HYPERBOLICITY PROBLEMS ON THE DOUADY SPACE AND ITS VARIANTS

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Introduction

Given an analytic space X, the Douady space $D(X)$ parametrizes all compact analytic subspaces of X. In his 1965's thesis, Douady constructed on it a natural complex structure. The relative version of the Douady space is due to Pourcin. When X is projective, Douady space coincides with Hilbert scheme and this construction is due to Grothendieck. We can also define Douady spaces variants which parametrize some special subspaces, e.g. the space $D'(X)$ (resp. $D'_m(X)$) of all connected and compact submanifolds of X (resp. of the same dimension m). We can also embed the morphism spaces in some suitable Douady spaces.

On the other hand, Kobayashi introduced in 1967 an intrinsic pseudo-distance d_X on every analytic space X. If d_X is a distance, X is called hyperbolic. It is called Brody-hyperbolic if it doesn't contain any non-constant holomorphic entire curve $f : \mathbb{C} \to X$. Brody-hyperbolicity is weaker than Kobayashi hyperbolicity. The two notions coincide on compact analytic spaces. Moreover, Eisenman introduced on every analytic space X of dimension n a k-dimensional measure $(1 \leq k \leq n)$ generalizing the Kobayashi metric. X is said to be k-measure hyperbolic if the Eisenman k-measure has positive volume over all k-dimensional analytic subspaces of X. It is strongly k-measure hyperbolic if the Eisenman k-measure has locally a lower bound.

In this context, the hyperbolicity of Douady spaces looks as a natural problem. The solution to this problem is negative in general even if X itself is hyperbolic. We will give two counter-examples for this. However, some interesting subspaces of the Douady space seem to be hyperbolic (at least for Brody-hyperbolicity). The first main result of this paper is the Brody-hyperbolicity of the Douady subspace $D'_1(X)$ parametrizing all subspaces of X which are algebraic curves. Explicitly, we have

Theorem 3.4. — *Let X be a 2-measure hyperbolic analytic space, then $D'_1(X)$ is Brody-hyperbolic.*

The proof of this theorem is based on the hyperbolicity of the Teichmüller space. In fact, we will construct for every $g \geq 1$, a holomorphic mapping $\Psi_g : \widetilde{D}'_{1,g}(X) \to T'_g$ from the universal covering space of the Douady space of genus g algebraic curves embedded in X into the Torelli space T'_g (which admits the Teichmüller space T_g as universal covering space). When X is 2-measure hyperbolic, the fibres of Ψ_g are all Brody-hyperbolic. So we can lift the property of Brody-hyperbolicity from T'_g to $\widetilde{D}'_{1,g}(X)$ and we conclude for $D'_{1,g}(X)$ by covering. Note that for the special cases $g = 0$ and $g = 1$, $D'_{1,g}(X)$ is discreet. It should be also noted that the Brody-hyperbolicity of the fibres of Ψ_g is deduced from the next general result on locally trivial analytic families which might have other applications:

Theorem 2.3. — *Let $\mathfrak{X} = (\mathcal{Y}, f, S)$ a locally trivial analytic family of k-dimensional analytic subspaces of an analytic space X. Suppose X $(k+1)$-measure hyperbolic, then S is Brody-hyperbolic.*

The techniques used to prove the theorem 3.4 (more precisely the case of the genus $g = 1$) can be generalized to study the subspace $D^T_d(X)$ of the Douady space $D(X)$ of a projective analytic space X, which parametrizes all d-dimensional connected compact submanifolds of X being complex tori (so abelian varieties). Here we use the Siegel moduli space in place of Torelli space in the case of curves. We obtain the following theorem:

Theorem 4.2. — *Let X be a projective analytic space. Suppose that X is $(d+1)$-measure hyperbolic, then $D^T_d(X)$ is discreet.*

We also prove the hyperbolicity of certain special subspaces of $D'_1(X)$ and $D^T_d(X)$ when X is non singular but without any condition of hyperbolicity on it.

The moduli space of holomorphic mappings.

Let X and Y be analytic spaces. X is supposed to be compact. Then the set $\mathrm{Hol}(X,Y)$ of all holomorphic mappings $f : X \to Y$ carries a natural complex structure induced by the one on Douady space $D(X \times Y)$. In fact, $\mathrm{Hol}(X,Y)$ can be realized as a Zariski open subspace of $D(X \times Y)$ by identifying every mapping $f : X \to Y$ with its graph Γ_f which is an analytic subspace of $X \times Y$ isomorphic to X and so it is compact.

When Y is (compact) hyperbolic, then $\mathrm{Hol}(X,Y)$ is (compact) hyperbolic. This is a theorem of Kobayashi. More generally, if we assume Y strongly k-measure hyperbolic, then we obtain the same property on $\mathrm{Hol}(X,Y)$ for strongly measure hyperbolic:

Theorem 5.3. — *Let X be a strongly measure hyperbolic compact analytic space and Y a strongly k-measure hyperbolic analytic space. Then $\mathrm{Hol}(X,Y)$ is also strongly k-measure hyperbolic.*

This theorem is a corollary of a more general result on locally trivial analytic families. Namely:

Theorem 2.4. — *Let X an analytic space and $\mathfrak{X} = (\mathcal{Y}, \pi, S)$ a locally trivial analytic family of k-dimensional compact complex subspaces of X. Suppose X strongly $(k+p)$-measure hyperbolic and the fibres of the family are measure hyperbolic. Then S is strongly p-measure hyperbolic.*

In particular, when $p = 1$, S is then hyperbolic. The proof of this theorem is based on an estimation on the Eisenman p-measure of S, the Eisenman-Kobayashi measure of one fibre and the Eisenman $(k+p)$-measure of X.

Next we consider the open subspace $\mathrm{Hol}_k(X,Y) \subset \mathrm{Hol}(X,Y)$ of holomorphic mappings $f : X \to Y$ such that rank $f \geq k$. We obtain the following:

Theorem 5.2. — *Let X a compact analytic space and Y an analytic space. Suppose Y $(k+1)$-measure hyperbolic, then $\mathrm{Hol}_k(X,Y)$ is Brody-hyperbolic.*

In particular, when Y is compact and of dimension $k-1$, we conclude that $\mathrm{Aut}(Y)$ is Brody-hyperbolic and then discreet because it is a complex Lie group. So, we find again a theorem of Kobayashi (see [K2], theorem 9.7).

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1. Preliminaries and examples

1.1. — We understand every analytic space to be complex, Hausdorff and of finite dimension. The existence of Douady spaces can be formulated as follows:

Theorem 1.1 (Douady [D]). — *Let X an analytic space. Then there exists an analytic space $D(X)$ and a subspace $R \hookrightarrow D(X) \times X$ such that:*

(a) R is flat over $D(X)$ and $\mathrm{pr}_1|_R$ is proper.

