K_X + (\frac{n(n-1)}{2})L$

has negative curvature. Then for any entire curve f in X either $f(\mathbb{C})$ is contained in the hypersurface $t=0$ or $f' \wedge f'' \wedge \cdots \wedge f^{(n)}$ vanishes identically.

In the sequel we give a simple proof of Siu's Theorem [Si87,90] just for the case of a Brody curve (note that thanks to Brody's theorem this is sufficient for proving hyperbolicity).

Proof . — Let us consider the section:

$$S: s \longrightarrow f'(s) \wedge t \circ f(s).f''(s)) \wedge \cdots \wedge (t \circ f(s))^{n-1}.f^{(n)}(s)$$

of the line bundle $f^\star F$ on $\mathbb{C}$. Then S is holomorphic by assumption on t and ∇, moreover it is bounded on $\mathbb{C}$; in fact, this follows directly from Lemma 1.6. From Lemma 1.1 and the fact that $f^\star(F)$ has negative curvature we deduce that S must vanish identically.

2. ENTIRE HOLOMORPHIC CURVES IN HYPERSURFACES OF $\mathbb{P}^n_{\mathbb{C}}$

In this section we give a slight improvement of a result due to Y.T. Siu [Si87] and A.M. Nadel [Na89] on the algebraic degeneracy of entire holomorphic curves contained in certain hypersurfaces of $\mathbb{P}^n_{\mathbb{C}}$. We generalize their result to the space $\mathcal{S}(d, k_0, k_1, k_2, \ldots, k_n)$ defined below.

2.1 Definition. — *Let d be a positive integer and choose integers k_i, $(i = 0, 1, \ldots, n)$ such that $d > k_i$ $(i = 0, 1, \ldots, n)$. We denote by $\mathcal{S}(d, k_0, k_1, \ldots, k_n)$ the vector space of all homogenous polynomials S of the form*

$$\sum_{i=0}^{n} S_i = 0$$

where $S_i = (Z_i)^{k_i} P_i$ and $P_i \in \mathbb{C}[Z_0, Z_1, \ldots, Z_n]$ are homogenous of degree k_i $(i = 0, 1, \ldots, n)$. We note that :

$$\dim_{\mathbb{C}} \mathcal{S}(d, k_0, k_1, \ldots, k_n) = \sum_{i=0}^{n} \binom{k_i + n}{k_i}.$$

S will said to be nondegenerate if the following determinant does not vanish identically

$$\begin{vmatrix} \dfrac{\partial S_0}{\partial Z_0} & \cdots & \dfrac{\partial S_0}{\partial Z_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial S_n}{\partial Z_0} & \cdots & \dfrac{\partial S_n}{\partial Z_n} \end{vmatrix}.$$

Remark. — The special case considered by A.M. Nadel [Na89] was the case of spaces $\mathcal{S}(d, k) = \mathcal{S}(d, k_0, k_1, \ldots, k_n)$ with $k_0 = k_1 = \ldots = k_n = k$. Using the same techniques as in Nadel's paper one gets the following.

2.2 Proposition. — *Let $\Sigma := \{S = 0\}$ be a smooth hypersurface in $\mathbb{P}^n_{\mathbb{C}}$, where $S = \sum_{i=0}^{n} S_i$ is a nondegenerate element of $\mathcal{S}(d, k_0, k_1, \ldots, k_n)$. Then one can constuct* *a meromorphic connection ∇_S on $\mathbb{P}^n_{\mathbb{C}}$ with pole order $\le 2+n+\sum_{i=0}^n k_i$ and with respect to which every hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ of the form*

$$\sum_{i=0}^n a_i S_i = 0,$$

is totally geodesic, where $a_0,\dots,a_n$ are complex constants (not all zero).

Proof. — We use the same construction as in [Na89], however a slightly refined calculation permits to improve the pole order of the meromorphic connection (Nadel's original lower bound was $3+n+(n+1)k$).

