SMOOTHING OF ISOLATED HYPERSURFACE SINGULARITIES AND THE QUILLEN METRICS

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dedicated to Atsuko

Abstract. — Let $\pi : X \to D = \{t \in \mathbb{C}; |t| < 1\}$ be a holomorphic map of complex manifolds which is smooth outside of finite number of points of $X_0 = \pi^{-1}(0)$. Let g_X be a Kähler metric of X, $g_{X/D}$ the induced metric on $TX/D = \mathrm{Ker}\, \pi_*$. Set $\lambda = \det R\pi_* \mathcal{O}_X$ for the determinant of direct images, and $\|\cdot\|_Q$ for the Quillen metric of λ associated to $g_{X/D}$.
The purpose of this article is to study the behavior of $\|\cdot\|_Q$ as $t \to 0$. We show that logarithmic singularity appears, and its coefficient is essentially determined by the Milnor number of $\mathrm{Sing}\, X_0$. Vector bundle case is also treated when the family is locally projective. Our method of computing the singularity of the Quillen metric heavily depends on the works of Bismut, Gillet and Soulé [B-G-S, 1–3], and Bismut, Bost [B-B], but an ideal of Morsification, which is familiar in the classical Picard-Lefschetz theory, is also crucial.

0. Introduction

In [Q], Quillen introduced a metric on the determinant of cohomology, which is called Quillen metric today, and calculated the curvature of the determinant in some cases. After Quillen, in the celebrated work [B-G-S 1], [B-G-S 2] and [B-G-S 3], Bismut, Gillet and Soulé generalized his result to arbitrary smooth morphism of Kähler manifolds, and established the Riemann-Roch-Grothendieck formula at the differential form level. Subsequently, their work has been developed in the context of Arakelov Geometry ([B-L], [F], [G-S 2], [S], etc.).

The other generalization was studied by Bismut and Bost ([B-B]). They established the curvature formula of the determinant bundle for degenerating family of Riemann surfaces with logarithmically divergent metrics, and obtained a refinement of Mumford's formula. In the different context, similar problem was treated by Wolpert [W]. As the Quillen metric is essentially a product of the Ray-Singer analytic torsion ([R-S]) and the L^2-inner product via the Hodge theory, it is also important to study the singularity of analytic torsion. He studied the case of degenerating family of Riemann surfaces with the Poincaré metric, and obtain the asymptotic formula of analytic torsion.

In studying the case of higher relative dimension, unfortunately not every family admits either logarithmically divergent or Einstein Kähler metrics, and therefore, it looks natural to consider the degenerating problem of the Quillen metric for the family equipped with the induced Kähler metric from the ambient space. In [B], Bismut studies this problem when the family is ordinary singular, and obtains a generalization of [B-B]. The purpose of this article is to study the problem for the other types of degenerating family.

Let $\pi : X \to D = \{t \in \mathbb{C}; |t| < 1\}$ be a proper surjective holomorphic map of complex manifolds.

Definition 0.1. — A family (π, X, D) is said to be a smoothing of isolated hypersurface singularities (IHS), if π is of maximal rank outside of finite number of points in $X_0 = \pi^{-1}(0)$. In particular, $\operatorname{Sing} X_0$ consists of isolated hypersurface singularities. When $\operatorname{Sing} X_0$ consists of A_1-singularities, *i.e.*, the singularity whose defining equation is given by $\{z_0^2 + \cdots + z_n^2 = 0\}$ in $\mathbb{C}^{n+1}$, the family is said to be A_1-singular.

In case $n = 1$, A_1-singular family is called f.s.o. in [B-B], and A_1-singular family is one of the natural higher dimensional generalization of f.s.o.

Definition 0.2. — Let (π, X, D) be a smoothing of IHS and $\widetilde{\pi} : \widetilde{X} \to D^2$ a proper holomorphic surjection of Kähler manifolds. Then $(\widetilde{\pi}, \widetilde{X}, D^2)$ is said to be a Morsification of (π, X, D) if the following conditions are satisfied:

1) If $i : D \hookrightarrow D \times \{0\}$ stands for an embedding of D into D^2, then $i^*\widetilde{X} = X$, $i^*\widetilde{\pi} = \pi$, and X is reduced in $\widetilde{X}$.

2) For any $s \in D^2 - \{0\}$, $\operatorname{Sing} X_s$ $(X_s = \pi^{-1}(s))$ consists of either empty set or A_1-singularities.

Our theorem is as follows:

Main Theorem. — *Let (π, X, D) be a smoothing of IHS which admits a Morsification g_X a Kähler metric of X, and $g_{X/D}$ the induced metric on $TX/D = \operatorname{Ker} \pi_*$. Let $\lambda(\mathcal{O}_X) = \det R\pi_*\mathcal{O}_X$ be the determinant of direct images, and $\|\cdot\|_Q$ the Quillen metric associated to $g_{X/D}$. Then,*

1) $\|\cdot\|_Q$ is a singular Hermitian metric of $\lambda(\mathcal{O}_X)$, and its curvature current is given by

$$c_1(\lambda(\mathcal{O}_X), \|\cdot\|_Q) = \frac{(-1)^{n+1}}{(n+2)!}\mu(\operatorname{Sing} X_0)\delta_0 + \pi_*(Td(TX/D, g_{X/D}))^{(1,1)}$$

where $n = \dim X/D$, $\mu(\operatorname{Sing} X_0) = \sum_{p \in \operatorname{Sing} X_0} \mu(p)$, $\mu(p)$ the Milnor number of $p \in \operatorname{Sing} X_0$, and δ_0 is the Dirac measure at 0. Furthermore, $\pi_*(Td(TX/D))^{(1,1)}$ is d-closed in the sense of current, and there exists $r > 1$ such that

$$\pi_*(Td(TX/D, g_{X/D}))^{(1,1)} \in L^r_{\mathrm{loc}}(D).$$

2) Let (E,h) *be a holomorphic Hermitian vector bundle on* X, $\lambda(E) = \det R\pi_* \mathcal{O}(E)$ *the determinant bundle, and* $\|\cdot\|_Q$ *the Quillen metric of* $\lambda(E)$ *associated to* $g_{X/D}$ *and* h. *Suppose that there exists a Morsification* $(\widetilde{\pi}, \widetilde{X}, D^2)$ *which is locally projective, and an extension* $\widetilde{E}$ *of* E *to* $\widetilde{X}$. *Then,* $\|\cdot\|_Q$ *is a singular Hermitian metric of* $\lambda(E)$ *and its curvature current is given by*

$$c_1(\lambda(E), \|\cdot\|_Q) = \frac{(-1)^{n+1}}{(n+2)!} r(E)\mu(\mathrm{Sing}\, X_0)\delta_0 + \pi_*(Td(TX/D, g_{X/D})ch(E,h))^{(1,1)}$$

where $r(E) = \mathrm{rank}\, E$, *and* $\pi_*(Td(TX/D)ch(E))^{(1,1)}$ *is* d*-closed in the sense of current and there exists* $r > 1$ *such that*

$$\pi_*(Td(TX/D, g_{X/D})ch(E,h))^{(1,1)} \in L^r_{\mathrm{loc}}(D).$$

We remark that the same sign conventions as [S] are used in the theorem for determinants of direct images and analytic torsions.

COROLLARY 0.1. — *In the situation of Main Theorem 1) let* $\sigma \in \Gamma(D, \lambda)$ *be a holomorphic section of determinant bundle such that* $\sigma(0) \neq 0$. *Then, as* $t \to 0$,

$$\log \|\sigma\|_Q(t) = \frac{(-1)^n}{(n+2)!}\mu(\mathrm{Sing}\, X_0)\log|t|^2 + a_0 + O(|t|^\alpha)$$

where $a_0 \in \mathbb{R}$ *and* $\alpha > 0$. *Similar formula also hods in the case 2).*

Combining Corollary 0.1 and Proposition 4.3, we have the following:

COROLLARY 0.2. — *In the situation of Main Theorem 1), we assume that* $\mathrm{Sing}\, X_0$ *consists of rational singularities. Let* $T(X_t)$ *be the Ray-Singer analytic torsion of* (X_t, g_t). *Then, as* $t \to 0$,

$$\log T(X_t) = \frac{(-1)^n}{(n+2)!}\mu(\mathrm{Sing}\, X_0)\log|t|^2 + O\Big(\log\log\frac{1}{|t|}\Big).$$

If every nontrivial root of Bernstein-Sato polynomial of $\mathrm{Sing}\, X_0$ *is strictly smaller than* -1, *then, as* $t \to 0$,

$$\log T(X_t) = \frac{(-1)^n}{(n+2)!}\mu(\mathrm{Sing}\, X_0)\log|t|^2 + a_0 + O(|t|^\alpha)$$

where $a_0 \in \mathbb{R}$ *and* $\alpha > 0$ *as above.*

We mention the relation of our theorem and classical Picard-Lefschetz theory for monodromy. In the Picard-Lefschetz theory, one can calculate the Milnor monodromy of IHS as follows. First take a nearby equation so that singularity of any singular fiber is of type A_1. (Such a process is originally called the Morsification). Then, the monodromy is given by a certain composition of each monodromy arising from the A_1-singularities. In this way, the problem reduces to A_1-singular case by considering a Morsification. Surprisingly, this idea works in computing the singularity of the Quillen metrics also, combined with the argument developed by Bismut and Bost ([B-B], § 10–12). In this sense, our theorem is an analogy of the Picard-Lefschetz theory for the Quillen metric.

We do not know whether there exist some obstructions for a smoothing of IHS to admit a Morsification. In view of Corollary 0.2 and our previous result [Y 1], it looks interesting to study the following value:

$$\log T(X_t) - \#\,\mathrm{Sing}\, X_0 \frac{(-1)^n}{(n+2)!} \log|t|^2 - \log T(X_0) \quad (t \to 0)$$

for A_1-singular family, where $T(X_0)$ is the analytic torsion of X_0 in the sense of [Y 1]. In the case $\dim X/D = 1$, it is treated in [B]. In view of [Br-Le], it also seems interesting to study the same problem when $\dim X/D = 1$, but $\mathrm{Sing}\, X_0$ does not necessaryly consists of nodes, because $T(X_0)$ has a sense also in this case ([Br-Le]). For the n-dimensional A_1-singularity, its contribution to the singularity of the Quillen metric is equal to $(-1)^{n+1}/(n+2)!$ by Main Theorem, and therefore its generating function is given by

$$\frac{1 - x - e^{-x}}{x^2}.$$

The author does not know the intrinsic reason of this formula, which must exist, and leave it for the future study.

This article is arranged as follows. In section 1, we study the relative Todd form for arbitrary family of hypersurfaces in the Euclidean space, and establish the vanishment of degree $(n+1, n+1)$-part. This is crucial to apply the argument of Bismut and Bost, which is done in section 3. In section 2, we recall some basic results of Morsification. From section 4 to 9, we prepare several techniques to establish Main Theorem for A_1-singular families. In section 4, we establish the continuity of eigenvalues of Laplacians in the parameter, which is a generalization of [J-W] and [Y 2]. In section 5, by a slight modification of the argument due to Cheng, Li and Yau ([C-L-Y], [L-Y]), we establish a Duhamel's principle applicable to degenerating family. In section 6, we establish an upper bound of the "trace" of the heat kernel under certain Sobolev type inequality. In general, it looks difficult to obtain an upper bound of the heat kernel itself for Schrödinger operators, only assuming Sobolev type inequality. In section 7, we establish an estimate of the error between the heat kernel and its parametrix, and in section 8 and 9, an asymptotic formula of analytic torsion for conic degenerating family, which refines [Y 1] is established. Using the result of section 8 and 9, we prove Main Theorem for special case in section 10-12, and general case in 13. In section 14, we treat examples. For the reader who does not have interest in technical details of analysis, we recommend to ship sections 4–9.

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1. Morsification and relative Todd form

LEMMA 1.1. — *Let V be a vector space over $\mathbb{R}$, J a complex structure on V, $V_{\mathbb{C}} = V \otimes \mathbb{C}$ its complexification, $V_{\mathbb{C}} = V_+ \oplus V_-$ the decomposition into eigenspaces of J. Clearly $V_- = \overline{V}_+$ by the complex conjugation. For $A = (a_1, \ldots, a_n) \in V_+ \otimes \mathbb{C}^n$, set $A^* = ^t (\overline{a}_1, \ldots, \overline{a}_n)$. Then,*

$$\det(I + xA^* \wedge A) = (1 + xA \wedge A^*)^{-1}_{\leq n} \quad (\in \Lambda V_{\mathbb{C}}[x]).$$

Here we consider $A^* \wedge A \in M(n, \Lambda V_{\mathbb{C}})$, the metrix of type (n, n) with values in $\Lambda V_{\mathbb{C}}$, and use a convention:

$$f(x)_{\leq n} = \sum_{i=0}^{n} a_i x_i \quad (f(x) = \sum_{i=0}^{\infty} a_i x_i).$$

Proof. — Noting $(A \wedge A^*)^k = -Tr(A^* \wedge A)^k$, we obtain

$$(1.1) \qquad (1 + xA \wedge A^*)^{-1} = \sum_{k=0}^{\infty} (TrA^* \wedge A)^k x^k,$$

and

(1.2)

$$\begin{aligned}
&\det(I + xA^* \wedge A)\big|_{x^k} \\
&= \left(\frac{d}{dx}\right)^k\Big|_{x=0} \frac{1}{k!} \sum_{\sigma \in G_n} \operatorname{sgn}\sigma (\delta_{1\sigma(1)} + x\overline{a}_1 \wedge a_{\sigma(1)}) \wedge \cdots \wedge (\delta_{n\sigma(n)} + x\overline{a}_n \wedge a_{\sigma(n)}) \\
&= \frac{1}{k!} \sum_{|I|=k} \sum_{\sigma \in G_n} \operatorname{sgn}\sigma \delta_{j_1\sigma(j_1)} \cdots \delta_{j_{n-k}\sigma(j_{n-k})} \overline{a}_{i_1} \wedge a_{\sigma(i_1)} \wedge \cdots \wedge \overline{a}_{i_k} \wedge a_{\sigma(i_k)} \quad (I \cap J = \emptyset) \\
&= \frac{1}{k!} \sum_{|I|=k} \sum_{\tau \in G_k} \operatorname{sgn}\tau \overline{a}_{i_1} \wedge a_{\tau(i_1)} \wedge \cdots \wedge \overline{a}_{i_k} \wedge a_{\tau(i_k)} \\
&= (TrA^* \wedge A)^k,
\end{aligned}$$

where $\det(I + xA^* \wedge A)\big|_{x^k}$ stands for the coefficient of x^k of the polynomial. Compairing (1.1) and (1.2), we get the formula. ■

LEMMA 1.2. — *For any* $F(x) = 1 + \sum_{i\geq 1} a_i x^i \in \mathbb{Q}[[x]]$ *and* $A \in V_+ \otimes \mathbb{C}^n$,

$$\det F(A^* \wedge A) = F(A \wedge A^*)^{-1}\{1 - (1 - F(A \wedge A^*)^{n+1}\}.$$

Proof. — Put $F(A^* \wedge A) = I + yA^* \wedge A$, $y = \sum_{i\geq 1} a_i (A \wedge A^*)^{i-1}$. Then, by Lemma 1.1, we get

$$\begin{aligned}
\det F(A^* \wedge A) &= (1 + yA \wedge A^*)^{-1}_{\leq n} \\
&= \sum_{k=0}^{n} (-1)^k (yA \wedge A^*)^k \qquad (1.3)\\
&= \sum_{k=0}^{n} (-1)^k \{F(A \wedge A^*) - 1\}^k \\
&= F(A \wedge A^*)^{-1}\{1 - (1 - F(A \wedge A^*)^{n+1}\}. \ \blacksquare
\end{aligned}$$

Put

$$U := D^{n+1} \times D \subset \mathbb{C}^{n+2}, \quad x = (x_0, \ldots, x_n) \in D^{n+1}, \quad \varepsilon \in D.$$

For $F(x, \varepsilon) \in \mathcal{O}(U)$, a nonconstant holomorphic function, put

$p : U \ni (x, \varepsilon) \to \varepsilon \in D$,

$\pi : U \ni (x, \varepsilon) \to (F(x, \varepsilon), \varepsilon) \in D^2$,

$TD^{n+1} := \operatorname{Ker} p_* = \{v \in TU; v = \sum_{i=0}^{n} \xi_i \frac{\partial}{\partial x_i}\} = T\mathbb{C}^{n+1}\big|_U$,

$TX \equiv TU/D^2 := \operatorname{Ker} \pi_* = \{v \in TU; v = \sum_{i=0}^{n} \xi_i \frac{\partial}{\partial x_i}, \sum_{i=0}^{n} \xi_i \frac{\partial F}{\partial x_i} = 0\} \subset TD^{n+1}$,

$N_{X/D^{n+1}} := \operatorname{Ker} p_* / \operatorname{Ker} \pi_*$,

$X_y := \pi^{-1}(y) \subset D^{n+1} \times \{y\}$ for any $y \in D^2$.

Then, for any $x \in X$,

$(TX)_x :\equiv (TU/D^2)_x = T_x X_{\pi(x)}$

$(N_{X/D^{n+1}})_x = (N_{X_{\pi(x)}/D^{n+1}})_x$.

Consider the following exact sequence:

$$0 \longrightarrow TU/D^2 \longrightarrow TD^{n+1} \longrightarrow N_{X/D^{n+1}} \longrightarrow 0. \qquad (1.4)$$

Let g_U be the Euclidean metric of U, $g_{TD^{n+1}}$ the Euclidean metric on TD^{n+1}, and define

$$g_{U/D^2} = g_U\big|_{TU/D^2} = g_{TD^{n+1}}\big|_{TU/D^2},$$

a Hermitian metric of TU/D^2 induced by the ambient Euclidean metric. Denote by R_{TU/D^2} the curvature form of $(TU/D^2, g_{U/D^2})$ defined on $U - \Sigma$, where

$$\Sigma = \{(x, \varepsilon) \in U; d_{D^{n+1}}F := \sum_{i=0}^{n} \frac{\partial F}{\partial x_i} dx^i = 0\}.$$

PROPOSITION 1.1. — $[Td(R_{TU/D^2})]^{(n+1,n+1)} \equiv 0$ *on* $U - \Sigma$.

Proof. — Let R_N be the curvature form of $N_{X/D^{n+1}}$ with respect to the metric induced by the identification:

$$N_{X/D^{n+1}} \cong (TU/D^2)^{\perp}.$$

Then we get

$$g_{TD^{n+1}} = g_{U/D^2} \oplus h.$$

Let $A \in A^0_U(\mathrm{End}(TU/D^2, (TU/D^2)^{\perp}))$ be the second fundamental form of (1.4).

By the curvature formula ([K], I (6.1)), we obtain

$$R_{TD^{n+1}|X} = \begin{pmatrix} R_{TU/D^2} - A^* \wedge A & -D'A^* \\ D''A & R_N - A \wedge A^* \end{pmatrix}. \tag{1.5}$$

Since $(TD^{n+1}|_X, g_{TD^{n+1}})$ is a flat vector bundle, we find $R_{TD^{n+1}|_X} \equiv 0$ which implies

$$R_{TU/D^2} = A^* \wedge A, \quad R_N = A \wedge A^*, \quad D''A = D'A^* = 0. \tag{1.6}$$

Apply Lemma 1.2 for R_{TU/D^2} and $F(x) = Td(x) = \frac{x}{1-e^{-x}}$. Then we have

$$\begin{aligned} Td(R_{TU/D^2}) &= \det Td(A^* \wedge A) \\ &= Td(R_N)^{-1}\{1 - (1 - Td(R_N))^{n+1}\}. \end{aligned} \tag{1.7}$$

In particular,

$$\begin{aligned} [Td(R_{TU/D^2})]^{(n+1,n+1)} &= [Td(R_N)^{-1}]^{(n+1,n+1)} - [Td(R_N)^{-1}(1 - Td(R_N))^{n+1}]^{(n+1,n+1)} \\ &= (-1)^{n+1}\Big\{\frac{1}{(n+2)!} - \frac{1}{2^{n+1}}\Big\}\Big(\frac{i}{2\pi}R_N\Big)^{n+1}. \end{aligned} \tag{1.8}$$

Consider the dual of (1.4):

$$0 \longrightarrow N^*_{X/D^{n+1}} \longrightarrow \Omega^1_{D^{n+1}} \longrightarrow \Omega^1_{U/D^2} \longrightarrow 0. \tag{1.9}$$

As is easily verified, $N^*_{X/D^{n+1}}$ is generated by

$$s = d_{D^{n+1}}F = \sum_{i=0}^{n} \frac{\partial F}{\partial x_i}(x, \varepsilon) dx^i \tag{1.10}$$

and therefore R_N is given by

$$R_N = -R^*_{N_{X/D^{n+1}}} = -\overline{\partial}\partial \log |d_{D^{n+1}}F|^2, \tag{1.11}$$

$$(1.12) \qquad |d_{D^{n+1}}F|^2 = \sum_{i=0}^{n} \left| \frac{\partial F}{\partial x_i}(x, \varepsilon) \right|^2.$$

Let $\nu : U - \Sigma \longrightarrow \mathbb{P}^n$ be the Grauss map:

$$(1.13) \qquad \nu : U - \Sigma \ni (x, \varepsilon) \longrightarrow \left[\frac{\partial F}{\partial x_0}(x, \varepsilon) : \cdots : \frac{\partial F}{\partial x_n}(x, \varepsilon) \right] \in \mathbb{P}^n.$$

Then, by (1.11) and (1.12), we find

$$(1.14) \qquad \frac{i}{2\pi} R_N = \nu^* \omega_{\mathbb{P}^n},$$

where $\omega_{\mathbb{P}^n} = \frac{i}{2\pi}\partial\overline{\partial} \log \|z\|^2$ is the Fubini-Study form of $\mathbb{P}^n$.

Therefore by (1.8), putting $C(n) = (-1)^{n+1}\{\frac{1}{(n+2)!} - \frac{1}{2^{n+1}}\}$, we get

$$(1.15) \qquad [Td(R_{TU/D^2})]^{(n+1,n+1)} = C(n)\nu^* \omega_{\mathbb{P}^n}^{n+1} = 0. \qquad \blacksquare$$

Let $\pi : X \to D$ be a smoothing of IHS, $\tilde{\pi} : \widetilde{X} \to D^2$ its Morsification, and g_X a Kähler metric of X such that $R_X \equiv 0$ on a neighborhood of Sing X_0. Then, for any $p \in$ Sing X_0, there exists a coordinate neighborhood $(U_p, (x_0, \dots, x_n))$ such that, on U_p, $g_X = \sum_{i=0}^{n} |dx_i|^2$.

Proposition 1.2. — *Choosing U_p sufficiently small, there exists a Kähler metric $g_{\widetilde{X}}$ and $(\widetilde{U}_p, (\tilde{x}_0, \dots, \tilde{x}_n, \varepsilon))$, a coordinate neighborhood of $p \in$ Sing X_0 in $\widetilde{X}$, such that*

1) $g_{\widetilde{X}} = \sum_{i=0}^{n} |d\tilde{x}_i|^2 + |d\varepsilon|^2$ *on* $\widetilde{U}_p$,

2) $U_p = \widetilde{U}_p \cap \{\varepsilon = 0\}$ *and* $\tilde{x}_i|_X = x_i$ *(in particular,* $g_{\widetilde{X}}|_{U_p} = g_X$*),*

3) $\tilde{\pi}(\tilde{x}, \varepsilon) = (F(\tilde{x}, \varepsilon), \varepsilon)$ *on* $\widetilde{U}_p$.

Proof. — Let (t, ε) be the coordinates of Δ^2. Since $X = \tilde{\pi}^{-1}(\{\varepsilon = 0\})$ is reduced and smooth, $d\tilde{\pi}^*\varepsilon \neq 0$ on a neighborhood of X in $\widetilde{X}$, and $\tilde{\pi}^*\varepsilon$ becomes one of the coordinates functions. Let $\widetilde{V}_p$ be a small Stein neighborhood around p, and $\tilde{x}_i \in \mathcal{O}(\widetilde{V}_p)$ an extension of x_i to $\widetilde{V}_p$.

By the construction, choosing $\widetilde{U}_p \subset\subset \widetilde{V}_p$ if necessary, we may assume that $(\widetilde{U}_p, (\tilde{x}_0, \dots, \tilde{x}_n, \varepsilon))$ becomes a local coordinates around p.

Since $\tilde{\pi}^*\varepsilon = \varepsilon$, $\tilde{\pi}$ is represented as follows

$$\tilde{\pi}(\tilde{x}, \varepsilon) = (F(\tilde{x}, \varepsilon), \varepsilon)$$

where $F \in \mathcal{O}(\widetilde{U}_p)$. This proves 2) and 3).

Let $g'_{\widetilde{X}}$ be a Kähler metric of $\widetilde{X}$ (which exists by the definition of Morsification), φ_p its potential on $\widetilde{V}_p$:

$$g'_{\widetilde{X}} = \partial\overline{\partial}\varphi_p. \tag{1.16}$$

By the Kählerness, there is a coordinate $(z_0, \dots, z_n)$ such that

$$g'_{\widetilde{X}} = \sum_{ij}(\delta_{ij} + O(\|z\|^2))dz_i d\overline{z}_j. \tag{1.17}$$

Therefore, the potential φ_p may be chosen as follows:

$$\varphi_p(z) = \|z\|^2 + O(\|z\|^4). \tag{1.18}$$

By Morse's lemma, there is a real coordinate $(u_1, \dots, u_{2n+4})$ such that:

$$\varphi_p(z) = \sum_{i=\emptyset}^{2n+4}(u_i)^2 = r^2. \tag{1.19}$$

Put $B_\varepsilon(p) := \{u \in \widetilde{V}_p; r < \varepsilon\}$.

Let $\chi_\varepsilon(t)$ be a nonnegative convex increasing function such that:

$$\chi_\varepsilon(t), \quad \chi'_\varepsilon(t), \quad \chi''_\varepsilon(t) \geq 0 \tag{1.20}$$

$$\chi_\varepsilon(t) \equiv 0 \quad (t \leq \varepsilon), \quad \chi_\varepsilon(t) = t \quad (t \geq 2\varepsilon). \tag{1.21}$$

Then, $\partial\overline{\partial}\chi_\varepsilon(\varphi_p) \geq 0$ on $\widetilde{V}_p$ and

$$\partial\overline{\partial}\chi_\varepsilon(\varphi_p) = \begin{cases} \partial\overline{\partial}\varphi_p = g'_{\widetilde{X}} & \text{on} \quad \widetilde{X} - \widetilde{V}_p \\ 0 & \text{on} \quad B_p(\varepsilon). \end{cases} \tag{1.22}$$

Let ρ_ε be a cut-off function such that:

$$\rho_\varepsilon = \begin{cases} 1 & \text{on} \quad B_p(2\varepsilon) \\ 0 & \text{on} \quad X - B_p(3\varepsilon). \end{cases} \tag{1.23}$$

Put

$$g_{\varepsilon,\delta} := \partial\overline{\partial}\rho_\varepsilon\Big(\sum_{i=0}^{n}|\tilde{x}_i|^2 + |\varepsilon|^2\Big) + \delta^{-1}\partial\overline{\partial}\chi_\varepsilon(\varphi_p). \tag{1.24}$$

Then,

$$g_{\varepsilon,\delta} = \begin{cases} g_{\widetilde{X}} & \text{on} \quad \widetilde{X} - \widetilde{W}_p \\ \partial\overline{\partial}\Big(\sum_{i=0}^{n}|\tilde{x}_i|^2 + |\varepsilon|^2\Big) & \text{on} \quad B_p(\varepsilon). \end{cases} \tag{1.25}$$

Choosing $\delta \ll 1$, we may assume $g_{\varepsilon,\delta} > 0$ on $\widetilde{X}$, and obtain a Kähler metric which satisfies 1). ■

Remark 1.1. — In view of the construction of $g_{\widetilde{X}}$ in Proposition 1.2, we may assume that $g_{\widetilde{X}}$ satisfies the conditions 1), 2) and 3) of Proposition 1.2 on a neighborhood of $\mathrm{Sing}\, X_0$, since the construction is local.

Proposition 1.3. — *Let $\pi : X \to D$ be a smoothing of IHS which admits a Morsification $\tilde{\pi} : \widetilde{X} \to D^2$, g_X a Kähler metric of X such that $R_X \equiv 0$ on a neighborhood of* $\mathrm{Sing}\, X_0$, *and $g_{\widetilde{X}}$ an extension of g_X to $\widetilde{X}$ constructed in Proposition 1.2 for any $p \in$* $\mathrm{Sing}\, X_0$.

Then, $Td(T\widetilde{X}/D^2, g_{\widetilde{X}/D^2})^{(n+1,n+1)}$ extends to a smooth $(n+1, n+1)$-form on X.

In particular, $\pi_*(Td(T\widetilde{X}/D^2, g_{\widetilde{X}/D^2}))^{(1,1)}$ extends to a smooth d-closed $(1,1)$-form on D^2.

Proof. — Let U be a neighborhood of $\mathrm{Sing}\, X_0$ in $\widetilde{X}$ for which there is a coordinates satisfying conditions 1), 2) and 3) of Proposition 1.2. Then, by Proposition 1.1,

$$[Td(R_{TU/D^2})]^{(n+1,n+1)} = 0 \tag{1.26}$$

on $U - \Sigma$. Therefore, by setting zero on Σ, it extends smoothly to U.

Since

$$\begin{aligned}\pi_*(Td(T\widetilde{X}/D^2, g_{\widetilde{X}/D^2}))^{(1,1)}(t,\varepsilon) &= \int_{X(t,\varepsilon)} Td(T\widetilde{X}/D^2, g_{\widetilde{X}/D^2})^{(n+1,n+1)} \\ &= \int_{X(t,\varepsilon)-U} Td(T\widetilde{X}/D^2, g_{\widetilde{X}/D^2})^{(n+1,n+1)}\end{aligned} \tag{1.27}$$

for $(t,\varepsilon) \in D^2 - \tilde{\pi}(\Sigma)$, we have

$$\pi_*(Td(T\widetilde{X}/D^2, g_{\widetilde{X}/D^2}))^{(1,1)} = \int_{\widetilde{X}-U/D^2} Td(T\widetilde{X}/D^2, g_{\widetilde{X}/D^2})^{(n+1,n+1)} \tag{1.28}$$

on $D^2 - \pi(\Sigma)$. As $\tilde{\pi} : \widetilde{X} - U \to D^2$ is a differentiably trivial family of complex manifolds with boundary, the right hand side extends to a smooth $(1,1)$-form on D^2 because of (1.26). Since it is d-closed on $D^2 - \Sigma$ by the curvature theorem ([B-G-S 1], Theorem 0.1), its smooth extension must also be d-closed. ∎

2. Local description of Morsification

Let $\pi : X \to D$ be a smoothing of IHS, $\tilde{\pi} : \widetilde{X} \to D^2$ its Morsification. Put

$$\Sigma := \{x \in \widetilde{X};\, x \in \mathrm{Sing}\, \widetilde{X}_{\pi(x)}\}. \tag{2.1}$$

By the definition of Morsification, $\mathrm{codim}_{\widetilde{X}}\, \Sigma = n+1$.