(b) If S is an analytic space, $Z \hookrightarrow S \times X$ a subspace having the properties stated in (a), then there exists an unique map $f : S \to D(X)$ such that $Z \simeq S \underset{D(X)}{\times} R$.

The analytic space $D(X)$ parametrizes compact subspaces of X and is called the *Douady space* of X. R is called the *universal family* over $D(X)$.

We have to consider some variants of $D(X)$ realized as open subspaces of it (see Grothendieck [Gr], IX, Corollaire 1.2). Namely, we define:

1) the subspace $D'(X)$ parametrizing compact connected complex submanifolds of X;

2) the subspace $D'_m(X)$ of $D'(X)$ parametrizing submanifolds of the same dimension m.

In particular, $D'_1(X)$ parametrizes submanifolds of X which are algebraic curves. It is called Douady space of embedded curves in X and can be written as follows:

$$D'_1(X) = \coprod_{g\geq 0} D'_{1,g}(X)$$

where $D'_{1,g}(X)$ is the subspace of $D'_1(X)$ parametrizing genus g curves in X.

1.2. — We remind the definition of the Eisenman measures. First, let X be an analytic space of dimension n, p a regular point of X, T_pX the holomorphic tangent space of X at p; $\bigwedge^k T_pX$ is the k-th exterior power of T_pX; D^k_pX is the set of decomposable elements in $\bigwedge^k T_pX$. If $\langle\ ,\ \rangle$ is a Hermitian metric on T_pX, then it can be extended to a Hermitian metric on $\bigwedge^k T_pX$ by putting

$$\langle \alpha,\beta\rangle \equiv \det\{\langle v_i, w_j\rangle\}$$

where $\alpha = v_1\wedge\cdots\wedge v_k$ and $\beta = w_1\wedge\cdots\wedge w_k$ from D^k_pX and extending this definition by linearity to arbitrary elements of $\bigwedge^k T_pX$. Let $\|y\|$ be the Hermitian metric on $\bigwedge^k TB^k$ (where B^k is the unit ball in $\mathbb{C}^k$) generated by the Bergman metric, and let 0 be the origin in B^k. Then (cf. Graham-Wu [G-W]):

Definition 1.2. — *For every $\alpha \in D^k_pX$ $(1 \leq k \leq n)$ the intrinsic Eisenman k-measure of α is $E^k_X(p,\alpha) = \inf\big\{\|y\|^2/y \in D^k_0 B^k$ and there exists a holomorphic mapping $f : B^k \to X$ with $f(0) = p$ and $f_*(y) = \alpha\big\}$.*

These measures can be also defined in the following way: $E^k_X(p,\alpha) = \inf\big\{R^{-2k}/$ there exists a holomorphic mapping $f : B^k \to X$ with $f(0) = p$ and $f_*\big(\frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_k}(0)\big) = R^{2k}\alpha\big\}$.

Observe that E^1_X is just the square of the Kobayashi-Royden infinitesimal metric (cf. Royden [R1]). In the following, we make a trivial but important observation; it is a consequence of the triviality of $E^k_{\mathbb{C}}$:

Proposition 1.2. — *Let X be a n-dimensional analytic space and Y a k-measure hyperbolic analytic space where $k \leq n+1$, is an integer. Then, every holomorphic map $F : \mathbb{C} \times X \to Y$ has rank less than k i.e.* rank $F \leq k-1$.

EXAMPLE 1.3. — Let X and Y analytic spaces of dimensions n and m respectively. Suppose that X is hyperbolic and Y is strongly measure hyperbolic but not hyperbolic. Then $X \times Y$ is strongly k-measure hyperbolic for every $k \geq m+1$ (see [G-W]). In this way, we can construct analytic spaces which are k-measure hyperbolic for k neither 1 nor the top dimension. For instance, $B^m \times F(d)$ is strongly $(n+1)$-measure hyperbolic of dimension $m+n$, where B^m is the unit ball in $\mathbb{C}^m$, $m > 1$, and $F(d)$ is the Fermat variety of degree $d > n+2$ in $\mathbb{P}^{n+1}(\mathbb{C})$, $n \geq 2$.

1.3. — We now give two examples showing that the Douady space $D(X)$ of an analytic space X is not in general hyperbolic even if X itself is hyperbolic.

EXAMPLE 1.4. — Let C be a compact Riemann surface of genus $g \geq 2$, and let $j : C \to \mathcal{J}(C)$ be a fixed Abel-Jacobi embedding of C into its Jacobian variety $\mathcal{J}(C) \cong \mathrm{Pic}(C)$. Let k be a positive integer. The k-th symmetric power $\mathrm{Sym}^k(C)$ is a connected complex manifold and can be identified with the space of effective divisors of degree k on C. So it can be considered as a connected component of the Douady space $D(C)$ of C. Let

$$\varphi_k : \mathrm{Sym}^k(C) \longrightarrow \mathrm{Jac}(C)$$

be the k-th Abel-Jacobi map defined by $\varphi_k(D) = j(p_1) + \cdots + j(p_k)$ for $D = p_1 + \cdots + p_k \in \mathrm{Sym}^k(C)$. Then (see [G-H], p. 228):

i) φ_k is holomorphic.

ii) $\varphi_k^{-1}(\varphi_k(D)) = |D| = \mathbb{P}(H^0(C,\mathcal{O}([D]))) \cong \mathbb{P}^{\dim |D|}$ where D is an effective divisor of degree k of C and $|D|$ the set of effective divisors of C, which are linearly equivalent to D.

iii) If $k > 2g-2$, then φ_k is an algebraic projective bundle (see [Ma]).

From *iii)*, we conclude that $\mathrm{Sym}^k(C)$ can't be hyperbolic for $k > 2g-2$ because of the non-hyperbolicity of $\mathrm{Jac}(C)$ and of the φ_k fibres.

However, the subspace of $\mathrm{Sym}^k(C)$ parametrizing the 0-folds of C is hyperbolic. Indeed, consider the natural projection

$$\pi_k : C^k := \overbrace{C \times \cdots \times C}^{k \text{ fois}} \longrightarrow \mathrm{Sym}^k(C),$$

then π_k is a branched covering and the branch locus is the set $D := \bigcup_{1\leq i\neq j\leq k} D_{ij}$ where $D_{ij} := \{(p_1,\ldots,p_k) \in C^k/p_i = p_j\} \subset C^k$. Moreover, $\pi_k(D)$ is the set of elements of $\mathrm{Sym}^k(C)$ corresponding to 0-dimensional analytic subspaces of C which are singular. Now, as $C^k \setminus D$ is hyperbolic and $\pi_k|_{C^k\setminus D}$ is an unbranched covering, then $\mathrm{Sym}^k(C) \setminus \pi_k(D)$ is hyperbolic.