Combining Siu's theorem and proposition 2.2, one gets

2.3 Theorem. — *Let Σ as above. Suppose that*

$$d > n+1+\frac{(n-1)(n-2)}{2}(2+n+\sum_{s=0}^n k_s)$$

then the image of an entire curve $f\colon \mathbb{C}\to\Sigma$ is not Zariski dense in Σ. More precisely, $f(\mathbb{C})$ is contained either in the pole set of ∇_S , or in the intersection $\Sigma\cap\Sigma'$ where $\Sigma'\neq\Sigma$ is another hypersurface defined by an element $S'\in\mathcal{S}(d,(k_0,k_1,\dots,k_n))$ which is again a linear combination of $S_i,(i=0,1,\dots,n)$.
In particular, Green and Griffiths' conjecture [GG78] is true for this class of hypersurfaces: an entire curve in a projective variety X of general type cannot be Zariski dense in X.

Proof. — We refer to Nadel [Na89] for details. The idea is that one of the hypersurfaces $\Sigma'\neq\Sigma$ in the linear system $\{\sum_{i=0}^n a_iS_i=0\}$ contains the $(n-2)$-nd order jet of f. Then, if $f(\mathbb{C})$ is not contained in the pole set of ∇_S, the Wronskian $f'\wedge f''\wedge\ldots\wedge f^{(n-1)}$ must vanish by Siu's theorem. It then follows from the fact that Σ' is totally geodesic that $f(\mathbb{C})\subset\Sigma'$.

Before we end this section we recall an important corollary of Theorem 2.3.

2.4 Corollary . — *Let d,k_0,k_1,k_2,k_3 be integers such that $d>9+\sum_{\mathrm{i}=0}^3 k_{\mathrm{i}}$. Let S be a nondegenerate element of $\mathcal{S}(d,k_0,k_1,k_2,k_3)$ such that $\Sigma=\{S=0\}$ is a smooth surface in $\mathbb{P}^3$. Then every entire curve in Σ is contained in an irreducible curve of degree $\le d^2$ and genus ≤ 1. In particular, Lang's conjecture is true for this class of algebraic surfaces in $\mathbb{P}^3$: any such surface which contains no rational or elliptic curve is hyperbolic.*

Proof. — By theorem 2.3, an entire curve must be contained in some irreducible projective curve of degree $\le d^2$, namely one of the irreducible components of the pole set of ∇ or of the intersection $\Sigma\cap\Sigma'$.

3. ALGEBRAIC FAMILIES OF SMOOTH HYPERBOLIC SURFACES IN $\mathbb{P}^3_{\mathbb{C}}$

In order to construct examples of hyperbolic surfaces Σ in $\mathbb{P}^3_{\mathbb{C}}$, we only need to apply corollary 2.4 and compute the genus of some special curves $\Sigma\cup\Sigma'$ when Σ' is either the pole set of the meromorphic connection or an element in the linear system $\{\sum_{i=0}^n a_iS_i=0\}$.

3.1 Theorem . — *Let $\varepsilon_0=d^{d/2}/2(d-2)^{d/2-1}$, and let ε_1 and $\varepsilon_2\in\mathbb{C}^*$ be such that :*

(1) $(\pm i\varepsilon_j)^d$ *and* $(\pm i\varepsilon_1)^d + (\pm i\varepsilon_2)^d$ *differ from* ε_0 *for* $j = 1, 2$,
(2) $(\pm i\varepsilon_1)^d + (\pm i\varepsilon_2)^d \neq 0$.
Then the surface $\Sigma \subset \mathbb{P}^3$ *defined in homogenous coordinates* Z_0, Z_1, Z_2, Z_3 *by*

$$Z_0^d + Z_1^{d-2}(Z_1^2 + \varepsilon_1^2 Z_0^2) + Z_2^{d-2}(Z_2^2 + \varepsilon_2^2 Z_0^2) + Z_3^d = 0$$

is hyperbolic for $d \geq 14$.

Remark. — The condition (1) above precisely ensures that Σ is smooth, whereas condition (2) will be needed in the proof of lemma 4.5 below.

As a corollary of theorem 3.1 and corollary 2.4 we have the following.

