ON THE Λ-ADIC KLINGEN-EISENSTEIN SERIES

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Introduction

The purpose of this paper is to construct a p-adic measure coming from the Klingen-Eisenstein series on the symplectic group

$$G = \mathrm{GSp}_{2m} = \left\{ \alpha \in \mathrm{GL}_{2m} \mid {}^t\alpha J_m \alpha = \nu(\alpha) J_m, \nu(\alpha) \in \mathrm{GL}_1 \right\},$$

over $\mathbf{Q}$ where

$$J_m = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}.$$

This measure takes values in a space of p-adic Siegel modular forms and defines a $\mathcal{L}$-adic Siegel modular form where Λ is the Iwasawa algebra $\mathbf{Z}_p[[X]]$ and $\mathcal{L}$ denotes the field of fractions of Λ.

More precisely, let $f \in \mathcal{S}_k^r(\Gamma)$ be a cusp form of degree r (with respect to a congruence subgroup Γ of Γ^r of level C). If $k > m + r + 1$ and $m \geq r$ then the Klingen-Eisenstein series is defined as the following absolutely convergent series

$$E_k^{m,r}(z,f,\chi) = \sum_{\gamma \in \Delta_{m,r} \cap \Gamma \backslash \Gamma} \chi(\det(d_\gamma)) f(\omega^{(r)}(\gamma z)) j(\gamma,z)^{-k}, \qquad (1)$$

with $z \in \mathfrak{H}^m$, $\omega(z)^{(r)}$ being the upper left corner of z of size $r \times r$, $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ and $\Delta_{m,r}$ denotes the set of elements in Γ^m having the form $\begin{pmatrix} * & * \\ 0_{m-r,m+r} & * \end{pmatrix}$ [Kl3]. This series turns out to be a modular form of weight k and of degree m on a congruence subgroup of the group Γ^m. M. Harris proved in [Har2]–[Har4] the validity of Garrett's conjecture: all the Fourier coefficients the modular form $E_k^{m,r}(z,f,\chi)$ belong to the field $\mathbf{Q}(f,\chi)$ generated by the Fourier coefficients of f and χ (at least for trivial χ). Explicit formulas for Fourier coefficients of the series $E_k^{m,r}(z,f)$ were given by Böcherer [Bö2] in general for $k > m + r + 1$ and by Kurokawa and Mizumoto [Kur-Miz], [Miz1], [Miz2] who treated the case $m = 2$, $r = 1$. It turned out that the most significant term in these formulas involves the special values of the standard zeta function of f twisted with a certain quadratic Dirichlet character attached to the matrix index ξ of a Fourier coefficient; as it was noticed above, these functions reduce to the (twisted) symmetric squares of the form f if $m = 2$, $r = 1$. The p-adic construction uses the identity of Böcherer [Bö1] generalised by Shimura [Shi10] which expresses these series as certain integrals involving the Petersson product of $f(w)$ with the pullbacks $E(\mathrm{diag}[z,w];\, k,\chi,N)$ of the Siegel-Eisenstein series

$$E(Z,k,\chi,N) = \sum_{\alpha\in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\alpha))\det(c_\alpha Z + d_\alpha)^{-k}, \qquad (2)$$

where Z is a variable in the Siegel upper half plane of degree $n = m + r$,

$$\mathfrak{H}^n = \left\{Z \in \mathrm{M}_n(\mathbf{C}) \mid {}^tZ = Z = X + iY,\ Y > 0\right\},$$

$\Gamma = \Gamma_0^n(N), \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}$, and P denotes the subgroup of $P \subset G_{\infty+}$, consisting of elements α with the condition $c_\alpha = 0$, and k is the weight (the above series converges absolutely for $k > n + 1$). The series (2) may be regarded as series of the type (1): they coinside with $E_k^{m+r,0}(Z,f,\chi)$ with a constant 1 as f.