EXAMPLE 1.5. — Let X be an analytic space. We recall that a Cartier divisor on X is an analytic subspace of X whose sheaf of ideals is locally generated by a single element which is not a zero divisor.

Let $\mathrm{Div}(X) = \{d \in D(X)/Y_d$ is a Cartier divisor on $X\}$, where Y_d is the subspace of X corresponding to d. Then, $\mathrm{Div}(X)$ is Zariski open in $D(X)$, and is union of $D(X)$ connected components when X is non singular (see Fujiki [Fu]). Let's define the map:

$$\begin{array}{rcl} F : \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) \\ d & \longmapsto & \text{associated line bundle.} \end{array}$$

Then F is projective and the fibre over $L \in \mathrm{Pic}(X)$ is identified with the projective space $\mathbb{P}(\Gamma(X,L))$, where $\Gamma(X,L)$ is the set of global sections of the sheaf associated to L. Consequently, $\mathrm{Div}(X)$ can't be hyperbolic even if X is so.

2. Locally trivial analytic families

2.1. — We will first recall a sufficient condition for a family to be locally trivial. It is a generalization to Grauert-Fischer theorem [G-F] (see Bingener [Bin]):

Theorem 2.1. — *Let $\mathfrak{X} := (\mathcal{Y},f,S)$ an analytic family of compact analytic spaces. Suppose that S is reduced and that the fibres of f are all isomorphic to a fixed analytic space Y_0. Then the family $\mathfrak{X}$ is locally trivial.*

Moreover, if the base S is a Stein space and is contractible, then the local triviality of any family over S is equivalent to global triviality by the next consequence of the Grauert-Oka principle:

Proposition 2.2. — *Let $\mathfrak{X} = (\mathcal{Y},f,S)$ a locally trivial analytic family of compact analytic spaces. Suppose that S is a contractible Stein space. Then $\mathfrak{X}$ is trivial.*

Proof. — Let Y_0 the fibre of f over a point $0 \in S$. Then the space $\mathrm{Isom}_S(\mathcal{Y},S \times Y_0)$ of S-isomorphisms from $\mathcal{Y}$ onto $S \times Y_0$ is defined as a relative space over S (in fact it is a subspace of the Douady space $D_S(\mathcal{Y} \times S \times Y_0))$. In our case $\mathrm{Isom}_S(\mathcal{Y}, S \times Y_0)$ is a principal holomorphic fibre bundle over S. The structure group is the complex Lie group $\mathrm{Aut}(Y_0)$ of automorphisms of Y_0. But S is Stein, so by Grauert theorem (see for instance Cartan [C1], theorems A and B), $\mathrm{Isom}_S(\mathcal{Y}, S \times Y_0)$ is holomorphically trivial if and only if it is topologically trivial. But the last property is verified because of the contractibility of S (see Steenrod [S], corollary 11.6, p. 53). Then we conclude that $\mathrm{Isom}_S(\mathcal{Y}, S \times Y_0)$ is holomorphically trivial. This implies that the morphism $\mathrm{Isom}_S(\mathcal{Y}, S \times Y_0) \to S$ has a holomorphic section, which means that there exists an S-isomorphism $\mathcal{Y} \simeq S \times Y_0$. In other words, $\mathcal{Y}$ is trivial over S, as desired. ∎

2.2. — Consider locally trivial analytic families of compact analytic subspaces of an analytic space X. Our goal is to study the hyperbolicity of the basis. We obtain the following theorem for Brody-hyperbolicity:

Theorem 2.3. — *Let $\mathfrak{X} = (\mathcal{Y}, f, S)$ a locally trivial analytic family of k-dimensional compact analytic subspaces of an analytic space X. Suppose X $(k+1)$-measure hyperbolic. Then S is Brody hyperbolic.*

Proof. — If S is not Brody-hyperbolic, then it exists a non-constant holomophic map

$$g = \mathbb{C} \longrightarrow S.$$

If we make the base-change corresponding to $g : \mathbb{C} \to S$ to the family $\mathfrak{X}$. We obtain a new family $\mathfrak{X}' = (\mathcal{Y}', p, \mathbb{C})$ where $\mathcal{Y}' = \mathbb{C} \underset{S}{\times} \mathcal{Y}$ and $p : \mathcal{Y}' \to \mathbb{C}$ the projection. $\mathfrak{X}'$ is also locally trivial and because $\mathbb{C}$ is Stein and contractible then $\mathfrak{X}'$ is trivial by proposition 2.2. Let Y be any fibre of p, then it exists an isomorphism $\mathcal{Y}' \simeq \mathbb{C} \times Y$ and we obtain the following cartesian diagram

$$\begin{array}{ccccc}
\mathbb{C} \times Y & \longrightarrow & \mathcal{Y} & \hookrightarrow & S \times X \\
{\scriptstyle p}\downarrow & \square & {\scriptstyle f}\downarrow & \circlearrowleft \swarrow & \downarrow \\
\mathbb{C} & \underset{g}{\longrightarrow} & S & & X
\end{array}$$

Let's note $F : \mathbb{C} \times Y \longrightarrow X$ the mapping induced by this diagram. As g is non-constant, F must be of a rank greater than k ($\mathrm{rank}\, F \geq k+1$). But X is $(k+1)$-measure hyperbolic, so by the proposition 1.2. This leads to a contradiction. Consequently, g is constant and S is Brody-hyperbolic.

2.3. — In this section, We will prove the following theorem:

Theorem 2.4. — *Let X be an analytic space and $\mathfrak{X} = (\mathcal{Y},\pi,S)$ a locally trivial analytic family of k-dimensional compact analytic subspaces of X. Suppose X is strongly $(k+p)$-measure hyperbolic and the fibres of π are measure hyperbolic. Then S is strongly p-measure hyperbolic.*

In particular, when $p = 1$, S is hyperbolic.