3.2 Corollary (Main result). — *Let* d *be a fixed integers and denote by* $\mathcal{P}_{3,d}$ *the projective space of all surfaces of degree* d *in* $\mathbb{P}^3_{\mathbb{C}}$ *then the set* $H_{3,d}$ *of hyperbolic surfaces in* $\mathbb{P}^3_{\mathbb{C}}$ *is nonempty for* $d \geq 14$.

Furthermore, let k_0, k_1, k_2, k_3 *be integers such that two of them* ≥ 2 *then for* $d > 9 + \sum_{i=0}^{3} k_i$, $H_{3,d}$ *contains a Zariski open subset of* $P_{3,d}$ *of dimension*

$$\sum_{i=0}^{3} \binom{k_i + 3}{k_i} - 1.$$

Proof of theorem 3.1 . — One can see that Σ is the zero set of an element $S \in \mathcal{S}(d, 0, 2, 2, 0)$. Hence by proposition 2.2 we can construct on $\mathbb{P}^3_{\mathbb{C}}$ a meromorphic connection ∇_S with respect to which Σ is totally geodesic. It is easy to see that the pole set of ∇ is the zero set of the holomorphic section of $\mathcal{O}_{\mathbb{P}^3}(9)$ given by

$$t = Z_0^2 Z_1 Z_2 Z_3 (dZ_1^2 + \varepsilon_1^2 (d-2) Z_0^2)(dZ_2^2 + \varepsilon_2^2 (d-2) Z_0^2).$$

By theorem 2.3, if $d \geq 14$ then either
(A) $f(\mathbb{C}) \subset \{t = 0\}$ or
(B) $f(\mathbb{C}) \subset \Sigma \cap \Sigma'$ where Σ' is defined by:

$$S' = a\ Z_0^d + b\ Z_1^{d-2}(Z_1^2 + \varepsilon_1^2 Z_0^2) + c\ Z_2^{d-2}(Z_2^2 + \varepsilon_2^2 Z_0^2) + e\ Z_3^d = 0.$$

Without loss of generality we can assume $e = 0$ since we can replace $S' = 0$ by $S' - e\ S = 0$. By lemma 3.3 below, the proof of theorem 3.1 is reduced to checking whether the geometric genus of finite number of explicit plane curves is actually ≥ 2 (this will be done in section 4).

3.3 Lemma . — *Suppose that* f *is given in* $\mathbb{P}^3$ *by entire holomorphic functions* $Z_0(t)$, $Z_1(t)$, $Z_2(t)$, $Z_3(t)$ $(t \in \mathbb{C})$, *then conditions (A) and (B) above imply one of the following 3 cases:*
1) $\exists\ \alpha \in \mathbb{C}$ *such that either* $[Z_0(t) : Z_1(t) : Z_3(t)]$ *or* $[Z_0(t) : Z_2(t) : Z_3(t)] \subset \mathbb{P}^2$ *is contained into a curve linearly isomorphic to the curve* $C_{1,\alpha} \subset \mathbb{P}^2$ *defined by:*

$$\alpha\ Z_0^d + Z_1^{d-2}(Z_1^2 + Z_0^2) + Z_3^d = 0.$$

2) $\exists\ \beta \in \mathbb{C}$ *such that* $[Z_0(t) : Z_1(t) : Z_2(t)] \subset \mathbb{P}^2$ *is contained into* $C_{2,\beta} \subset \mathbb{P}^2$ *defined by:*

$$\beta\ Z_0^d + Z_1^{d-2}(Z_1^2 + \varepsilon_1^2 Z_0^2) + Z_2^{d-2}(Z_2^2 + \varepsilon_2^2 Z_0^2) = 0.$$

3) f is a constant curve.

Proof . — Suppose first that condition (A) is satisfied. Then one of the following situations is true

- $Z_0(t) = 0$: then $S(f(t)) = 0$ $(*)$ implies $Z_1^d(t) + Z_2^d(t) + Z_3^d(t) = 0$, thus $[Z_1(t) : Z_2(t) : Z_3(t)]$ is a constant curve in $\mathbb{P}^2$ for $d > 3$, since is a Fermat curve of genus > 1, thus f is constant and we are in case 3).
- $Z_3(t) \equiv 0$: then $(*)$ implies 2).
- either $Z_1(t)$ or $Z_2(t)$ is a constant multiple of $Z_0(t)$: then $(*)$ implies 1). One can check that in case (A) all curves $C_{1,\alpha}$ and $C_{2,\beta}$ are in fact smooth with our restrictions on ε_1 and ε_2. We will not explain this in detail because we will consider a more general setting.