LEMMA 2.1. — $\Sigma - \operatorname{Sing} X_0$ *is a manifold.*

Proof. — Let (z, ε) be the coordinate of Prop. 1.2. Then, $\tilde{\pi}(z,\varepsilon)=(F(z,\varepsilon),\varepsilon)$. Since $p \in \Sigma - \operatorname{Sing} X_0$ is a nondegenerate critical point of $F(\cdot,\varepsilon)$, $\left(\frac{\partial F}{\partial z_0},\dots,\frac{\partial F}{\partial z_n},\varepsilon\right)$ is also a local coordinate in $\widetilde{X}$ around p. Clearly, $\Sigma \cap B_0(p) = \left\{\frac{\partial F}{\partial z_0} = \cdots = \frac{\partial F}{\partial z_n} = 0\right\}$ is a manifold. ■

LEMMA 2.2. — $\tilde{\pi} : \Sigma - \operatorname{Sing} X_0 \to \tilde{\pi}(\Sigma - \operatorname{Sing} X_0)$ *is of maximal rank.*

Proof. — By lemma 2.1, there exists a local coordinate (w, ε) around $p \in \Sigma - \operatorname{Sing} X_0$ such that

$$\Sigma \cap B = \{(w,\varepsilon); w = 0\}, \quad \tilde{\pi}(w,\varepsilon) = (F(w,\varepsilon),\varepsilon).$$

Therefore, rank $\tilde{\pi}_* = 1$. ■

Let

$$\Sigma = \bigcup_{i\in I} \Sigma_i, \quad \tilde{\pi}(\Sigma) = \bigcup_{j\in J} \Delta_j \tag{2.2}$$

be the irreducible decompositions. Put

$$I_k = \{i \in I; \tilde{\pi}(\Sigma_i) = \Delta_k\}, \tag{2.3}$$

$$I = I_1 \cup \cdots \cup I_{|J|}, \quad I_k \cap I_\ell = \emptyset \quad (k \neq \ell), \tag{2.4}$$

$$\Sigma_{(j)} := \bigcup_{i\in I_j} \Sigma_i. \tag{2.5}$$

Then, $\tilde{\pi} : \Sigma_{(j)} \to \Delta_j$ is an analytic covering. As $\tilde{\pi} : \Sigma_{(j)} - \operatorname{Sing} X_0 \to \Delta_j - \{0\}$ is unbranched by lemma 2.2, $\#\tilde{\pi}^{-1}(x)(x \in \Delta_j - \{0\})$ is constant. Set

$$n_j := \#\tilde{\pi}^{-1}(x) \quad (x \in \Delta_j - \{0\}), \tag{2.6}$$

and define a divisor of D^2 by

$$\Delta := \sum_{j\in J} n_j \Delta_j. \tag{2.7}$$

PROPOSITION 2.1. — *For* $0 < |\varepsilon| \ll 1$,

$$\#\Delta \cap D_\varepsilon := \sum_{j\in J} n_j \cdot \#(\Delta_j \cap D_\varepsilon) = \sum_{p\in \operatorname{Sing} X_0} \mu(p)$$

where $D_\varepsilon = D \times \{\varepsilon\}$, *and* $\mu(p)$ *is the Milnor number.*

Proof. — As

$$\sum_{j\in J} n_j \cdot \#(\Delta_j \cap D_\varepsilon) = \sum_{j\in J} \#(\Sigma_{(j)} \cap \tilde{\pi}^{-1}(D_\varepsilon)) \qquad (2.8)$$
$$= \#(\Sigma \cap \tilde{\pi}^{-1}(D_\varepsilon)),$$

it is sufficient to show

$$\#(\Sigma \cap \tilde{\pi}^{-1}(D_\varepsilon)) = \sum_{p\in \mathrm{Sing}\, X_0} \mu(p). \qquad (2.9)$$

But, for any $p \in \mathrm{Sing}\, X_0$, there is a small neighborhood $U(p)$ in $\widetilde{X}$ and $\varepsilon_0 > 0$ such that, if $0 < |\varepsilon| < \varepsilon_0$,

$$\#(\Sigma \cap \tilde{\pi}^{-1}(D_\varepsilon) \cap U(p)) = \mu(p), \qquad (2.10)$$

because $\tilde{\pi} : \tilde{X} \to D^2$ is a Morsification (cf. [D] and [L]). Clearly, (2.9) follows from (2.10). ∎

Let $f_j \in \mathcal{O}(D^2)$ (prime in $\mathcal{O}_{\mathbb{C}^2,0}$) be a defining equation of Δ_j:

$$\Delta_j = \{(t,\varepsilon) \in D^2; f_j(t,\varepsilon) = 0\}. \qquad (2.11)$$

Then,

$$f := \prod_{j\in J} f_j^{n_j} \in \mathcal{O}(D^2) \qquad (2.12)$$

is a defining equation of Δ.

Since $|\Delta| \cap (D \times \{0\}) = (0,0)$, there exists an integer $a \in \mathbb{Z}_+$ such that

$$f(t,0) = b(t) \cdot t^a \quad (b(0) \neq 0). \qquad (2.13)$$

Proposition 2.2.

$$a = \mu(\mathrm{Sing}\, X_0) = \sum_{p\in \mathrm{Sing}\, X_0} \mu(p).$$

Proof. — Let $a_j \in \mathbb{Z}_+$ be the integer such that

$$f_j(t,0) = b_j(t) t^{a_j} \quad (b_j(0) \neq 0). \qquad (2.14)$$

Clearly

$$a = \sum_{j\in J} n_j a_j. \qquad (2.15)$$

By Weierstrass preparation theorem, we may assume that each $f_j(t,\varepsilon)$ is a Weierstrass polynomial in t:

$$f_j(t,\varepsilon) = t^{a_j} + c_j(\varepsilon) t^{a_j - 1} + \cdots + c_{ja_j}(\varepsilon) \in \mathcal{O}_{\mathbb{C},0}[t]. \qquad (2.16)$$

Let $d_j(\varepsilon)$ be the discriminant of $f_j(t,\varepsilon)$. Suppose $d_j(\varepsilon) = 0$ in $\mathcal{O}_{\mathbb{C},0}$. Then,

$$N_j := \{(t,\varepsilon); f_j(t,\varepsilon) = \frac{d}{dt} f_j(t,\varepsilon) = 0\} \tag{2.17}$$

is a divisor of D^2, *i.e.*, $\dim N_j = 1$. This contradicts that f_j is prime in $\mathcal{O}_{\mathbb{C}^2,0}$.

Therefore, $d_j(\varepsilon) \neq 0$ in $\mathcal{O}_{\mathbb{C},0}$ and, for generic $\varepsilon \in D$, $|\varepsilon| \ll 1$, the equation $f_j(t,\varepsilon) = 0$ has mutually different a_j-th roots. Thus we have

$$\begin{aligned} a &= \sum_{j \in J} n_j \#(\Delta_j \cap D_\varepsilon) \\ &= \mu(\operatorname{Sing} X_0). \end{aligned} \tag{2.18}$$

■

3. Reduction to the A_1-singular family

We assume the following proposition in this section:

PROPOSITION 3.1. — *Let $\pi : X^{n+1} \to D$ be an A_1-singular family, g_X a Kähler metric of X such that $R_X \equiv 0$ on a neighborhood of $\operatorname{Sing} X_0$, $g_{X/D}$ the Hermitian metric of TX/D induced by g_X. Let (E,h) be a holomorphic Hermitian vector bundle on X such that $R_E \equiv 0$ on a neighborhood of $\operatorname{Sing} X_0$. Let $\lambda(E) := \det R\pi_* E$ be the determinant bundle and $\|\cdot\|_Q$ the Quillen metric associated to $g_{X/D}$ and h. If $\dim H^q(X_t, E_t)$ is a constant function and $\dim H^q(X_0, \mathcal{O}(E_0)) = \dim H^{0,q}_{(2)}(X_0, E_0)$ for any $q \geq 0$, then $\|\cdot\|_Q$ is a singular Hermitian metric on $\lambda(E)$ and*

$$c_1(\lambda(E), \|\cdot\|_Q) = a(n) r(E) \# \operatorname{Sing} X_0 \delta_0 + \pi_*(Td(TX/D) ch(E))^{(1,1)} \tag{3.1}$$

where $a(n) \in \mathbb{Q}$ is a constant which depends only on n. Here $H^{0,q}_{(2)}(X_0, E_0)$ is defined as follows:

$L^{0,q}_{(2)}(X_0, E_0)$: the Hilbert space of L^2 $(0,q)$-forms with values in E_0,

$$H^{0,q}_{(2)}(X_0, E_0) := \{f \in L^{0,q}_{(2)}(X_0, E_0); \bar\partial_{\min} f = (\bar\partial^*)_{\min} f = 0\}$$

is the space of L^2-harmonic $(0,q)$-forms.

Assuming Proposition 3.1, we prove the following theorem.

THEOREM 3.1. — *Let $\pi : X \to D$ be a smoothing of IHS which admits a Morsification $\tilde\pi : \widetilde{X} \to D^2$, g_X a Kähler metric of X such that $R_X \equiv 0$ on a neighborhood of $\operatorname{Sing}(X_0)$, and $g_{X/D}$ the induced metric on TX/D. Let (E,h) be a holomorphic Hermitian vector bundle on X whose curvature vanishes around $\operatorname{Sing} X_0$. Suppose (E,h) admits an extension $(\widetilde{E}, \tilde h)$ to $\widetilde{X}$ such that*

1) $\dim H^q(\widetilde{X}_y, \widetilde{E}_y)$ *is a constant function on* $D^2 - \{0\}$.

2) $\dim H^q(\widetilde{X}_y, \widetilde{E}_y) = \dim H^{0,q}_{(2)}(\widetilde{X}_y, \widetilde{E}_y)$ *for any* $y \in D^2 - \{0\}$.

Let $\lambda(E)$ *and* $\|\cdot\|_Q$ *be the same as before. Then,*

1) $\|\cdot\|_Q$ *is a singular Hermitian metric on* D,

2) $c_1(\lambda(E), \|\cdot\|_Q) = a(n)r(E)\mu(\mathrm{Sing}\, X_0)\delta_0 + \pi_*(Td(TX/D)ch(E))^{(1,1)}$

where $\mu(\mathrm{Sing}\, X_0) := \sum_{p \in \mathrm{Sing}\, X_0} \mu(p)$.

By the anomaly formula ([B-G-S 1], Theorem 0.3, Theorem 1.29), if g_X and g'_X are Kähler metrics of X such that $g_X = g'_X$ on a neighborhood of $\mathrm{Sing}\, X_0$, then

$$c_1(\lambda(E), \|\cdot\|_Q; g_{X/D}) - c_1(\lambda(E), \|\cdot\|_Q; g'_{X/D})$$

is represented by a smooth $(1,1)$-form on D (cf. [B-B], Theorem 5.1). Therefore, by Proposition 1.2, it is sufficient to prove the assertion when g_X admits an extension $g_{\widetilde{X}}$ to $\widetilde{X}$ so that it satisfies the conditions of Proposition 1.2. In the sequel, we fix such an extension $g_{\widetilde{X}}$ and consider $\lambda(\widetilde{E}) := \det R\pi_* \widetilde{E}$, and $\|\cdot\|_{Q,\widetilde{E}}$, the Quillen metric of $\lambda(\widetilde{E})$ associated to $g_{\widetilde{X}/D^2}$ and $\tilde{h}$.

Proposition 3.2. — *Under the situation of Theorem 3.1,*

1) $\|\cdot\|_{Q,\widetilde{E}}$ *is a singular Hermitian metric on* D^2,

2) $c_1(\lambda(\widetilde{E}), \|\cdot\|_{Q,\widetilde{E}}) = a(n)r(E)\delta_\Delta + \pi_*(Td(T\widetilde{X}/D^2)ch(\widetilde{E}))^{(1,1)}$

where Δ *is the divisor defined in section 2.*

For the proof, we need the following lemma:

Lemma 3.1. — *Let* $\varphi \in L^1_{\mathrm{loc}}(D^2 - \{0\})$, ψ *a* d*-closed real* C^∞ $(1,1)$*-form on* D^2, *and* Δ *a divisor of* D^2. *Suppose the following equation holds as currents on* $D^2 - \{0\}$:

$$\frac{i}{2\pi}\bar{\partial}\partial\varphi = a\delta_\Delta + \psi \quad (a \in \mathbb{R}).$$

Then, $\varphi \in L^1_{\mathrm{loc}}(D^2)$ *and the above equation holds as currents on* D^2.

Proof. — By the $\partial\bar{\partial}$-Poincaré lemma, there is $f \in C^\infty(D^2, \mathbb{R})$ such that

$$\psi = \frac{i}{2\pi}\bar{\partial}\partial f. \tag{3.1}$$

Let $g \in \mathcal{O}(D^2)$ be a defining equation of Δ. Then,

$$\delta_\Delta = -\frac{i}{2\pi}\bar{\partial}\partial \log|g|^2. \tag{3.2}$$

Therefore, on $D^2 - \{0\}$,

$$\bar{\partial}\partial\{\varphi + a\log|g|^2 - f\} = 0. \tag{3.3}$$

By Hartogus's theorem, $\varphi + a\log|g|^2 - f$ has a smooth extension to D^2, and (3.3) holds on D^2. ∎

Proof of Proposition 3.2. — Let $\sigma \in \lambda(\widetilde{E})$ be a holomorphic section which does not vanish at 0. Let Δ be the divisor defined in section 2. Then, on $D^2 - |\Delta|$, the following equality holds by the curvature formula ([B-G-S], Theorem 0.1):

$$\frac{i}{2\pi}\bar{\partial}\partial \log\|\sigma\|^2_{Q,\widetilde{E}} = \pi_*(Td(T\widetilde{X}/D^2)ch(\widetilde{E}))^{(1,1)}. \tag{3.4}$$

By Proposition 1.3, Proposition 3.1 and [B-B], Proposition 10.2, the following equality holds as currents on $D^2 - \{0\}$:

$$\frac{i}{2\pi}\bar{\partial}\partial \log\|\sigma\|^2_{Q,\widetilde{E}} = a(n)r(\widetilde{E})\delta_\Delta + \pi_*(Td(T\widetilde{X}/D^2)ch(\widetilde{E}))^{(1,1)}. \tag{3.5}$$

Then, by Lemma 3.1, $\log\|\sigma\|_{Q,\widetilde{E}} \in L^1_{\mathrm{loc}}(D^2)$ and the above equality holds on D^2. ∎

Proof of Theorem 3.1. — Let $\sigma \in \Gamma(D, \lambda(E))$, $\sigma(0) \neq 0$ and $\tilde{\sigma} \in \Gamma(D^2, \lambda(\widetilde{E}))$ its extension. Since $\|\cdot\|_{Q,\widetilde{E}}$ coincides with $\|\cdot\|_{Q,E}$ restricted on $D\times\{0\}$, we obtain, on $D\times\{0\}$

$$\|\sigma\|_Q = \|\tilde{\sigma}\|_{Q,\widetilde{E}}. \tag{3.6}$$

Let $f \in \mathcal{O}(D^2)$ be the defining equation of Δ. By Proposition 3.2,

$$\log\|\sigma\|^2_Q = -a(n)r(E)\log|f(t,0)|^2 + g \quad (g \in C^\infty(D)). \tag{3.7}$$

By Proposition 2.2, we have

$$\log|f(t,0)|^2 = \mu(\mathrm{Sing}\,X_0)\log|t|^2 + h \quad (h \in C^\infty(D)). \tag{3.8}$$

Therefore, combining (3.8), (3.9) and (3.10), we get

$$\log\|\sigma\|^2_Q = -a(n)r(E)\mu(\mathrm{Sing}\,X_0)\log|t|^2 + \varphi \quad (\varphi \in C^\infty(D)). \tag{3.9}$$ ∎

4. Smoothing of IHS and spectrum of Laplacians

As before, let $\pi : X^{n+1} \to D$ be a smoothing of IHS, g_X a Kähler metric of X, $g_t := g_{X|X_t}$ the induced metric on X_t, and $\Box^{p,q}_t$ (resp. Δ^r_t) the Laplacian acting on (p,q)-forms (resp. r-forms) on (X_t, g_t). Let

$$\sigma(\Box^{p,q}_t) = \{\lambda^{p,q}_1(t) \leq \lambda^{p,q}_2(t) \leq \cdots\},\ \sigma(\Delta^r_t) = \{\lambda^r_1(t) \leq \lambda^r_2(t) \leq \cdots\}$$

be the spectrum of $\Box_t^{p,q}$ and Δ_t^r respectively. When $t = 0$, we consider $\Box_0^{p,q}$ (resp. Δ_0^r) to be the Friedrichs extension of the Laplacian acting on forms on $X_{0,\mathrm{reg}}$. Put $H_{(2)}^{p,q}(X_0) := \operatorname{Ker}\Box_0^{p,q}$ and $H_{(2)}^r(X_0) := \operatorname{Ker}\Delta_0^r$.

THEOREM 4.1. — *Under the above situation,*

1) $\sigma(\Delta_0^r)$ *is discrete for* $r < n$,

2) $\lim_{t\to 0}\sigma(\Delta_t^r) = \sigma(\Delta_0^r) \quad (r < n)$,

3) $\lim_{t\to 0}\sigma(\Box_t^{p,q}) = \sigma(\Box_0^{p,q}) \quad (p+q < n)$,

4) $\dim H_{(2)}^r(X_0) = \dim H^r(X_t) \quad (r < n-1)$,

5) $\dim H_{(2)}^{p,q}(X_0) = \dim H^{p,q}(X_t) \quad (r < n-1)$.

COROLLARY 4.1. — *For p, q with $p+q < n-1$, there exists $C > 0$ such that*

$$\lambda^{p,q}_{h^{p,q}+1}(t) \geq C \quad \text{for} \quad t \in D$$

where $h^{p,q} = \dim H^{p,q}(X_t)$.

LEMMA 4.1 ([O 1]). — *Set* $\delta(x) := \operatorname{dis}(x, \operatorname{Sing} X_0)$ *for* $x \in X$. *Then for any r-forms* $(r < n)\psi$ *of* X_t,

$$\left\|\frac{1}{\delta(\log\frac{1}{\delta} + A)}\psi\right\|_{2,t} \leq C\big(\|d\psi\|_{2,t} + \|\delta_t\psi\|_{2,t} + \|\psi\|_{2,t}\big)$$

where A, and $C > 0$ are constants independent of t.

Proof. — For simplicity, we prove the case when $g_X = \partial\bar\partial\|z\|^2$ on a neighborhood U of $\operatorname{Sing} X_0$, where $z = (z_0, \ldots, z_n)$ is a local coordinates centered at each $p \in \operatorname{Sing} X_0$. Put $U_p(r) := \{z \in U_p; \|z\| < r\}$. We may assume $U_p \cap U_{p'} = \emptyset$ if $p \neq p'$. As $\delta(z) = \|z\|$ for $z \in U_p$, by the argument of cut off function, it is sufficient to show the following inequality:

$$\left\|\frac{1}{\|z\|^2\log\frac{1}{\|z\|^2}}\psi\right\|_{2,t} \leq C\big(\|d\psi\|_{2,t} + \|\delta_t\psi\|_{2,t}\big) \tag{4.1}$$

for $\psi \in A^r(X_t \cap U_p(\frac{1}{2})) \quad (r < n)$.

Using the Jacobi identity (cf. [O-T], Proposition 1.5, for $F = \log(-\log\|z\|^2)$), we have

$$(\sqrt{-1}[\partial\bar\partial\log(-\log\delta^2), \Lambda]\psi, \psi)_t \leq 4\left\|\frac{1}{\delta\log\delta}\psi\right\|_{2,t}\big(\|d\psi\|_{2,t} + \|\delta_t\psi\|_{2,t}\big) \tag{4.2}$$

where Λ stands for the adjoint of exterior multiplication of $\sqrt{-1}\partial\bar\partial\|z\|^2$.

The eigenvalues of $A := \partial\bar\partial\log(-\log\|z\|^2)$ with respect to $\partial\bar\partial\|z\|^2$ are given by

$$\lambda_0 = \lambda = \frac{-1}{\delta^2(\log\delta)^2},\ \lambda_1 = \cdots = \lambda_n = \mu = \frac{1}{\delta^2\log\delta}. \tag{4.3}$$

Let $V = T'_Z X_t \subset T'_Z \mathbb{C}^{n+1}$ be an n-dimensional subspace, and $A_V, A_{V^\perp}$ the restriction of A to $V^\perp$ respectively. Clearly

$$TrA = TrA_V + TrA_{V^\perp} \tag{4.4}$$

by the identification of quadratic forms with the Hermitian matrix corresponding to them. By the above formula, we get

$$-TrA_V \geq (n-1)(-\mu) + (-\lambda). \tag{4.5}$$

Let $\lambda_1(t) \leq \cdots \leq \lambda_n(t)$ be the eigenvalues of A_V. Then,

$$\begin{aligned} \Gamma_{p,q}(V) := \inf_{\substack{|I|=p \\ |J|=q}} \Big\{ \sum_{i\in I} \lambda_i(t) + \sum_{j\in J} \lambda_j(t) - \sum_{k=1}^{n} \lambda_k(t) \Big\} \\ \geq (n-1)(-\mu) + (-\lambda) + (p+q) \inf_i \lambda_i(t). \end{aligned} \tag{4.6}$$

If $p + q \leq n - 1$, by (4.3) and (4.6), we obtain

$$\Gamma_{p,q}(V) \geq \frac{1}{\delta^2 (\log \delta)^2}. \tag{4.7}$$

Combining (4.2) and (4.7), we have the desired estimate, since (cf. [De])

$$(\sqrt{1}[\partial\bar\partial F, \Lambda]\psi, \psi) \geq (\sqrt{-1}\Gamma_{p,q}(V)\psi, \psi). \tag{4.8}$$

■

Lemma 4.2 (The Rellich lemma). — *Let $L^r_{1,2}(X_0)$ be the completion of $A^r_0(X_{0,\mathrm{reg}})$ by the norm $\|\varphi\|^2_{1,2} := \|\varphi\|^2_2 + \|d\varphi\|^2_2 + \|\delta\varphi\|^2_2$. Then, the inclusion*

$$L^r_{1,2}(X_0) \hookrightarrow L^r_2(X_0)$$

is compact.

Proof. — Put

$$(X_0)_\varepsilon := X_0 - (\mathrm{Sing}\, X_0)_\varepsilon, \quad (\mathrm{Sing}\, X_0)_\varepsilon := \bigcup_{p\in \mathrm{Sing}\, X_0} U_p(\varepsilon). \tag{4.9}$$

Let $\{f_n\}_{n\in\mathbb{N}} \subset L^r_{1.2}(X_0)$ be a bounded sequence:

$$\|f_n\|_{1,2} \leq M < +\infty. \tag{4.10}$$

In view of [Y 2], Prop. 5.1, it is sufficient to show

$$\|f_n\|_{L^2((\mathrm{Sing}\, X_0)_\varepsilon)} \leq C\varepsilon \log \varepsilon^{-1}. \tag{4.11}$$

But this follows from Lemma 4.1. ■

Corollary 4.2. — *$\sigma(\Delta^r_0)$ and $\sigma(\Box^{p,q}_0)(p + q = r < n)$ consists of discrete eigenvalues with finite multiplicities.*

The following is a slight generalization of [Y 2], Theorem 1.1.

Lemma 4.3. — *For every small $\varepsilon \ll 1$, there exists $0 < \gamma(\varepsilon) < \varepsilon$ and a family of into-diffeomorphisms for $|t| < \gamma(\varepsilon)$:*

$$f_{\varepsilon,t} : X_{0,\varepsilon} \hookrightarrow X_t$$

by which the following conditions are satisfied:

1) On $X_{0,\varepsilon}$ $\quad |f_{\varepsilon,t}^* g_t - g_0| \leq C|t| g_0$.

2) $X_t - f_{\varepsilon,t}(X_{0,\varepsilon}) \subset (\operatorname{Sing} X_0)_{2\varepsilon} \cap X_t$

3) For $\varphi \in A_0^r(X_{0,\varepsilon})$, $\quad \frac{1}{2}\|\varphi\|_{1,2,X_0}^2 \leq \|(f_{\varepsilon,t}^{-1})^* \varphi\|_{1,2X_t}^2 \leq 2\|\varphi\|_{1,2,X_0}^2$.

Let $\{\rho_\varepsilon\}(\varepsilon \ll 1)$ be a family of cut-off functions on X_0:

$$\rho_\varepsilon(z) := \begin{cases} 1 \quad (z \in X_{0,\sqrt{\varepsilon}}) \\ \dfrac{-2}{\log \varepsilon} \displaystyle\int_\varepsilon^{\|z\|} \frac{dr}{r} \quad (z \in U_p(\sqrt{\varepsilon}), \|z\| \geq \varepsilon) \\ 0 \quad (z \in U_p(\sqrt{\varepsilon}), \|z\| \leq \varepsilon). \end{cases} \tag{4.12}$$

For any r-form $\varphi \in A^r(X_t)$, put

$$\varphi_{\varepsilon,t} := \rho_\varepsilon f_{\varepsilon,t}^* \varphi \in C_0^r(X_{0,\varepsilon}) \cap L_{1,2}^r(X_0) \tag{4.13}$$

if $|t| < \gamma(\varepsilon)$.

Let $\{t_i\}$, $t_i \to 0$ be a given sequence. For simplicity, we use X_i instead of X_{t_i}. Let $\{\varphi_k(i)\}_{k\in\mathbb{N}}$ be a complete orthonormal system of $L_2^r(X_i)$ which consists of eigenfunction of Δ_i^r:

$$\Delta_i^r \varphi_k(i) = \lambda_k^r(i)\varphi_k(i), \quad (\varphi_k(i), \varphi_\ell(i))_i = \delta_{k\ell} \tag{4.14}$$

where $(\cdot,\cdot)_i$ stands for the inner product of $L_2^r(X_i)$. In the sequel of the proof, we use the same notations as [Y 2], § 5.

Proposition 4.1. — *For every $N \geq 0$, there is a subsequence $\{i(\nu)\}$ such that the followings hold for $0 \leq k \leq N$:*

1) $\lim_{\nu\to\infty} \lambda_k^r(i(\nu)) = \lambda_k^r$

2) $|t_{i(\nu)}| \leq \gamma(\frac{1}{\nu})$ *and*

$$s - \lim_{\nu\to\infty} \psi_k(i(\nu)) = \varphi_k \quad in \quad L_2^r(X_0)$$
$$w - \lim_{\nu\to\infty} \psi_k(i(\nu)) = \varphi_k \quad in \quad L_{1,2}^r(X_0)$$

where

$$\psi_k(i(\nu)) := \rho_{\frac{1}{\nu}} f_{\frac{1}{\nu},t_{i(\nu)}}^* \varphi_k(i(\nu)).$$

For the proof, we need the following:

LEMMA 4.4. — *Suppose Proposition 4.1 is true for N. Then,*

$$\lim_{i\to\infty} \sup \lambda_{N+1}(i) \leq \lambda_{N+1}.$$

Proof. — In view of [Y 2], Lemma 5.2, it is enough to show that if $\{\varphi_{n+1,i}\}$ approximates φ_{N+1}:

$$\|\varphi_{N+1,i} - \varphi_{N+1}\|_{L_{1,2}} \leq \frac{1}{i}, \quad \operatorname{supp} \varphi_{N+1,i} \subset X_{0,\frac{1}{2}}, \tag{4.15}$$

putting

$$\chi_{N+1}(i) := (f_{\frac{1}{2},t_i}^{-1})^* \varphi_{N+1,i} \quad (|t_i| < \gamma(\frac{1}{i})), \tag{4.16}$$

then, for $k \leq N$,

$$\lim_{i\to\infty} (\chi_{N+1}(i), \varphi_k(i))_i = 0. \tag{4.17}$$

By Lemma 4.3,

$$\begin{aligned} |(\chi_{N+1}(i), \varphi_k(i))_i - (\varphi_{N+1,i}, \psi_k(i))_0| &\leq C|t_i|\, \|\varphi_{N+1,i}\|_2 \|\psi_k(i)\|_2 \\ &\leq C|t_i| \to 0. \end{aligned} \tag{4.18}$$

By Prop. 4.1 for N,

$$|(\varphi_{N+1,i}, \psi_k(i))| \leq |(\varphi_{N+1,i}, \varphi_k)| + \|\varphi_k - \psi_k(i)\| \to 0 \tag{4.19}$$

which combined with (4.18), yields (4.17). ■

Proof of Proposition 4.1. — We prove by induction. Assume for $k \leq N$,

$$\lim_{i\to\infty} \lambda_k^r(i) = \lambda_k^r, \quad \lim_{i\to\infty} \psi_k(i) = \varphi_k \tag{4.20}$$

strongly in L_2^r and weakly in $L_{1,2}^r$, where

$$\psi_k(i) := \rho_{\frac{2}{i}} f_{\frac{1}{i},t_i}^* \varphi_k(i) \quad (|t_i| < V_i). \tag{4.21}$$

In view of [Y 2], proof of Proposition 5.2, it is sufficient to show that for some subsequence $\{i(\nu)\}$, there exists an eigenfunction of Δ_0^r, say $\psi(\neq 0)$, such that $\psi_{N+1}(i(\nu))$ converges to ψ strongly in L_2^r and weakly in $L_{1,2}^r$.