Let $D(s,f,\chi)$ be the standard zeta function of $f \in \mathcal{S}_k^r(\Gamma)$ as above (with local factors of degree $2r+1$) and χ be a Dirichlet character. Then the essential fact for our construction is the following identity:

$$\Lambda(k,\chi)D(k - r,f,\eta)E_k^{m,r}(z,f,\chi) = \langle f'(w), E_k^{m+r,0}(\mathrm{diag}[z,w])\rangle. \qquad (3)$$

Here $\Lambda(k,\chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, η is a certain Dirichlet character, $E_k^{m+r,0}$ a series of type (2) transformed by a suitable element of G^{m+r}, f' an easy transform of f, $(z,w) \in \mathfrak{H}_m \times \mathfrak{H}_r$ (see [Shi10, (7.4), p.572]). Our construction is based on the fact the series (2) produce a p-adic measure (the Siegel-Eisenstein measure), see [Pa3]. In the simpliest situation this measure depends on the variables (x,ξ) which belong to $\mathbf{Z}_p^\times \times A_{n,p}$ where $\mathbf{Z}_p^\times$ is the p-adic unit group, and $A_{n,p}$ is a free $\mathbf{Z}_p$-module of rank $n(n+1)/2$ formed by half integral symmetric matrices of size n over $\mathbf{Z}_p$. The Siegel-Eisenstein measure defines a Λ-adic modular form whose special values are (involuted) Siegel-Eisenstein series [Pa3] described in Section 3. A general theory of Λ-adic forms was developped by H. Hida in the elliptic modular case, then it was extended by himself to $G = \mathrm{GL}_n$ and to cohomological automorphic forms. Using pullback of the Siegel-Eisenstein Λ-adic modular forms to $(z,w) \in \mathfrak{H}^2 \times \mathfrak{H}^1$ and a Λ-adic version of the Petersson scalar product over the variable w with Λ-adic form representing a family of Hida we construct then a Klingen-Eisenstein Λ-adic modular form (of the variable z). Recently a serious attempt to extend Hida's theory to the Siegel modular case was made by K. Buecker (Dissertation of Cambridge University, UK, 1994) under the direction of Prof. R. Taylor. Very interesting families of Siegel modular forms were constructed by J. Tilouine and E. Urban [Ti-U]. If the techniques of Λ-adic forms work for $G = \mathrm{GSp}_{2r}$ as well as for GL_2, one can expect to obtain also a much more general construction of non-Archimedean standard L-functions of several variables (at least in the p-ordinary case).

Content of the paper

1. Modular Forms, Involution and q-expansion

We put $G^n = Sp_{2n}(\mathbf{Q})$ and $G^{m,r} = Sp_{2m}(\mathbf{Q}) \times Sp_{2r}(\mathbf{Q})$. We define the congruence subgroup $\Gamma_0^n(p^\alpha)$ in a usual way and put $\Gamma_0^{m,r}(p^\alpha) = \Gamma_0^m(p^\alpha) \times \Gamma_0^r(p^\alpha)$. Let B_n be the semigroup of symmetric, semi-definite, half-integral matrices of size n. We put $B_{m,r} = B_m \times B_r$.

Let G be one of the groups G^n $(n = 1,2,3)$ or $G^{2,1}$. Let $\Gamma_0(p^\alpha)$ be the one of $\Gamma_0^n(p^\alpha)$ $(n = 1,2,3)$ or $\Gamma_0^{2,1}(p^\alpha)$ according to G. Let B be the one of B_n $(n = 1,2,3)$ or $B_{2,1}$ according to G.

$$\begin{aligned} &M_k^{2,1}(\Gamma_0^{2,1}(p^\alpha),\psi) \\ &\quad = \{ f \in M_k^{2,1}(\Gamma_1(p^\alpha)) \mid f|_k(g_2,g_1) = \psi(\det(d_2)\cdot d_1) f \ \ \text{for any} \ \ (g_2,g_1) \in \Gamma_0^{2,1}(p^\alpha) \} \end{aligned}$$