Suppose now that condition (B) is satisfied. Then the curve f satisfies the following two equations:

$(*)$ $Z_0^d(t) + Z_1^{d-2}(t)(Z_1^2(t) + \varepsilon_1^2 Z_0^2(t)) + Z_2^{d-2}(t)(Z_2^2(t) + \varepsilon_2^2 Z_0^2(t)) + Z_3^d(t) = 0,$

$(*')$ $a\ Z_0^d(t) + b\ Z_1^{d-2}(t)(Z_1^2(t) + \varepsilon_1^2 Z_0^2(t)) + c\ Z_2^{d-2}(t)(Z_2^2(t) + \varepsilon_2^2 Z_0^2(t)) = 0.$

We will discuss three cases:

- $b = 0$ or $c = 0$: then either $Z_1(t)$ or $Z_2(t)$ is a constant multiple of Z_0 because a polynomial of two variables decomposes into linear factors, so in this case (B) implies 1).
- $b, c \neq 0$ and $b = c$: by factorizing b in $(*')$ (B) implies 2).
- $b, c \neq 0$ and $b \neq c$: then replacing $(*')$ by $(*') - c(*)$ (B) implies 1).

4. GEOMETRIC GENUS OF PLANE CURVES

In this section we compute the geometric genus of some plane curves, especially those arising from lemma 3.3 .

Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ and $\nu\colon \bar{C} \longrightarrow C$ the normalisation of C. We define δ to be the sheaf supported on the set of singular points of C, which fits in the exact sequense

$$0 \to \mathcal{O}_C \to f_*\mathcal{O}_{\bar{C}} \to \delta \to 0.$$

The associated exact sequence of cohomology gives the geometric genus

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{p \in \mathrm{sing} C} \dim(\delta_p).$$

Now we have the following formula for monomial singularities.

4.1 Proposition . — *With the above notations, suppose that C is given in local analytic coordinates x and y around a singular point p by $f(x,y) := x^a - y^b = 0$ then*

$$\delta_p := \dim \delta_p = \frac{(a-1)(b-1)}{2} + \frac{\gcd(a,b) - 1}{2}.$$

Proof. — Let $\mu := \dim \mathbb{C}\{x,y\}/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ the Milnor number of the singularity at p. Then we conclude using the following formula (see [Mi68])

$$\delta_p = \frac{\mu + r - 1}{2},$$

where r is the number of irreducible branch at p.

4.2 Proposition . — *Let $\alpha \in \mathbb{C}$ then the curve C defined in inhomogenous coordinates (X, Y) of $\mathbb{P}^2$ by :*

$$\alpha + X^{d-2}(X^2 + 1) + Y^d = 0$$

is hyperbolic for $d \geq 5$.

Proof . — The case $\alpha = 0$ is done in Lemma 4.3, assume now that $\alpha \neq 0$ and that $d \geq 5$. It is easy to check that singular points of C satisfy:

$$\begin{cases} X^{d-2} = -\frac{\alpha d}{2} \\ X^2 = -\frac{d-2}{d} \\ Y = 0 \end{cases}.$$

We will distinguish 2 cases:
(1) d is odd

- if $\alpha \neq \mathrm{i}/\varepsilon_0$ and $\alpha \neq -\mathrm{i}/\varepsilon_0$ then C is smooth so $g = \frac{(d-2)(d-1)}{2} \geq 2$.
- if $\alpha = \pm \mathrm{i}^d/\varepsilon_0$, by symmetry we can suppose $\alpha = \mathrm{i}^d/\varepsilon_0$. Then we have only one singular point $p = (X_0 = \mathrm{i}(\frac{d-2}{d})^{\frac{1}{2}}, 0)$. Using a Taylor expansion around p, one can check that the singularity of C is given locally by $x^2 + y^d = 0$, and since d is odd it follows that C is algebraically irreducible. Thus by proposition 4.1 the genus is