By Lemma 4.3 and definition of ρ_ε, we get

$$\begin{aligned} \|\psi_{N+1}(i)\|_{1,2}^2 &\leq 2\Big\{\|d\rho_{\frac{2}{i}}\varphi_{N+1}(i)\|_{2,i}^2 + \|\delta_i\rho_{\frac{2}{i}}\varphi_{N+1}(i)\|_{2,i}^2 + \|\rho_{\frac{2}{i}}\varphi_{N+1}(i)\|_{2,i}^2\Big\} \\ &\leq 2\Big\{\|\varphi_{N+1}(i)\|_{1,2,i}^2 + \Big\|\frac{1}{\delta\log\delta}\varphi_{N+1}(i)\Big\|_{2,i}^2\Big\} \\ &\leq C(1 + \lambda_{N+1}(i)), \end{aligned} \tag{4.22}$$

which, combined with Lemma 4.4, yields

$$\|\psi_{N+1}(i)\|_{1,2}^2 \leq C(1 + 2\lambda_{N+1}). \tag{4.23}$$

By the Rellich lemma, one can find a subsequence $\{i(\nu)\}$ and $\psi \in L^r_{1,2}(X_0)$ such that

$$\lim_{\nu\to\infty} \psi_{N+1}(i(\nu)) = \psi \tag{4.24}$$

strongly in L^r_2 and weakly in $L^r_{1,2}$. We may also assume that $\lambda_{N+1}(i(\nu))$ converges to $\lambda\in\mathbb{R}_+$.

By Lemma 4.3,

$$\|\psi_{N+1}(i)\|_2^2 \geq \frac{1}{2}\left\|\rho_{\frac{2}{i}}\varphi_{N+1}(i)\right\|^2_{2,i} \geq \frac{1}{2}\|\varphi_{N+1}(i)\|^2_{L^2(X_{t_i}-K_i)} \tag{4.25}$$

where $K_i := (\mathrm{Sing}\, X_0)_{\sqrt{\frac{2}{i}}} \cap X_{t_i}$. Since

$$\begin{aligned}\|\varphi_{N+1}(i)\|^2_{L^2(K_i)} &\leq \frac{1}{i}\log i \left\|\frac{1}{\delta\log\delta^{-1}}\varphi_{N+1}(i)\right\|^2_{L^2(K_i)}\\ &\leq C\frac{1}{i}\log i(1+\lambda_N),\end{aligned} \tag{4.26}$$

combined with (4.25), we obtain

$$\begin{aligned}\|\psi_{N+1}(i)\|_2^2 &\geq \frac{1}{2}\Big\{1 - C(1+\lambda_{N+1})\frac{1}{i}(\log i + 1)\Big\}\\ &\geq \frac{1}{4}.\end{aligned} \tag{4.27}$$

This shows $\psi \neq 0$. Since $\Delta_i\varphi_{N+1}(i) = \lambda_{N+1}(i)\varphi_{N+1}(i)$, for any $\chi \in A^r_0(X_0, \mathrm{reg})$,

$$(d\psi, d\chi) + (\delta\psi, \delta\chi) = \lambda(\psi, \chi). \tag{4.28}$$

By the same manner as [O 2], for any $f \in L^r_{1,2}(X_0)$,

$$\begin{aligned}(d\psi, df) + (\delta\psi, \delta f) &= \lim_{\varepsilon\to 0}\big\{(d\psi, d(\rho_\varepsilon f)) + (\delta\psi, \delta(\rho_\varepsilon f))\big\}\\ &= \lim_{\varepsilon\to 0}\lambda(\psi, \rho_\varepsilon f)\\ &= \lambda(\psi, f).\end{aligned} \tag{4.29}$$

This implies $\Delta_0\psi = \lambda\psi$ and proves the proposition. ∎

Proposition 4.2 (Ohsawa [O 1]).

$$H^r_{(2)}(X_0) \cong H^r(X_{0,\mathrm{reg}}) \quad (r < n).$$

In particular,

$$H^r_{(2)}(X_0) \times H_0^{2n-r}(X_{0,\mathrm{reg}}) \longrightarrow \mathbb{C}$$

is a perfect pairing.

Proof of Theorem 4.1. — 1) follows from Corollary 4.2, 2) from Proposition 4.1, since $\{t_i\}$, $t_i \to 0$ is an arbitrary sequence, and 3) from 2) since (p, q)-forms of X_t converges (p, q)-forms of X_0.

Proof of 4). — Put $b_r := \dim H^r(X_t) \quad (t \neq 0)$. By 2), $\dim H^r_{(2)}(X_0) \geq b_r$. Suppose that strict inequality hold in this inequality. Then, by Proposition 4.1, one can find a sequence $\{t_i\}$, $\{\varphi_{b_r+1}(t_i)\}$ and $\psi \in L^r_{1,2}(X_0)$ such that

1) $\Delta^r_0 \psi = 0, \quad \|\psi\|_2 = 1$.

2) $|t_i| < \gamma\left(\frac{2}{i}\right)$ and $\psi_{b_r+1}(i) := \rho_{\frac{2}{i}} f^*_{\frac{1}{2},t_i} \varphi_{b_r+1}(t_i)$ converges to ψ weakly in $L_{1,2}$ and strongly in L_2.

3) $\Delta^r_{t_i} \varphi_{b_r+1}(t_i) = \lambda_{b_r+1}(t_i) \varphi_{b_r+1}(t_i), \quad \|\varphi_{b_r+1}(t_i)\|_{2,t_i} = 1$.

4) Either $\varphi_{b_r+1}(t_i) \in \operatorname{Im} d$ or $\operatorname{Im} \delta$.

Suppose there are infinite t_i's with $\varphi_{b_r+1}(t_i) \in \operatorname{Im} d$. Then, we may assume $\varphi_{b_r+1}(t_i) \in \operatorname{Im} d$ for every i by choosing a subsequence. Choose $\tau \in H^r_0(X_{0,\mathrm{reg}})$ arbitrary. By Proposition 4.1, we get

$$\begin{aligned} \int_{X_{0,\mathrm{reg}}} \psi \wedge \tau &= \lim_{i \to \infty} \int_{X_{0,\mathrm{reg}}} \psi_{b_r+1}(i) \wedge \tau \\ &= \lim_{i \to \infty} \int_{X_{t_i}} \varphi_{b_r+1}(t_i) \wedge (f^{-1}_{\frac{1}{i},t_i})^* \tau \\ &= 0, \end{aligned} \tag{4.30}$$

since $\varphi_{b_r+1}(t_i)$ is d-exact and τ is d-closed which, combined with Proposition 4.2, implies $\psi = 0$. Contradiction.

Suppose there are only finitely many t_i's with $\varphi_{b_r+1}(t_i) \in \operatorname{Im} d$. Then, consider $\left\{ d\varphi_{b_r+1}(t_i) / \sqrt{\lambda_{b_r+1}(t_i)} \right\}$ instead. As, $r \leq n-2$, we can apply Proposition 4.1 to above sequence and obtain contradiction in the same way. ∎

Proof of 5). — By 3),

$$\dim H^{p,q}_{(2)}(X_0) \geq \dim H^{p,q}(X_t). \tag{4.31}$$

As

$$H^r_{(2)}(X_0) = \bigoplus_{p+q=r} H^{p,q}_{(2)}(X_0) \quad (r < n), \tag{4.32}$$

5) follows from 4) and (4.31). ∎

COROLLARY 4.3. — *Let* $\pi : \widetilde{X}_0 \to X_0$ *be a desingularization. Then,* $\dim H^p(X_t, \mathcal{O}_{X_t}) = \dim H^p(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0})$ *for* $p < n-1$.

Proof. — By Theorem 4.1, we get

$$\dim H^p(X_t, \mathcal{O}_{X_t}) = \dim H^{p,0}_{(2)}(X_0). \tag{4.33}$$

Since an element of $H^{p,0}_{(2)}(X_0)$ is a holomorphic p-form on $X_{0,\mathrm{reg}}$ it extends to a holomorphic p-form on $\widetilde{X}_0$ if $p < n-1$. Therefore,

$$\dim H^{p,0}_{(2)}(X_0) = \dim H^p(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0}). \tag{4.34}$$
■

THEOREM 4.2. — *Let $\pi : X^{n+1} \to D$ be a smoothing of IHS. Suppose* Sing X_0 *is rational. Then,*

$$\dim H^q(X_t, \mathcal{O}_{X_t}) = \dim H^q(X_0, \mathcal{O}_{X_0})$$

for any $q \geq 0$.

Proof. — When $q < n-1$, the assertion follows from Corollary 4.3, since $\dim H^q(X_0, \mathcal{O}_{X_0}) = \dim H^q(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0})$ if Sing X_0 is rational.

Case $q = n$. Since Sing X_0 is rational,

$$\dim H^n(X_0, \mathcal{O}_{X_0}) = \dim H^0(X_0, K_X|_{X_0}), \tag{4.35}$$

it is sufficient to show that $\dim H^0(X_t, K_X|_{X_t})$ is a constant function on D. By Takegoshi's theorem (cf. [T], Theorem 6.11), we know

$$r : \Gamma(X, K_X) \longrightarrow \Gamma(X_t, K_X|_{X_t}) \longrightarrow 0 \tag{4.36}$$

is surjective for any $t \in D$. By the theorem of cohomology and base change (cf. [H], Theorem 12.11), we get

$$\dim H^0(X_t, K_X|_{X_t}) = \dim \mathcal{O}_{D_t/m_t} \pi_* K_X \otimes \mathcal{O}_{D_t/m_t}. \tag{4.37}$$

Since $\pi_* K_X$ is torsion-free and $\dim D = 1$, $\pi_* K_X$ is locally free. Therefore, the right hand side of (4.37) does not depend on $t \in D$. ■

Case $q = n-1$. As $\pi : X \to D$ is flat, $\Sigma_q (-1)^q \dim H^q(X_t, \mathcal{O}_{X_t})$ is constant on D. Therefore $\dim H^{n-1}(X_t, \mathcal{O}_{X_t})$ is constant because $\dim H^q(X_t, \mathcal{O}_{X_t})$ is constant for $q \neq n-1$.

PROPOSITION 4.3. — *Let $\pi : X^{n+1} \to D$ be a smoothing of IHS, g_X a Kähler metric of X, $g_{X/D}$ the induced metric on TX/D. Suppose* Sing X_0 *is rational. Then,*

1) If $\mu^{0,q}(t)$ stands for the first nonzero eigenvalue of $\Box^{0,q}_t$, then there exists a constant $C > 0$ such that $\mu^{0,q}(t) \geq C > 0$ for any $t \in D^*$ and $q \geq 0$.

2) If $\sigma \in \Gamma(D(\varepsilon), R^q \pi_* \mathcal{O}_C)$ with $\sigma(0) \neq 0$, then there exist $0 < C_1 \leq C_2 < +\infty$ and $k \geq 0$ such that

$$0 < C_1 (\log |t|)^{-k} \leq \|\sigma(t)\|_2 \leq C_2 < +\infty \quad \textit{for any} \quad t \in D^*.$$

If (π, X, b) is A_1-singular, then we may choose $k = 0$.

Proof of 1). — For $q < n-1$, the assertion follows from Corollary 4.1. As $\mu^{0,n-1}(t) \leq \mu^{0,n}(t)$, it is sufficient to prove the assertion for $q = n-1$. By Theorems 4.1 and 4.2, it is enough to show $\dim H^{0,n-1}_{(2)}(X_0) = \dim H^{n-1}(X_0, \mathcal{O}_{X_0})$. Since $\overline{H^{0,n-1}_{(2)}(X_0)} \cong H^{n-1,0}(\widetilde{X}_0)$ by the rationality of $\operatorname{Sing} X_0$ (cf. [vS-S], Definition (1.5)), and $H^{n-1}(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0}) \cong H^{n-1}(X_0, \mathcal{O}_{X_0})$ by the rationality again, using the Hodge symmetry, we have

$$\dim H^{0,n-1}_{(2)}(X_0) = \dim H^{n-1}(X_0, \mathcal{O}_{X_0}). \qquad \blacksquare$$

Proof of 2). — Let $\varphi_\sigma \in H^{0,q}(\pi^{-1}(D(\varepsilon)))$ be a representative of σ. By the Serre duality $(R^q\pi_*\mathcal{O}_X)^* \cong R^{n-q}\pi_*\omega_{X/D}$ there exists $\psi \in H^{n+1,n-q}(\pi^{-1}(D(\varepsilon')))$ $(\varepsilon' \leq \varepsilon)$ such that

$$\pi_*(\varphi_\sigma \wedge \psi) = \int_{X_t} \varphi_\sigma \wedge \psi = dt \quad (|t| < \varepsilon'). \tag{4.38}$$

If H_t stands for the harmonic projection, then

$$\int_{X_t} H_t\varphi_\sigma \wedge \psi = dt \quad (|t| < \varepsilon). \tag{4.39}$$

By Barlet's theorem ([Ba]), there exists an asymptotic expansion

$$\int_{X_t} |\psi/dt|^2 = a_0|t|^{-\alpha}(\log|t|)^k(1 + O(|t|^\beta)) \tag{4.40}$$

for some $\alpha \geq 0, \beta > 0$ and $k \in \mathbb{Z}_{\geq 0}$.

Consider the Mellin transform:

$$\int_D |t|^s dt \wedge d\bar{t} \int_{X_t \cap U} |\psi/dt|^2 = \int_{\pi^{-1}(D)\cap U} |f|^s \chi \, dv \tag{4.41}$$

where U is a neighborhood of $\operatorname{Sing} X_0$, f a defining equation, χ an appropriate smooth function, and dv a Lebesgue measure. As is well-known (cf. [Ba], [Ko],§ 10 and references therein), we find that $\alpha - 1$ is the largest root of Bernstein-Sato polynomial $b_f(s)$ of f, and $k+1$ its multiplicity. Here we use the same definition of b-function as [Ko], § 10. By [Ko], 10.8, Remark, we have $\alpha = 0$ if $\operatorname{Sing} X_0$ is rational, which, combined with (4.39), (4.40) and Cauchy-Schwartz inequality, yields the desired lower bound. When (π, X, D) is A_1-singular, as the b-function of A_1-singularity is $b(s) = (s+1)\left(s + \frac{n+1}{2}\right)$, we obtain $k = 0$ for $n > 1$, and get uniform lower bound. Since

$$\begin{aligned} \|H_t\varphi_\sigma\|^2_{2,t} &\leq \|\varphi_\sigma\|^2_{2,t} \\ &\leq (\sqrt{-1})^q \int_{X_t} \varphi_\sigma \wedge \overline{\varphi}_\sigma \wedge \omega_X^{n-q} \\ &\leq |\zeta_\sigma|_\infty \operatorname{vol}(X_t), \end{aligned} \tag{4.42}$$

we also obtain the uniform upper bound. $\blacksquare$

Let us consider an analogue of Theorem 4.1 for vector bundle valued Laplaciens. Since Lemmas 4.1, 4.2, 4.3 also hold if the fiber metric of the vector bundle is flat on a neighborhood of Sing X_0, we have the following theorem.

Theorem 4.3. — *Let $\pi : X \to D$ be a smoothing of IHS, g_X a Kähler metric of X, $g_{X/D}$ the induced metric on TX/D and (E, h) a holomorphic Hermitian vector bundle. Let $\Box_t^{0,q}$ be the $\overline{\partial}$-Laplacian acting on $A^{0,q}(E_t)$ and $\sigma(\Box_t^{0,q}) = \{\lambda_1^{0,q}(t) \le \lambda_2^{0,q}(t) \le \cdots\}$ its spectrum, where $\Box_0^{0,q}$ is considered to be the Friedrichs extension.*

Suppose $R_n \equiv 0$ on a neighborhood of Sing X_0. Then for $q < n$

$$\lim_{t\to 0} \sigma(\Box_t^{0,q}) = \sigma(\Box_0^{0,q}).$$

In case of A_1-singular family, we have the following:

Theorem 4.4. — *Let $\pi : X \to D$ be a A_1-singular family, g_X a Kähler metric of X, $g_{X/D}$ the induced metric on TX/D. Let (E, h) be a holomorphic Hermitian vector bundle such that*

1) $R_n \equiv 0$ *on a neighborhood of* Sing X_0,

2) $\dim H^q(X_t, \mathcal{O}_{X_t}(E_t)) = 0$ *for $q < n$ and $t \in D$.*

Then,

1) $H^q_{(2)}(X_0, E_0) = 0$ *for $q < n$,*

2) $\dim H^n_{(2)}(X_0, E_0) = \dim H^n(X_0, \mathcal{O}_{X_0}(E_0))$,

where $H^{0,q}_{(2)}(X_0, E_0) = \{f \in L^{0,q}_{1,2}(E); \overline{\partial} f = 0, \overline{\partial}^* f = 0\}$.

Proof. — For simplicity we prove the case Sing $X_0 = \{0\}$. Let $p : \widetilde{X_0} \to X_0$ be the natural resolution and $Y := p^{-1}(0)$ the exceptional set. Since $(\mathcal{O}_{X_0,0}, 0)$ is an A_1-singularity, Y is a hyperquadric in $\mathbb{P}^n$. Let ζ be a local coordinate transversal to Y, *i.e.*, , $Y \cap U = \{\zeta = 0\}$. Then,

$$p^* g_{X_0} \sim |d\zeta|^2 + |\zeta|^2 g_Y, \quad g_{\widetilde{X_0}} \sim |d\zeta|^2 + g_Y \tag{4.43}$$

where $g_{\widetilde{X_0}}$ is a fixed Kähler metric of $\widetilde{X_0}$ and g_Y a Kähler metric of Y.

1) $(q < n-1)$. Suppose there exists $\varphi \in H^{0,q}_{(2)}(X_0, E_0)$, $\varphi \neq 0$. Put $\tilde{\varphi} := p^*\varphi$. Since

$$c_1|\zeta|^{2(n-q)}\omega_{\widetilde{X_0}}^{n-q} \le p^*\omega_{X_0}^{n-q} \le c_2|\zeta|^{2(n-q-1)}\omega_{\widetilde{X_0}}^{n-q}, \tag{4.44}$$

we get

$$\int_U |\zeta|^{2(n-q)}|\tilde{\varphi}|^2 dv_{\widetilde{X}_0} = \Big|\int_U |\zeta|^{2(n-q)}\tilde{\varphi}\wedge\overline{\tilde{\varphi}}\wedge\omega_{\widetilde{X}_0}^{n-q}\Big| \le \int_U |\varphi|^2 dv_{X_0} < \infty \tag{4.45}$$

which implies $\tilde{\varphi} \in L^{0,q}_{(2)}(\widetilde{X}_0, p^*E\otimes[Y]^{n-q})$ with respect to some Hermitian metric of $p^*E\otimes[Y]^{n-q}$. We show $\overline{\partial}\tilde{\varphi} = 0$ in the sense of current as an element of $L^{0,q}_{(2)}(\widetilde{X}_0, p^*E\otimes[Y]^{n-q})$. Since the problem is local, we may assume $\tilde{\varphi} \in L^{0,q}_{(2)}(D^n)$ and $\overline{\partial}\tilde{\varphi} = 0$ on $D^n - \Delta$, $\Delta = \{0\}\times D^{n-1}$.

Let $\chi \in A_0^{n,n-q-1}(D^n)$. By definition, we get

$$\langle\overline{\partial}\tilde{\varphi}, \chi\rangle := (-1)^{q+1}\int_{D^n}\tilde{\varphi}\wedge\overline{\partial}\chi. \tag{4.46}$$

Let ρ_ε be the same as in Lemma 4.3 and put $\tilde{\rho}_\varepsilon := p^*\rho_\varepsilon$. Then, by Stokes's theorem, we obtain

$$\begin{aligned}\langle\overline{\partial}\tilde{\varphi}, \chi\rangle &= (-1)^{q+1}\lim_{\varepsilon\to 0}\int_{D^n}\tilde{\varphi}\wedge\tilde{\rho}_\varepsilon\overline{\partial}\chi \\ &= (-1)^{q+1}\lim_{\varepsilon\to 0}\Big\{\int_{D^n}\tilde{\varphi}\wedge\overline{\partial}(\rho_\varepsilon\chi) - \int_{D^n}\tilde{\varphi}\wedge\overline{\partial}\rho_\varepsilon\wedge\chi\Big\} \\ &= (-1)^{q}\lim_{\varepsilon\to 0}\int_{D^n}\tilde{\varphi}\wedge\overline{\partial}\rho_\varepsilon\wedge\chi,\end{aligned} \tag{4.47}$$

and

$$\Big|\int_{D^n}\tilde{\varphi}\wedge\overline{\partial}\rho_\varepsilon\wedge\chi\Big| \le c\|\tilde{\varphi}\|_2\|\chi\|_\infty\Big\{\int_{\varepsilon\le|\zeta|\le\sqrt{\varepsilon}}\frac{\sqrt{-1}d\zeta\wedge d\overline{\zeta}}{|\log\varepsilon|^2|\zeta|^2}\Big\}^{\frac{1}{2}} \le \frac{c}{\sqrt{|\log\varepsilon|}}\|\tilde{\varphi}\|_2\|\chi\|_\infty, \tag{4.48}$$

we conclude $\overline{\partial}_{\max}\overline{\varphi} = 0$.

Suppose $H^q(\widetilde{X}_0, p^*E\otimes[Y]^{n-q}) = 0$.

Then, by Hodge theory, one can find g such that

$$\tilde{\varphi} = \overline{\partial}_{\max}g, \quad g \in L^{0,q-1}_{1,2}(\widetilde{X}_0, p^*E\otimes[Y]^{n-q}). \tag{4.49}$$

Noting $p^*E\big|_Y \cong \mathbb{C}^r$ $(r = \operatorname{rank} E)$, consider the following exact sequence of sheaves on $\widetilde{X}_0$ for every $k \ge 1$:

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}_0}(p^*E\otimes[Y]^{k-1}) \longrightarrow \mathcal{O}_{\widetilde{X}_0}(p^*E\otimes[Y]^{k}) \longrightarrow \mathcal{O}_Y([Y]^k\big|_Y\otimes\mathbb{C}^r) \longrightarrow 0. \tag{4.50}$$

Since

$$[Y]\big|_Y = -2H, \quad H = \mathcal{O}(1), \tag{4.51}$$

we know $H^q(Y, \mathcal{O}_Y([Y]^k\big|_Y)^r) = 0$ for $q < n-1$ which, combined with the long exact sequence of cohomology yields

$$H^q(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0}(p^*E \otimes [Y]^{k-1})) \cong H^q(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0}(p^*E \otimes [Y]^k)) \tag{4.52}$$

for any $q < n-1$ and $k \geq 1$. By the hypothesis 2), we obtain $H^q(\widetilde{X}_0, \mathcal{O}_{\widetilde{X}_0}(p^*E \otimes [Y]^k)) = 0$ for any $k \geq 1$ and $q < n-1$. Therefore we can find such g as (4.49). Since $[Y]$ is trivial on $p^{-1}(X_{0,\mathrm{reg}})$, $(p^{-1})^*g$ is considered to be a E-valued $(0, q-1)$-form on $X_{0,\mathrm{reg}}$. Then, for a neighborhood V of 0, we get

$$\begin{aligned} \|(p^{-1})^*g\|_{2,V}^2 &= \Big|\int_{p^{-1}(V)} g \wedge \overline{g} \wedge p^*\omega_{X_0}^{n-q+1}\Big| \\ &\leq c\Big|\int_{p^{-1}(V)} |\zeta|^{2(n-q)} g \wedge \overline{g} \wedge \omega_{\widetilde{X}_0}^{n-q+1}\Big| \\ &\leq c\|g\|^2_{L^2(\widetilde{X}_0, p^*E \otimes [Y]^{n-q})}, \end{aligned} \tag{4.53}$$

which implies, by putting $\psi := (p^{-1})^*g$, we obtain

$$\varphi = \overline{\partial}_{\max}\psi, \quad \psi \in L_2^{0,q-1}(X_0, E_0). \tag{4.54}$$

Therefore by Lemma 4.1, we have

$$\begin{aligned} \|\varphi\|_2^2 &= \lim_{\varepsilon \to 0} |(\overline{\partial}_{\max}\psi, \rho_\varepsilon\varphi)| \\ &= \lim_{\varepsilon \to 0} |(\psi, - * \partial\rho_\varepsilon * \varphi)| \\ &\leq c \lim_{\varepsilon \to 0} \|\psi\|_2 \|(\delta \log \delta)^{-1}\varphi\|_{2,(\mathrm{Sing}\, X_0)_\varepsilon} \\ &= 0. \end{aligned} \tag{4.55}$$

■

2) Let $f \in H^{0,n}_{(2)}(X_0, E_0)$. Since the metric of E_0 is Euclidean on a neighborhood of $\mathrm{Sing}\, X_0$, $\overline{\partial}^* f = 0$ implies $\partial f = 0$. Therefore, $\overline{f}$ is holomorphic on a neighborhood of $\mathrm{Sing}\, X_0$, which, combined with the rationality, implies an isomorphism

$$H^{0,n}_{(2)}(X_0, E_0) \cong H^{0,n}(\widetilde{X}_0, p^*E_0). \tag{4.56}$$

By the rationality again, we have

$$\dim H^n_{(2)}(X_0, E_0) = \dim H^n(X_0, \mathcal{O}(E_0)). \tag{4.57}$$

■

1) $(q = n-1)$. Since $\mathrm{Sing}\, X_0$ is A_1, Hardy's inequality holds uniformly in t (cf. Appendix): for any $f \in A^{0,n}(X_t, E_t)$,

$$\Big\|\frac{1}{\delta^2} f\Big\|_{2,t} \leq C\big(\|\overline{\partial} f\|_{2,t} + \|\overline{\partial}^* f\|_{2,t} + \|f\|_{2,t}\big) \tag{4.58}$$

where C is independent of t. Therefore, Proposition 4.1 holds also for $(0,n)$-forms. Suppose $H^{0,n-1}_{(2)}(X_0, E_0) \neq 0$. Then, by Theorem 4.3, $\lambda_1^{0,n-1}(t)$ converges to zero. As $\lambda_1^{0,n-2}(t)$

is uniformly bounded from below, $\lambda_1^{0,n}(t) \to 0$ since $\lambda_1^{0,n-1}(t) = \min\{\lambda_1^{0,n-2}(t), \lambda_1^{0,n}(t)\}$. By Proposition 4.1 for $(0, n)$-forms (which holds because of (4.58)), we get

$$\dim H^n_{(2)}(X_0, E_0) > \dim H^n(X_t, \mathcal{O}_{X_t}(E_t)) \quad (t \neq 0). \tag{4.59}$$

Since $\pi : X \to D$ is a flat family and $H^q(X_t, \mathcal{O}_{X_t}(E_t))=0$ for any $q < n$, $\dim H^n(X_t, \mathcal{O}(E_t))$ is constant on D, which, combined with 2) and (4.59), yields a contradiction. ■

PROPOSITION 4.4. — *Under the situation of Theorem 4.4,*

1) $\lambda_1^{0,q}(t) \geq C > 0$ *for any* $t \in D$ *and* $q < n$.

2) *If* $\sigma \in T(D(\varepsilon), \det R\pi_* \mathcal{O}(E))$, $\sigma(0) \neq 0$, *then there exist* $0 < C_1 \leq C_2 < +\infty$ *such that, on* $D(\varepsilon)$ $C_1 \leq \|\sigma\|_2(t) \leq C_2$.

Proof. — *1)* follows from Theorem 4.4 and *2)* is similar to Proposition 4.3. ■

5. A Duhamel's principle for the heat kernel

Let (M, g) be a complete Riemannian manifold of dimension n. Let (E, h, ∇) be a Hermitian vector bundle on M with a Hermitian connection, $H = \nabla^*\nabla + Q$ a semipositive self-adjoint Schrödinger operator on E, where $Q \in \Gamma(M, \mathrm{End}\ E)$ with $Q^* = Q$. Fix a point $p \in M$ and a constant $\rho > 0$ such that

(1) $\rho < \frac{1}{2} i_p$ where i_p is the injectivity radius at p,

(2) $|\nabla^l g_{ij}(x)| + |\nabla^l h(x)| + |\nabla^l Q(x)| \leq C_l$

for any $x \in B(p, 2\rho)$. Let $K(t, x, y)$ be the heat kernel of H.

THEOREM 5.1 (cf. [C-L-Y], Theorem 3). — *Let* A_p *be a differential operator of order* k *at* p. *Then, for* $t \in (0, \sqrt{R}]$, $R \leq \rho$,

$$\Big|\int_{M-B(p,R)} \mathrm{tr}\{A_p K(t, p, x)\} K(t, x, p)\Big| \leq C\, t^{-\frac{n+k}{2}} \exp \gamma\Big(-\frac{R^2}{t} + t\Big)$$

where $C, \gamma > 0$ *are constants which depends only on* ρ, C_l $(l \leq k+4)$, *and* A_p.

The norm on $\mathrm{End}(E_x, E_y)$ is defined by $|A|^2 = \mathrm{Tr}\ AA^*$ as usual.

LEMMA 5.1. — *Let* $d(x, y)$ *be the distance function on* M. *Put*

$$\rho(x, y, t) := \frac{1}{4t} d(x, y)^2 - C_0 t, \quad g_p(y, s) := -2\rho(y, p, (1+2\delta)t - s).$$

Then, for any $y \in M$ and $s \in [0, (1+2\delta)t]$,

$$\frac{1}{2}|\nabla_y g|^2 + g_s + 2C_0 = 0.$$

Proof. — See [C-L-Y], Theorem 3, and [L-Y], § 3.

LEMMA 5.2. — *For any $x, y \in B(p,\rho)$ and $t \in [0, \sqrt{\rho}]$,*

$$|\nabla_y^k K(t,x,y)| \le B_k t^{-\frac{n+k}{2}} \exp\Big(-\frac{\gamma d(x,y)}{t}\Big)$$

where $B_k, \gamma > 0$ are constants which depends only on ρ and C_k.

Proof. — See [C-G-T], Example 2.1.

In the sequel, we assume $0 < s \le \tau \le t$. Fix $\sigma_p \in E_p$ with $|\sigma_p| = 1$, and put

$$\begin{aligned} F(y,s) &:= \int_{M-B(p,R)} K(s,y,x)\{A_p K(t,p,x)\}^* \sigma_p dx \\ &= \int_{M-B(p,R)} K(s,y,x)\chi_{M-B(p,R)}(x)\{A_p K(t,p,x)\}^* \sigma_p dx \end{aligned} \tag{5.1}$$

where $\chi_{M-B(p,R)}$ is the characteristic function of $M - B(p,R)$. Let $\phi(\ge 0) \in C_0^\infty(M)$ be a cut-off function which satisfies $\phi = 1$ on $B(p, \frac{R}{2})$, $\phi = 0$ on $M - B(p,R)$, and $|\nabla\phi| \le CR^{-1}$ on $B(p,R) - B(p,\frac{R}{2})$.