The proof is based on an estimation on Eisenman p-measure of S. To show this, let first the following commutative diagram defined by $\mathfrak{X}$:

$$\begin{array}{ccc} \mathcal{Y} & \overset{i}{\hookrightarrow} & S \times X \\ \pi\downarrow & \circlearrowleft \quad \swarrow_{\mathrm{proj}_1} & \downarrow \mathrm{proj}_2 \\ S & & X \end{array}$$

where i is an embedding and proj_1, proj_2 are the projections. Let $h : \mathcal{Y} \to X$ be the composition $\mathrm{proj}_1 \circ i$, s a regular point of S and Y a fibre of π over s. Finally, let $y \in Y$ be a regular point, $v \in \overset{k}{\bigwedge} T_y Y$ and $\xi \in \overset{p}{\bigwedge} T_s S$. Then:

Lemma 2.5.

$$E_X^{k+p}\Big(h(y); \overset{k+p}{\bigwedge} dh(y; v \otimes \xi)\Big) \leq E_Y^k(y; v) \cdot E_S^p(s; \xi).$$

Proof. — Let $\varepsilon > 0$. By definition of E_S^p and E_Y^k, it exists:

– a holomorphic map $f : B^p \to S$ such that $f(0) = S$ and $\overset{p}{\bigwedge} df(0; \overset{p}{\bigwedge} e_p) = r^0 \cdot \xi$, where $e_p = \left(\frac{\partial}{\partial z_1}|_0; \cdots; \frac{\partial}{\partial z_k}|_0\right) \in T_0 B^p$ and r a positive real number verifying:

$$r^{-p} \leq E_S^p(s; \xi) + \varepsilon.$$

– a holomorphic map $g : B^k \to Y$ such that $g(0) = y$ and $\overset{k}{\bigwedge} dg(0; \overset{k}{\bigwedge} e_k) = R^k \cdot v$, where $e_k = \left(\frac{\partial}{\partial z_1}|_0; \cdots; \frac{\partial}{\partial z_1}|_0\right) \in T_0 B^p$ and r a positive real number verifying:

$$r^{-p} \leq E_S^p(y; v) + \varepsilon.$$

B^p and B^k are respectively the p-dimensional and the k-dimensional unit balls.

Making the base change $f : B^p \to S$ to the family $\mathfrak{X} = (\mathcal{Y},\pi,S)$, we obtain the following cartesian diagram:

$$\begin{array}{ccccc} B^p \underset{S}{\times} \mathcal{Y} & \overset{p}{\longrightarrow} & \mathcal{Y} & \overset{i}{\hookrightarrow} & S \times X \\ \pi_1 \downarrow & \square & \pi \downarrow & \circlearrowleft \swarrow_{\mathrm{proj}_1} & \downarrow \mathrm{proj}_2 \\ B^p & \underset{f}{\longrightarrow} & S & & X \end{array}$$

where p, π_1 are the projections. If we put $\mathcal{Y}_1 = B^p \underset{S}{\times} \mathcal{Y}$, then the family $\mathfrak{X}_1 = (\mathcal{Y}_1, \pi_1, B^p)$ is locally trivial (because $\mathfrak{X}$ is so). But B^p is Stein and contractible, thus by proposition 2.2, $\mathfrak{X}_1$ is (globally) trivial. So we have an isomorphism

$$\mathcal{Y}_1 \cong Y \times B^p.$$

Consequently, we obtain the following exact sequence

$$0 \longrightarrow T_y Y \longrightarrow T_y \mathcal{Y}_1 \longrightarrow T_0 B^p \longrightarrow 0. \tag{1}$$

Immediately we have

$$\dim T_y \mathcal{Y}_1 = \dim T_y Y + \dim T_0 B^p = k+p.$$

Let $h_1 = \mathcal{Y}_1 \to X$ be the composition $p \circ h$ and $dh_1(y, \cdot) : T_y \mathcal{Y}_1 \to T_{h(y)} X$ its jacobian map at the point y (here $h_1(y)$ is identified with $h(y)$). From the exact sequence (1) we obtain an isomorphism

$$\psi : \bigwedge^{k+p} T_y \mathcal{Y}_1 \xrightarrow{\sim} \bigwedge^{k} T_y Y \otimes \bigwedge^{p} T_0 B^p.$$

We define a linear map

$$\varphi : \bigwedge^{k} T_y Y \otimes \bigwedge^{p} T_0 B^p \longrightarrow \bigwedge^{k+p} T_{h(y)} X$$

by putting $\varphi = \bigwedge^{k+p} dh_1(y, \cdot) \circ \psi$, then φ is injective and we have

$$\varphi\left(v \otimes \bigwedge^{p} e_p\right) = r^p \cdot \bigwedge^{k+p} dh(y; v \otimes \xi).$$

In particular, $\varphi\left(v \otimes \bigwedge^{p} e_p\right)$ is independent of f.

On the other hand, because $\mathfrak{X}_1$ is trivial, we can find a holomorphic map $G : B^{k+p} \to X$ such that $G(0) = h(y)$ and

$$\begin{aligned} \bigwedge^{k+p} dG\left(0; \bigwedge^{k+p} e_{k+p}\right) &= R^k \cdot \varphi\left(v \otimes \bigwedge^{p} e_p\right) \\ &= R^k \cdot r^p \cdot \bigwedge^{k+p} dh(y; v \otimes \xi) \end{aligned}$$

(we can take $G = G_1|_{B^{k+p}}$ where $G_1 : B^k \times B^p$ defined by $g(a,b) = (g(a), b)$). Thus

$$E_X^{k+p}\left(h(y); \bigwedge^{k+p} dh(y; v \otimes \xi)\right) \leq E_Y^k(y; v) \cdot E_S^p(s; \xi) + \varepsilon'$$

where $\varepsilon' = \varepsilon(\varepsilon + E_Y^k(y; v) + E_S^p(s; \xi)$. Finally, when $\varepsilon \to 0$, we obtain the lemma. ■

Proof of theorem 2.4. — Let $o \in S$ be a regular point and Y the fibre of π over 0. Because $\mathfrak{X}$ is locally trivial, we can find a smooth neighbourhood $\mathcal{U}$ of 0 in S such that Y is the fibre of π over every point of $\mathcal{U}$. Let $y \in Y$ be a regular point and v a decomposable element of $\bigwedge^{k} T_y Y$ such that $E_Y^k(y; v) > 0$. According to lemma 2.5, we have for every $s \in \mathcal{U}$ and $\xi \in T_s S$:

$$E_X^{k+p}(h(y); \bigwedge^{k+p} dh(y; v \otimes \xi)) \leq E_Y^k(y; v) \cdot E_S^p(S; \xi).$$

As X is strongly $(k+p)$-measure hyperbolic and $\bigwedge^{k+p} dh(y; v \otimes \xi) \neq 0$, we can easily find a lower bound of $E_S^p(s; \xi)$ on a small neigbourhood $\mathcal{U}' \subset \mathcal{U}$ of 0 in S. The theorem is then proved. ■

3. Brody-hyperbolicity of embedded curves Douady spaces

3.1. — In all this section, X is an analytic space of dimension n. Our goal is to prove that if X is 2-measure hyperbolic, then $D_1'(X)$ is Brody-hyperbolic. To do this, we will first construct for every $g \geq 1$ a holomorphic mapping defined on the universal covering space $\widetilde{D'_{1,g}}(X)$ of $D'_{1,g}(X)$ into the Torelli space T_g'.