$$g = \frac{(d-2)(d-1)}{2} - \frac{d-1}{2} = \frac{(d-1)(d-3)}{2} \geq 2.$$

(2) d is even

- if $\alpha \neq (-1)^{\frac{d}{2}}/\varepsilon_0$, then C is smooth and thus $g \geq 2$.
- if $\alpha = (-1)^{\frac{d}{2}}/\varepsilon_0$., then C has two singular points $p_1 = (i(\frac{d-2}{d})^{\frac{1}{2}}, 0)$ and $p_2 = (-\mathrm{i}(\frac{d-2}{d})^{\frac{1}{2}}, 0)$. By symmetry, singularities at both points are isomorphic. As in case (1), C is given near the singular points by $x^2 + y^d = 0$. This is a reducible singularity which can be decomposed into two factors $x + \mathrm{i}y^{\frac{d}{2}}$ and $x - \mathrm{i}y^{\frac{d}{2}}$.

We claim that C is algebraically irreducible. Otherwise, by symmetry, it would decompose into two irreducible components C_1 and C_2 of degrees d_1 and d_2 respectively, clearly $d_1 + d_2 = d$. Now apply Bezout's theorem : $(C_1 \cdot C_2) = d_1 d_2$, where $(C_1 \cdot C_2)$ is the intersection number between the two irreducible components (cf. [Be78]). This number is equal to

$$(C_1 \cdot C_2)_{p_1} + (C_1 \cdot C_2)_{p_2} = (\text{by symmetry}) \quad 2(C_1 \cdot C_2)_{p_1},$$

where $(C_1 \cdot C_2)_{p_1}$ is by definition :

$$\dim \mathbb{C}\{x, y\}/(x + \mathrm{i}y^{\frac{d}{2}}, x - \mathrm{i}y^{\frac{d}{2}}) = \dim \mathbb{C}\{x, y\}/(x, y^{\frac{d}{2}}) = \frac{d}{2}.$$

So we have to resolve the following system:

$$\begin{cases} d_1 + d_2 = d \\ d_1 d_2 = d \end{cases}$$

which implies $d_1 = d_2 = 2$, and this is not allowed. Thus C is irreducible as claimed and thanks to proposition 4.1

$$g = \frac{d(d-2)}{2} - 2\frac{d}{2} = \frac{d(d-4)}{2} \geq 2.$$

4.3 Lemma. — *The plane curve C defined in inhomogenous coordinates (X, Y) of $\mathbb{P}^2$ by :*

$$X^{d-2}(X^2 + 1) + Y^d = 0$$

is hyperbolic when $d \geq 5$.

Proof. — Suppose that $d \geq 5$. It is easy to check that the only singular point of C is $p = (0, 0)$. The singularity around this point is of the type $x^{d-2} + y^d = 0$. Let us investigate two cases.

•d is odd:

Then the curve C is irreducible analytically (thus also algebraically) and proposition 4.1 gives us the genus:

$$g = \frac{(d-2)(d-1)}{2} - \frac{(d-1)(d-3)}{2} = \frac{d-1}{2} \geq 2.$$

•d is even :

In this case the singularity decomposes into two factors $x^{\frac{d}{2}-1} + \mathrm{i}y^{\frac{d}{2}} = 0$ and $x^{\frac{d}{2}-1} - \mathrm{i}y^{\frac{d}{2}} = 0$. So analytically C decomposes on some neighbourhood of p into two irreducible components $X^{\frac{d}{2}-1}\sqrt{X^2+1} + \mathrm{i}Y^{\frac{d}{2}} = 0$ and $X^{\frac{d}{2}-1}\sqrt{X^2+1} - \mathrm{i}Y^{\frac{d}{2}} = 0$. These analytic branches can not of course be the zero sets of restrictions of some polynomials. Therefore C is algebraically irreducible and by proposition 4.1 we get:

$$g = \frac{(d-2)(d-1)}{2} - \frac{(d-1)(d-3)}{2} - \frac{1}{2} = \frac{d-2}{2} \geq 2.$$

The proof is now complete.