2. Λ-adic modular forms

Let $\chi = \omega^a$ with some a $(0 \le a < p-1)$. Let G be one of the groups G^n $(n = 1,2,3)$ or $G^{2,1}$. Let $\Gamma_0(p^\alpha)$ be the one of the groups $\Gamma_0^n(p^\alpha)$ $(n = 1,2,3)$ or $\Gamma_0^{2,1}(p^\alpha)$ according to G. Let B be the one of B_n $(n = 1,2,3)$ or $B_{2,1}$ accoding to G. Let $\mathcal{P}$ be a Zariski dense subset of Spec $\Lambda(\bar{\mathbf{Q}}_p)$. We call an element F of $\Lambda[[q^B]]$ a Λ-adic modular forms on G with character χ with respect to P if $F(\epsilon(u)u^k - 1)$ gives a q-expansion of modular forms in $M_k(\Gamma_0(p^\alpha),\epsilon\chi\omega^{-k})$ for all (k,ϵ) such that $P_{k,\epsilon} \in \mathcal{P}$. We shall take $\mathcal{P} = \mathcal{P}(5) = \{P_{k,\epsilon}; k \ge 5\}$. We denote by $M(G,\chi;\Lambda)$ the Λ-submodule of $\Lambda[[q^B]]$ generated by Λ-adic modular forms on G.

We use the symbol $M^n(\chi;\Lambda)$ $(n = 1,2,3)$ or $M^{2,1}(\chi;\Lambda)$ for $M(G,\chi;\Lambda)$ according to the group G. Then obviously

$$A[[q^{B_{2,1}}]] = A[[q^{B_2}]]\widehat{\otimes_A}A[[q^{B_1}]].$$

We let the Hecke operator $T_1(p)$ act on $M^{2,1}(\Gamma_0^{2,1}(p^\alpha),\psi;\mathcal{O})$. We put

$$e_1 = \lim_{n\to\infty} T_1(p)^{n!}$$

and

$$M^{2,1-\mathrm{ord}}(\Gamma_0^{2,1}(p^\alpha),\psi;\mathcal{O}) = e_1 M^{2,1}(\Gamma_0^{2,1}(p^\alpha),\psi;\mathcal{O}).$$

We call $f \in M^{2,1}(\Lambda,\chi)$ 1-ordinary if its specialization at $P_{k,\epsilon}$ is in $M^{2,1-\mathrm{ord}}(\Gamma_0^{2,1}(p^\alpha),\psi;\mathcal{O})$ for any $P_{k,\epsilon} \in \mathcal{P}$. We denote by $M^{2,1-\mathrm{ord}}(\Lambda,\chi)$ the space of 1-ordinary Λ-adic modular forms.

Proposition.

$$M^{2,1-\mathrm{ord}}(\chi;\Lambda) = M^2(\chi;\Lambda) \otimes_\Lambda M^{\mathrm{ord}}(\chi;\Lambda).$$

Proof. — Let $f^i \in M^{\mathrm{ord}}(\chi;\Lambda)$ $(i = 1,2,\dots,n)$ be a basis. Define Λ-adic linear forms

$$l^i : M^{\mathrm{ord}}(\chi;\Lambda) \to \Lambda \quad (i = 1,2,\dots,n) \quad \text{such that} \quad l^i(f^j) = \delta_{i,j}.$$

Let $P = P_{k,\varepsilon} \in Pc$, $G \in M^{2,1-\mathrm{ord}}(\chi;\Lambda)$, $G_P \in M_k^{2,1-\mathrm{ord}}(\Gamma_0(p^\alpha),\chi\varepsilon\omega^{-k};\mathcal{O})$. We have that

$$M_k^{2,1-\mathrm{ord}}(\Gamma_0(p^\alpha),\chi\varepsilon\omega^{-k};\mathcal{O}) = M_k^{\mathrm{ord}}(\Gamma_0(p^\alpha),\chi\varepsilon\omega^{-k};\mathcal{O}) \otimes_{\mathcal{O}} M_k^2(\Gamma_0(p^\alpha),\chi\varepsilon\omega^{-k};\mathcal{O}),$$

and there exist g_P^i such that $G_P = \sum_{i=1}^n f_P^i \otimes g_P^i$. Put

$$R_P^i = l_P^i \otimes \mathrm{id} \,:\, G_P \mapsto (l_P^i \otimes \mathrm{id})(\sum_{j=1}^n f_P^j \otimes g_P^j) = 1 \otimes g_P^i = g_P^i.$$