LEMMA 5.3 (cf. [C-L-Y]).

$$\frac{1}{2}\int_M \phi^2 e^g |F|^2 dy\Big|_{s=\tau} \le \frac{1}{2}\int_M \phi^2 e^g |F|^2 dy\Big|_{s=0} - 2\int_0^\tau ds \int_M \phi\, e^g \langle \nabla_y\phi \otimes F, \nabla F\rangle dy.$$

Proof. — Since F satisfies the heat equation:

$$\Big(\frac{\partial}{\partial s} + H_y\Big)F(y,s) = 0, \tag{5.2}$$

by the same manner as [C-L-Y], p. 1039, we get

$$\begin{aligned} \frac{1}{2}\int_M \phi^2 e^g |F|^2 dy\Big|_{s=\tau} \le& \frac{1}{2}\int_M \phi^2 e^g |F|^2 dy\Big|_{s=0} \\ &- 2\int_0^\tau ds \int_M \phi\, e^g \langle \nabla_y\phi \otimes F, \nabla F\rangle dy \\ &+ \frac{1}{2}\int_0^\tau ds \int_M \phi^2 e^g |F|^2 \Big(g_s + \frac{1}{2}|\nabla_y g|^2 - 2\frac{\langle F, QF\rangle}{|F|^2}\Big) dy. \end{aligned} \tag{5.3}$$

As $|\langle F, QF\rangle| \le C_0|F|^2$, by Lemma 5.1, we obtain

$$\begin{aligned}
&\int_0^\tau ds \int_M \phi^2 e^g |F|^2 \Big(g_s + \frac{1}{2}|\nabla_y g|^2 - 2\frac{\langle F, QF\rangle}{|F|^2}\Big) dy \\
&= \int_0^\tau ds \int_{B(R)} \phi^2 e^g |F|^2 \Big(g_s + \frac{1}{2}|\nabla_y g|^2 - 2\frac{\langle F, QF\rangle}{|F|^2}\Big) dy \\
&\le \int_0^\tau ds \int_{B(R)} \phi^2 e^g |F|^2 \Big(g_s + \frac{1}{2}|\nabla_y g|^2 + 2C_0\Big) dy = 0,
\end{aligned} \tag{5.4}$$

which, combined with (5.3), yields the assertion. ∎

LEMMA 5.4. — *For* $0 < \tau \le t \le \sqrt{\rho}/2$,

$$\Big|\int_0^\tau ds \int_M \phi\, e^g \langle \nabla_y \phi \otimes F, \nabla F\rangle dy\Big| \le C\, t^{-(\frac{n}{2}+k)} \exp\Big(-\frac{\gamma R^2}{t} + \gamma t\Big).$$

Proof. — By the semigroup property of the heat kernel, we get

$$\begin{aligned}
F(y,s) &= \int_{M-B(p,R)} K(s,y,x)\{A_p K(t,p,x)\}^* \sigma_p dx \\
&= \{A_p(t+s,p,y)\}^* \sigma_p - \int_{B(p,R)} K(s,y,x)\{A_p K(t,p,x)\}^* \sigma_p dx
\end{aligned} \tag{5.5}$$

and

$$\nabla_y F(y,s) = \nabla_y \{A_p(t+s,p,y)\}^* \sigma_p - \int_{B(p,R)} \nabla_y K(s,y,x)\{A_p K(t,p,x)\}^* \sigma_p dx. \tag{5.6}$$

By Lemma 5.2, we obtain

$$\Big|\int_{B(p,R)} K(s,y,x)\{A_p K(t,p,x)\}^* \sigma_p dx\Big| \le C \int_{B(p,R)} s^{-\frac{n}{2}} t^{-\frac{n+k}{2}} e^{-\frac{\gamma d(y,x)^2}{s}} e^{-\frac{\gamma d(x,p)^2}{t}} dx. \tag{5.7}$$

By an appropriate coordinates centered at p, we have

$$t\, d(y,x)^2 + s\, d(x,p)^2 \ge C\,(t+s)\big|x - \frac{t}{t+s} y\big|^2 + C\, \frac{st}{t+s} d(y,p)^2, \tag{5.8}$$

which, combined with (5.7), yields

$$\Big|\int_{B(p,R)} K(s,y,x)\{A_p K(t,p,x)\}^* \sigma_p dx\Big| \le C\,(t+s)^{-\frac{n}{2}} t^{-\frac{k}{2}} e^{-\frac{\gamma d(y,p)}{t+s}}. \tag{5.9}$$

Since

$$\big|\{A_p K(t+s,p,y)\}^*\big| \le C\,(t+s)^{-(\frac{n}{2}+k)} e^{-\frac{\gamma d(y,p)}{t+s}}, \tag{5.10}$$

by Lemma 5.2, we have

$$\begin{aligned}
|F(y,s)| &\le \big|\{A_p K(t+s,p,y)\}^* \sigma_p\big| + \Big|\int_{B(p,R)} K(s,y,x)\{A_p K(t,p,x)\}^* \sigma_p dx\Big| \\
&\le C\,(t+s)^{-\frac{n}{2}} t^{-\frac{k}{2}} e^{-\frac{\gamma d(y,p)}{t+s}}.
\end{aligned} \tag{5.11}$$

Similarly, we get

$$|\nabla_y F(y,s)| \le C\,(t+s)^{-\frac{n}{2}} s^{-\frac{1}{2}} t^{-\frac{k}{2}} e^{-\frac{\gamma d(y,p)}{t+s}}. \tag{5.12}$$

Since

$$g(y,s) \le -\frac{d(y,p)^2}{2\delta t} + C_0(1+2\delta)t \quad (s \le t), \tag{5.13}$$

we obtain

$$\begin{aligned} &\Big|\int_M \phi\, e^g \langle \nabla_y \phi \otimes F, \nabla F\rangle dy\Big| \\ &\le C \int_{B(p,R)-B(p,R/2)} e^{-\frac{d(y,p)^2}{2\delta t}+Ct} |F|\,|\nabla F|\,|\nabla\phi| dy \\ &\le C\,R^{-1} e^{-\frac{R^2}{8\delta t}+Ct} \int_{B(p,R)-B(p,R/2)} t^{-k} s^{-\frac{1}{2}} (t+s)^{-n} e^{-\frac{\gamma R^2}{4(t+s)}} dy \\ &\le C\,R^{n-1} s^{-\frac{1}{2}} t^{-(n+k)} e^{-\frac{\gamma R^2}{t}+Ct}. \end{aligned} \tag{5.14}$$

Finally, we have

$$\begin{aligned} \Big|\int_0^\tau ds \int_M \phi\, e^g \langle \nabla_y \phi \otimes F, \nabla F\rangle dy\Big| &\le C\sqrt{\tau} t^{-(n+k)} R^{n-1} e^{-\frac{\gamma R^2}{t}+Ct} \\ &\le C\, t^{-(\frac{n}{2}+k)} e^{-\frac{\gamma R^2}{2t}+Ct}. \end{aligned} \tag{5.15}$$

■

Proof of Theorem 5.1. — Combining Lemma 5.3 and 5.4, we get

$$\begin{aligned} \frac{1}{2}\int_M \phi^2 e^g |F|^2 dy\Big|_{s=\tau} &\le \frac{1}{2}\int_{B(p,R)} \phi^2 e^{g(y,0)} |F(y,0)|^2 dy + C\, t^{-(\frac{n}{2}+k)} e^{-\frac{\gamma R^2}{t}+Ct} \\ &\le C\, t^{-(\frac{n}{2}+k)} e^{-\frac{\gamma R^2}{t}+Ct}, \end{aligned} \tag{5.16}$$

since $F(y,0) = \chi_{M-B(p,R)}(y) K(t,y,p)\sigma_p$ which vanishes on $B(p,R)$. Suppose $\sqrt{t} < R$. Then, as $\sqrt{t/4} < R/2$, we obtain

$$\int_{B(p,\sqrt{t/4})} e^g |F|^2 dy\Big|_{s=\tau} \le \int_M \phi^2 e^g |F|^2 dy\Big|_{s=\tau} \tag{5.17}$$

and, for $y \in B(p,\sqrt{t/4})$,

$$g(y,t) \ge -\frac{1}{32\delta} - (1+2\delta)C_0 t \tag{5.18}$$

which, combined with (5.16), yields

$$\int_{B(p,\sqrt{t/4})} |F|^2 dy\Big|_{s=\tau} \le C\, t^{-(\frac{n}{2}+k)} e^{-\frac{\gamma R^2}{t}+\gamma t} \quad (\sqrt{t} < R). \tag{5.19}$$

Since $|F(y,s)|$ satisfies the following inequality ([C-L-Y], Lemma 4.6):

$$\Big(\Delta^- - \frac{\partial}{\partial s}\Big)|F(y,s)| \ge -\kappa\,|F(y,s)| \tag{5.20}$$

for $(y, s) \in B(p, \rho) \times [0, \sqrt{\rho}]$, where $\kappa(\geq 0)$ is a constant which depends only on ρ and C_k $(k \leq 4)$, by applying the Moser iteration argument to $|F(y, s)|$, we have

$$|F(p, t)| \leq C\, t^{-\frac{n+2}{2}} \Big(\int_0^t d\tau \int_{B(p, \sqrt{t/4})} |F(y, \tau)|^2 dy \Big)^{\frac{1}{2}} \tag{5.21}$$

which, combined with (5.19), yields

$$\begin{aligned} |F(p, t)|^2 &\leq C\, t^{-\frac{n}{2}-1} \int_0^t d\tau \int_{B(p, \sqrt{t/4})} |F(y, \tau)|^2 dy \\ &\leq C\, t^{-\frac{n}{2}-1} \int_0^t t^{-(\frac{n}{2}+k)} e^{-\frac{\gamma R^2}{t} + \gamma t} d\tau \\ &\leq C\, t^{-(n+k)} e^{-\frac{\gamma R^2}{t} + \gamma t}. \end{aligned} \tag{5.22}$$

Therefore, we have

$$\Big| \int_{M - B(p,R)} K(t, p, x) \{A_p K(t, p, x)\}^* \sigma_p \Big| \leq C\, t^{-\frac{n+k}{2}} \exp \gamma \Big(- \frac{R^2}{t} + t \Big). \tag{5.23}$$

Let $\{\sigma_1, \cdots, \sigma_r\}$ be a orthonormal basis of E_p $(r = rank\ E)$. Then, by (5.23), we get

$$\begin{aligned} &\Big| \int_{M - B(p,R)} \mathrm{tr}\{A_p K(t, p, x)\} K(t, x, p) \Big| \\ &= \sum_i \int_{M - B(p,R)} \langle K(t, p, x) \{A_p K(t, p, x)\}^* \sigma_i, \sigma_i \rangle dx \\ &\leq Cr\, t^{-\frac{n+k}{2}} \exp \gamma \Big(- \frac{R^2}{t} + t \Big). \end{aligned} \tag{5.24}$$

■

As an application of Theorem 5.1, we prove an effective Duhamel's principle for the trace of the heat kernel.

Let (M, g) and (M', g') be Riemannian manifolds of dimension n, $D \Subset \Omega \Subset M$ and $D' \Subset \Omega' \Subset M'$ are domains of M and M' respectively. Let (E, h, ∇) and (E', h', ∇') be Hermitian vector bundles with Hermitian connections on M and M' respectively. Suppose that there is a diffeomorphism $\phi : \Omega \cong \Omega'$ such that

(1) $\phi(D) = D'$

(2) $\phi^* E' = E, \quad \phi^* g' = g, \quad \phi^* h' = h, \quad \phi^* \nabla' = \nabla \quad on \quad \Omega.$

Let $H = \nabla^* \nabla + Q$ and $H' = \nabla'^* \nabla' + Q'$ be self-adjoint semipositive Schrödinger operators such that $\phi^* Q' = Q$. Let $K(t, x, y)$ and $K'(t, x, y)$ be their heat kernels respectively.

THEOREM 5.2. — *Let* $\rho, C_k > 0$ $(k \geq 0)$ *be constants such that*

(1) $\mathrm{dist}(D, \partial \Omega) > 2\rho$, $\quad i_p > 2\rho$ *for any* $p \in \partial \Omega$,

(2) $|\nabla^k g| + |\nabla^k h| + |\nabla^k Q| \leq C_k$ *for any* $x \in \partial\Omega_\rho := \{y \in M; d(y, \partial\Omega) \leq \rho\}$.

Then, for $t \leq \sqrt{\rho}$,

$$\Big|\int_D \{\mathrm{tr}\ K(t,x,x) - \mathrm{tr}\ K'(t,x,x)\}dx\Big| \leq C\, e^{-\frac{\gamma}{t}}$$

where $C, \gamma > 0$ *are constants which depends only on* ρ, C_k, $(k \leq 4)$.

Proof. — In the sequel, we identify $K'(t,x,y)$ with a kernel on $\Omega \times \Omega$ via ϕ. By Stokes's theorem, there exists a first order differential operator B defined on $E\big|_{\partial\Omega}$ such that

$$\int_\Omega \{\langle \nabla^*\nabla s, s'\rangle - \langle s, \nabla^*\nabla s'\rangle\} dv_\Omega = \int_{\partial\Omega} \{\langle B\, s, s'\rangle - \langle s, B\, s'\rangle\} dv_{\partial\Omega}. \tag{5.25}$$

Choose $\sigma \in E_x$. By Duhamel's principle, we get

$$\begin{aligned} \langle \sigma, \{K(t,x,x) - K'(t,x,x)\}\sigma\rangle = \int_0^t ds \int_{\partial\Omega} \Big\{ &\langle B_z K(s,z,x)\sigma, K'(t-s,z,x)\sigma\rangle \\ &- \langle K(s,z,x)\sigma, B_z K'(t-s,z,x)\sigma\rangle \Big\} dz \end{aligned} \tag{5.26}$$

which, integrated on D, yields

$$\begin{aligned} &\Big|\int_D \mathrm{tr}\{K(t,x,x) - K'(t,x,x)\} dx\Big| \\ &\leq r\int_0^t ds \int_{\partial\Omega} dz \int_D |B_z K(s,z,x)|\, |K'(t-s,z,x)| dx \\ &+ r\int_0^t ds \int_{\partial\Omega} dz \int_D |K(s,z,x)|\, |B_z K'(t-s,z,x)| dx \\ &\leq r\int_0^t ds \Big\{\int_{\partial\Omega} dz \int_{M-B(z,\rho)} |B_z K(s)|^2 dx\Big\}^{\frac{1}{2}} \Big\{\int_{\partial\Omega} dz \int_{M'-B(z,\rho)} |K'(t-s)|^2 dx\Big\}^{\frac{1}{2}} \\ &+ r\int_0^t ds \Big\{\int_{\partial\Omega} dz \int_{M-B(z,\rho)} |K(s)|^2 dx\Big\}^{\frac{1}{2}} \Big\{\int_{\partial\Omega} dz \int_{M'-B(z,\rho)} |B_z K'(t-s)|^2 dx\Big\}^{\frac{1}{2}}. \end{aligned} \tag{5.27}$$

Since

$$\int_{\partial\Omega} dz \int_{M-B(z,\rho)} |B_z K(s,z,x)|^2 dx = \int_{\partial\Omega} dz \int_{M-B(z,\rho)} \mathrm{tr}\, \{B_z^* B_z K(s,z,x)\} K(s,x,z) dx, \tag{5.28}$$

and similar formula holds for the other terms, we obtain the assertion of the theorem by applying Theorem 5.1.

∎

6. Sobolev inequality and lower bound of the spectrum

Let (M, g) be an n-dimensional Riemannian manifold with finite volume, (E, h, ∇) a Hermitian vector-bundle with a Hermitian connection, $H := \nabla^*\nabla + Q$ a self-adjoint Schrödinger operator, where $Q \in \Gamma(M, \operatorname{End} E)$ with $Q^* = Q$. Put $\widehat{M} := M \times M$ and $\pi_i : \widehat{M} \to M$ for the projection to the i-th factor. Define $\widehat{E} := \pi_1^* E \otimes \pi_2^* E$, and $\widehat{H} := H \otimes 1 + 1 \otimes H$ on $L^2(\widehat{E})$.

THEOREM 6.1. — *Suppose the Sobolev inequality holds for $\widehat{H}$, i.e.,*

$$\|f\|^2_{\frac{2\nu}{\nu-1}} \leq A(\widehat{H} f, f) \quad \text{for any } f \in \Gamma_0(\widehat{M}, \widehat{E})$$

where $A > 0$, $\nu > 1$ are constants.

Then, for the Friedrichs extensions of H (which is also denotes by H),

1) $\sigma(H)$ consists of discrete eigenvalues,

2) e^{-tH} is a trace class operator and the following upper bound holds for any $t > 0$;

$$\operatorname{Tr} e^{-tH} \leq C t^{-\nu}$$

where $C = \operatorname{Const}(\operatorname{vol}(M), A, \nu, r)$.

Proof.

1) For any $f \in \Gamma_0(M, E)$, put

$$\hat{f}(x, y) := f(x) \otimes f(y) \in \Gamma_0(\widehat{M}, \widehat{E}). \tag{6.1}$$

Since

$$\|\hat{f}\|^2_{\frac{2\nu}{\nu-1}} = \|f\|^4_{\frac{2\nu}{\nu-1}} \tag{6.2}$$

and

$$\begin{aligned}(\widehat{H}\hat{f}, \hat{f}) &= (Hf \otimes f + f \otimes Hf, f \otimes f) = 2(Hf, f)\|f\|^2 \\ &\leq (Hf, f)^2 + \|f\|_2^4,\end{aligned} \tag{6.3}$$

combined with the Sobolev inequality for $\widehat{H}$, we get

$$\|f\|^2_{\frac{2\nu}{\nu-1}} \leq (Hf, f) + \|f\|_2^2. \tag{6.4}$$

Then, by the same argument as lemma 4.2, the Rellich lemma holds for H, which implies the discreteness of $\sigma(H)$.

2) Let $\{\varphi_i\}_{i=1}^\infty$ be a complete orthonormal basis of $L^2(E)$ such that

$$H\varphi_i = \lambda_i \varphi_i. \tag{6.5}$$

Then the heat kernel of H is represented by

$$K(t,x,y) = \sum_i e^{-t\lambda_i}\varphi_i(x)\langle\ ,\varphi_i(y)\rangle. \tag{6.6}$$

Put

$$\widehat{K}_N(t,x,y) := \sum_{i\le N} e^{-t\lambda_i}\varphi_i(x)\otimes\varphi_i(y) \in L^2(\widehat{M},\widehat{E}). \tag{6.7}$$

Then,

$$\begin{aligned}\operatorname{Tr}_N e^{-tH} &:= \sum_{i\le N} e^{-t\lambda_i}\\ &= \int_{\widehat{M}} \left|\widehat{K}_N(\frac{t}{2},x,y)\right|^2 dv_{\widehat{M}}.\end{aligned} \tag{6.8}$$

Since $\widehat{K}_N$ satisfies the heat equation

$$\frac{\partial}{\partial t}\widehat{K}_N + \frac{1}{2}\widehat{H}\widehat{K}_N = 0, \tag{6.9}$$

we have (cf. [C-K-S])

$$\begin{aligned}-\frac{d}{dt}\operatorname{Tr}_N e^{-tH} &= \left(\widehat{H}\widehat{K}_N\left(\frac{t}{2}\right),\widehat{K}_N\left(\frac{t}{2}\right)\right)_{\widehat{M}}\\ &\ge \frac{1}{A}\left\|\widehat{K}_N\left(\frac{t}{2}\right)\right\|^2_{\frac{2\nu}{\nu-1}}\\ &\ge \frac{1}{A}\left\|\widehat{K}_N\left(\frac{t}{2}\right)\right\|_2^{2(1+\frac{1}{\nu})}\cdot\left\|\widehat{K}_N\left(\frac{t}{2}\right)\right\|_1^{-\frac{2}{\nu}}\\ &\ge \frac{1}{A}\left(\operatorname{Tr}_N e^{-tH}\right)^{1+\frac{1}{\nu}}\cdot\left\|\widehat{K}_N\left(\frac{t}{2}\right)\right\|_1^{-\frac{2}{\nu}}.\end{aligned} \tag{6.10}$$

Since

$$\left\|\widehat{K}_N\left(\frac{t}{2}\right)\right\|_1 \le r\cdot\operatorname{vol}(M) \tag{6.11}$$

by lemma 6.1 below, we obtain

$$\frac{d}{dt}\left(\operatorname{Tr}_N e^{-tH}\right)^{-\frac{1}{\nu}} \ge \frac{1}{A\nu}\left(r\cdot\operatorname{vol}(M)\right)^{-\frac{2}{\nu}}. \tag{6.12}$$

As $\operatorname{Tr}_N e^{-tH} = N$ at $t=0$, solving (6.12), we have

$$\operatorname{Tr}_N e^{-tH} \le \left\{N^{-\frac{1}{\nu}} + \frac{1}{A\nu}(r\cdot\operatorname{vol}(M))^{-\frac{2}{\nu}}t\right\}^{-\nu}. \tag{6.13}$$

Taking the limit $N\to\infty$ of (6.13), 2) follows. ■

LEMMA 6.1. — *For any $N\ge 1$,*

$$\int_{\widehat{M}} \left|\widehat{K}_N(t,x,y)\right| dv_{\widehat{M}} \le r\cdot\operatorname{vol}(M).$$

Proof. — Let $\big|\widehat{K}_N(t,x,y)\big|_{\rm op}$ be the operator norm of $\widehat{K}_N(t,x,y) \in \mathrm{End}(E_y, E_x)$, where we identify $\widehat{K}_N$ with the corresponding integral kernel. Then, there exist $e, f \in L^\infty(E)$ such that

1) $|e(x)| = |f(x)| = 1$ a.e. $x \in M$;

2) $\big|\widehat{K}_N(t,x,y)\big|_{\rm op} = \big\langle \widehat{K}_N(t,x,y)e(y), f(x)\big\rangle$ a.e. $(x,y) \in \widehat{M}$.

Then,

$$\begin{aligned}\int_{\widehat{M}} \big|\widehat{K}_N(t,x,y)\big|_{\rm op} dv_{\widehat{M}} &= \int_{\widehat{M}} \big\langle \widehat{K}_N(t,x,y)e(y), f(x)\big\rangle dxdy \\ &= \sum_{i\le N} e^{-t\lambda_i}(\varphi_i, f)(e,\varphi_i) \\ &\le \Big\{\sum_i \big|(\varphi_i, f)\big|^2\Big\}^{\frac{1}{2}} \Big\{\sum_i \big|(e,\varphi_i)\big|^2\Big\}^{\frac{1}{2}} \\ &= \|f\|_2 \cdot \|e\|_2 = \mathrm{vol}(M).\end{aligned} \tag{6.14}$$

Since

$$\big|\widehat{K}_N(t,x,y)\big| \le r\big|\widehat{K}_N(t,x,y)\big|_{\rm op}, \tag{6.15}$$

we obtain the assertion, combining (6.14) and (6.15). ■

Corollary 6.1. — *Under the situation of theorem 6.1,*

$$\lambda_k \ge Ck^{\frac{1}{\nu}}$$

when $C = \mathrm{Const}(A, \nu, r, \mathrm{vol}(M))$.

Proposition 6.1. — *Suppose the following inequalities hold on M:*

1) For any $f \in C_0^\infty(M)$, $\|f\|_{\frac{2\nu}{\nu-1}} \le C_0\|df\|_2$.

2) For any $s \in T_0(M,E)$, $|(Qs,s)| \le C_1(Hs,s)$.

Then the Sobolev inequality holds for $\widehat{H}$, i.e., for any $\varphi \in \Gamma_0(\widehat{M}, \widehat{E})$,

$$\|\varphi\|_{\frac{4\nu}{2\nu-1}} \le C(\widehat{H}\varphi, \varphi)$$

where $C = \mathrm{Const}(C_0, C_1, \mathrm{vol}(M), r)$.

Proof. — For $\varphi \in \Gamma_0(\widehat{M}, \widehat{E})$, put

$$\varphi(x,y) = \sum a_{ij}\varphi_i(x) \otimes \varphi_j(y). \tag{6.16}$$

Then, we get

$$\begin{aligned}\big|((Q\otimes 1)\varphi, \varphi)\big| &= \Big|\sum_j \int_M \langle Q(x)\sum_i a_{ij}\varphi_i(x), \sum_k a_{kj}\varphi_k(x)\rangle\, dx\Big| \\ &\le \sum_j C_1\Big(H\sum_i a_{ij}\varphi_i(x), \sum_k a_{kj}\varphi_k(x)\Big) \\ &\le C_1\big((H\otimes 1)\varphi, \varphi\big).\end{aligned} \tag{6.17}$$

Similarly, we obtain

$$\big|(1\otimes Q\varphi,\varphi)\big| \le C_1(1\otimes H\varphi,\varphi). \tag{6.18}$$

Therefore, for $\Delta_E := \nabla^*\nabla\otimes 1 + 1\otimes\nabla^*\nabla$,

$$\begin{aligned}(\Delta_{\widehat{E}}\varphi,\varphi) &= (\widehat{H}\varphi,\varphi) - ((Q\otimes 1 + 1\otimes Q)\varphi,\varphi)\\ &\le (1+C_1)(\widehat{H}\varphi,\varphi).\end{aligned} \tag{6.19}$$

Let $\nabla_{\widehat{E}}$ be the induced connection of $\widehat{E}$ by that of E. Then $\nabla_{\widehat{E}} = \nabla_E\otimes 1 + 1\otimes\nabla_E$, and $\Delta_{\widehat{E}} = \nabla_{\widehat{E}}^*\nabla_{\widehat{E}}$. Thus, by (6.19), we have

$$\|\nabla_{\widehat{E}}\varphi\|_2^2 \le (1+C_1)(\widehat{H}\varphi,\varphi). \tag{6.20}$$

Since 1) implies the Sobolev inequality on $\widehat{M}$ (cf. [Da]), *i.e.*, for any $\psi\in C_0^\infty(\widehat{M})$,

$$\|\psi\|_{\frac{4\nu}{2\nu-1}} \le C_2\|d\psi\|_2, \tag{6.21}$$

where $C_2 = \mathrm{Const}(C_0, \mathrm{vol}(M), \nu)$, combined with Kato's inequality, we obtain

$$\begin{aligned}\|\varphi\|^2_{\frac{4\nu}{2\nu-1}} &\le C_2\|\nabla_{\widehat{E}}\varphi\|_2^2\\ &\le C_2(1+C_1)(\widehat{H}\varphi,\varphi).\end{aligned} \tag{6.22}$$

■

THEOREM 6.2. — *Let $\pi : X \overset{n+1}{\longrightarrow} D$ be a smoothing of IHS. Suppose* $\mathrm{Sing}\,X_0$ *consists of homogeneous singularities, i.e., for any $p\in\mathrm{Sing}\,X_0$, there exists a homogeneous polynomial $F(z)\in\mathbb{C}[z]$ such that $(\mathcal{O}_{X_0,p},p) \overset{\sim}{\longrightarrow} (\mathbb{C}\{z\}/(F),0)$. Put $\widehat{X}_t := X_t\times X_t$, $\widehat{\Lambda}^r := \pi_1^*\Lambda^r\otimes\pi_2^*\Lambda^r$ as before. Let g_X be a Kähler metric of X such that $g_X = \partial\bar\partial\|z\|^2$ with respect to the coordinate z above. Then there exists $\nu > 1$ such that, for any $\varphi\in\Gamma(\widehat{X}_t,\widehat{\Lambda}^r)$, $(r < n)$,*

$$\|\varphi\|^2_{t,\frac{2\nu}{\nu-1}} \le C\big\{\big(\widehat{\Delta}_t^r\varphi,\varphi\big) + \|\varphi\|_2^2\big\}.$$

Proof. — When $r < n-1$, we can verify the condition 2) of Proposition6.1 in the same way as Lemma 4.1 by using $F = \log\|z\|^2$ in (4.2). The condition 1) follows from [L-T], and the assertion follows from Proposition 6.1. For the case $r = n-1$, see Appendix. ■

COROLLARY 6.2. — *Under the situation of Theorem 6.2,*

$$\lim_{t\to 0}\mathrm{Tr}\,e^{-s\Delta_t^r} = \mathrm{Tr}\,e^{-s\Delta_0^r}$$

for any $s > 0$ and $r < n$.

7. Heat kernels on asymptotically flat manifolds

Definition 7.1. — *Let $(X, g, 0)$ be a complete Riemannian manifold of dimension m (> 2) with a (fixed) point 0. Put $|x| := \mathrm{dist}(0, x) = d(0, x)$, and i_x for the injectivity radius at x.*

$(X, g, 0)$ is said to be an asymptotically flat (AF) manifold if the following conditions are satisfied:

1) There is a constant $c > 0$ such that, for any $y \in X$,

$$i_y \geq c(1 + |y|) =: j_y\,,$$

2) If $B(y, j_y)$ stands for the metric ball of radius j_y centered at y, and $x{=}(x_1, \ldots, x_m)$ for the normal coordinates on $B(y, j_y)$, write $g(x) = \sum_{ij} g_{ij}(x) dx^i\, dx^j$ on $B(y, j_y)$. Then, for any $x \in B(y, j_y)$ and multiindex $\alpha > 0$,

$$C_0^{-1} I \leq (g_{ij}(x)) \leq C_0 I, \ \ |\partial^\alpha g_{ij}(x)| \leq K_\alpha (1 + |y|)^{-|\alpha|}.$$

Let (E, h, ∇) be a Hermitian vector bundle of rank r with a Hermitian connection on $(X, g, 0)$, $s = \{s_1, \ldots, s_r\}$ and unitary frame on $B(y, j_y)$. Put

$$h_{ij}(x) := h(s_i, s_j)(x), \ \ \nabla s_i(x) = \sum_j \omega_{ij}(x) s_j(x)$$

and $\Omega = (\Omega_{ij})$ for the curvature with respect to s.