Theorem 3.1. — *The universality of T_g' furnishes a natural holomorphic map*

$$\psi_g : \widetilde{D'_{1,g}}(X) \longrightarrow T_g'$$

for every $g \geq 1$, defined by: $\psi_g(d) =$ the isomorphism class of the curve parametrized by d.

Proof. — Let

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & D'_{1,g}(X) \times X \\ \pi \big\downarrow & \circlearrowleft & \swarrow \\ D'_{1,g}(X) & & \end{array}$$

be the universal family over $D'_{1,g}(X)$. It's locally a genus g Torelli curve. The local Torelli structures make the sheaf $R^1\pi_*(\underset{\sim}{\mathbb{C}})$ a local system over $D'_{1,g}(X)$. If we make the base change, corresponding to the universal covering $\widetilde{D'_{1,g}}(X) \to D'_{1,g}(X)$, to the universal family over $D'_{1,g}(X)$, then we obtain a new family over $\widetilde{D'_{1,g}}(X)$ according to the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{Y}' & \longrightarrow & \mathcal{Y} & \hookrightarrow & D'_{1,g}(X) \\
{\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & \circlearrowleft \; \swarrow & \big\downarrow \\
\widetilde{D}'_{1,g}(X) & \longrightarrow & D'_{1,g}(X) & & X
\end{array}$$

The $R^1\pi_*(\underset{\sim}{\mathbb{C}})$ pre-image is isomorphic to the sheaf $R^1\pi'_*(\underset{\sim}{\mathbb{C}})$ over $\widetilde{D}'_{1,g}(X)$ (see Grothendieck [Gr], VIII, corollaire 1.4). So, $R^1\pi'_*(\underset{\sim}{\mathbb{C}})$ is also a local system over $\widetilde{D}'_{1,g}(X)$. According to Deligne ([De], chap. 1, corollaire 1.4), such sheaf is given by a complex representation of finite dimension of the fundamental group $\pi_1(\widetilde{D}'_{1,g}(X))$. But $\widetilde{D}'_{1,g}(X)$ is simply connected; thus $\pi_1(\widetilde{D}'_{1,g}(X))$ is trivial and the local system $R^1\pi'_*(\mathbb{C})$ is constant. By the universal coefficient theorem, we deduce that the sheaf $R^1\pi'_*(\mathbb{Z})$ over $\widetilde{D}'_{1,g}(X)$ is isomorphic to the sheaf $\underset{\sim}{\mathbb{Z}}^{2g}$. The choice of such an isomorphism defines a Torelli structure on $\mathcal{Y}'$ over $\widetilde{D}'_{1,g}(X)$ (except for the special case of the genus $g = 1$). By the universality property of the Torelli space T'_g, it exists a unique (up to isomorphism) holomorphic map

$$\psi_g : \widetilde{D}'_{1,g}(X) \longrightarrow T'_g$$

defined by $\psi_g(d) =$ the isomorphism class of the fibre $\pi'^{-1}(d) \in T'_g$ of π' over d.

Returning now to the special case when the genus $g = 1$, and considering a covering $(\widetilde{U}_i)_{i\in I}$ of $\widetilde{D}'_{1,1}(X)$ such that $\pi'^{-1}(\widetilde{U}_i)$ carries a Torelli structure over $\widetilde{U}_i$ for every $i \in I$, we can construct for every $i \in I$ a holomorphic map

$$\psi_1^i : \widetilde{U}_i \longrightarrow T'_1$$

by the universality of the Torelli space T'_1. Thus, to obtain a global holomorphic map on $\widetilde{D}_{1,1}(X)$ into T'_1 it suffices to prove that for every $i,j \in I$ we have

$$\psi_1^i = \psi_1^j \text{ on } \widetilde{U}_i \cap \widetilde{U}_j.$$

Let's put $\widetilde{U}_{ij} = \widetilde{U}_i \cap \widetilde{U}_j$, $\widetilde{V}_i = \pi'^{-1}(\widetilde{U}_i)$, $\widetilde{V}_j = \pi'^{-1}(\widetilde{U}_j)$ and $\widetilde{V}_{ij} = \widetilde{V}_i \cap \widetilde{V}_j$. Let $\varepsilon_i : \widetilde{V}_i \to \widetilde{U}_i$ (resp. $\varepsilon_j : \widetilde{V}_j \to \widetilde{U}_j$) be the section defined by the genus 1 curve $\widetilde{V}_i$ (resp. $\widetilde{V}_j$) over $\widetilde{U}_i$ (resp. $\widetilde{U}_j$). Then $\widetilde{V}_{ij}$ is in two ways (corresponding to ε_i and ε_j) a genus 1 Torelli curve over $\widetilde{U}_{ij}$. These two structures are equivalent. To show this, let's define the holomorphic map

$$\begin{array}{rcl}
\tau_{ij} : \widetilde{V}_{ij} & \longrightarrow & \widetilde{V}_{ij} \\
b & \longmapsto & \text{“}b - \varepsilon_i \circ \pi_{ij}(b) + \varepsilon_j \circ \pi_{ij}(b)\text{”}
\end{array}$$

where $\pi_{ij} : \widetilde{V}_{ij} \to \widetilde{U}_{ij}$ is the restriction of π' to $\widetilde{V}_{ij}$. We see that τ_{ij} is an $\widetilde{U}_{ij}$-automorphism of $\widetilde{V}_{ij}$ and that on every fibre of π_{ij}, τ_{ij} is a translation. In particular, it induces the identity on the first cohomology groups of the fibres. Thus τ_{ij} is an automorphism of genus 1 Torelli curves over $\widetilde{U}_{ij}$, which implies that $\psi_i = \psi_j$ on $\widetilde{U}_{ij}$. We then obtain a holomorphic mapping

$$\psi_1 : \widetilde{D}'_{1,1}(X) \longrightarrow T'_1$$

defined by $\psi_1 = \psi_1^i$ on $\widetilde{U}_i$ for every $i \in I$. This concludes the proof. ■

Proposition 3.2. — *Suppose that X is 2-measure hyperbolic. Then the fibres of ψ_g are Brody-hyperbolic for every $g \geq 1$.*