This gives an $\mathcal{O}$-module homomorphism

$$R_P^i = l_P^i \otimes \mathrm{id} \,:\, M_k^{2,1-\mathrm{ord}}(\Gamma_0(p^\alpha),\chi\varepsilon\omega^{-k};\mathcal{O}) \to M_k^2(\Gamma_0(p^\alpha),\chi\varepsilon\omega^{-k};\mathcal{O}).$$

We want now to construct a Λ-adic lift of R_P^i or of g_P^i. We use the following patching lemma of A. Wiles:

Lemma. — *Suppose that for each $P \in \mathcal{P}$ we are given a $g_P \in \Lambda/P\Lambda$ such that they all are compatible (i.e. g_P and g_Q map to the same $g_{P+Q} \in \Lambda/(P+Q)\Lambda$). Then there exist $g \in \Lambda$ such that*

$$\forall P \in \mathcal{P} \quad g_P \equiv g \bmod P \in \Lambda/P\Lambda.$$

(See [Wi], [Hi4], p. 232).

3. The Siegel-Eisenstein Λ-adic modular form

The construction of this form uses normalized Siegel-Eisenstein series. We introduce here normalized Siegel-Eisenstein series in order to give a precise statement on algebraic properties of these Fourier coefficients. Let χ be a Dirichlet character mod$N > 1$ such that $\chi(-1) = (-1)^k$.

$$\begin{aligned} G_k^*(z) = G^*(z,k,\chi,N) &= N^{nk/2} i^{nk} 2^{-n(k+1)} \pi^{-nk} \Gamma(1_n,0)^{-1} \\ &\quad \times L_N^*(k,\chi) \prod_{i=1}^{[n/2]} L_N^*(2k-2i,\chi^2) E(-(Nz)^{-1}) \det(\sqrt{N}z)^{-k} \\ &= N^{nk/2} \tilde{\Gamma}(k,0) L_N(k,\chi) \prod_{i=1}^{[n/2]} L_N(2k-2i,\chi^2) E^*(Nz), \end{aligned} \tag{3.1}$$

with

$$\begin{aligned} E^*(Nz) &= E(-(Nz)^{-1}) \det(Nz)^{-k} = N^{-kn/2} E|W(N), \\ \tilde{\Gamma}(k,0) &= i^{nk} 2^{-n(k+1)} \pi^{-nk} \Gamma(1_n,0)^{-1} \Gamma((k+\delta)/2) \prod_{j=1}^{[n/2]} \Gamma(k-j) \\ &= i^{nk} 2^{-n(k+1)} \pi^{-nk} \times \begin{cases} \Gamma_n(k)\Gamma(k - {}^{n,2,+\mu)}/_2), & \text{if } n \text{ is even;} \\ \Gamma_n(k), & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{3.2}$$

In this article we are interested only in the case when n is odd. Then one has

$$\begin{aligned} G_k^+(z) &= G_k^-(z) = G_k^*(z) \\ &= i^{nk} 2^{-n(k+1)} \pi^{-nk} \Gamma_n(k) L_N(k,\chi) \prod_{i=1}^{[n/2]} L_N(2k-2i,\chi^2) E|W(N) \end{aligned} \tag{3.3}$$

where

$$\Gamma_n(s) = \pi^{n(n-1)/4} \prod_{j=0}^{n-1} \Gamma(s-(j/2)) \tag{3.4}$$

Here

$$W(N) = \begin{pmatrix} 0_n & -1_n \\ N1_n & 0_n \end{pmatrix}$$

denotes the principal involution of level N, and the gamma factor $\Gamma(1_n,s)$, is defined in [Pa3].