4.4 Proposition . — *Let $\varepsilon_1, \varepsilon_2 \in \mathbb{C}^*$ be such that :*

(1) $(\pm \mathrm{i}\varepsilon_1)^d + (\pm \mathrm{i}\varepsilon_2)^d \neq (\pm \mathrm{i}\varepsilon_j)^d \;\; j = 1, 2$

(2) $(\pm \mathrm{i}\varepsilon_1)^d + (\pm \mathrm{i}\varepsilon_2)^d \neq 0.$

Then the plane curve C defined in inhomogenous coordinates (X, Y) of $\mathbb{P}^2$ by

$$\beta + X^{d-2}(X^2 + \varepsilon_1^2) + Y^{d-2}(Y^2 + \varepsilon_2^2) = 0$$

is hyperbolic whenever $d \geq 5$.

Proof . — The case $\beta = 0$ is treated in Lemma 4.5 below. Assume now that $\beta \neq 0$ and that $d \geq 5$. One can check that the singular points of C satisfy

$$(a_1)\begin{cases} Y = 0 \\ dX^2 + \varepsilon_1^2(d-2) = 0 \\ 2\varepsilon_1^2 X^{d-2} + \beta d = 0 \end{cases} \quad \text{or} \quad (\mathrm{a}_2)\begin{cases} X = 0 \\ dY^2 + \varepsilon_2^2(d-2) = 0 \\ 2\varepsilon_2^2 Y^{d-2} + \beta d = 0 \end{cases}$$

or

$$(b)\begin{cases} dX^2+\varepsilon_1^2(d-2)=0\\ dY^2+\varepsilon_2^2(d-2)=0\\ 2(\varepsilon_1^2X^{d-2}+\varepsilon_2^2Y^{d-2})+\beta d=0\end{cases}.$$

(a_j) has solutions iff $\beta=(\pm\mathrm{i})^d\varepsilon_j/\varepsilon_0$ $(j=1,2)$ and (b) has solutions when $\beta=((\pm\mathrm{i})^d\varepsilon_1+(\pm\mathrm{i})^d\varepsilon_2)/\varepsilon_0$. When these conditions are not satisfied the curve is smooth, so its genus is $g\geq 2$. With assumptions above on ε_1 and ε_2 (a_1) (a_2) and (b) are non compatible so we can forget e.g. (a_2) and distinguish two cases :
(1) d is odd :

- if $\beta=\pm(\mathrm{i}\varepsilon_1)^d/\varepsilon_0$ then only equation (a_1) has solutions. By symmetry we can suppose $\beta=(\mathrm{i}\varepsilon_1)^d/\varepsilon_0$. Then the only solution of (a_1) is $p=(X_0=\mathrm{i}\varepsilon_1(\frac{d-2}{d})^{\frac12},0)$. Using a Taylor expansion around p, we can see that the singularity at p is given locally by $x^2+y^{d-2}=0$ which is irreducible since d is odd, so C is locally analytically irreducible. Hence it is also algebraically irreducible, and by proposition 4.1 the genus is

$$g=\frac{(d-2)(d-1)}{2}-2(\frac{d-3}{2})=\frac{d^2-5d+8}{2}\geq 2.$$

- if $\beta=(\pm\mathrm{i}\varepsilon_1)^d+(\pm\mathrm{i}\varepsilon_2)^d/\varepsilon_0$ then only (b) has solutions. By symmetry and condition (2) above we can suppose $\beta=(\mathrm{i}\varepsilon_1)^d+(\mathrm{i}\varepsilon_2)^d/\varepsilon_0$. Then the only solution of (b) is $p=(X_0=\mathrm{i}\varepsilon_1(\frac{d-2}{d})^{\frac12},Y_0=\mathrm{i}\varepsilon_2(\frac{d-2}{d})^{\frac12})$. Using a Taylor expansion around p, it is clear that the singularity is a node so C is algebraically irreducible. Indeed we would have otherwise $C=C_1\cup C_2$ where C_1 and C_2 are irreducible components. By Bezout's theorem, we would have $(C_1,C_2)=1$ since the intersection is normal at a single point, and C_1,C_2 would have then degree 1, contradiction. Thus

$$g=\frac{(d-2)(d-1)}{2}-1\geq 2.$$

(2) d is even :