Definition 7.2. — *A vector bundle (E, h, ∇) is said to be asymptotically flat* (AF)*, if, for any $y \in X$, there is a suitable choice of frame $\{s_1, \ldots, s_r\}$ on $B(y, j_y)$ such that $h_{ij}(y) = \delta_{ij}$ and*

$$C^{-1} I \leq (h_{ij}(x)) \leq CI,$$

$$|\partial^\alpha h_{ij}(x)| \leq K_\alpha (1 + |y|)^{-|\alpha|},$$

$$|\partial^\beta \omega_{ij}(x)| \leq K_\beta (1 + |y|)^{-(|\beta|+1)}$$

for any $x \in B(y, j_y)$, $\alpha > 0$ and $\beta \geq 0$ where C and K_α are constants independent of x, y. A section $Q \in \Gamma(X, \mathrm{End}\, E)$ is said to be asymptotically flat (AF)*, if, when $Q = (Q_{ij}(x))$ with respect to $\{s_i \otimes s_j^*\}$ on $B(y, j_y)$,*

$$|\partial^\alpha F_{ij}(x)| \leq K_\alpha (1 + |y|)^{-(|\alpha|+2)}$$

for any $x \in B(y, j_y)$ and $\alpha \geq 0$.

Put $\Delta := \nabla^* \nabla$ for the Laplacian and $H := \Delta + Q$, $(Q^* = Q)$ to define a self-adjoint Schrödinger operator on E. In the sequel, we assume that $(X, g, 0)$, (E, h, ∇) and Q are AF in the above sense.

Following [B-G-V], pp. 82–87, let $u_i(x, y) \in \Gamma(B(y, j_y) \times B(y, j_y), \pi_1^* E_1 \otimes \pi_2^* E)$ be the function constructed in [Y 1], (2.4), (2.5). Put

$$p_k(r, x, y) := T(u_0 + tu_1 + \cdots t^k u_k) \tag{7.1}$$

$$T(t, x, y) := (4\pi t)^{-\frac{m}{2}} \exp\Big\{ - \frac{d(x, y)^2}{4t} \Big\} \tag{7.2}$$

$$F_k(t, x, y) := K(t, x, y) - p_k(t, x, y) \tag{7.3}$$

where $K(t, x, y)$ is the heat kernel of H. Note that for any $y \in X$, $F_k(t, \cdot, y)$ is defined on $B(y, j_y)$.

LEMMA 7.1. — *For any $(x, t) \in B(y, j_y) \times \mathbb{R}_+$,*

$$\Big(\frac{\partial}{\partial t} + H_x\Big) F_k(t, x, y) = (4\pi t)^{-\frac{m}{2}} t^{k-\frac{m}{2}} e^{-\frac{d(x,y)^2}{4t}} B_x u_k$$

where B is the same differential operator as in [B-G-V], (2.2).

Proof. — See [B-G-V].

LEMMA 7.2. — *For any $x \in B(y, j_y)$ and $i \geq 0$, multiindex $\alpha \geq 0$,*

$$|\partial^\alpha u_i(x, y)| \leq C_\alpha (1 + |y|)^{-|\alpha| - 2i}.$$

Proof. — See [Y 1], Proposition 2.2.

LEMMA 7.3. — *For any $p \in X$, $x, y \in B(p, j_p)$, and $0 \leq t \leq \frac{1}{4} j_p^2$,*

$$|K(t, x, y)| \leq Ct^{-\frac{m}{2}} \exp\Big(- \frac{\gamma d(x, y)^2}{t} \Big)$$

where $C, \gamma > 0$ are constants independent of p, x, y, t.

Proof. — See [Y 1], Proposition 2.1.

LEMMA 7.4. — *For $k > \frac{m}{2} + 4$, $F_k(\cdot, \cdot, y)$ extends to a C^2 function by setting $F_k(t, x, y) \equiv 0$ for $t \leq 0$.*

Proof. — See [C-Y] .

Let $\mathcal{X}_y \in C^\infty(X)$ be a cut-off function such that $\mathcal{X}_y \geq 0$, $\mathcal{X}_y = 1$ on $B(y, \frac{1}{2} j_y)$, $\mathcal{X}_y = 0$ on $X - B(y, j_y)$ and $|dX_y| \leq 4 j_y^{-1}$. Put

$$G_k(t, x, y) := \mathcal{X}_y(x) \Big(\frac{\partial}{\partial t} + H_x\Big) F_k(t, x, y) \tag{7.4}$$

$$(7.5)\qquad H_k(t,x,y) := \int_0^t d\tau \int_X K(t-\tau,x,z)\; G_k(\tau,z,y)\; dv.$$

LEMMA 7.5. — *For* $t \in [0, 1+|y|^2]$,

$$\sup_{[0,t]\times B(y,\frac{1}{2}j_y)} |F_k(\cdot,\cdot,y) - H_k(\cdot,\cdot,y)| \le C\Big\{ \sup_{[0,t]\times\partial B(y,j_y)} |F_k(\cdot,\cdot,y)| + \sup_{[0,\tau]\times\partial B(y,j_y)} |H_k(\cdot,\cdot,y)| \Big\}$$

where $C = \mathrm{Const}(c, C_\alpha, K_\alpha,\ (|\alpha| \le 2))$.

Proof. — See [Y 1], Lemma 2.3.

LEMMA 7.6. — *For* $t \in [0, 1+|y|^2]$,

$$\sup_{[0,t]\times\partial B(y,\frac{1}{2}j_y)} |F_k(\cdot,\cdot,y)| \le C t^{k+1-\frac{m}{2}} (1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8t}\Big).$$

Proof. — By definition and Lemmas 7.2, 7.3, for $(s,x) \in [0,t]\times\partial B(y,j_y)$, we get

$$\begin{aligned}(7.6)\qquad |F_k(s,x,y)| &\le C s^{-\frac{m}{2}} e^{-\frac{\gamma d(x,y)^2}{s}} \big(1 + s(1+|y|^2)^{-1} + \cdots + s^k(1+|y|^2)^{-k}\big)\\ &\le C s^{-\frac{m}{2}} \sum_{i=0}^{k} s^{-\frac{m}{2}+i} (1+|y|^2)^{-i} \exp\Big(-\frac{c\gamma(1+|y|^2)}{2s}\Big)\\ &\le C s^{k+1-\frac{m}{2}} (1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{2s}\Big)\end{aligned}$$

since $(1+|y|^2)/s \ge 1$. As $s^{k+1-\frac{m}{2}} \exp(-c\gamma(1+|y|^2)/2s)$ is an increasing function in s, we obtain the estimate. ∎

In the sequel, we assume $k > \frac{m}{2} + 4$.

LEMMA 7.7. — *For* $t \in [0, 1+|y|^2]$,

$$\sup_{[0,t]\times\partial B(y,\frac{1}{2}j_y)} |H_k(\cdot,\cdot,y)| \le C t^{k+1-\frac{m}{2}} (1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{4s}\Big).$$

Proof. — For $(s,x) \in [0,t]\times\partial B(y,\frac{1}{4}j_y)$,

$$\begin{aligned}(7.7)\qquad |H_k(s,x,y)| &= \Big| \int_0^s d\tau \int_{B(y,j_y)} K(s-\tau,x,z) G_k(\tau,z,y)\, dv \Big|\\ &\le \int_0^s d\tau \int_{B(y,\frac{1}{4}j_y)} |K(s-\tau)||G_k(\tau)|\, dv\\ &\qquad + \int_0^s d\tau \int_{B(y,j_y)-B(y,\frac{1}{4}j_y)} |K(s-\tau)||G_k(\tau)|\, dv.\end{aligned}$$

Put I_1 and I_2 for the first term and second term of the right hand side respectively. Since $d(x,z) \geq d(z,y) = \|s\|^2 = \sum_{i=1}^{m}(z_i)^2$ and $d(x,z) \geq \frac{1}{4}j_y$ for $z \in B(y, \frac{1}{4}j_y)$, by Lemma 7.2 and 7.3, we get

$$
\begin{aligned}
(7.8)\quad I_1 &\leq C\int_0^s d\tau \int_{B(y,\frac{1}{4}j_y)} (s-\tau)^{-\frac{m}{2}} e^{-\frac{\gamma d(y,z)^2}{2(s-\tau)}} e^{-\frac{\gamma d(x,z)^2}{(s-\tau)}} e^{-\frac{\gamma d(y,z)^2}{\tau)}} \\
&\quad \times \tau^{k-\frac{m}{2}}(1+|y|^2)^{-(k+1)}\, dv \\
&\leq C(1+|y|^2)^{-(k+1)} s^{-\frac{m}{2}} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8s}\Big) \\
&\quad \times \int_0^s \tau^k\, d\tau \int_{\|z\|\leq \frac{1}{4}j_y} \Big\{\frac{\tau(s-\tau)}{s}\Big\}^{-\frac{m}{2}} \exp\Big(-\frac{\gamma s\|s\|^2}{2\tau(s-\tau)}\Big)\, dz \\
&\leq Cs^{k+1-\frac{m}{2}}(1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8s}\Big).
\end{aligned}
$$

Since $d(x,z) \geq \frac{1}{4}j_y$ for $z \in B(y,j_y) - B(y,\frac{1}{4}j_y)$, in the same way as the above, we obtain

$$
\begin{aligned}
(7.9)\quad I_2 &\leq C\int_0^s d\tau \int_{B(y,j_y)-B(y,\frac{1}{4}j_y)} |K(s-\tau)| \\
&\quad \times \exp\Big(-\frac{c\gamma(1+|y|^2)}{4\tau}\Big)\tau^{k-\frac{m}{2}}(1+|y|^2)^{-(k+1)}\, dv \\
&\leq C(1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8s}\Big)\int_0^s \tau^{k-\frac{m}{2}} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8\tau}\Big)\, d\tau \\
&\leq Cs^{k+1-\frac{m}{2}}(1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8s}\Big).
\end{aligned}
$$

Therefore, combining (7.7) and (7.8), we have

$$
\begin{aligned}
(7.10)\qquad I_1 + I_2 &\leq Cs^{k+1-\frac{m}{2}}(1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8s}\Big) \\
&\leq Ct^{k+1-\frac{m}{2}}(1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8t}\Big)
\end{aligned}
$$

for $0 \leq s \leq t \leq 1+|y|^2$. ■

Combining Lemmas 7.5, 7.6 and 7.7, we obtain the following

PROPOSITION 7.1. — *For $t \in [0, 1+|y|^2]$,*

$$
\sup_{[0,t]\times B(y,\frac{1}{2}j_y)} |H_k(\cdot,\cdot,y) - F_k(\cdot,\cdot,y)| \leq Ct^{k+1-\frac{m}{2}}(1+|y|^2)^{-(k+1)} \exp\Big(-\frac{c\gamma(1+|y|^2)}{8t}\Big).
$$

PROPOSITION 7.2. — *For $t \in [0, 1+|y|^2]$,*

$$
|H_k(t,y,y)| \leq Ct^{k+1-\frac{m}{2}}(1+|y|^2)^{-(k+1)}.
$$

Proof. — Put

$$I = \int_0^t d\tau \int_X |K(t-\tau, y, z)|\,|G_k(\tau, z, y)|\,dv. \tag{7.11}$$

By definition, Lemmas 7.1, 7.2 and 7.3, we get

$$\begin{aligned}
|H_k(t,y,y)| &\leq I \\
&\leq C(1+|y|^2)^{-(k+1)} \int_0^t \tau^{k-\frac{m}{2}}\,d\tau \int_{B(y,j_y)} |K(t-\tau,y,z)| \exp\Big(-\frac{d(y,z)^2}{4\tau}\Big)\,dv \\
&\leq C(1+|y|^2)^{-(k+1)} \int_0^t \tau^{k-\frac{m}{2}}\,d\tau \int_{\mathbb{R}^m} (t-\tau)^{-\frac{m}{2}} \exp\Big(-\gamma\big(\frac{1}{\tau}+\frac{1}{t-\tau}\big)\|z\|^2\Big)\,dz \qquad (7.12)\\
&\leq C(1+|y|^2)^{-(k+1)} t^{-\frac{m}{2}} \int_0^t \tau^k d\tau \int_{\mathbb{R}^m} \Big\{\frac{\tau(t-\tau)}{t}\Big\}^{-\frac{m}{2}} \exp\Big(-\frac{\gamma t\|z\|^2}{\tau(t-\tau)}\Big)\,dz \\
&\leq Ct^{k+1-\frac{m}{2}}(1+|y|^2)^{-(k+1)}.
\end{aligned}$$

∎

Theorem 7.1. — *Let $(X, g, 0)$ be an asymptotically flat manifold of dimension m and (E, h, ∇) an asymptotically flat vector bundle. Let $H = \nabla^*\nabla + Q$ be a self-adjoint Schrödinger operator on E with asymptotically flat potential, $K(t, x, y)$ its heat kernel and $p_k(t, x, y)$ its parametrix as before . Then, for any $k \geq 0$ and $t \in [0, 1 + |x|^2]$,*

$$\big|K(t,x,x) - p_k(t,x,x)\big| \leq C_k t^{k+1-\frac{m}{2}}(1+|x|^2)^{-(k+1)}$$

where C_k is a constant which is independent of t and x.

Proof. — First we prove the theorem for $k > \frac{m}{2} + 4$. By Propositions 7.1 and 7.2, we get

$$\begin{aligned}
\big|K(t,x,x) - p_k(t,x,x)\big| &= \big|F_k(t,x,x)\big| \\
&\leq \sup_{[0,t]\times B(x,\frac{1}{2}j_x)} \big|H_k(\cdot,\cdot,x) - F_k(\cdot,\cdot,x)\big| + \big|H_k(t,x,x)\big| \qquad (7.13)\\
&\leq Ct^{k+1-\frac{m}{2}}(1+|x|^2)^{-(k+1)}
\end{aligned}$$

which proves the theorem in this case. When $k \leq \frac{m}{2} + 4$, fix $N > \frac{m}{2} + 4$, Then, since

$$F_k(t,x,) = F_N(t,x,x) + (4\pi t)^{-\frac{m}{2}}\big(t^{k+1}u_{k+1}(x,x) + \cdots + t^N u_N(x,x)\big), \tag{7.14}$$

we have

$$\begin{aligned}
\big|F_k(t,x,x)\big| &\leq \big|F_N(t,x,x)\big| + Ct^{-\frac{m}{2}}\big(t^{k+1}(1+|x|^2)^{-(k+1)} + \cdots + t^N(1+|x|^2)^{-N}\big) \\
&\leq C\Big\{\sum_{i=0}^{N-k}\Big(\frac{t}{1+|x|^2}\Big)^2\Big\} t^{k+1-\frac{m}{2}}(1+|x|^2)^{-(k+1)} \qquad (7.15)\\
&\leq Ct^{k+1-\frac{m}{2}}(1+|x|^2)^{-(k+1)}
\end{aligned}$$

as $t \leq 1 + |x|^2$. ∎

8. Conic degeneration and asymptotics of analytic torsion

In this section, we prove a refinement of our previous theorem (cf. [Y 1], Theorem 6.1).

DEFINITION 8.1. — *Let* $(X^m, g, 0)$ *be an* AF *manifold. It is said to be asymptotically conical* (AC), *if there exist a compact Riemannian manifold* (N, g_N), *a compact set* $K \subset X^m$, *and a diffeomorphism*

$$\varphi : [1, \infty) \times N \longrightarrow X - K$$

such that, for some $\delta > 0$,

$$\varphi^* g_X = dr^2 + r^2 g_N + O(r^{-\delta}).$$

The cone $C(N) = (N \times \mathbb{R}_+,\ dr^2 + r^2 g_N = g_{C(N)})$ *is said to be the tangent cone of* (X, g). *We often identify* $C(N)$ *with its metric completion. Denote by* $K^q(t, x, y)$ *the heat kernel of the Laplacian on* q*-forms (when* X *is a complex manifold, we use* $(0, q)$ *for the Laplacian on* $(0, q)$*-forms). Let*

$$K^q(t, x, x) \sim (4\pi t)^{-\frac{m}{2}} \sum_{i=0}^{\infty} u_i^q(x) t^i \ \ (t \to 0) \tag{8.1}$$

be its asymptotic expansion.

Put $B(r) := K \cup \varphi^{-1}([1, r]) \subset X$ for $r \gg 1$. Then, $B(r)$ is a manifold with boundary N; $\partial B(r) \xrightarrow{\sim} N$. Therefore, for a manifold M' with boundary N, one can obtain a compact manifold M by patching M' and $B(r)$ along the boundary:

$$M = M' \cup_N B(r). \tag{8.2}$$

Note that M does not depend on the choice of r $(\gg 1)$.

DEFINITION 8.2 (cf. [J-W]). — *Let* M *be a compact manifold,* $M' \subset M$ *a submanifold with boundary* N, *and* $\{g_\varepsilon\}_{0<\varepsilon\ll 1}$ *a family of Riemannian metrics of* M *which depends smoothly in* ε. *Then the family* $\{(M, g_\varepsilon)\}$ *is said to be a conic degenerating family with tangent cone* $C(N)$ *iff there exists an* AC *manifold* (X, g) *with tangent cone* $C(N)$ *and a Riemannian metric* g_0 *of* M' *such that*

1) $M = M' \cup_N B(\varepsilon^{-1})$, $\quad g_\varepsilon = \varepsilon^{2g}$ *on* $B(\varepsilon^{-1})$,

2) $\lim_{\varepsilon \to 0} g_\varepsilon = g_0$ *in the* C^∞*-topology, and* g_0 *extends to a smooth Riemannian metric on* $M_0 := M' \cup_N C_{0,1}(N)$.

Furthermore, if there is a smooth family of complex structures $\{I_\varepsilon\}$ *on* M, *a complex structure* I *on* X, *and* I_0 *on* M_0 *such that*

4) $(M, I_\varepsilon, g_\varepsilon)$ *is Kähler,*

5) $I_\varepsilon = I$ *on* $B(\varepsilon^{-1})$ *and* $I_\varepsilon \to I_0$*,*

then the family is said to be a conic degeneration of Kähler manifolds.

Let $\Box_\varepsilon^{0,q}$ be the Laplacian on $(0,q)$-forms on $(M, I_\varepsilon, g_\varepsilon)$, $\sigma(\Box_\varepsilon^{0,q}) = \{0 \le \lambda_1^{0,q}(\varepsilon) \le \lambda_2^{0,q}(\varepsilon) \le \cdots\}$ its spectrum, $K_\varepsilon^{0,q}(t,x,y)$ its heat kernel, and

$$\operatorname{tr} K_\varepsilon^{0,q}(t,x,x) \sim (4\pi t)^{-n} \sum_{i=0}^{\infty} u_i^{0,q}(x,\varepsilon) t^i \tag{8.3}$$

$$\operatorname{Tr} e^{-t\,\Box_\varepsilon^{0,q}} \sim (4\pi t)^{-n} \sum_{i=0}^{\infty} u_i^{0,q}(\varepsilon) t^i \quad \left(u_i^{0,q}(\varepsilon) = \int_M u_i^{0,q}(x,\varepsilon)\,dx\right) \tag{8.4}$$

be their asymptotic expansions as $t \to 0$.

When $\varepsilon = 0$, $\Box_0^{0,q}$ is considered to be the Friedrichs extension. Then its spectrum consists of discrete eigenvalues and it is a trace class operator and admits an asymptotic expansion as $t \to 0$ (cf. [C] and [B-S]).

Let n be the complex dimension of M. Let $P^{0,q}(t,x,y)$ be the heat kernel of the Laplacian on $(0,q)$-forms on $C(N)$. By using the polar coordinates $x = (r,z)$ of $C(N)$,

$$f^{0,q}(t,r,z) := \operatorname{tr} P^{0,q}(t,x,x) \sim (4\pi t)^{-n} \sum_{j=0}^{\infty} a_j^{0,q}(z) r^{-2j} t^j \tag{8.5}$$

is the pointwise asymptotic expansion as $t \to 0$ (cf. [C]).

Put

$$\begin{aligned} D^{0,q}(N) :=& \frac{1}{2}\int_0^1 \frac{du}{u} \int_N \left\{ f^{0,q}(u,1,z) - (4\pi u)^{-n} \sum_{j=0}^{n} a_j^{0,q}(z) u^j \right\} dv_N \\ &+ \frac{1}{2}\int_1^\infty \frac{du}{u} \int_N \left\{ f^{0,q}(u,1,z) - (4\pi u)^{-n} \sum_{j=0}^{n-1} a_j^{0,q}(z) u^j \right\} dv_N \end{aligned} \tag{8.6}$$

$$a_j^{0,q}(1) := \int_N a_j^{0,q}(z)\,dv_N.$$

By Cheeger ([C], Theorem 2.1), the following holds.

Proposition 8.1. — *For* $t \in (0,1]$*,*

$$\Big| \operatorname{Tr} e^{-t\,\Box_0^{0,q}} - (4\pi t)^{-n} \sum_{i=0}^{n-1} u_i^{0,q}(0) t^i - \int_{M'} (4\pi)^{-n} u_n^{0,q}(x,0)\,dv - D^{0,q}(N) - \frac{1}{2}(4\pi)^{-n} a_n^{0,q}(1) \log\frac{1}{t} \Big| \le Ct$$

where $C > 0$ *is a constant independent of* t*.*

Clearly, Proposition 8.1 also holds for the Laplacian on $(0,q)$-forms with coefficients in holomorphic Hermitian vector bundle (E,h) if $(E,h) \cong (\mathbb{C}^r, \text{flat metric})$ on a neighbourhood of Sing M_0.

Our main concern is the determinant of Laplacian (cf. [S]) *i.e.*,

$$\log\det\Box^{0,q} := -\frac{d}{ds}\Big|_{s=0}\zeta^{0,q}(s), \tag{8.7}$$

$$\zeta^{0,q}(s) = \frac{1}{t(s)}\int_0^\infty t^{s-1}\big(\operatorname{Tr} e^{-t\,\Box^{0,q}} - \dim H^q(M,\mathcal{O}(E))\big)\,dt, \tag{8.8}$$

$$\log T := \sum_{q=0}^{n}(-1)^{q+1}q\log\det\Box^{0,q}. \tag{8.9}$$

THEOREM 8.1. — *Let $\{(M,I_\varepsilon,g_\varepsilon)\}$ be a conic degenerating family of Kähler manifolds with tangent cone $C(N)$, $\{(E,J_\varepsilon,h_\varepsilon)\}$ a family of holomorphic vector bundle on the family such that $(E,J_\varepsilon,h_\varepsilon) = (\mathbb{C}^n,h)$, h a flat metric outside of M', and converges to a holomorphic Hermitian vector bundle (E_0,J_0,h_0) on M_0. Suppose the Sobolev inequality is uniform in ε for $\{(\widehat{M}, I_\varepsilon\times I_\varepsilon, g_\varepsilon\otimes g_\varepsilon)\}$ i.e., for any $\varphi\in\Gamma(\widehat{M}_\varepsilon,\widehat{\Lambda}^{0,q})$,*

$$\|\varphi\|_{\varepsilon\frac{2\nu}{\nu-1}} \le C\big\{(\widehat{\Box}^{0,q}_\varepsilon\varphi,\varphi)_\varepsilon + \|\varphi\|^2_{\varepsilon,2}\big\}$$

where $\nu>1$, $C>0$ are constants independent of ε,φ.

Then, as $\varepsilon\to 0$, the following asymptotic formula holds:

$$\log\det\Box^{0,q}_\varepsilon = (4\pi)^{-n}a^{0,q}_n(1)(\log\varepsilon)^2 + 2i^{0,q}(X)\log\varepsilon + \sum_{0<\lambda_i(\varepsilon)<1}\log\lambda^{0,q}_i(\varepsilon) + o(\log\varepsilon)$$

where

$$i^{0,q}(X) = D^{0,q}(N) - \lim_{T\to\infty}\Big(\int_{B(T)}(4\pi)^{-n}u^{0,q}_n(x)\,dv_X - (4\pi)^{-n}a^{0,q}_n(1)\log T\Big) - \frac{1}{2}(4\pi)^{-n}\Gamma'(1)a^{0,q}_n(1).$$

Here (X,g) is the AC manifold associated to the family $\{(M,I_\varepsilon,g_\varepsilon)\}$.

In the sequel, we omit $(0,q)$ for simplicity. Put

$$\begin{aligned} I(T) := &\int_{X-B(\sqrt{T})}\Big\{\operatorname{tr}K(T,x,x) - (4\pi T)^{-n}\sum_{i=0}^{n}u_i(x)T^i\Big\}\,dv_X \\ &+ \int_{B(\sqrt{T})}\Big\{\operatorname{tr}K(T,x,x) - (4\pi T)^{-n}\sum_{i=0}^{n-1}u_i(x)T^i\Big\}\,dv_X. \end{aligned} \tag{8.10}$$

THEOREM 8.2. — *Under the situation of Theorem 8.1,*

$$\lim_{T\to\infty}I(T) = D(N).$$

LEMMA 8.1. — *Under the situation of Theorem 8.1, there exist* $C_0, C_1 > 0$ *such that*

1) $\#\{\lambda_i(\varepsilon); \lambda_i(\varepsilon) \leq 1\} \leq C_0$;

2) $\sum_{\lambda > 1} \exp(-t\lambda_i(\varepsilon)) \leq C_1 t^{-\nu} \quad (t > 0)$.

Proof. — By the uniformity of the Sobolev inequality, using Theorem 6.1 and its corollary, there exists $C > 0$ such that

$$\lambda_k(\varepsilon) + \frac{1}{2} \geq C k^{\frac{1}{\nu}}. \tag{8.11}$$

Assertions follow immediately from (8.11). ∎

LEMMA 8.2. — *Let* $F_k(t, x, y)$ *be the same as (7.3). Then, for* $k \geq n$ *and* $t \in [0, 1]$,

$$\left| \int_{X - B(\varepsilon^{-1})} F_k(\varepsilon^{-2} t, x, x)\, dv_X \right| \leq C_k t^{k+1}.$$

Proof. — Since $t \in [0, 1]$ and $x \in X - B(\varepsilon^{-1})$, we have $0 \leq \varepsilon^{-2} t \leq 1 + |x|^2$, and apply Theorem 7.1 to yield

$$\begin{aligned} \int_{X - B(\varepsilon^{-1})} |F_k(\varepsilon^{-2} t, x, x)|\, dv &\leq C_k (\varepsilon^{-2} t)^{k+1} \int_{X - B(\varepsilon^{-1})} (1 + |x|^2)^{-(k+1+n)}\, dv \\ &\leq C_k (\varepsilon^{-2} t)^{k+1} \int_{\varepsilon^{-1}}^{\infty} r^{-2(k+1+n)} r^{2n-1}\, dr \\ &\leq C_k t^{k+1}. \end{aligned} \tag{8.12}$$

∎

Put

$$\begin{aligned} A(\varepsilon, t) := \operatorname{Tr} \exp(-t \square_\varepsilon) &- (4\pi t)^{-n} \sum_{i=0}^{n-1} u_i(\varepsilon) t^i \\ &- \int_{B(\varepsilon^{-1}) - B(\varepsilon, \sqrt{t})} (4\pi)^{-n} u_n(x)\, dv_X - I(\varepsilon^{-2} t) \\ &- \int_{M'} (4\pi)^{-n} u_n(x, \varepsilon)\, dv_\varepsilon. \end{aligned} \tag{8.13}$$

PROPOSITION 8.2. — *For any* $t \in [0, 1]$,

$$|A(\varepsilon, t)| \leq C t$$

where C *is a constant independent of* ε.

Proof. — Let $L_\varepsilon(t,x,y) = \varepsilon^{-2n}K(\varepsilon^{-2}t,x,y)$ be the heat kernel of the Laplacian on $(0,q)$-forms on $(X,\varepsilon^2 g_X)$. By Duhamel's principle (Theorem 5.2), we get

$$\begin{aligned}\int \operatorname{tr} K_\varepsilon(t,x,x)\,dv_\varepsilon &= \int_{B(\varepsilon^{-1})} \operatorname{tr} K_\varepsilon(t,x,x)\,dv_\varepsilon + \int_{M'} \operatorname{tr} K_\varepsilon(t,x,x)\,dv_\varepsilon \\ &= \int_{B(\varepsilon^{-1})} \operatorname{tr} L_\varepsilon(t,x,x)\,dv_\varepsilon + \int_{M'} \operatorname{tr} K_\varepsilon(t,x,x)\,dv_\varepsilon + O(e^{-\frac{c}{t}}) \\ &= \int_{B(\varepsilon^{-1})} \operatorname{tr} K(\varepsilon^{-2}t,x,x)\,dv_\varepsilon + \int_{M'} \operatorname{tr} K_\varepsilon(t,x,x)\,dv_\varepsilon + O(e^{-\frac{c}{t}})\end{aligned} \tag{8.14}$$

which, combined with the definition of $A(\varepsilon,t)$, yields

$$\begin{aligned}A(\varepsilon,t) &= \int_{B(\varepsilon^{-1})} \Big\{ \operatorname{tr} K(\varepsilon^{-2}t,x,x) - (4\pi\varepsilon^{-2}t)^{-n}\sum_{i=0}^{n-1} u_i(x)(\varepsilon^{-2}t)^i \Big\}\,dv_X \\ &\quad - \int_{B(\varepsilon^{-1})-B(\varepsilon^{-1}\sqrt{t})} (4\pi t)^{-n}u_n(x)\,dv - I(\varepsilon^{-2}t) \\ &\quad + \int_{M'} \Big\{ \operatorname{tr} K_\varepsilon(t,x,x) - (4\pi t)^{-n}\sum_{i=0}^{n} u_i(x)t^i \Big\}\,dv_\varepsilon + O(e^{-\frac{c}{t}}) \\ &= \int_{B(\varepsilon^{-1})-B(\varepsilon^{-1}\sqrt{t})} F_n(\varepsilon^{-2}t,x,x)\,dv + \int_{B(\varepsilon^{-1}\sqrt{t})} F_{n-1}(\varepsilon^{-2}t,x,x)\,dv \\ &\quad - I(\varepsilon^{-2}t) + O(t) \\ &= \int_{B(\varepsilon^{-1})-B(\varepsilon^{-1}\sqrt{t})} F_n(\varepsilon^{-2}t,x,x)\,dv - \int_{X-B(\varepsilon^{-1}\sqrt{t})} F_n(\varepsilon^{-2}t,x,x)\,dv + O(t) \\ &= \int_{X-B(\varepsilon^{-1}\sqrt{t})} F_n(\varepsilon^{-2}t,x,x)\,dv + O(t)\end{aligned} \tag{8.15}$$

where $O(t)$, $O(e^{-\frac{c}{t}})$ are uniform in ε.