We now pass to a p-adic construction. Let $A_{n,p} = A_n \otimes \mathbf{Z}_p$ be a free $\mathbf{Z}_p$-module of rank $n(n+1)/2$ of half integral symmetric matrices of size n over $\mathbf{Z}_p$. We construct certain p-adic measures on the compact group $Y = A_{n,p} \times \mathbf{Z}_p^\times$, with values in the completed semi-group ring $R = {}_p[[q^{B_n}]] \otimes \mathbf{Q}$ of the multiplicative semi-group q^{B_n}, where B_n is the additive group of positive semi definite half integral matrices $\xi = (\xi_{ij}) \geq 0$, $2\xi_{ij} \in \mathbf{Z}$,

$\xi_{ii} \in \mathbf{Z}$, ${}_p$ being the ring of integers of the Tate field $\mathbf{C}_p$. This measure will be characterized by its integrals on the discrete subset $Z_p \subset Y$ formed by elements of the type $((k,\chi),\psi)$ for sufficiently large $k \in \mathbf{Z}$, Dirichlet charcters χ on $\mathbf{Z}_p^\times$, and additive characters ψ of $A_{n,p}$. We use the following simple observation: if for a fixed element

$$G(q) = \sum_{\xi \in B_n} g(\xi) q^\xi \in {}_p[[q^{B_n}]] \otimes \mathbf{Q}$$

and for an open compact subset $U \subset A_{n,p}$ we put

$$G(q; U) = \sum_{\xi \in B_n \cap U} g(\xi) q^\xi \in {}_p[[q^{B_n}]] \otimes \mathbf{Q}, \tag{3.5}$$

then we obtain a measure μ_G on $A_{n,p}$ defined by $\mu_G(U) = G(q; U)$. Now we observe that the coefficients of the normalised Eisenstein series can be represented as certain p-adic integrals.

Consider again the normalized Eisentein series $G_k^+(z) = G^+(z,k,\chi,N)$. Under the above assumption on k these series are holomorphic Siegel modular forms with cyclotomic Fourier coefficients:

$$G^+(z,k,\chi,N) = \sum_{A_n \ni \xi > 0} b^+(\xi,k,\chi) e_n(\xi z),$$

with

$$b^+(\xi,k,\chi) = \begin{cases} 2^{-n\kappa} \det(\xi)^{k-\kappa} L_M^+(k-(n/2),\chi\omega_\xi) M(\xi,\chi,k) & \text{for } n \text{ even}, \\ 2^{-n\kappa} \det(\xi)^{k-\kappa} M(\xi,\chi,k) & \text{for } n \text{ odd}. \end{cases} \tag{3.6}$$

Here $\kappa = (n+1)/2$, and $L_M^+(k-(n/2),\chi\omega_\xi)$ is the value of a normalised Dirichlet L-function, and the integral factor

$$M(\xi,\chi,k) = \prod_{q \in P(\xi)} M_q(\xi,\chi(q)q^{-k}) \tag{3.7}$$

is a finite Euler product, extended over primes q in the set $P(\xi)$ of prime divisors of the number N and of all elementary divisors of the matrix ξ. The important property of the product is that for each q we have that $M_q(\xi,t) \in \mathbf{Z}[t]$ is a polynomial with integral coefficients. The explicit form of it is not important for our construction.

For the p-adic constuction we put $M = Np$ and

$$G_p^+(z,k,\chi,M) = \sum_{\substack{A_n \ni \xi > 0 \\ (\det \xi, M) = 1}} b^+(\xi,k,\chi) e_n(\xi z) \tag{3.8}$$

Theorem. — *Let n be odd. Then there exists a measure μ_{E-S} on $Y = \{(\xi,x) \in A_{n,p} \times \mathbf{Z}_p^\times\}$ with values in $R = {}_p[[q^{B_n}]] \otimes \mathbf{Q}$ which is uniquely defined by the following* *properties: for all pairs (k,χ) with $k \in \mathbf{Z}$ sufficiently large, $2k > n$, and a Dirichlet character $\chi : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$, $\chi \bmod M$ with M divisible by p one has:*

$$\int_Y \det(\xi)^{k-\kappa} x_p^{k-(n/2)} \chi(x) \mu_{E-S}(\xi,x) = G_p^+(z,k,\chi^{-1},M) \in R. \tag{3.9}$$