- if $\beta=(i\varepsilon_1)^d/\varepsilon_0$ then only (a_1) has solutions $p_1=(\mathrm{i}\varepsilon_1(\frac{d-2}{d})^{\frac12},0)$ and $p_2=(-\mathrm{i}\varepsilon_1(\frac{d-2}{d})^{\frac12},0)$. By symmetry we have the same type of singularity around both points and as in case (1) we can check that C can be given around p_1 by $x^2+y^{d-2}=0$, which decomposes into two factors (d even).

We claim that C is algebraically irreducible. Otherwise, by symmetry, it would decompose into two irreducible components, say C_1 and C_2, with respective degrees d_1 and d_2. Apply now Bezout's theorem : $(C_1\cdot C_2)=2(C_1\cdot C_2)_{p_1}=d_1d_2$. Since $(C_1\cdot C_2)_{p_1}=\frac{d-2}{2}$ we have to solve the conditions:

$$\begin{cases} d_1+d_2=d\\ d_1d_2=d-2\end{cases}\Rightarrow\ (d_1-1)(d_2-1)+1=0.$$

This leads to a contradiction. Now by proposition 4.1 the genus of C is:

$$g=\frac{(d-2)(d-1)}{2}-2\frac{d-2}{2}=\frac{(d-2)(d-3)}{2}\geq 2.$$

- if $\beta=(\mathrm{i}\varepsilon_1)^d+(\mathrm{i}\varepsilon_2)^d/\varepsilon_0$, then (c) only has solutions : $\pm\mathrm{i}\varepsilon_j(\frac{d-2}{d})^{\frac12}$ $(j=1,2)$. By symmetry, singularities of C around those points are isomorphic. As in case (1), these points are nodes. We will prove now that C is irreducible. Otherwise, by symmetry, C factorizes into two irreducible components, say C_1 and C_2, with respective degrees d_1 and d_2. By Bezout's theorem $d_1 d_2 = (C_1 \cdot C_2) = 4$, which is not possible when $d \geq 5$. Now by proposition 4.1 we get

$$g = \frac{(d-2)(d-1)}{2} - 4 \geq 2.$$

Remark on the proof. — When $\beta \neq 0$ conditions (1) and (2) are superfluous. We used them only to give a simpler proof (however, we will see in lemma 4.5 below that when $\beta = 0$ condition (2) is essential).

4.5 Lemma . — *Let ε_1 and ε_2 be $\in \mathbb{C}^*$ such that $(\pm i\varepsilon_1)^d + (\pm i\varepsilon_2)^d \neq 0$. Then the curve C given in inhomogenous coordinates (X, Y) of $\mathbb{P}^2$ by*

$$X^{d-2}(X^2 + \varepsilon_1^2) + Y^{d-2}(Y^2 + \varepsilon_2^2) = 0$$

is hyperbolic when $d \geq 4$.

Proof. — Suppose $d \geq 4$. With the above conditions on ε_1 and ε_2, it is easy to see that there is only one singular point $p = (0, 0)$. The singularity of C at this point is given by $x^{d-2} + y^{d-2} = 0$ which is an ordinary multiple singularity.
We claim that C is irreducible. Otherwise $C = C_1 \cup C_2$ where C_1 and C_2 are two curves of degrees d_1 and d_2 and multiplicities at p equal to m_1 and m_2. Now by Bezout theorem $d_1 d_2 = (C_1 \cdot C_2) = (C_1 \cdot C_2)_p = m_1 m_2$ (because the tangent cone of C the union of distinct lines). Since, $m_i \leq d_i$ $(i = 1, 2)$, we must have $m_i = d_i$ $(i = 1, 2)$ contradiction. Finally, by proposition 4.1

$$g = \frac{(d-2)(d-1)}{2} - \frac{(d-2)(d-3)}{2} = d - 2 \geq 2.$$

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