The assertion follows from (8.15) by applying Lemma 8.2 with $k=n$. ∎

Put

$$B(\varepsilon,t) := A(\varepsilon,t) + I(\varepsilon^{-2}t). \tag{8.16}$$

Proposition 8.3. — *For any $t \in (0,1]$,*

$$\begin{aligned}\lim_{\varepsilon\to 0} B(\varepsilon,t) &= \operatorname{Tr}\exp(-t\,\square_0^{0,q}) - (4\pi t)^{-n}\sum_{i=0}^{n-1} u_i(0)t^i \\ &\quad - \int_{M'} (4\pi)^{-n}u_n(x,0)\,dv_0 - \frac{1}{2}(4\pi)^{-n}a_n(1)\log\frac{1}{t}.\end{aligned}$$

Proof. — By Corollary 6.2, we have

$$\lim_{\varepsilon\to 0} \operatorname{Tr}\exp(-t\,\square_\varepsilon) = \operatorname{Tr}\exp(-t\,\square_0). \tag{8.17}$$

Since $\{(M, g_\varepsilon)\}$ is a conic degeneration, we get

$$\lim_{\varepsilon \to 0} u_i(\varepsilon) = u_i(0) \;\; (i < n), \tag{8.18}$$

$$\lim_{\varepsilon \to 0} \int_{M'} (4\pi)^{-n} u_n(x, \varepsilon) dv_\varepsilon = \int_{M'} (4\pi)^{-n} u_n(x, 0)\, dv_0, \;\; (i < n). \tag{8.19}$$

As

$$u_n(x) = r^{-2n} a_n(z) + O(r^{-2(n+\delta)}) \tag{8.20}$$

in the polar coordinates of $C(N)$,

$$\lim_{\varepsilon \to 0} \int_{B(\varepsilon^{-1}) - B(\varepsilon^{-1}\sqrt{t})} (4\pi)^{-n} u_n(x)\, dv = \frac{1}{2} (4\pi)^{-n} a_n(1) \log \frac{1}{t}. \tag{8.21}$$

■

PROPOSITION 8.4. — *For any $t \in (0, 1]$,*

$$\lim_{\varepsilon \to 0} |I(\varepsilon^{-2} t) - D(N)| \leq Ct$$

where C is a constant independent of ε and t.

Proof. — Since $I(\varepsilon^{-2} t) = B(\varepsilon, t) - A(\varepsilon, t)$, applying Proposition 8.2,

$$\begin{aligned} |I(\varepsilon^{-2} t) - D(N)| &= |B(\varepsilon, t) - A(\varepsilon, t) - D(N)| \\ &\leq |B(\varepsilon, t) - D(N)| + |A(\varepsilon, t)| \\ &\leq |B(\varepsilon, t) - D(N)| + Ct. \end{aligned} \tag{8.22}$$

Therefore, by Propositions 8.1 and 8.3, we have

$$\lim_{\varepsilon \to 0} |I(\varepsilon^{-2} t) - D(N)| \leq Ct.$$

■

Proof of Theorem 8.2. — Let μ be an arbitrary given number, $\{T_n\}$, $T_n \to \infty$ an arbitrary sequence. Put $\varepsilon_n := \sqrt{\mu / T_n}$. By Proposition 8.4,

$$\lim_{n \to \infty} |I(\varepsilon_n^{-2} \mu) - D(N)| \leq C\mu \tag{8.23}$$

Therefore, there exists $n(\mu)$ such that, for $n > n(\mu)$

$$|I(\varepsilon_n^{-2} \mu) - D(N)| \leq 2C\mu, \tag{8.24}$$

which implies

$$|I(T_n) - D(N)| \leq 2C\mu. \tag{8.24}$$

■

PROPOSITION 8.5.

$$\lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon^{-2}} \int_{\varepsilon^2}^{1} \frac{dt}{t} I(\varepsilon^{-2} t) = D(N).$$

Proof. — As

$$\frac{1}{\log \varepsilon^{-2}} \int_{\varepsilon^2}^{1} \frac{dt}{t} I(\varepsilon^{-2} t) = \int_0^1 I(\varepsilon^{-2\sigma})\, d\sigma \tag{8.26}$$

by setting $\sigma = 1 - \frac{\log t}{\log \varepsilon^{-2}}$, the assertion follows from Theorem 8.2, applying the Lebesgue convergent theorem. ∎

Proof of Theorem 8.1. — Put

$$\delta_0(\varepsilon) := \int_0^{\varepsilon^2} \Big\{ \operatorname{Tr} \exp(-t\,\Box_\varepsilon) - (4\pi t)^{-n} \sum_{i=0}^{n} u_i(\varepsilon) t^i \Big\} \frac{dt}{t} \tag{8.27}$$

$$\delta_1(\varepsilon) := \int_{\varepsilon^2}^{1} \Big\{ \operatorname{Tr} \exp(-t\,\Box_\varepsilon) - (4\pi t)^{-n} \sum_{i=0}^{n} u_i(\varepsilon) t^i \Big\} \frac{dt}{t} \tag{8.28}$$

$$\delta_2(\varepsilon) := (4\pi t)^{-n} \sum_{i=0}^{n-1} \frac{u_i(\varepsilon)}{n-i} - (4\pi)^{-n} \Gamma'(1) u_n(\varepsilon) + \Gamma'(1) h^q(E) \tag{8.29}$$

$$\delta_3(\varepsilon) := \int_1^{\infty} \frac{dt}{t} \operatorname{Tr} \exp(-t\,\Box_\varepsilon) - h^q(E). \tag{8.30}$$

Then,

$$\zeta_\varepsilon'(0) = \delta_0(\varepsilon) + \delta_1(\varepsilon) + \delta_2(\varepsilon) + \delta_3(\varepsilon). \tag{8.31}$$

By Duhamel's principle (Theorem 5.2) and Theorem 7.1, we get

$$\begin{aligned} |\delta_0(\varepsilon)| \leq & \Big| \int_0^{\varepsilon^2} \frac{dt}{t} \int_{B(\varepsilon^{-1})} \Big\{ \operatorname{tr} K(\varepsilon^{-2} t, x, x) - (4\pi\varepsilon^{-2} t)^{-n} \sum_{i=0}^{n} u_i(x) (\varepsilon^{-2} t)^i \Big\}\, dv_X \Big| \\ & + \Big| \int_0^{\varepsilon^2} \frac{dt}{t} \int_{M'} \Big\{ \operatorname{tr} K_\varepsilon(t, x, x) - (4\pi t)^{-n} \sum_{i=0}^{n} u_i(x, \varepsilon) t^i \Big\}\, dv_\varepsilon \Big| \\ & + C \int_0^{\varepsilon^2} e^{-\frac{\varepsilon}{t}} \frac{dt}{t} \\ \leq & C \int_0^{\varepsilon^2} \varepsilon^{-2} t \frac{dt}{t} + C \int_0^{\varepsilon^2} t \frac{dt}{t} + C e^{-\frac{C}{\varepsilon^2}} \leq C. \end{aligned} \tag{8.32}$$

By Propositions 8.2 and 8.5,

$$\begin{aligned} \delta_1(\varepsilon) = & \int_\varepsilon^1 \frac{dt}{t} A(\varepsilon, t) - \int_{\varepsilon^2}^1 \frac{dt}{t} \int_{B(\varepsilon\sqrt{t})} (4\pi)^{-n} u_n(x)\, dv_X + \int_{\varepsilon^2}^1 \frac{dt}{t} I(\varepsilon^{-2} t) \\ = & - \int_1^{\varepsilon_2} \frac{dt}{t} \int_{B(\sqrt{t})} (4\pi)^{-n} u_n(x)\, dv_X + (D(N) + o(1)) \log \varepsilon^{-2} + O(1) \\ = & - (4\pi)^{-n} a_n(1) \Big(\log \frac{1}{\varepsilon} \Big)^2 + D(N) \log \varepsilon^{-2} \\ & - \int_1^{\varepsilon^{-2}} \frac{dt}{t} \Big\{ \int_{B(\sqrt{t})} (4\pi)^{-n} u_n(x)\, dv - \frac{(4\pi)^{-n}}{2} a_n(1) \log t \Big\} + o(\log \varepsilon) \end{aligned} \tag{8.33}$$

$$= -(4\pi)^{-n}a_n(1)\Big(\log\frac{1}{\varepsilon}\Big)^2$$
$$+\Big\{D(N) - \lim_{T\to\infty}\Big(\int_{B(T)}(4\pi)^{-n}u_n(x)\,dv_X - (4\pi)^{-n}a_n(1)\log T\Big)\Big\}\log\varepsilon^{-2}$$
$$+\,o(\log\varepsilon).$$

We obtain

$$\delta_2(\varepsilon) = (4\pi)^{-n}\sum_{i=0}^{n-1}\frac{u_i(0)}{n-1} - (4\pi)^{-n}\Gamma'(1)a_n(1)\log\frac{1}{\varepsilon} + \Gamma'(1)h^q(E) \tag{8.34}$$

by the definition of conic degeneration. And by Lemma 8.1, we have

$$\begin{aligned}\delta_3(\varepsilon) &= \sum_{0<\lambda\leq 1}\int_1^\infty\frac{dt}{t}\exp(-t\lambda_i(\varepsilon)) + \sum_{\lambda>1}\int_1^\infty\frac{dt}{t}\exp(-t\lambda_i(\varepsilon))\\ &= \sum_{0<\lambda\leq 1}\log\frac{1}{\lambda_i(\varepsilon)} + O(1).\end{aligned} \tag{8.35}$$

Combining (8.31)–(8.35), we obtain the formula. ■

To describe the behaviour of analytic torsion, introduce the following function.

Definition 8.3. — *Let (M^n, g) be a compact Kähler manifold with possibly conical singularities (*cf. *[Y 1]), (E, h) a holomorphic Hermitian vector bundle. Define*

$$Z(s) := \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)\big\{\zeta^{0,q}(s) + \dim H^{0,q}_{(2)}(E)\big\}.$$

Proposition 8.6. — *$Z(s)$ is regular at $s = 0$.*

Proof. — See [Y 1] and [Y 3]. Note that the assertion is equivalent to the following:

$$\begin{aligned}\operatorname*{Res}_{s=0} Z(s) &= \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)a_n^{0,q}(0)\\ &= \sum_{q=0}^{n}(-1)^{q+1}q a_n^{0,q}(0)\\ &= 0.\end{aligned} \tag{8.36}$$

■

Theorem 8.3. — *Under the situation of Theorem 8.1 with the uniform Sobolev inequality for $q < n$,*

$$\log T(\varepsilon) = (Z(0,0) - Z(0,\varepsilon))\log\varepsilon^2 + \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)\sum_{0<\lambda_i\leq 1}\log\lambda_i^{0,q}(\varepsilon) + o(\log\varepsilon),$$

where $Z(0,\varepsilon)$ stands for the Z-function for $(M_\varepsilon, E_\varepsilon) = ((M, I_\varepsilon, g_\varepsilon), (E, J_\varepsilon, h_\varepsilon))$.

Proof. — By Theorem 8.1 and Proposition 8.6,

$$\begin{aligned}\log T(\varepsilon) =& \Big(\sum_{q=0}^{n-1}(-1)^{q+1}(q-n)D^{0,q}(N) \\ & - \lim_{T\to\infty}(4\pi)^{-n}\int_{B(T)}\sum_{q=0}^{n-1}(-1)^{q+1}(q-n)u_n^{0,q}(x)\,dv\Big)\log\varepsilon^2 \\ & + \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)\sum_{0<\lambda_i\leq 1}\log\lambda_i^{0,q}(\varepsilon) + o(\log\varepsilon).\end{aligned} \tag{8.37}$$

But we have

$$\begin{aligned}&\sum_{q=0}^{n-1}(-1)^{q+1}(q-n)D^{0,q}(N) - \lim_{T\to\infty}(4\pi)^{-n}\int_{B(T)}\sum_{q=0}^{n-1}(-1)^{q+1}(q-n)u_n^{0,q}(x)\,dv \\ &= \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)\Big\{D^{0,q}(N) + (4\pi)^{-n}\int_{M'}u_n^{0,q}(x,0)\,dv_0\Big\} \\ &\quad - \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)(4\pi)^{-n}u_n^{0,q}(\varepsilon) + O(\varepsilon) \\ &= Z(0,0) - Z(0,\varepsilon) + O(\varepsilon),\end{aligned} \tag{8.38}$$

since by Propositions 8.1 and 8.6,

$$Z(s,0) = \frac{1}{\Gamma(s)}f(s) + \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)\Big\{D^{0,q}(N) + (4\pi)^{-n}\int_{M'}u_n^{0,q}(x)\,dv\Big\} \tag{8.39}$$

for some $f \in \mathcal{O}_{\mathbb{C},0}$, and by [G],

$$Z(s,\varepsilon) = \frac{1}{\Gamma(s)}f_\varepsilon(s) + \sum_{q=0}^{n-1}(-1)^{q+1}(q-n)(4\pi)^{-n}u_n^{0,q}(\varepsilon) \tag{8.40}$$

for some $f_\varepsilon \in \mathcal{O}_{\mathbb{C},0}$. ∎

9. Rationality of $Z(0)$ for IHHS

Let $(M^n, g, 0)$ be a Kähler manifold with a conical singularity. In this section, we always assume that $(\mathcal{O}_{M,0}, 0)$ is an isolated homogeneous hypersurface singularity (IHHS), and g is Euclidean around 0, *i.e.*, there exist a neighborhood U of 0, an embedding $i : (U,0) \hookrightarrow (\mathbb{C}^{n+1}, 0)$, and a homogeneous polynomial $F(z) \in \mathbb{C}[z_0, \dots, z_n]$ such that

1) $(U, 0) \cong \{z \in \mathbb{C}^{n+1};\ F(z) = 0, \|z\| < 1\}$.

2) $g = i^* \partial\bar{\partial}\|z\|^2$.

Put $Y := \{[z] \in \mathbb{P}^n; F(z) = 0\}$. As $\dim \operatorname{Sing} M = 0$, Y is non-singular. Let $L := \mathcal{O}_{\mathbb{P}^n}(-1)$ be the tautological line bundle of $\mathbb{P}^n$ and $L_Y := L_{|Y}$ its restriction to Y. Since L_Y is negative, one can contract its zero section Z_Y ($\cong Y$) to obtain a complex space $(C_Y, 0)$ with an isolated singularity. The blowing-down map is denoted by $p : L_Y \to C_Y$. Then, $(C_Y, 0)$ is nothing but the affine cone over $Y : C_Y = \{z \in \mathbb{C}^{n+1}; F(z) = 0\}$.

Since the construction of the resolution $p : L_Y \to C_Y$ is local, one obtain a natural resolution $\tilde{p} : (\widetilde{M}, Y) \to (M, 0)$.

Theorem 9.1. — *There exist $f_n(x_1, \ldots, x_n) \in \mathbb{C}[x]$ and $g_n(y_1, \ldots, y_{n-1}, z) \in \mathbb{Q}[y, z]$ such that, for $(M, g, 0)$ as above,*

1) $Z(0, \widetilde{M}) = f_n(c(\widetilde{M}))[\widetilde{M}]$

2) $Z(0, M) - Z(0, \widetilde{M}) = g_n(c(Y), c_1(L_Y))[Y]$.

where $c_i(\cdot)$ stands for the i-th Chern class.

First we remark 1) is essentially proved in the work of Bismut, Gillet and Soulé ([B-G-S 2], [G-S 2], [F], [Y3]). Therefore, we only prove 2). For the proof, we construct a family of Kähler metrics $\{g_\alpha\}$ ($\alpha \leq 2$) on $\widetilde{M}$ as follows:

$$g_\alpha = \begin{cases} p^* g & \text{on } \widetilde{M} - p^{-1}(U) \\ \partial\bar{\partial}\|z\|^2 + \alpha^2 \partial\bar{\partial} \log \|z\|^2 & \text{on } p^{-1}(V), \end{cases} \tag{9.1}$$

where $V := U \cap \mathbb{B}(\frac{1}{2})$ by the embedding. Put

$$(X, g_X) = \left(L_Y, \partial\bar{\partial}\|z\|^2 + \partial\bar{\partial} \log \|z\|^2\big|_{C_Y}\right). \tag{9.2}$$

Then it is a AC kähler manifold in the sense of § 8 with tangent cone $(C_Y, \partial\bar{\partial}\|z\|^2\big|_{C_Y})$. By coordinate change $w = \alpha^{-1} z$,

$$\partial\bar{\partial}\|z\|^2 + \alpha^2 \partial\bar{\partial} \log \|z\|^2 = \alpha^2 \left(\partial\bar{\partial}\|w\|^2 + \partial\bar{\partial} \log \|w\|^2\right) = \alpha^2 g_X. \tag{9.3}$$

Therefore, the family $\{(\widetilde{M}, g_\alpha)\}$ is a conic degeneration of Kähler manifold. Note that for this family, complex structure is constant. By Theorem 6.2, the Sobolev inequality holds uniformly for the family, and one can check there appears no small eigenvalues for this family as in [Y 1], § 5:

$$\sum_{0 < \lambda_i^{0,q}(\alpha) \leq 1} \lambda_i^{0,q}(\alpha) \geq C > 0. \tag{9.4}$$

By Theorem 8.3, we obtain the following:

Proposition 9.1. — *As $\alpha \to 0$,*

$$\log T(\alpha) = \left(Z(0, 0) - Z(0, \alpha)\right) \log \alpha^2 + o(\log \alpha).$$

Let $\mu(t)$ be a positive function on $\mathbb{R}_+$ such that $\mu(t) = t$ for $t \leq \frac{1}{4}$ and $\mu(t) = \frac{1}{2}$ for $t \geq \frac{3}{4}$. Put

$$G_{t,\varepsilon'} := g_{\mu(|\varepsilon|^2)+t} \tag{9.5}$$

for $\varepsilon \in \mathbb{P}^1$ and $t \in (0,1]$. For $\pi : \mathcal{M} := \widetilde{M} \times \mathbb{P}^1 \to \mathbb{P}^1$, a trivial family on $\mathbb{P}^1$, $G_t := \{G_{t,\varepsilon}\}_{\varepsilon\in\mathbb{P}^1}$ defines a Hermitian metric on $T\widetilde{M}$ ($\subset T\mathcal{M}$). Let R_t be the curvature of $(T\widetilde{M}, G_t)$. Combined with the theorem of Bismut, Gillet and Soulé (cf. [B-G-S]) and Proposition 9.1, we obtain the following:

Proposition 9.2.

$$Z(0,0) - Z(0,1) - \int_{\mathcal{M}} Td(R_0)^{(n+1,n+1)} = 0.$$

Proof. — Since the determinant bundle $\det R\pi_*\mathcal{O}_{\mathcal{M}}$ is trivial, its degree is zero. By Proposition 9.1, the Quillen metric defines a generalized Hermitian metric on the determinant line bundle. Then, as $\pi_*(Td(R_0))^{(1,1)}$ extends to a smooth $(1,1)$-form on $\mathbb{P}^1$ (cf. [Y 1], Proposition 1.1), by the same argument as [B-B], Proposition 10.1, the curvature current of the determinant bundle is represented as follows:

$$c_1 = -\Big(Z(0,0) - Z(0,1)\Big)\delta_0 + \pi_*(Td(R_0))^{(1,1)}. \tag{9.6}$$

Note that $Z(0,\varepsilon) = Z(0,1)$ for any $\varepsilon \neq 0$ by Theorem 9.1 1). Integrating (9.6) on $\mathbb{P}^1$, we obtain

$$\begin{aligned} \deg(\det R\pi_*\mathcal{O}_{\mathcal{M}}) &= Z(0,0) - Z(0,1) - \int_{\mathcal{M}} Td(R_0)^{(n+1,n+1)} \\ &= 0 \end{aligned} \tag{9.7}$$

■

Lemma 9.1.

$$\int_{\mathcal{M}} Td(R_0) = -\lim_{\delta\to 0}\int_0^1 dt \int_{p^{-1}(U)}\int_{|\varepsilon|=\delta} \bar\partial Td'(R_t; G_t^{-1}\frac{d}{dt}G_t)$$

where $Td'(A;B) := \frac{d}{du}\Big|_{u=0} Td(\frac{i}{2\pi}(A+uB))$.

Proof. — By the Bott-Chern formula ([B-C], Proposition 3.15), we get

$$Td(R_1) - Td(R_0) = \bar\partial\partial \int_0^1 Td'(R_t; G_t^{-1}\frac{d}{dt}G_t) \tag{9.8}$$

on $\pi^{-1}(\mathbb{P}^1) - D(\delta))$ for any $\delta > 0$. Therefore, we obtain

$$\begin{aligned} \int_{\mathbb{P}^1 - D(\delta)}\int_{\widetilde{M}} Td(R_1) - Td(R_0) &= \int_{\mathbb{P}^1-D(\delta)}\int_{\widetilde{M}} \bar\partial\partial \int_0^1 Td'(R_t; G_t^{-1}\frac{d}{dt}G_t)dt \\ &= -\int_0^1 \int_{\mathbb{P}^1-D(\delta)}\int_{\widetilde{M}} d(\bar\partial Td'(R_t; G_t^{-1}\frac{d}{dt}G_t)) \\ &= \int_0^1 dt \int_{\widetilde{M}\times\partial D(\delta)} \bar\partial Td'(R_t; G_t^{-1}\frac{d}{dt}G_t). \end{aligned} \tag{9.9}$$

By taking the limit $\delta \to 0$, we have

$$\int_{\mathcal{M}} Td(R_1) - Td(R_0) = \lim_{\delta\to 0}\int_0^1 dt \int_{p^{-1}(U)} \int_{|\varepsilon|=\delta} \bar\partial Td'(R_t; G_t^{-1}\frac{d}{dt}G_t) + \lim_{\delta\to 0}\int_0^1 dt \int_{(M-U)\times\partial D(\delta)} \bar\partial Td'(R_t; G_t^{-1}\frac{d}{dt}G_t). \tag{9.10}$$

Since $G_{t,\varepsilon}$ is non-degenerate on $(M-U)\times\mathbb{P}^{-1}$, we get

$$\left|\bar\partial Td'(R_t; G_t^{-1}\frac{d}{dt}G_t)\right| \le C \tag{9.11}$$

on $[0,1]\times(M-U)\times\mathbb{P}^{-1}$, which yields

$$\begin{aligned} \Big|\int_{(M-U)\times\partial D(\delta)} \bar\partial Td'(R_t; G_t^{-1}\frac{d}{dt}G_t)\Big| &\le C\cdot \mathrm{vol}\Big((M-U)\times\partial D(\delta)\Big) \\ &\le C\delta. \end{aligned} \tag{9.12}$$

As G_1 defines a non-degenerate Hermitian metric on $T\widetilde{M}$, we obtain

$$0 = \deg(\det R\pi_*\mathcal{O}_{\mathcal{M}}) = \int_{\mathcal{M}} Td(R_1), \tag{9.13}$$

which, combined with (9.10) and (9.12), yields the formula. ∎

Now, regard $p^{-1}(U)$ as an open set of L_Y on which the following exact sequence holds:

$$0 \longrightarrow \pi^* L_Y \longrightarrow TL_Y \longrightarrow \pi^* TY \longrightarrow 0. \tag{9.14}$$

Note that $\pi^* L_Y$ has a holomorphic section corresponding to the Euler vector field which vanishes exactly along Z_Y. By the definition (9.3), the metric of TL_Y is represented by the following form with respect to (9.14):

$$g_{TL_Y} = \pi^* g_{L_Y} \oplus (1+\|z\|^2)\pi^* g_{TY} \tag{9.15}$$

where g_{L_Y} is the induced metric from L, and g_{TY} the restriction of the Fubini-Study metric of $\mathbb{P}^n$. Identify $(\pi^* L)^{\perp}$ with $\pi^* TY$ by the projection.

Lemma 9.2. — *On $p^{-1}(U)$,*

$$G_t^{-1}\frac{d}{dt}G_t = \frac{1}{\|z\|^2+|\varepsilon|^2+t}P$$

where $P : TL_Y \to (\pi^* L)^{\perp}$ is the orthogonal projection.

Proof. — By (9.1) and (9.3), on $p^{-1}(U)$,

$$\begin{aligned} G_t &= \frac{\partial\|z\|^2\bar\partial\|z\|^2}{\|z\|^2} + (\|z\|^2+|\varepsilon|^2+t)\partial\bar\partial\log\|z\|^2 \\ &= \pi^* g_{L_Y} + (\|z\|^2+|\varepsilon|^2+t)\pi^* g_{TY}. \end{aligned} \tag{9.16}$$

By differentiating (9.16) by t, we obtain the formula. ∎

PROPOSITION 9.3.

$$\int_{\mathcal{M}} Td(R_0)^{(n+1,n+1)} = -\int_{L_Y} \frac{1}{1+\|z\|^2} Td'(R;P)^{(n,n)}$$

where R is the curvature of TL_Y with respect to the metric (9.3).

Proof. — Put

$$Td'(R_t; G_t^{-1}\frac{d}{dt}G_t) = A_{n,n} + A_{n-1,n}\wedge\partial|\varepsilon|^2 + A_{n,n-1}\wedge\bar{\partial}|\varepsilon|^2 + A_{n-1,n-1}\wedge\partial\bar{\partial}|\varepsilon|^2 \quad (9.17)$$

for some $A_{p,q}(\varepsilon) \in A^{p,q}(L_Y)$. Then,

$$\begin{aligned}
I(\delta) &:= \int_0^1 \int_{p^{-1}(U)} \int_{|\varepsilon|=\delta} \bar{\partial} Td'(R_t; G_t^{-1}\frac{d}{dt}G_t) \\
&= \int_0^1 dt \int_{p^{-1}(U)} \int_{|\varepsilon|=\delta} \bar{\partial}(A_{n,n} + A_{n-1,n}\wedge\partial|\varepsilon|^2 + A_{n,n-1}\wedge\bar{\partial}|\varepsilon|^2 \\
&\qquad\qquad + A_{n-1,n-1}\wedge\partial\bar{\partial}|\varepsilon|^2) \\
&= \int_0^1 dt \int_{p^{-1}(U)} \int_{|\varepsilon|=\delta} \bar{\partial}_\varepsilon A_{n,n} + \bar{\partial}_L A_{n,n-1}\wedge\bar{\partial}|\varepsilon|^2 \qquad (9.18)\\
&= \int_0^1 dt\Big\{ \int_{|\varepsilon|=\delta} \bar{\partial}_\varepsilon\Big(\int_{p^{-1}(U)} A_{n,n}\Big) + \int_{p^{-1}(U)} d_L A_{n,n-1}(\delta)\Big\} \\
&= \int_0^1 dt\Big\{ \int_{|\varepsilon|=\delta} \bar{\partial}_\varepsilon\Big(\int_{p^{-1}(U)} A_{n,n})\Big) + \int_{\partial p^{-1}(U)} A_{n,n-1}(\delta)\Big\}
\end{aligned}$$

where $A_{n,n-1}(\delta) := \int_{|\varepsilon|=\delta} A_{n,n-1}\wedge\bar{\partial}|\varepsilon|^2$. Since $A_{n,n} = Td'(R_t|_{\widetilde{M}\times\{\varepsilon\}}; G_t^{-1}\frac{d}{dt}G_t)$ and

$$G_t = \partial\bar{\partial}\|z\|^2 + (|\varepsilon|^2 + t)\partial\bar{\partial}\log\|z\|^2 \quad (9.19)$$

on $p^{-1}(U)$ for $\delta \ll 1$, we have

$$\begin{aligned}
\int_{p^{-1}(U)} A_{n,n} &= \int_{\|z\|\le 1} Td'(R_t|_{\widetilde{M}\times\{\varepsilon\}}; \frac{1}{\|z\|^2+|\varepsilon|^2+t}P) \\
&= \int_{\|w\|\le \frac{1}{\sqrt{|\varepsilon|^2+t}}} \frac{1}{(|\varepsilon|^2+t)(1+\|w\|^2)} \frac{i}{2\pi} Td'(R;P) \qquad (9.20)\\
&= \frac{1}{|\varepsilon|^2+t}\{f(\infty) - f(|\varepsilon|^2+t)\}
\end{aligned}$$

where

$$f(r) = \int_{\|w\|\ge 1/\sqrt{r}} \frac{1}{1+\|w\|^2}\frac{i}{2\pi} Td'(R;P). \quad (9.21)$$

As (L_Y, g_{TY}) is asymptotically conical, we find $|f'(r)| \leq C$ as $r \to 0$. Therefore,

$$\begin{aligned}
&\int_0^1 dt \int_{|\varepsilon|=\delta} \overline{\partial}_\varepsilon \Big(\int_{p^{-1}(U)} A_{n,n} \Big) \\
&\quad = \int_0^1 dt \int_{|\varepsilon|=\delta} \overline{\partial}_\varepsilon \Big\{ \frac{1}{|\varepsilon|^2 + t} (f(\infty) - f(|\varepsilon|^2 + t)) \Big\} \\
&\quad = -2\pi i f(\infty) \int_0^1 \frac{\delta^2 dt}{(t+\delta^2)^2} + 2\pi i \int_0^1 f(\delta^2 + t) \frac{\delta^2 dt}{(t+\delta^2)^2} \\
&\qquad + 2\pi i \int_0^1 \frac{f'(\delta^2 + t)}{\delta^2 + t} \delta^2 dt \\
&\quad = -2\pi i f(\infty) + 2\pi i \int_0^\infty \frac{f(\delta^2(1+s))}{(1+s)^2} ds + O(\delta^2 \log \delta^2) \\
&\quad = -2\pi i f(\infty) + o(1) \qquad (\delta \to 0).
\end{aligned} \tag{9.22}$$

Since G_t is a non-degenerate family of metrics on $\partial p^{-1}(U)$, and is of the form (9.19), we get $|A_{n,n-1}(z,\varepsilon)| \leq C$ for any $(z,\varepsilon) \in \partial p^{-1}(U) \times D(1/4)$. Therefore, we have

$$\begin{aligned}
\Big| \int_{\partial p^{-1}(U)} A_{n,n-1}(\delta) \Big| &\leq \int_{\partial p^{-1}(U)} \int_{|\varepsilon|=\delta} |A_{n,n-1} \wedge \overline{\partial} |\varepsilon|^2 | \\
&\leq C\delta^2.
\end{aligned} \tag{9.23}$$

Then, the assertion follows from Lemma 9.1, (9.18), (9.22) and (9.23). ∎

Let us now consider (9.14) as the following exact sequence on $C_Y - \{0\}$:

$$0 \longrightarrow S \longrightarrow TC_Y \longrightarrow \pi^* TY \longrightarrow 0 \tag{9.24}$$

where S is the trivial bundle generated by the Euler vector field.