Proof. — Proof of the theorem is given in [Pa3] and uses arguments analogous to those in [Pa1], p.115–116. Notice that the Fourier coefficients of the right hand side of (3.9) contain the finite Euler product $M(\xi,\chi,k) = \prod_{q\in P(\xi)} M_q(\xi,\chi(q)q^{-k})$, and it has the form of a finite linear combination of terms of the type $\chi(b)b^{-k} = (\chi(b)b^k)^{-1}$ with $(b,p) = 1$ whose coefficients are integers independent of χ and k. So one obviously constructs μ_{E-S} term by term.

The explicit formulas for the Fourier coefficients show that this measure takes values in $\mathbf{Z}_p[[q^{B_n}]]$ and its restriction to $\mathbf{Z}_p^\times \xrightarrow{\sim} \Delta \times \Gamma$, $\Gamma = \langle u \rangle$ defines an element $E_\Lambda(\chi) \in M^3(\chi;\Lambda)$ which is a power series in $\Lambda[[q^{B_n}]]$ and which is called the Siegel-Eisenstein Λ-adic modular form whose special values $X = \varepsilon(u)u^k - 1$ are given in terms of (3.9). More precisely, we normalize this Λ-adic form by the condition:

$$E_\Lambda(\chi)(\varepsilon(u)u^k - 1) = G_p^+(z,k,\chi^{-1}\varepsilon^{-1}\omega^k,M), \quad (M = p^\alpha).$$

4. The pullback of Λ-adic modular forms

Let $\pi : B_3 \to B_2 \times B_1$ be a map given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, a_{33} \right).$$

We note that for each $(\xi_2,\xi_1) \in B_{2,1}$ the fiber $\pi^{-1}(\xi_2,\xi_1)$ is a finite set. For $F = \sum a(\xi_3,F)q^{\xi_3} \in \Lambda[[q^{B_3}]]$ we define $\Phi(F) \in \Lambda[[q^{B_{2,1}}]]$

$$a(\xi_2,\xi_1,\Phi(F)) = \sum_{\xi_3\in\pi^{-1}(\xi_2,\xi_1)} a(\xi_3,F).$$

Φ is a Λ-module homomorphism.

Proposition. — *$\Phi^3_{2,1}$ maps $M^3(\chi;\Lambda)$ into $M^{2,1}(\chi;\Lambda)$.*

Let $E_\Lambda(\chi) \in M^3(\chi;\Lambda)$ be the (involuted) Λ-adic Siegel-Eisenstein seires of degree three with character χ constructed in Section 3. Then $\Phi^3_{2,1}(E_\Lambda(\chi)) \in M^{2,1}(\chi;\Lambda)$.

5. Λ-adic Petersson inner product

This section follows Hida's idea of algebraic Petersson inner product([Hi4], p. 222). Let $S^{\mathrm{ord}}(\chi;\Lambda)$ denote the space of Λ-adic cusp forms with character χ. Let $h^{\mathrm{ord}}(\chi;\Lambda)$ be the p-adic Hecke algebra with character χ. We note that $S^{\mathrm{ord}}(\chi;\Lambda)$ and $h^{\mathrm{ord}}(\chi;\Lambda)$ are the Λ-dual of each other by the paring

$$\langle f,h\rangle = a(1,f|h).$$

We put

$$D = h^{\mathrm{ord}}(\chi;\mathcal{L}) = h^{\mathrm{ord}}(\chi;\Lambda)\otimes_{\Lambda}\mathcal{L}.$$