Proof. — Let H be any fibre of ψ_g ($H = \psi_g^{-1}(d)$ for $d \in T'_g$) and let C be a genus g algebraic curve representing the isomorphism class d. Making the base change, corresponding to the natural embedding $H \hookrightarrow \widetilde{D}'_{1,g}(X)$, to the family $\mathcal{Y}'$ over $\widetilde{D}'_{1,g}(X)$, we obtain a genus g curve $\mathcal{Y}_H$ over H. The fibres of this curve are all isomorphic to C so $\mathcal{Y}_H$ constitutes a locally trivial family over H by theorem 2.1. Now, we apply the theorem 2.3 to conclude. ■

Furthermore, if X is strongly 2-measure hyperbolic and $g \geq 2$ then $\mathcal{Y}_H$ (see the proof of proposition 3.2) is a locally trivial family with hyperbolic fibres (so, they are in particular measure-hyperbolic). Applying the theorem 2.4, we obtain:

Proposition 3.3. — *Suppose that $g \geq 2$ and X strongly 2-measure hyperbolic. Then the fibres of ψ_g are hyperbolic.*

3.2. — We now prove the following main theorem:

Theorem 3.4. — *Let X be a 2-measure hyperbolic analytic space. Then $D'_1(X)$ is Brody-hyperbolic*

Proof. — We have

$$D'_1(X) = \coprod_{g \geq 0} D'_{1,g}(X).$$

So we only have to prove that for every $g \geq 0$, $D'_{1,g}(X)$ is Brody-hyperbolic. Let

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & D'_{1,g}(X) \times X \\ {\scriptstyle \pi_g}\downarrow & \circlearrowleft \quad \swarrow {\scriptstyle \mathrm{proj}_1} & \downarrow {\scriptstyle \mathrm{proj}_2} \\ D'_{1,g}(X) & & X \end{array} \tag{1}$$

be the universal family over $D'_{1,g}(X)$. It is easy to see that the map $F : \mathcal{Y} \to X$ induced by the diagram (1) is of rank at least 2.

Case 1 : $g = 0$

We will prove that $D'_{1,0}(X)$ is discreet. The map π_0 is a proper flat morphism with all its fibres isomorphic to $\mathbb{P}^1(\mathbb{C})$. Then by theorem 2.1, $(\mathcal{Y}, \pi_0, D'_{1,0}(X))$ is a locally trivial family. Assume that $D'_{1,0}(X)$ is not discreet, then there exists a non-constant holomorphic map $f : \Delta \to D'_{1,0}(X)$ from the unit disc Δ of $\mathbb{C}$ to $D'_{1,0}(X)$. Without loss of generality, we can suppose that the pull-back $\mathfrak{X}' = (\mathcal{Y}', \pi'_0, \Delta)$ of $\mathfrak{X} = (\mathcal{Y}, \pi_0, D'_{1,0}(X))$ by the base change

f, is trivial, *i.e.* $\mathcal{Y}$ is isomorphic to $\mathbb{P}^1(\mathbb{C}) \times \Delta$. Let $G : \mathbb{P}^1(\mathbb{C}) \times \Delta \to \mathcal{Y}$ be the map induced by the base change f. Then $F \circ G$ is of rank 2. By Proposition 1.2, this cannot hold because X is 2-measure hyperbolic. Consequently, $D'_{1,0}(X)$ is discreet.

Case 2 : $g = 1$

$D'_{1,1}(X)$ is also discreet. Indeed, suppose the opposite, then there exists a non-constant holomorphic map $f : \Delta \to D'_{1,1}(X)$. Since $\mathfrak{X} = (\mathcal{Y}, \pi_1, D'_{1,1}(X))$ is locally a Torelli curve of genus 1, then we can choose f such that the pull-back $\mathfrak{X}' = (\mathcal{Y}', \pi_1, \Delta)$ of $\mathfrak{X}$ by the base change f, is a Torelli curve of genus 1. By the construction of the universal family over the Torelli space T'_1, there exists a surjective holomorphic map $g : \mathbb{C} \times \Delta \to \mathcal{Y}'$. Furthermore, the map $f^*F : \mathcal{Y}' \to X$, induced from F by the base change f, has the rank at least 2. Thus $f^*F \circ G : \mathbb{C} \times \Delta \to X$ is of rank 2, which is impossible by proposition 2.1 since X is 2-measure hyperbolic. Then $D'_{1,1}(X)$ is discreet.

Case 3 : $g \geq 2$

Suppose that there exists a holomorphic map of $\mathbb{C} \to \widetilde{D}'_{1,g}(X)$. Then the composition $\psi_g \circ f$ is constant because T'_g is hyperbolic. So the image of f belongs to one fibre of ψ_g. But according to proposition 3.2, the ψ_g fibres are Brody-hyperbolic. Then f must be constant. Consequently, $\widetilde{D}'_{1,g}(X)$ is Brody-hyperbolic and we conclude for $D'_{1,g}(X)$ by covering. ■

EXAMPLE 3.5. — Let S be a complex surface of general type. It's well known that S is measure hyperbolic (see Lang [L] or Kobayashi [K2]). So theorem 3.4 proves that $D'_1(S)$ is Brody-hyperbolic.

3.3. — In this section, we suppose that X is a complex manifold. Let Y be a compact complex submanifold of X. $\mathcal{T}_X, \mathcal{T}_Y$ and $\mathcal{N}_{Y/X}$ are respectively the sheaves of holomorphic sections germs of the tengent bundles TX, TY and of the normal bundle $N_{Y/X}$. The exact sequence

$$0 \longrightarrow TY \longrightarrow TX_{|Y} \longrightarrow N_{Y/X} \longrightarrow 0$$

induces the following cohomology exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{T}_Y) \longrightarrow H^0(Y, \mathcal{T}_{X/Y}) \longrightarrow H^0(Y, \mathcal{N}_{Y/X}) \xrightarrow{\delta_Y} H^1(Y, \mathcal{T}_Y) \longrightarrow \cdots$$

On the other hand, $H^0(Y, \mathcal{N}_{Y/X})$ coincides with the tangent space of the Douady space $D(X)$ at the point $[Y]$, corresponding to Y in $D(X)$. $H^1(Y, \mathcal{T}_Y)$ is canonically isomorphic to the tangent space at 0 of the local moduli variety $(M,0)$ of Y (cf. [Gr], IX, Proposition 2.2).

Definition 3.6. — *We say that* $[Y] \in D(X)$ *verifies the condition* (P) *if:*

1. *the map* δ_Y *is injective.*
2. *the semi-universal deformation of* Y *is universal.*

Lemme 3.7. — (P) *is an open condition in* $D(X)$ *i.e. the subset of* $D(X)$ *corresponding to submanifolds* Y *of* X *verifying the condition* (P)*, is open in* $D(X)$.