Let $A \in A^{1,0}(\mathrm{End}(S, S^\perp))$ be the second fundamental form associated to (9.24). Then, the curvature R_h of (TL_Y, h) is represented as follows (cf. (1.5)):

$$R_h = \begin{pmatrix} R_S - A^* \wedge A & -\partial A^* \\ \overline{\partial} A & R(\pi^* TY) - A \wedge A^* \end{pmatrix}. \tag{9.25}$$

Identify $Z = (z_0, \dots, z_n)$ with the Euler vector field $\sum z_i \frac{\partial}{\partial z_i}$, and put $e := Z/\|Z\|$.

Lemma 9.3. — *Suppose $h = \partial\overline{\partial}\|z\|^2\big|_{C_Y}$. Then, on $C_Y - \{0\}$,*

$$R_S = \pi^* R_L = A^* \wedge A, \quad \overline{\overline{\partial} A} = \partial A^* = 0$$
$$R(\pi^* TY) = \pi^* R_{TY} + \pi^* R_L \otimes I = \pi^* R(TY \otimes L)$$

where R_L is the curvature of $(L, \|z\|^2)$ and R_{TY} of $(TY, \partial\overline{\partial} \log \|z\|^2)$.

Proof. — Since

$$\partial\overline{\partial}\|z\|^2 = \frac{\partial\|z\|^2 \overline{\partial}\|z\|^2}{\|z\|^2} + \|z\|^2 \partial\overline{\partial} \log \|z\|^2, \tag{9.26}$$

$$R(S) = \pi^* R_L, \quad R(\pi^* TY) = \pi^* R(TY \otimes L), \tag{9.27}$$

we find

$$R_h e = (R(S) - A^* \wedge A)e + (\overline{\partial} A)e. \tag{9.28}$$

Thus, it is sufficient to show $R_h e = 0$.

Let $\mathbb{C}^{n+1} = T\mathbb{C}^{n+1}\big|_{C_Y}$ be the trivial bundle, and consider the following exact sequence:

$$0 \longrightarrow TC_Y \longrightarrow \mathbb{C}^{n+1} \longrightarrow N \longrightarrow 0 \tag{9.29}$$

where $N = N_{C_Y/\mathbb{C}^{n+1}}$ is the normal bundle of C_Y. Let $P_{C_Y} : T\mathbb{C}^{n+1} \to TC_Y$ be the orthogonal projection. Then, by definition,

$$R_h e = P_{C_Y} dP_{C_Y} de \tag{9.30}$$

where d stands for the exterior differential on $\mathbb{C}^{n+1}$. Since C_Y is defined by a single homogeneous polynomial F, we get

$$P_{C_Y} = I - \frac{\overline{\nabla F} \cdot \nabla F^t}{\|\nabla F\|^2}, \quad \nabla F^t = \Big(\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n}\Big). \tag{9.31}$$

Note $TC_Y = \{v \in T\mathbb{C}^{n+1}; \nabla F^t \cdot v = 0\}$. As

$$de = \frac{dZ}{\|Z\|} - \frac{Zd\|Z\|}{\|Z\|^2}, \tag{9.32}$$

we have

$$\begin{aligned} P_{C_Y} de &= de - \frac{\overline{\nabla F}}{\|\nabla F\|^2} \nabla F^t dZ + \frac{\overline{\nabla F}}{\|\nabla F\|^2} \nabla F^t \cdot Zd\|Z\| \\ &= de - \frac{\overline{\nabla F}}{\|\nabla F\|^2} dF + \deg(F) \cdot F \frac{\overline{\nabla F}}{\|\nabla F\|^2} d\|Z\| \\ &= de, \end{aligned} \tag{9.33}$$

since $dF = F = 0$ on C_Y, and therefore we obtain $P_{C_Y} dP_{C_Y} de = 0$. ∎

By a straightforward computation, we have the following.

LEMMA 9.4. — *Put $a(z) := \frac{1+\|z\|^2}{\|z\|^2}$. Then, when $h = g_{TL_Y}$ and A the same as in Lemma 9.3, the curvature R of (TL_Y, h) is represented as follows:*

$$R = \begin{pmatrix} \pi^* R_L - \frac{1}{a} A^* \wedge A & \partial \log a \wedge A^* \\ -\frac{1}{a} \overline{\partial} \log a \wedge A & \pi^* R(TY \otimes L) + \overline{\partial}\partial \log a \otimes I - \frac{1}{a} A \wedge A^* \end{pmatrix}$$

For the convenience of notations, put

$$x := \frac{1}{1 + \|z\|^2}, \quad u := \frac{\partial \|z\|^2}{\|z\|^2(1 + \|z\|^2)}, \quad v := \frac{\overline{\partial}\|z\|^2}{1 + \|z\|^2}, \quad R_\delta = R + \delta P, \tag{9.34}$$

and

$$\mathrm{Tr}\, R_\delta^k = A_{0,0}(k) + uA_{1,0}(k) + vA_{0,1}(k) + uvA_{11}(k). \tag{9.35}$$

LEMMA 9.5. — *For any $k \geq 0$,*

$$A_{10}(k) = A_{01}(k) = 0.$$

Proof. — By Lemma 9.4,

$$\begin{aligned} R_\delta = &\begin{pmatrix} x\pi^* R_L & 0 \\ 0 & \pi^* R_{TY} + (1-x)\pi^* R_L \otimes I + (1-x)A \wedge A^* + \delta I \end{pmatrix} \\ &+ u \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix} - xv \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + uv \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{9.36}$$

Since

$$A_{10}(k) = \frac{d}{du}\Big|_{u=v=0} \mathrm{Tr}\, R_\delta^k = k\, \mathrm{Tr}\, \frac{d}{du}\Big|_{u=v=0} R_\delta \cdot R_\delta^{k-1}, \tag{9.37}$$

and

$$\frac{d}{du}\Big|_{u=v=0} R_\delta = \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix}, \quad R_\delta\Big|_{u=v=0} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{9.38}$$

we have $A_{10}(k) = 0$. Similarly, we get $A_{01}(k) = 0$. ■

LEMMA 9.6. — *For any $k \geq 0$,*

$$A_{00}(k), A_{11}(k) \in \mathbb{Z}[\pi^* R_L, \mathrm{Tr}(\pi^* R_Y)^k, \mathrm{Tr}(\pi^* R_Y)^l A \wedge A^*, x, \delta]_{k,l \geq 0}.$$

Proof. — Put $\Lambda := \mathbb{Z}[\pi^* R_L, \mathrm{Tr}(\pi^* R_Y)^k, \mathrm{Tr}(\pi^* R_Y)^l A \wedge A^*, x, \delta]$. As

$$R_\delta\Big|_{u=v=0} = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix}, \tag{9.39}$$

we have

$$A_{00}(k) = R_{11}^k + \mathrm{Tr}\, R_{22}^k \equiv \mathrm{Tr}\, R_{22}^k \pmod{\Lambda}. \tag{9.40}$$

Since $R_{22} = \pi^* R_{TY} + (1-x)A \wedge A^* + \lambda I$ ($\lambda \in \Lambda$), we get

$$\begin{aligned} \mathrm{Tr}\, R_{22}^k &\in \sum_{i \leq k} \Lambda \cdot \mathrm{Tr}(\pi^* R_{TY} + (1-x)A \wedge A^*)^i \\ &\in \sum_{|I|=|J| \leq k} \Lambda \cdot \mathrm{Tr}(\pi^* R_{TY})^{i_1}(AA^*)^{j} \cdots (\pi^* R_{TY})^{i_l}(AA^*)^{j_l}. \end{aligned} \tag{9.41}$$

But

$$\begin{aligned} &\mathrm{Tr}(\pi^* R_{TY})^{i_1}(AA^*)^{j_1} \cdots (\pi^* R_{TY})^{i_l}(AA^*)^{j_l} \quad (j_1, \cdots .j_l \geq 1) \\ &\quad = (A^*A)^{|J|-l} \wedge A^*(\pi^* R_{TY})^{i_1} A \wedge \cdots \wedge A^*(\pi^* R_{TY})^{i_l} A \\ &\quad = (A^*A)^{|J|-l} \wedge \mathrm{Tr}(\pi^* R_{TY})^{i_1} AA^* \cdots \mathrm{Tr}(\pi^* R_{TY})^{i_l} AA^* \\ &\quad \in \Lambda. \end{aligned} \tag{9.42}$$

Therefore, $A_{00}(k) \in \Lambda$. Similarly, we have

$$\begin{aligned} R_{11}(k) &= \frac{\partial^2}{\partial u \partial v}\Big|_{u=v=0} \operatorname{Tr} R_\delta^k \\ &= k \operatorname{Tr}\left(\frac{\partial^2}{\partial u \partial v}\Big|_{u=v=0} R_\delta\right) R_\delta^{k-1} \\ &\quad + k(k-1) \sum_{i+j=k-2} \operatorname{Tr}\left(\frac{\partial}{\partial u}\Big|_{u=v=0} R_\delta\right) R_\delta^i \left(\frac{\partial}{\partial v}\Big|_{u=v=0} R_\delta\right) R_\delta^j. \end{aligned} \tag{9.43}$$

As

$$\frac{\partial^2}{\partial u \partial v}\Big|_{u=v=0} R_\delta = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \tag{9.44}$$

$$\operatorname{Tr}\left(\frac{\partial^2}{\partial u \partial v}\Big|_{u=v=0} R_\delta\right) R_\delta^{k-1} = \operatorname{Tr} R_{22}^{k-1} \in \Lambda. \tag{9.45}$$

Similarly, since

$$\frac{d}{du}\Big|_{u=v=0} R_\delta = \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix}, \quad \frac{d}{dv}\Big|_{u=v=0} R_\delta = -x \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \tag{9.46}$$

we obtain

$$\begin{aligned} &\operatorname{Tr}\left(\frac{\partial}{\partial u}\Big|_{u=v=0} R_\delta\right) R_\delta^i \left(\frac{\partial}{\partial v}\Big|_{u=v=0} R_\delta\right) R_\delta^j \\ &\qquad = -x \operatorname{Tr}\begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11}^i & 0 \\ 0 & R_{22}^i \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} R_{11}^j & 0 \\ 0 & R_{22}^j \end{pmatrix} \\ &\qquad = -x(\pi^* R_L)^j \operatorname{Tr} R_{22}^i A A^*. \end{aligned} \tag{9.47}$$

As before, we can show $\operatorname{Tr} R_{22} A A^* \in \Lambda$ and complete the proof. ∎

PROPOSITION 9.4.

$$\operatorname{Tr}(\pi^* R_Y)^k A \wedge A^* \in \mathbb{Q}[\pi^* R_L, \operatorname{Tr} \pi^* R_Y^k]_{k \geq 0}.$$

Proof. — Let A_{C_Y} be the second fundamental form of the exact sequence (9.29) with respect to the Euclidean metric. Then, since $\mathbb{C}^{n+1}$ is flat, in the same way as (1.6), we get

$$R_{C_Y} = A_{C_Y}^* \wedge A_{C_Y}, \quad R_N = A_{C_Y} \wedge A_{C_Y}^* \tag{9.48}$$

where R_{C_Y} and R_N are the curvature of TC_Y and N respectively, which, combined with Lemma 9.3, yields

$$R(\pi^* TY) - A \wedge A^* = A_{C_Y}^* \wedge A_{C_Y}. \tag{9.49}$$

Put $\Lambda_0 := \mathbb{Z}[\pi^* R_L, \operatorname{Tr} \pi^* R_Y^k]_{k \geq 0}$. taking the trace of the both hand sides of (9.49), by Lemma 9.3, we get

$$\begin{aligned} \Lambda_0 \ni \operatorname{Tr} R(\pi^* TY) - \operatorname{Tr} A \wedge A^* &= \operatorname{Tr} A_{C_Y}^* \wedge A_{C_Y} \\ &= -R_N. \end{aligned} \tag{9.50}$$

and therefore,

$$\operatorname{Tr}(A_{C_Y}^* \wedge A_{C_Y})^k = -R_N^k \in \Lambda_0. \tag{9.51}$$

Now, we prove the proposition by induction. When $k = 0$, the assertion is clear from Lemma 9.3. Suppose $\operatorname{Tr}(\pi^* R_{TY})^k A \wedge A^* \in \Lambda_0$ for $k \leq m$. Combining (9.49), (9.51) and Lemma 9.3, we obtain

$$\operatorname{Tr}(\pi^* R_{TY} - A \wedge A^*)^{m+2} = \operatorname{Tr}(A_{C_Y}^* \wedge A_{C_Y} - \pi^* R_L \otimes I)^{m+2} \in \Lambda_0. \tag{9.52}$$

By th same computation as in Lemma 9.6, we get

(9.53)

$$\operatorname{Tr}(\pi^* R_{TY} - A \wedge A^*)^{m+2} = \operatorname{Tr} \pi^* R_{TY}^{m+2} - (m+2) \operatorname{Tr}(\pi^* R_{TY})^{m+1} A \wedge A^* + \sum_{|I| \leq m} a(I) \operatorname{Tr}(\pi^* R_{TY})^{i_1} A \wedge A^* \cdots \operatorname{Tr}(\pi^* R_{TY})^{i_l} A \wedge A^*$$

for some $a(I) \in \mathbb{Z}[\pi^* R_L]$. Therefore, by the hypothesis and (9.52), $\operatorname{Tr}(\pi^* R_{TY})^{m+1} A \wedge A^* \in \Lambda_0$. ■

Corollary 9.1. — *For any $k \geq 0$,*

$$A_{00}(k), A_{11}(k) \in \mathbb{Q}[\pi^* R_L, \pi^* \operatorname{Tr} R_Y^l, x, \delta]_{\ell \geq 1}$$

Theorem 9.2. — *For any $n \geq 1$, there exists $f_n(y_0, \ldots, y_n, z, x, w, \delta) \in \mathbb{Q}[y, z, x, w, \delta]$ such that, for any smooth projective hypersurface Y in $\mathbb{P}^{n+1}$,*

$$Td(R_\delta) = f_n(\pi^* c(TY), \pi^* c_1(L), \frac{1}{1 + \|z\|^2}, \tau, \delta)$$

where

$$\tau = \frac{\sqrt{-1}}{2\pi} uv = \frac{\sqrt{-1}}{2\pi} \frac{\partial \|z\|^2 \wedge \bar{\partial} \|z\|^2}{\|z\|^2 (1 + \|z\|^2)^2}.$$

Proof. — By the Chern-Weil theory, there exists a polynomial $g(x_1, \ldots, x_n) \in \mathbb{Q}[x]$ such that

$$Td(R_\delta) = g(\operatorname{Tr} R_\delta, \ldots, \operatorname{Tr} R_\delta^n). \tag{9.54}$$

By Corollary 9.1, there exist $a_k(y, z, x, \delta), b_k(y, z, x, \delta) \in \mathbb{Q}[y, z, x, \delta]$ such that

$$\operatorname{Tr} R_\delta^k = a_k(\pi^* c(Y), \pi^* c_1(L), x, \delta) + \tau b_k(\pi^* c(Y), \pi^* c_1(L), x, \delta). \tag{9.55}$$

The assertion follows from (9.54) and (9.55). ■

Proof of Theorem 9.1.

By Proposition 9.2 and 9.3, we get

$$Z(0,0) - Z(0,1) = \int_L \frac{1}{1 + \|z\|^2} Td'(R; P)^{(n,n)}. \tag{9.56}$$

Since $R_\delta = R + \delta P$, by theorem 9.2, we have

$$\begin{aligned} Td'(R;P) &= \frac{d}{d\delta}\Big|_{\delta=0} Td(R_\delta) \\ &= \frac{d}{d\delta}\Big|_{\delta=0} f_n(\pi^* c(Y), \pi^* c_1(L), \frac{1}{1+\|z\|^2}, \tau, \delta), \end{aligned} \tag{9.57}$$

and therefore

(9.58)

$$Td'(R;P)^{(n,n)} = \frac{d}{d\tau}\Big|_{\tau=0} \frac{d}{d\delta}\Big|_{\delta=0} f_{n-1}(\pi^* c(Y), \pi^* c_1(L), \frac{1}{1+\|z\|^2}, \tau, \delta)^{(n-1,n-1)}\tau,$$

which, combined with (9.56), yields

$$\begin{aligned} Z(0,0) - Z(0,1) &= \int_{\mathbb{C}} \tau \frac{\partial^2 f_{n-1}}{\partial\tau\partial\delta}(\pi^* c(Y), \pi^* c_1(L), \frac{1}{1+\|z\|^2}, 0, 0)[Y] \\ &= \int_0^1 \frac{\partial^2 f_{n-1}}{\partial\tau\partial\delta}(\pi^* c(Y), \pi^* c_1(L), x, 0, 0)\,dx[Y], \end{aligned} \tag{9.59}$$

as

$$\int_{\mathbb{C}} \frac{1}{(1+\|z\|^2)^k}\tau = \frac{1}{1+k} = \int_0^1 x^k\,dx. \tag{9.60}$$

■

10. Proof of Proposition 3.1

In this section, we always consider the situation of Proposition 3.1.

Proposition 10.1. — *Suppose, for any $p \in \operatorname{Sing} X_0$, there exists a coordinate $(z_0, \ldots, z_n)$ centered at p such that, on a neighborhood of p,*

(1) $\pi(z) = a_0 z_0^2 + \cdots + a_n z_n^2 \quad (a_0 \cdots a_n \neq 0)$,

(2) $g_X = |dz_0|^2 + \cdots + |dz_n|^2$.

Then, Proposition 3.1 holds.

Proof. — By the hypothesis, Theorem 4.3 and Proposition 4.4, there exists $\sigma \in \Gamma(D(\varepsilon), \lambda(E))$ such that, for $t \in D(\varepsilon)$,

$$0 < C_1 \leq \|\sigma(t)\|_2 \leq C_2 < +\infty, \tag{10.1}$$

and

$$\Big|\sum_{q=0}^{n} \sum_{0<\lambda<1} \log \lambda_i^{0,q}(t)\Big| \leq C_3. \tag{10.2}$$

Set

$$Y_p := \{z \in \mathbb{P}^n ; a_0 z_0^2 + \cdots + a_n z_n^2 = 0\}. \tag{10.3}$$

Then, the family $\{(X_t, g_t)\}$ is a conic degeneration with tangent cone $(C_{Y_p}, g_{C_{Y_p}})$ corresponding to each singularity, and the uniform Sobolev inequality holds for this family by Theorem 6.2. Applying Theorem 8.3 and 9.1 to $\{(X_t, g_t)\}$, we obtain

$$\log T(t) = (Z(0,0) - Z(0,1)) \log |t| + o(\log |t|). \tag{10.4}$$

Note that $\varepsilon^2 = |t|$ for this family, since X_t is given by $\{a_0 z_0^2 + \cdots + a_n z_n^2 = t\}$ on a neighborhood of the singularity.

Combining (10.1), (10.2) and (10.4), we conclude that the Quillen metric is a singular Hermitian metric on D. By Proposition 1.3, $\pi_*(Td(TX/D, g_{X/D}))^{(1,1)}$ extends to a smooth $(1,1)$-form on D. Therefore, by the same argument as [B-B], Proposition 10.1, the curvature current of $(\lambda(E), \|\cdot\|_Q)$ is given by

$$c_1 = -\frac{1}{2}(Z(0,0) - Z(0,1))\delta_0 + \pi_*(Td(TX/D, g_{X/D}))^{(1,1)}. \tag{10.5}$$

By (8.37), $Z(0,0) - Z(0,1)$ is localizable, *i.e.*, by setting

$$W_t := \{[w] \in \mathbb{P}^{n+1} ; a_0 w_0^2 + \cdots + a_n w_n^2 - t w_{n+1}^2 = 0\}, \tag{10.6}$$

$$Z(0,0) - Z(0,1) = r(E) \sum_{p \in \mathrm{Sing}\, X_0} \{g_{n-1}(c(Y_p), c_1(L)) + f_n(c(\tilde{W}_0)) - f_n(c(W_1))\}. \tag{10.7}$$

Since any pair (Y_p, L_{Y_p}) is isomorphic to a specific quadric Y ($a_i = 1$ for any i) and $L_Y = \mathcal{O}_{\mathbb{P}^n}(-1)\big|_Y$, and the similar holds to $\tilde{W}_0$, W_1, we get

$$g_{n-1}(c(Y_p), c_1(L)) + f_n(c(\tilde{W}_0)) - f_n(c(W_1)) = a(n) \in \mathbb{Q} \tag{10.8}$$

which, combined with (10.5-6), yields the assertion. ■

Lemma 10.1. — *Let g'_X be a Kähler metric of X such that*

1) $g'_X = \partial\bar{\partial}\|z\|^2 + \sum_{i,j} A_{ij}(z) dz_i d\bar{z}_j, \quad |A_{ij}(0)| = 0 \quad (0 \leq i, j \leq n),$

2) $\pi(z) = a_0 z_0^2 + \cdots + a_n z_n^2$

on a neighborhood of any singular point of X_0. Let g_X be the same metric as Proposition 10.1, $\widetilde{Td}(TX/D; g'_X, g_X)$ the secondary class associated to the Todd genus with respect to g'_X and g_X. Then,

$$\Big| \int_{X_t} \widetilde{Td}(TX/D; g'_X, g_X)^{(n.n)} \Big| \leq C$$

where C is a constant independent of $t \in D$.

Proof. — Since the problem is local, it is sufficient to show

$$|\int_{X_t\cap U}\widetilde{Td}(TX/D;g'_X,g_X)^{(n.n)}|\leq C \tag{10.9}$$

where U is a neighborhood of Sing X_0. Put

$$g_\varepsilon := \varepsilon g_X + (1-\varepsilon)g'_X, \quad g_{\varepsilon/D} := g_\varepsilon\big|_{TX/D}. \tag{10.10}$$

By the Bott-Chern formula, we get

$$\widetilde{Td}(TX/D;g'_X,g_X) = \int_0^1 Td'(R_\varepsilon; g_\varepsilon^{-1}\frac{d}{d\varepsilon}g_\varepsilon)d\varepsilon \tag{10.11}$$

where R_ε is the curvature of $(TX/D, g_{\varepsilon/D})$. Set

$$\begin{aligned} V_t &= \{z\in\mathbb{C}^{n+1}; a_0z_0+\cdots+a_nz_n^2 = t\},\\ \phi_t &: V_1\cap\mathbb{B}(|t|^{-\frac12})\ni z\to\sqrt{t}z\in V_t\cap\mathbb{B}(1). \end{aligned} \tag{10.12}$$

Then, we obtain

$$\phi_t^*g_\varepsilon(z) = |t|\Big(\partial\overline{\partial}\|z\|^2+(1-\varepsilon)\sum_{i,j}A_{ij}(\sqrt{t}z)dz_id\bar{z}_j\Big), \tag{10.13}$$

and

$$\frac{d}{d\varepsilon}\phi_t^*g_\varepsilon = -|t|\sum_{i,j}A_{ij}(\sqrt{t}z)dz_id\bar{z}_j \tag{10.14}$$

which yields, for any $z\in V_1\cap\mathbb{B}(|t|^{-\frac12})$,

$$\big|\phi_t^*g_\varepsilon^{-1}\frac{d}{d\varepsilon}\phi_t^*g_\varepsilon(z)\big|_{\phi_t^*g_\varepsilon} = -|t|\sum_{i,j}|A_{ij}(\sqrt{t}z)|\leq C|t|^{\frac12}\|z\|. \tag{10.15}$$

Put

$$G_\varepsilon(z) := \partial\overline{\partial}\|z\|^2+(1-\varepsilon)\sum_{i,j}A_{ij}(\sqrt{t}z)dz_id\bar{z}_j. \tag{10.16}$$

Since $\phi_t^*g_\varepsilon = |t|G_\varepsilon$, we have $R_{\phi_t^*g_\varepsilon} = R_{G_\varepsilon}$ (cf. [Y 1], proposition 1.1). Identify G_ε with the matrix $I+(1-\varepsilon)A(\sqrt{t}z)$. Then,

$$\big|\frac{\partial^2}{\partial z_i\partial\bar{z}_j}G_\varepsilon(z,t)\big|\leq C|t| \tag{10.17}$$

for any $(z,t)\in V_1\cap\mathbb{B}(|t|^{-\frac12})\times D$, which, combined with (10.15), yields

$$\begin{aligned} |\int_{X_t\cap U}\widetilde{Td}(TX/D;g'_X,g_X)^{(n.n)}| &= \sharp\,\mathrm{Sing}\,X_0\Big|\int_{V_1\cap\mathbb{B}(|t|^{-\frac12}}\widetilde{Td}(TV_1/D;\phi_t^*g_0,\phi_t^*g_1)^{(n.n)}\Big|\\ &\leq C\int_{W_1\cap\mathbb{B}(|t|^{-\frac12})}|R_{G_\varepsilon}|^n|\phi_t^*g_\varepsilon^{-1}\frac{d}{dt}\phi_t^*g_\varepsilon\big|_{\phi_t^*g_\varepsilon}dv\\ &\leq C\int_0^{|t|^{-\frac12}}|t|^{n+\frac12}r^{2n}dr\\ &\leq C. \end{aligned} \tag{10.18}$$

■

THEOREM 10.1. — *Proposition 3.1 holds.*

Proof. — Let g_X be a given metric. Since the metric is kählerian, for each $p \in \operatorname{Sing} X_0$, there is a coordinate neighborhood $(U_p, z = (z_0, \dots, z_n))$ centered at p for which the assumption of Lemma 10.1 holds. Let g'_X be a Kähler metric of X which coincides with g_X on $X - \cup_{p \in \operatorname{Sing} X_0} U_p$, and is of the form

$$g'_X = \partial\bar{\partial}\|z\|^2 \quad (V_p \subset\subset U_p). \tag{10.19}$$

Let $\sigma \in \Gamma(D(\varepsilon), \lambda(E))$, $\sigma(0) \neq 0$ be a holomorphic section, $\|\sigma\|_Q$ and $\|\sigma\|'_Q$ its Quillen norm with respect to $g_{X/D}$ and $g'_{X/D}$ respectively. By the anomally formula ([B-G-S 1], Theorem 0.2),

$$\log \frac{\|\sigma\|'_Q}{\|\sigma\|_Q} = \pi_*(\widetilde{Td}(TX/D; g_{X/D}, g'_{X/D}) \operatorname{ch}(E))^{(0,0)}, \tag{10.20}$$

whose right hand side is bounded on D by Lemma 10.1. Therefore, by Proposition 10.1, we have

$$\log \|\sigma\|'_Q = -a(n) r(E) \sharp \operatorname{Sing} X_0 \log |t|^2 + O(1). \tag{10.21}$$

Since $\partial\bar{\partial} \log \|\sigma\|'_Q$ extends to a smooth $(1,1)$-form on D by Proposition 1.3, we have the following equation on D in the sense of current:

$$c_1(\lambda(E), \|\cdot\|_Q) = a(n) r(E) \sharp \operatorname{Sing} X_0 \delta_0 + \pi_*(Td(TX/D) \operatorname{ch}(E))^{(1,1)}. \tag{10.22}$$

■

11. Evaluation of the universal constant

In this section, we evaluate the constant $a(n)$ in Proposition 3.1.

THEOREM 11.1.

$$a(n) = \frac{(-1)^{n+1}}{(n+2)!}.$$

Let $\pi : X^{n+1}(d) \to D$ be a family of projective hypersurfaces defined as follows:

$$X^{n+1}(d) = \{([x], t) \in \mathbb{P}^{n+1} \times D; x_0^d + \cdots x_n^d - t x_{n+1}^d = 0\}, \quad \pi([x], t) = t. \tag{11.1}$$

By calculation, $X_t(d)$ is non-singular for $t \neq 0$, and $X_0(d)$ has only one singularity at $([0:1], 0)$. Let $(\mathbb{C}^{n+1}, (z_0, \dots, z_n))$ be an affine coordinate of $\mathbb{P}^{n+1}$. Then, this also becomes a local coordinates of $X(d)$, and π is expressed as follows:

$$\pi(z) = z_0^d + \cdots + z_n^d. \tag{11.2}$$

Let g_d be a Kähler metric of $X(d)$ such that

$$g_d = \partial\overline{\partial}\|z\|^2 \tag{11.3}$$

on a neighborhood of the origin expressed in the above coordinate.

Let $L = \pi_1^*\mathcal{O}_{\mathbb{P}^{n+1}}(-1)$ be a line bundle on $X(d)$, and $h_0(z) = 1 + \|z\|^2$ the natural Hermitian metric of L (expressed in the affine coordinates). Let $\chi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-negative convex increasing function such that $\chi(t) \equiv 0$ for $t \in [0, \log 2]$, $\chi(t) = t$ for $t \in [\log 5, \infty)$, and put

$$\phi(z) := \chi(\log(1 + \|z\|^2)) - \log(1 + \|z\|^2), \quad h(z) := e^{\phi} h_0(z). \tag{11.4}$$

Proposition 11.1. — *There exists a non-negative function $\lambda(\not\equiv 0)$ on $\mathbb{P}^{n+1}$ such that, for any $t \in D$ and any $f \in A^{0,q}_{X_t}(L)$ $(q < n)$,*

$$\int_{X_t} \lambda(z)|f|^2 dv_{X_t} \leq (\Box^{0,q}_t f, f).$$

Proof. — Let R_h be the curvature of (L, h). Then,

$$\begin{aligned} c_1(L,h)(= \frac{i}{2\pi}R_h) &= \frac{i}{2\pi}\overline{\partial}\partial\chi(\log(1+\|z\|^2)) \\ &= \chi'(h_0(z))\frac{i}{2\pi}\overline{\partial}\partial\log(1+\|z\|^2) \\ &\quad + \chi''(h_0(z))\frac{i}{2\pi}\overline{\partial}\log(1+\|z\|^2)\wedge\partial\log(1+\|z\|^2) \\ &= -\chi'(h_0(z))\omega_{\mathbb{P}^{n+1}} - \chi''(h_0(z))\frac{i}{2\pi}\frac{\partial\|z\|^2\overline{\partial}\|z\|^2}{(1+\|z\|^2)^2} \end{aligned} \tag{11.5}$$

where $\omega_{\mathbb{P}^{n+1}}$ is the Fubini-Study form of $\mathbb{P}^{n+1}$. Therefore, as $\chi', \chi'' \geq 0$ by definition, we get

$$-(\chi' + c\chi'')\omega_{\mathbb{P}^{n+1}} \leq c_1(L,h) \leq -\chi'\omega_{\mathbb{P}^{n+1}} \tag{11.6}$$

for some $c > 0$. Since $g_{X/D}$ is uniformly quasi-isometric to the induced Fubini-Study metric, by (11.6), we have

$$-C_1(\chi' + c\chi'')\Omega_{X/D} \leq c_1(L,h)\big|_{X/D} \leq -C_2\chi'\Omega_{X/D} \tag{11.7}$$

where $\Omega_{X/D}$ is the relative Kähler form associated to $g_{X/D}$.