Since D is semi-simple there is a non degenerate pairing $(\cdot,\cdot)$ on D given by

$$(h,g) = Tr_{D/\mathcal{L}}(hg).$$

By duality we have the dual paring $(\cdot,\cdot)$ on $S^{\mathrm{ord}}(\chi;\mathcal{L})$. If $F\in S^{\mathrm{ord}}(\chi;\Lambda)$ is the normalized eigenform then

$$c(F,G) = \frac{(F,G)}{(F,F)} \in \mathcal{L}$$

is well defined. We write $M(\chi;\Lambda) = M^1(\chi;\Lambda)$. Let $M^{\mathrm{ord}}(\chi;\Lambda)$ denote the ordinary part and let π_{ord} denote the projection to the ordinary part. The Hecke algebra $H^{\mathrm{ord}}(\chi;\Lambda)$ acts on $M^{\mathrm{ord}}(\chi;\Lambda)$. $H^{\mathrm{ord}}(\chi;\mathcal{L}) = H^{\mathrm{ord}}(\chi;\Lambda)\otimes_{\Lambda}\mathcal{L}$ is a direct sum of $h^{\mathrm{ord}}(\chi;\mathcal{L})$ and the Hecke algebra over $\mathcal{L}$ corresponding to Eisenstein series. Let 1_{cusp} denote the idempotent corresponding to $h^{\mathrm{ord}}(\chi;\mathcal{L})$. We define a Λ-bilinear map $\langle\cdot,\cdot\rangle : S^{\mathrm{ord}}(\chi;\Lambda)\times M(\chi;\Lambda)\to\mathcal{L}$ by

$$\langle F,G\rangle = (F,1_{cusp}\cdot\pi_{\mathrm{ord}}(G)).$$

6. Λ-adic kernel function

Let $G\in M^{2,1}(\chi;\Lambda)$. We write

$$G = \sum_{j\in J} g_j\otimes h_j$$

with $g_j\in M^2(\chi;\Lambda)$ and $h_j\in M^1(\chi;\Lambda)$. We define a Λ-linear map $R(G) : S^{\mathrm{ord}}(\chi;\Lambda)\to M^2(\chi;\Lambda)\otimes\mathcal{L}$ by

$$R(G) = \sum_{j\in J} g_j\langle\cdot,h_j\rangle. \tag{6.1}$$

(see also Section 6). In the definition (6.1) we used the Λ-bilinear map $\langle\cdot,\cdot\rangle : S^{\mathrm{ord}}(\chi;\Lambda)\times M(\chi;\Lambda)\to\mathcal{L}$ of the previous section.

7. The Λ-adic Böcherer-Shimura formula

We may call

$$E_\Lambda(F,\chi) = \frac{1}{\langle F,F\rangle} R(\Phi(E_\Lambda(\chi))(F) \in M^2(\chi;\Lambda)\otimes_\Lambda \mathcal{L}$$

the Λ-adic Klingen-Eisenstein series for a Λ-adic cusp form $F \in S^{\mathrm{ord}}(\chi;\Lambda)$.

Now our task is to investigate a precise relation of

$$\frac{R(\Phi(E_\Lambda(\chi))(F)}{(F,F)} \tag{7.1}$$

with the classical Klingen-Eisenstein series. We give here only an expected formula:

Let $m = 2$, $r = 1$. For all $P = P_{k,\varepsilon} \in \mathcal{P}$ we have that the specialization of (7.1) at P is given by

$$\begin{aligned}&\Lambda_p(k,\omega^{-k}\chi\varepsilon)\frac{D(k-r,f_{k,\varepsilon})}{\langle f_{k,\varepsilon},f_{k,\varepsilon}\rangle}E_k^{m,r}(z,f_{k,\varepsilon},\chi) = \\ &\Lambda_p(k,\omega^{-k}\chi\varepsilon)\frac{\langle f'_{k,\varepsilon}(w),E_k^{m+r,0}(\mathrm{diag}[z,w],\omega^{-k}\chi\varepsilon)\rangle}{\langle f_{k,\varepsilon},f_{k,\varepsilon}\rangle}\end{aligned} \tag{7.2}$$

where $\Lambda_p(k,\omega^{-k}\chi\varepsilon)$ is an explicitely given elementary factor.

Proof of (7.2) (in case $m = 2$, $r - 1$) is based on the identity (3), the definition of $E_\Lambda(\chi)$, of the kernel function

$$R(G) : S^{\mathrm{ord}}(\chi;\Lambda) \to M^2(\chi;\Lambda)\otimes\mathcal{L}$$

of the Section 6.

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