Proof. — The second part of (P) is open by [Bin], Satz 7.1. Let

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & D(X) \times X \\ {\scriptstyle p}\downarrow & \swarrow & \\ D(X) & & \end{array}$$

be the universal family over $D(X)$. Let Y be a compact complex submanifold of X and note $0 := [Y] \in D(X)$ (we use the same notation as for the point corresponding to Y in the local moduli variety $(M,0)$ of Y). The universality of $(M,0)$ provides us with a holomorphic map of germs $\varphi_0 : (D(X),0) \to (M,0)$ from the germ $(D(X),0)$ of $D(X)$ at 0 into $(M,0)$. Then the differential $T_0\varphi_0$ at 0 is nothing other than the map δ_Y defined above. Since δ_Y is injective, φ_0 is an embedding. Thus there exists an open neigbourhood U of 0 in $D(X)$ such that for every $a \in U$, the map $\varphi_a : (D(X),a) \to (M,a)$ is an embedding. Now by [Bin], Satz 7.1, the universality property is open, so the differential $T_a\varphi_a$ of φ_a at a coincides with δ_{Y_a}, where Y_a is the fibre of p over a. Consequently, $\delta_{Y_a} : H^0(Y_a,\mathcal{N}_{Y_a/X}) \to H^1(Y_a,\mathcal{T}_{Y_a})$ is injective. This proves the lemma. ■

Let now R_g be the subset of $D'_{1,g}(X)$ which parametrizes algebraic curves Y of genus g embedded in X such that δ_Y is injective. Since semi-universal deformation of such curves are universal, then every element of R_g verifies the condition (P) and consequently, R_g is open in $D'_{1,g}(X)$. Furthermore, we have:

Theorem 3.8. — *Let* X *be a complex manifold and* $g \geq 1$ *an integer. Then* R_g *is hyperbolic.*

Proof. — Let $\psi_g : \widetilde{D}'_{1,g}(X) \to T'_g$ be the holomorphic map defined in theorem 3.1. Since δ_Y is injective for all Y in R, the restriction ${\psi_g}_{|\widetilde{R}} : \widetilde{R} \to T'_g$ is a local embedding where $\widetilde{R}$ is the pre-image of R by the universal covering $\widetilde{D}'_{1,g}(X) \to D'_{1,g}(X)$. But T'_g is hyperbolic, then by [K2], Theorem 3.4, 1) $\widetilde{R}$ (and consequently R) is hyperbolic. ■

4. Brody-hyperbolicity of embedded abelian varieties Douady spaces

4.1. — The techniques used to prove theorem 3.4 are valid to be used also in the case of Douady space of a projective analytic space X submanifolds which are abelian varieties of the same dimension d (let's call this Douady space of d-dimensional abelian subvarieties of X and note it $D_d^T(X)$). By Grothendieck [Gr], IV, proposition 5.9, and using a result of Mumford ([M1], proposition 6.16), $D_d^T(X)$ is an open analytic subspace of the Douady space $D(X)$ which is identified with the Hilbert scheme $\mathrm{Hilb}(X)$.

4.2. — As for the theorem 3.4, we first construct a holomorphic mapping with Brody hyperbolic fibres from the universal covering space $\widetilde{D_d^T}(X)$ of $D_d^T(X)$ into the Siegel moduli space S_d. By definition $S_d = \{Z \in M_d(\mathbb{C}) / {}^tZ = Z \text{ and } \mathrm{Im}\, Z > 0\}$ where $M_d(\mathbb{C})$ is the vector space of complex (d,d)-matrices. S_d is a fine moduli space of d-dimensional polarized abelian varieties with a fixed type D and symplectic basis and there is a universal family over it (see [L.B] for definitions and details).

Proposition 4.1. — *The universality of S_d furnishes a natural holomorphic mapping*

$$\varphi_d : \widetilde{D_d^T}(X) \longrightarrow S_d.$$

defined by $\varphi_d(Z) =$ the isomorphism class of the abelian variety parametrized by Z.

Proof. — Let

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & D_d^T(X) \times X \\ {\scriptstyle \pi}\downarrow & \circlearrowleft & \swarrow \\ D_d^T(X) & & \end{array}$$

be the universal family over $D_d^T(X)$. Making the base change corresponding to the universal covering $\widetilde{D_d^T}(X) \to D_d^T(X)$ to this family, we obtain a new family $\mathfrak{X}' = (\mathcal{Y}', \pi', \widetilde{D_d^T}(X))$. $\mathfrak{X}'$ is locally a family of polarized abelian varieties with fixed type and symplectic basis. This is a consequence of theorem 6.14 of Mumford [M1] and because $R^1\pi'_*(\underset{\sim}{\mathbb{Z}})$ is a local system. In the same way as in the proof of theorem 3.4, we can show that $R^1\pi'_*(\underset{\sim}{\mathbb{Z}})$ is constant. So the types and the symplectic bases of over local families are the same. Consequently, there exist, locally on $\widetilde{D_d^T}(X)$, holomorphic maps into the Siegel space S_d being a moduli space of polarized d-dimensional abelian varieties with the same type and symplectic basis. Glueing them by the same method as in the proof of theorem 3.4 (case $g = 1$), we obtain a global holomorphic map

$$\varphi_d : \widetilde{D}_d^T(X) \longrightarrow S_d\,. \qquad \blacksquare$$

4.2. — Let X a projective complex manifold and R_d the subset of $D_d^T(X)$ parametrizing abelian varieties Y embedded in X such that the map δ_Y defined in 3.3 is injective. Since semi-universal deformation fo complex tori are universal, then every element of R_d verifies the condition (P) defined in 3.3. Consequently, R_d is open in $D_d^T(X)$ by lemma 3.7. The following theorem is an immediate application of proposition 4.1.

Theorem 4.2. — *Let X be a projective complex manifold. Then R_d is hyperbolic.*

Proof. — Let $\varphi_d : \widetilde{D}_d^T(X) \to S_d$ the holomorphic map defined by proposition 4.2 and $\widetilde{R}_d$ the pre-image of R_d by the universal covering $\widetilde{D}_d^T(X) \to D_d^T(X)$. By the injectivity of δ_Y for all Y in R_d, the map $\varphi_{d|\widetilde{R}_d} : \widetilde{R}_d \to S_d$ is a local embedding. But S_d is hyperbolic, then $\widetilde{R}$ (and consequently R) is hyperbolic by [K2], theorem 3.4, 1). $\blacksquare$

4.3. — We now prove that $D_d^T(X)$ is discreet:

Theorem 4.3. — *Let X be a projective analytic space. Suppose that X is $(d+1)$-measure hyperbolic, then $D_d^T(X)$ is discreet.*

Proof. — Let

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & D_d^T(X) \times X \\ {\scriptstyle \pi_d}\downarrow & \circlearrowleft \quad \swarrow{\scriptstyle \mathrm{proj}_1} & \downarrow{\scriptstyle \mathrm{proj}_2} \\ D_d^T(X) & & X \end{array} \qquad (1)$$

the universal family over $D_d^T(X)$. The map $F : \mathcal{Y} \to X$ induced by the diagram (1) is of rank at least $k+1$.

Suppose that $D_d^T(X)$ is not discreet, then there exists a non-constant holomorphic map $f : \Delta \to D_d^T(X)$ from the unit disc Δ in $\mathbb{C}$ into $D_d^T(X)$. Since the universal family $\mathfrak{X} = (\mathcal{Y}, \pi_d, D_d^T(X))$ over $D_d^T(X)$ is locally a family of polarized abelian varieties of a certain type D with symplectic base (because X is projective), then we can choose f such that the pull-back $\mathfrak{X}' = (\mathcal{Y}', \pi'_d, \Delta)$ of $\mathfrak{X}$ by the base change f is a family of polarized abelian varieties of type D with symplectic base. By the construction of the universal family over the siegel space S_d, we can construct a surjective holomorphic map $G : \mathbb{C}^d \times \Delta \to \mathcal{Y}'$.

Furthermore, the pull-back $f^*F : \mathcal{Y}' \to X$ of F by the base change f is of rank at least $k+1$, thus $f^*F \circ G : \Delta \times \mathbb{C}^d \to X$ is of rank $d+1$, which is impossible by the proposition 2.1 since X is $(d+1)$-measure hyperbolic. Consequently, $D_d^T(X)$ is discreet. ■

Remark 4.3. — If we take $d = 1$ in the last theorem, we find again the part of the theorem 3.4 concerning the special case of genus 1. In fact, the condition for X to be projective is not necessary (see Grothendieck [Gr], VIII, corollaire 2.2).

5. Moduli spaces of holomorphic mappings

5.1. — Let X be a compact analytic space and Y an analytic space. According to Douady [D], the space $\mathrm{Hol}(X,Y)$ of holomorphic mapping $f : X \to Y$ carries a structure of an open analytic subspace of the Douady space $D(X \times Y)$ and has the following two universal properties:

– the canonical mapping $\Phi : X \times \mathrm{Hol}(X,Y) \longrightarrow Y$ is holomorphic.
$$(x,f) \longmapsto f(x)$$

– if $\varphi : X \times T \to Y$ is holomorphic for analytic space T, then the map $\widetilde{\varphi} : T \to \mathrm{Hol}(X,Y)$ defined by $\widetilde{\varphi}(t) = \varphi(\cdot,t) \in \mathrm{Hol}(X,Y)$ is holomorphic.

Y hyperbolicity gives us some informations about the space $\mathrm{Hol}(X,Y)$. We recall the following Kobayashi theorem [K2]:

Theorem 5.1.

1) If Y is (complete) hyperbolic, then each connected component of $\mathrm{Hol}(X,Y)$ *is (complete) hyperbolic.*

2) If Y is compact hyperbolic, then $\mathrm{Hol}(X,Y)$ *is compact.*

5.2. — More generally, suppose that Y is $(k+1)$-measure hyperbolic for an integer k and consider the open subspace of $\mathrm{Hol}(X,Y)$ of holomorphic mappings $f : X \to Y$ with rank at least k, which we note $\mathrm{Hol}_k(X,Y)$. Then we have:

Theorem 5.2. — *If X is a compact analytic space and Y a $(k+1)$-measure hyperbolic analytic space, then* $\mathrm{Hol}_k(X,Y)$ *is Brody-hyperbolic.*

Proof. — Suppose that there exists a non-constant holomorphic mapping

$$f : \mathbb{C} \to \mathrm{Hol}_k(X,Y).$$

Let $F : \mathbb{C}\times X \to Y$ be the induced mapping defined by $F(t,x) = f(t)(x)$. According to 5.1, F is holomorphic. Let's fix a regular point $(t_0,x_0) \in \mathbb{C} \times X$ such that $\operatorname{rank}_{(t_0,x_0)} F = \operatorname{rank} F$ and $\operatorname{rank}_{x_0} f(t_0) = \operatorname{rank} f(t_0)$. Choosing connected smooth neighbourhoods $U \subset \mathbb{C}$ of t_0 and $V \subset X$ of x_0 and noting $G := F|_{U\times V}$, then G is of constant rank on $U \times V$. So by the constant rank theorem, $F(U \times V) = G(U \times V)$ is a connected complex manifold. As f is non-constant, $f(t_0)(V)$ is properly contained in $F(U \times V)$. Thus

$$\dim_{F(t_0,x_0)} f(t_0)(V) < \dim_{F(t_0,x_0)} F(U \times V),$$

which implies that

$$k \leq \operatorname{rank} f(t_0) < \operatorname{rank} F.$$

But this is impossible because Y is $(k+1)$-measure hyperbolic ($F : \mathbb{C} \times X \to Y$ must have rank less than $(k+1)$). So f must be constant and consequently, $\mathrm{Hol}_k(X,Y)$ is Brody-hyperbolic. ■

In particular, when Y is compact measure hyperbolic of dimension k, $\mathrm{Hol}_k(Y,Y)$ coincides with the group $\mathrm{Aut}(Y)$ of Y automorphisms. Theorem 5.2 implies that $\mathrm{Hol}_{k-1}(Y,Y)$ (and so $\mathrm{Hol}_k(Y,Y)$) is Brody-hyperbolic. But $\mathrm{Aut}(Y)$ is a complex Lie group. Thus $\mathrm{Aut}(Y)$ is discreet. So we find again theorem 9.7 in [K2] and theorem 1 in [Wo].

5.3. — In this section, we assume that Y is strongly k-measure hyperbolic. Then we obtain the same property on $\mathrm{Hol}(X,Y)$ for X strongly measure hyperbolic. Namely we have:

Theorem 5.3. — *Let X be a strongly measure hyperbolic compact analytic space of dimension n and Y a strongly k-measure hyperbolic analytic space. Then* $\mathrm{Hol}(X,Y)$ *is strongly k-measure hyperbolic.*

Proof. — The embedding of $\mathrm{Hol}(X,Y)$ in the Douady space $D(X \times Y)$ induces an analytic family of n-dimensional compact analytic subspaces of $X \times Y$ over $\mathrm{Hol}(X,Y)$. The fibres of this family are all isomorphic to X, thus it is a locally trivial family. Moreover, $X \times Y$ is strongly $(n+k)$-measure hyperbolic. Then by theorem 2.4, $\mathrm{Hol}(X,Y)$ is strongly k-measure hyperbolic, as desired. ■

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