Now we apply Kodaira-Nakano formula (cf. [K] and [De]) to obtain

$$(\Box^{0,q}_t f, f) - (\overline{\Box}^{0,q}_t f, f) = (\sqrt{-1}[R_h, \Lambda_t]f, f) \tag{11.8}$$

where $\overline{\Box}^{0,q}_t = \partial^*\partial + \partial\partial^*$, and Λ_t is the adjoint of the exterior multiplication of Ω_t. By the formula of Gigant ([K], III, (3.6) and [De]) and (11.7), we have

$$\langle\sqrt{-1}[R_h, \Lambda_t]f, f\rangle \geq C_3\chi'|f|^2, \tag{11.9}$$

which, combined with (11.8), yields

$$C_3 \int_{X_t} \chi'(h_0(z))|f(z)|^2 dv_{X_t} \le (\Box_t^{0,q} f, f)_t. \tag{11.10}$$

■

Put $\lambda_i^{0,q}(t)$ for the i-th eigenvalue of $\Box_t^{0,q}$ as before.

PROPOSITION 11.2. — *For any $q < n$ and $t \in D$,*

(1) $H^{0,q}_{(2)}(X_0.L_0) = 0$,

(2) $\lambda_1^{0,q}(t) \ge C > 0$

where C is a constant independent of t.

Proof. — Clear by Theorem 4.3 and Proposition 11.1. ■

Since $H^q(X_t, \mathcal{O}(L_t)) = 0$ for any $t \in D$ and $q < n$, det $R^q\pi_*\mathcal{O}(L)$ is trivial and gives trivial contribution to the Quillen norm for the above range of q. By the flatness of the family, $H^n(X_t, \mathcal{O}_{X_t}(L))$ has a constant dimension in t, and $R^n\pi_*\mathcal{O}(L)$ becomes a locally free sheef on D whose dual is given by $\pi_*\omega_{X(d)/D}(H)$, where $H = L^{-1}$ is the hyperplane bundle, and $\omega_{X(d)/D}$ is the relative dualizing sheef. Regard $\pi_*\omega_{X(d)/D}(H)$ as a vector bundle on D, and construct a holomorphic frames as follows.

First consider the following exact sequence of sheefs for any hypersurface S in $\mathbb{P}^{n+1}$:

$$0 \longrightarrow \Omega^{n+1}_{\mathbb{P}^{n+1}}(H) \longrightarrow \Omega^{n+1}_{\mathbb{P}^{n+1}}(\log S(H)) \longrightarrow \Omega^n_S(H) \longrightarrow 0 \tag{11.11}$$

in which, the residue map gives the following isomorphism:

$$\mathrm{Res}: \quad H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(\log S(H))) \cong H^0(S, \Omega^n_S(H)). \tag{11.12}$$

Identify $V := H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(\log S(H)))$ as the space of meromorphic $(n+1)$-forms on $\mathbb{P}^{n+1}$ with logarithmic poles along S and H. By computing the transition relation, we obtain the following.

LEMMA 11.1. — *A basis of V is given by*

$$\left\{ \frac{z^e dz_0 \wedge \cdots \wedge dz_n}{F(z)}; |e| \le d - (n+1) \right\}$$

where $z^e = z_0^{e_0} \cdots z_n^{e_n}$, $|e| = \sum e_i$ $(e_i \ge 0)$, and $F(z)$ is the defining equation of S in $\mathbb{C}^{n+1}$.

Proof. — Since it is straightforward, we leave it to the reader. ■

Consider the family $(\pi, X(d))$. By computing the residue map, we obtain the following.

PROPOSITION 11.3. — *A basis of $\pi_* \omega_{X(d)/D}(H)$ is given by*

$$\left\{ \tau^e = \frac{z^e}{z_0^{d-1}} dz_1 \wedge \cdots \wedge dz_n; |e| \leq d-(n+1) \right\}.$$

Put

$$\sigma'(t) := \wedge_{|e| \leq d-(n+1)} \tau^e \in \wedge^{max} H^0(X_t, \Omega^n_{X_t}(H)). \tag{11.13}$$

Then, σ' is a non-zero holomorphic section of $\det \pi_* \omega_{X/D}(H)$ which does not vanish at $t=0$.

PROPOSITION 11.4. — *As $t \to 0$,*

$$\log \|\sigma'(t)\|_2 = -\frac{1}{d} \binom{d}{n+2} \log |t|^2 + O(\log \log \frac{1}{|t|}).$$

Proof. — Set

$$\psi_t : \mathbb{P}^{n+1} \ni [z_0 : \cdots : z_{n+1}] \to [t^{\frac{1}{d}} z_0 : \cdots : t^{\frac{1}{d}} z_{n+1}] \in \mathbb{P}^{n+1} \tag{11.14}$$

which induces an isomorphism $\psi_t : X_1(d) \to X_t(d)$. As

$$\psi_t^* \tau^e(t) = t^{\frac{n+|e|-(d-1)}{d}} \tau^e(1), \tag{11.15}$$

we find

$$\|\tau^e(t)\|_2^2 = \|\psi_t^* \tau^e(t)\|_2^2 = |t|^{\frac{2(n+|e|-(d-1))}{d}} \|\tau^e(1)\|_2^2. \tag{11.16}$$

Since

$$\|\sigma'(t)\|_2^2 \sim \prod_{|e| \leq d-(n+1)} \|\tau^e(t)\|_2^2, \tag{11.17}$$

substituting (11.16) to (11.17), we get

(11.18)

$$\begin{aligned}
\log \|\sigma'(t)\|_2^2 &= \sum_{|e| \leq d-(n+1)} \|\tau^e(t)\|_2^2 + O(1) \\
&= \frac{1}{d} \sum_{|e| \leq d-(n+2)} \{n + |e| - (d-1)\} \log |t|^2 \\
&\qquad\qquad + \sum_{|e| = d-(n+1)} \log \|\tau^e(t)\|_2^2 + O(1) \\
&= -\frac{1}{d} \binom{d}{n+2} \log |t|^2 + \binom{d-1}{n} \log \log \frac{1}{|t|} + O(1; d)
\end{aligned}$$

where $O(1; d)$ is a term bounded in t, but may not in d. ∎

PROPOSITION 11.5. — *$(\pi, X^{n+1}(d), D)$ admits a Morsification to which L extends.*

Proof. — Define a family over D^2 as follows:

$$\tilde{X}(d) := \{([z], t, \varepsilon) \in \mathbb{P}^{n+1} \times D^2; F_\alpha(z, t, \varepsilon) = 0\}, \quad \tilde{\pi}([z], t, \varepsilon) = (t, \varepsilon), \tag{11.19}$$

$$F_\alpha(z, t, \varepsilon) := z_0^d + \cdots + z_n^d - d\varepsilon^{d-1}(\alpha_0^{d-1} z_0 + \cdots + \alpha_n^{d-1} z_n) z_{n+1}^{d-1} - t z_{n+1}^d \tag{11.20}$$

where $\alpha = (\alpha_0, \ldots, \alpha_n)$ satisfies the following condition:

for any $k = (k_0, \ldots, k_n) \in (\mathbb{Z}/(d-1)\mathbb{Z})^{n+1}$,

$$\sum_{i=0}^{n} \alpha_i^d \exp\left(\frac{2\pi k i}{d-1}\right) \neq 0. \tag{11.21}$$

By the direct computation, we can show that $\tilde{\pi} : \tilde{X}(d) \to D^2$ is a Morsification of $X(d)$ to which L extends naturally.

LEMMA 11.2. — *Let $S^n(d)$ be a projective hypersurface of degree d. Then, for any $I = (i_1, \ldots, i_k)$, $|I| = n$, there exists $C_I(n) > 0$ which depends only on I and n such that*

$$|c_{i_1}(S) \cdots c_{i_{k-1}}(S) c_1(H)^{i_k}[S]| \leq C_I(n) d^{n+1}.$$

Proof. — See [D], § 3, (3.7).

Proof of Theorem 11.1.

Let $\pi : X^{n+1}(d) \to D$ be the same as before, and $\sigma \in \Gamma(D, \det R^n\pi_*\mathcal{O}(L))$ such that $\sigma^{-1} = \sigma'$. We compute $\|\sigma\|_Q$ in two ways.

1) Let $T(t)$ be the analytic torsion of $(X_t(d), L)$. By Theorem 8.3, Theorem 9.1 and Proposition 11.2, there exist $f_n(x) \in \mathbb{Q}[x]$ and $g_n(y, z) \in \mathbb{Q}[y, z]$ whose coefficients depend only on n such that

$$\log T(t) = \{f_n(c(X_t)) + g_n(c(Y), c_1(L_Y))\} \log|t|^{\frac{2}{d}} + o(\log|t|) \tag{11.22}$$

where $Y = \{[z] \in \mathbb{P}^n; z_0^d + \cdots + z_n^d = 0\}$. By Proposition 1.4, we know

$$\log\|\sigma(t)\|_2^2 = -\log\|\sigma'(t)\|_2^2 = \frac{1}{d}\binom{d}{n+2}\log|t|^2 + o(\log|t|). \tag{11.23}$$

Since $\lambda(\mathcal{O}(L)) \cong (\det R^n\pi_*\mathcal{O}(L))^{(-1)^n}$, $\sigma^{(-1)^n}$ is considered to be a section of $\lambda(\mathcal{O}(L))$ which does not vanish at $0 \in D$, whose Quillen norm is given by

$$\begin{aligned} \log\|\sigma^{(-1)^n}(t)\|_Q^2 &= \log T(t) + \log\|\sigma^{(-1)^n}(t)\|_2^2 \\ &= \frac{1}{d}\left\{h_n(d) + (-1)^n\binom{d}{n+2}\right\}\log|t|^2 + o(\log|t|) \end{aligned} \tag{11.24}$$

where $h_n(d) \in \mathbb{Q}[d]$ is a polynomial of degree smaller than $n+1$ by Lemma 11.2.

2) By Proposition 11.5, $(\pi.X(d), D)$ admits a Morsification. Since L and its extension is also negative, the conditions 1) and 2) of Theorem 3.1 are satisfied by vanishing theorem and Theorem 4.4. Therefore, we can apply Theorem 3.1 to yield

$$\log \|\sigma^{(-1)^n}(t)\|_Q^2 = -a(n)\mu(\operatorname{Sing} X_0)\log|t|^2 + O(1). \tag{11.25}$$

Since $\operatorname{Sing} X_0$ is homogeneous of multiplicity d, by [D], p.13, we get

$$\mu(\operatorname{Sing} X_0) = (d-1)^{n+1} \tag{11.26}$$

which, combined with (11.25), yields

$$\log \|\sigma^{(-1)^n}(t)\|_Q^2 = -a(n)(d-1)^{n+1}\log|t|^2 + O(1). \tag{11.27}$$

Comparing the coefficient of $\log|t|$ in (11.24) and (11.27), we have

$$\frac{1}{d}h_n(d) + (-1)^n\frac{1}{d}\binom{d}{n+2} = -a(n)(d-1)^{n+1}. \tag{11.28}$$

Since $\deg h_n(d) \leq n+1$, we have

$$\begin{aligned} a(n) &= \lim_{d\to\infty}\frac{-1}{d^{n+1}}\left\{\frac{1}{d}h_n(d) + (-1)^n\frac{1}{d}\binom{d}{n+2}\right\} \\ &= \frac{(-1)^{n+1}}{(n+2)!}. \end{aligned} \tag{11.29}$$

■

THEOREM 11.2. — *Let (π, X, D) be a smoothing of IHS which admits a Morsification, g_X a Kähler metric of X, and $g_{X/D}$ the induced metric on TX/D. Let $\lambda(\mathcal{O}_X) = \det R\pi_*\mathcal{O}_X$ be the determinant of direct images, and $\|\cdot\|_Q$ the Quillen metric associated to $g_{X/D}$. Suppose that g_X is Euclidean flat on a neighborhood of $\operatorname{Sing} X_0$. Then, $\|\cdot\|_Q$ is a singular Hermitian metric of $\lambda(\mathcal{O}_X)$ whose curvature current is given by*

$$c_1(\lambda(\mathcal{O}_X), \|\cdot\|_Q) = \frac{(-1)^{n+1}}{(n+2)!}\mu(\operatorname{Sing} X_0)\delta_0 + \pi_*(Td(TY/D^2))^{(1,1)}$$

where $\pi_*(Td(TY/D^2))^{(1,1)}$ is a smooth d-closed $(1,1)$-form on D.

Proof. — Clear from Theorem 10.1, Theorem 3.1 and Theorem 11.1. ■

12. Higher rank case

We treat vector bundle case in this section. In higher rank case, we must impose some extra conditions on the family.

THEOREM 12.1. — *Let $\pi : X^{n+1} \to D$ be a smoothing of IHS, g_X a Kähler metric of X such that $R_X \equiv 0$ on a neighborhood of $\operatorname{Sing} X_0$, $g_{X/D}$ the induced metric on TX/D. Let*

(E, h) *be a holomorphic Hermitian vector bundle such that* $R_E \equiv 0$ *on a neighborhood of* $\operatorname{Sing} X_0$. *Suppose that* (π, X, D) *admits a Morsification* $p : Y \to D^2$ *such that*

(1) (p, Y, D^2) *is locally projective,*

(2) E *is a restriction of a holomorphic vector bundle* F *on* Y.

Then, the Quillen metric $\|\cdot\|_Q$ *associated to* $g_{X/D}$ *and* h *is a singular Hermitian metric of* $\lambda(E)$, *and its curvature current is given by*

$$c_1(\lambda(E), \|\cdot\|_Q) = \frac{(-1)^{n+1}}{(n+2)!} r(E)\mu(\operatorname{Sing} X_0)\delta_0 + \pi_*(Td(TX/D)\operatorname{ch}(E))^{(1,1)}.$$

For the proof, we need the following.

Proposition 12.1. — *Under the situation of Theorem 12.1, there exists an exact sequence of vector bundles on a neighborhood of* Y_0

$$0 \longrightarrow E_{n+1} \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E \longrightarrow 0$$

such that

(1) $E_i \cong L^{n_i} \otimes \mathbb{C}^{m_i}$ *for some* $n_i, m_i \geq 1 \quad (1 \leq i \leq n)$,

(2) $H^q(Y_s, \mathcal{O}_{Y_s}(E_n)) = 0$ *for any* $0 \leq q < n$ *and any* $s \in D^2$,

where $L = i^*\mathcal{O}_{\mathbb{P}^N}(-1)$, *and* $i : Y \hookrightarrow \mathbb{P}^N \times D^2$ $(\pi_2 \circ i = p)$ *is an embedding of* Y.

Proof. — Put $H = L^{-1}$. Then, by the positivity of H, there exists $n_1 \gg 1$ and $\varepsilon_1 > 0$ such that $E \otimes H^{n_1}$ is generated by global sections on $Y_{\varepsilon_1} := p^{-1}(D^2_{\varepsilon_1})$. Therefore, one can choose $s_1, \ldots, s_{m_1} \in \Gamma(Y_{\varepsilon_1}, E \otimes H^{n_1})$ so that

$$i_1 : \mathbb{C}^{m_1} \ni \xi = (\xi_1, \ldots, \xi_{m_1}) \to \sum \xi_i s_i \in E \otimes H^{n_1} \tag{12.1}$$

is surjective at any $y \in Y_{\varepsilon_1}$, and therefore

$$j_1 : (L^{n_1}) \otimes \mathbb{C}^{m_1} \to E \to 0 \tag{12.2}$$

is surjective. Set $E_1 := L^{n_1} \otimes \mathbb{C}^{m_1}$ and $F_1 = \operatorname{Ker} j_1$. Inductively, we can define ε_i $(1 \leq i \leq n)$, $E_i := L^{n_i} \otimes \mathbb{C}^{m_i}$ and F_i, and obtain an exact sequence on Y_ε $(\varepsilon = \min\{\varepsilon_i\})$:

$$0 \longrightarrow F_n \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E \longrightarrow 0. \tag{12.3}$$

Set $E_{n+1} := F_n$. Since $0 \longrightarrow F_1 \longrightarrow E_1 \longrightarrow E \longrightarrow 0$ is exact, $H^0(Y_s, F_1) = 0$ for any $s \in D^2_\varepsilon$. Again, since $0 \longrightarrow F_2 \longrightarrow E_2 \longrightarrow F_1 \longrightarrow 0$ is exact, $H^0(Y_s, F_1) = H^1(Y_s, F_2) = 0$ for any $s \in D^2_\varepsilon$. Inductively, we can show $H^q(Y_s, F_i) = 0$ for any $s \in D^2_\varepsilon$, which yields 2) of the assertion, since $E_{n+1} = F_n$. ∎

Proof of Theorem 12.1. — Let $\mathcal{E} = \{E_i\}_{i \geq 0}$ $(E_0 = E)$ be the exact sequence constructed in Proposition 12.1. By Theorem 4.4 and Proposition 12.1, the conditions 1) and

2) of Theorem 3.1 are satisfied for E_i $(i \geq 1)$, and therefore, Theorem 12.1 holds for any E_i $(i \geq 1)$. Choose $\sigma_i \in \Gamma(D^2_\varepsilon, \lambda(E_i))$, $\sigma_i(0) \neq 0$ for any $i \geq 0$ and put

$$\sigma := \bigotimes_{i\geq 0} \sigma_i^{(-1)^i}. \tag{12.4}$$

By the anomaly formula ([B-G-S 1], Theorem 0.3), we get

$$\log \|\sigma\|_Q = \sum_{i=0}^{n} (-1)^i \log \|\sigma_i\|_{Q,i} = \pi_*(Td(TY/D^2; g_{Y/D^2})\widetilde{\mathrm{ch}}(\mathcal{E}))^{(0,0)} \tag{12.5}$$

where $\widetilde{\mathrm{ch}}(\mathcal{E})$ stands for the Bott-Chern secondary class for $\mathcal{E}$. Since $R_{E_i} \equiv 0$ on a neighborhood of Sing X_0, $Td(TY/D^2; g_{Y/D^2})\widetilde{\mathrm{ch}}(\mathcal{E})$ vanishes on a neighborhood of Sing X_0, and the right hand side of (12.5) extends to a smooth function on D. Thus, in the sense of current,

$$\frac{i}{2\pi}\bar{\partial}\partial \log \|\sigma\|_Q = \sum_{i=0}^{n} (-1)^i \pi_*(Td(TY/D^2; g_{Y/D^2}) \,\mathrm{ch}(E_i))^{(1,1)}. \tag{12.6}$$

Since Theorem 12.1 holds for E_i $(i > 0)$, we have

(12.7)

$$\begin{aligned}
&c_1(\lambda(E), \|\cdot\|_Q) \\
&= \sum_{i=0}^{n} (-1)^i \pi_*(Td(TY/D^2) \,\mathrm{ch}(E_i))^{(1,1)} \\
&\quad - \sum_{i=1}^{n} \frac{(-1)^{n+i+1}}{(n+2)!} r(E_i)\mu(\mathrm{Sing}\, X_0)\delta_0 - \sum_{i=1}^{n} (-1)^i \pi_*(Td(TY/D^2) \,\mathrm{ch}(E_i))^{(1,1)} \\
&= \frac{(-1)^{n+1}}{(n+2)!} r(E)\mu(\mathrm{Sing}\, X_0)\delta_0 + \pi_*(Td(TY/D^2) \,\mathrm{ch}(E))^{(1,1)}.
\end{aligned}$$

■

13. Proof of the Main Theorem

We prove the following theorem in this section.

THEOREM 13.1. — *Let $\pi : X^{n+1} \to D$ be a smoothing of IHS, g_X a Kähler metric of X, and $g_{X/D}$ the induced metric on TX/D. Let $\lambda(\mathcal{O}_X) = \det \pi_* R\mathcal{O}_X$ be the determinant of direct images, $\|\cdot\|_Q$ its Quillen metric associated to $g_{X/D}$.*

If (π, X, D) admits a Morsification, then $\|\cdot\|_Q$ is a singular Hermitian metric on $\lambda(\mathcal{O}_X)$, and its curvature current is given by

$$c_1(\lambda(\mathcal{O}_X), \|\cdot\|_Q) = \frac{(-1)^{n+1}}{(n+2)!}\mu(\mathrm{Sing}\, X_0)\delta_0 + \pi_*(Td(TX/D; g_{X/D}))^{(1,1)}.$$

Here, $\pi_*(Td(TX/D; g_{X/D}))^{(1,1)} \in L^r_{loc}(D, \wedge^{1,1}) \cap \partial\bar{\partial} C^\alpha(D)$ *as current on* D *for some* $r > 1$ *and* $\alpha > 0$*. In particular, for* $\sigma \in \Gamma(D, \lambda(\mathcal{O}_X)), \sigma(0) \neq 0$*, the following asymptotic formula holds:*

$$\log \|\sigma(t)\|_Q = \frac{(-1)^n}{(n+2)!}\mu(\operatorname{Sing} X_0) \log |t|^2 + C_0 + O(|t|^\alpha).$$

For the proof, we need some lemmas.

Lemma 13.1. — *Let* (X, g) *be a Kähler manifold. Then, for any* $p \in X$*, there exists a Kähler metric* g' *such that*

(1) $R_{g'} \equiv 0$ *on a neighborhood of* p

(2) $[\omega_g] = [\omega_{g'}]$

where ω_g *is the Kähler class of* g*.*

Proof. — By the Kählerness, there exists a coordinates $(z_0, \dots, z_n)$ centered at p such that

$$\omega_g = \frac{\sqrt{-1}}{2}(\partial\bar{\partial}\|z\|^2 + \sum_{i,j} a_{ij}(z) dz_i \wedge d\bar{z}_j), \quad a_{ij}(0) = \partial_k a_{ij}(0) = 0. \tag{13.1}$$

Since $d\omega_g = 0$, by the $\partial\bar{\partial}$-Poincaré lemma, there exists a function $\psi \in C^\infty(\mathbb{B}(1))$ such that

$$\omega_g = \frac{\sqrt{-1}}{2}\partial\bar{\partial}(\|z\|^2 + \psi(z)), \quad \nabla^k\psi(0) = 0 \quad (k \leq 4). \tag{13.2}$$

Let $\rho(t)$ be a cut-off function such that $\rho(t) \geq 0$, $\rho(t) = 0$ for $t \leq \frac{1}{2}$, $\rho(t) = 1$ for $t \geq 1$ and $|\rho'(t)| \leq C$. Put $\rho_\varepsilon(z) := \rho(\|z\|/\varepsilon)$ and

$$\omega_\varepsilon := \partial\bar{\partial}(\|z\|^2 + \rho_\varepsilon\psi). \tag{13.3}$$

Then, we can write $\omega_\varepsilon := \partial\bar{\partial}\|z\|^2 + \rho_\varepsilon\partial\bar{\partial}\phi + \tau_\varepsilon$ where τ_ε satisfies

$$\begin{aligned} |\tau_\varepsilon| &\leq |\nabla\rho_\varepsilon||\nabla\psi| + |\nabla^2\rho_\varepsilon||\psi| \\ &\leq C(\varepsilon^{-1} \cdot \varepsilon^2 + \varepsilon^{-2} \cdot \varepsilon^4) \leq C\varepsilon^2. \end{aligned} \tag{13.4}$$

Since $\partial\bar{\partial}\|z\|^2 + \rho_\varepsilon\partial\bar{\partial}\phi = (1 - \rho_\varepsilon)\partial\bar{\partial}\|z\|^2 + \rho_\varepsilon\omega_g$, we get $\omega_\varepsilon \geq c\omega_1$ when $\varepsilon \ll 1$, and corresponding Hermitian tensor g_ε defines a Kähler metric on X which coincides with g outside of $\mathbb{B}(1)$ and satisfies 1). As

$$\omega - \omega_\varepsilon = \partial\bar{\partial}\{(1 - \rho_\varepsilon)\psi\}, \tag{13.5}$$

we obtain $[\omega] = [\omega_\varepsilon]$. ■

Lemma 13.2. — *Let* V_+ *and* A *be the same as in Lemma 1.1, and assume* $\dim_{\mathbb{C}} V_+ = n$*. Let* $h = \sum_{i=1}^n \theta\bar{\theta}^i$ *be a Hermitian metric on* V_+*,* $\omega = \sqrt{-1}\sum_i \theta^i \wedge \bar{\theta}^i$ *its associated 2-form. Then,* *1) For any $S = \sqrt{-1}\sum_i S_{ij}\theta^i \wedge \overline{\theta}^i \geq 0$,*

$$C(n,k)^{-1}|S^k|\omega^n \leq S^k \wedge \omega^{n-k} \leq C(n,k)|S^k|\omega^n$$

where $C(n,k) > 0$ is a constant which depends only on n and k. In particular, for $S = \sqrt{-1}A \wedge A^*$,

$$(\sqrt{-1}A \wedge A^*)^k \wedge \omega^{n-k} = \frac{(n-k)!}{n!}\Big(\sum_{|I|=k}|a_I|^2\Big)\omega^n$$

where $a_I = a_{i_1} \wedge \cdots \wedge a_{i_k}$ for $I = (i_1, \dots, i_k)$.

2) For $G \in GL(\mathbb{C}^n)$, set $\omega_G := \sqrt{-1}\sum_i G\theta^i \wedge \overline{G\theta}^i$. Then,

$$C_0(n,k,G)S^k \wedge \omega^{n-k} \leq S^k \wedge \omega_G \leq C_1(n,k,G)S^k \wedge \omega^{n-k}$$

for $S \geq 0$ where $C_i(n,k,G)$ $(i = 0,1)$ are constants which depends only on n, k and entries of G.

Proof. — Since 2) follows from 1), we only prove 1). As there is an unitary matrix U such that

(13.6) $$\omega = \sqrt{-1}\sum_i U\theta^i \wedge \overline{U\theta}^i, \quad S = \sqrt{-1}\sum_i \lambda_i U\theta^i \wedge \overline{U\theta}^i, \quad \lambda_i \geq 0,$$

we get by direct computation

(13.7) $$S^k \wedge \omega^{n-k} = \frac{(n-k)!}{n!}\Big(\sum_{|I|=k}\lambda_I\Big)\omega^n, \quad \lambda_I = \lambda_{i_1}\cdots\lambda_{i_k}$$

which yields the former part of 1). When $S = \sqrt{-1}A \wedge A^*$, since

(13.8) $$(\sqrt{-1})^k a_{i_1} \wedge \bar{a}_{i_1} \wedge \cdots \wedge a_{i_k} \wedge \bar{a}_{i_k} \wedge \omega^{n-k} = \frac{(n-k)!}{n!}|a_I|^2\omega^n,$$

we obtain the latter part of 1). ■

LEMMA 13.3. — *Let V, A be the same as in Lemma 13.2. Let $B \in \wedge^{1,1} \otimes \mathrm{End}(\mathbb{C}^n)$, $\theta \in \mathrm{End}(\mathbb{C}^n)$ where $\wedge^{p,p} := (\wedge^p V_+) \wedge (\wedge^p V_-)$. Set*

$$\Lambda := \mathbb{Z}[A^* \wedge A, B, \theta], \quad M := \mathbb{Z}[B, \theta] \subset \Lambda, \quad M_A := \{A\mu A^* \in \wedge^* V_{\mathbb{C}}; \mu \in M\}.$$

Then,

$$\mathrm{Tr}\,\Lambda \subset \mathbb{Z}[M_A, A \wedge A^*],$$

i.e., for any $\lambda \in \Lambda$, there exist $\mu_1, \dots, \mu_k \in M_A$, and a polynomial $f(x_1, \dots, x_k, y) \in \mathbb{Z}[x_1, \dots, x_k, y]$ such that

$$\mathrm{Tr}\,\lambda = f(\mu_1, \dots, \mu_k, A \wedge A^*).$$

Proof. — Let $\lambda \in \Lambda$ be a monomial, *i.e.*, there exist $\mu_0, \dots, \mu_k \in M$ such that $\lambda = \mu_0(A^* \wedge A)^{a_0} \cdots \mu_k(A^* \wedge A)^{a_k}$ $(a_k \geq 1)$. Then, there exist $m \geq 0$ and $b \in \{0, 1\}$ such that

$$\lambda = (A \wedge A^*)^m \mu_0(A^* \wedge A)\mu_1(A^* \wedge A) \cdots \mu_k(A^* \wedge A)^b$$

since $(A^* \wedge A)^l = (A \wedge A^*)^{l-1}(A^* \wedge A)$. When, $b \neq 0$, we get

$$\begin{aligned}\operatorname{Tr}\lambda &= (A \wedge A^*)^m \operatorname{Tr}\mu_0(A^* \wedge A)\mu_1(A^* \wedge A) \cdots \mu_k(A^* \wedge A) \\ &= \pm(A \wedge A^*)^m(A \wedge \mu_0 A^*)(A \wedge \mu_1 A^*) \cdots (A \wedge \mu_k A^*) \in \mathbb{Z}[A \wedge A^*, M_A].\end{aligned} \tag{13.9}$$

When $b = 0$, we have

$$\begin{aligned}\operatorname{Tr}\lambda &= \pm(A \wedge A^*)^m \operatorname{Tr}\mu_k\mu_0(A^* \wedge A)\mu_1(A^* \wedge A) \cdots \mu_{k-1}(A^* \wedge A) \\ &= \pm(A \wedge A^*)^m(A \wedge \mu_k\mu_0 A^*)(A \wedge \mu_1 A^*) \cdots (A \wedge \mu_{k-1} A^*) \in \mathbb{Z}[A \wedge A^*, M_A].\end{aligned} \tag{13.10}$$

As any element of Λ is represented by a $\mathbb{Z}$-linear combination of monomials, the assertion holds. ∎

LEMMA 13.4. — *Let $B_i \in \operatorname{End}(\mathbb{C}^n) \otimes \wedge^{p_i, p_i}$. Suppose*

$$(AB_{i_1}A^*) \cdots (AB_{i_k}A^*) \wedge \operatorname{Tr}B_{j_1} \cdots \operatorname{Tr}B_{j_l} \wedge (A \wedge A^*)^m \in \wedge^{n,n}.$$

Then, there exists $C = C(k, l, m, n)$ such that

$$\begin{aligned}&|(AB_{i_1}A^*) \cdots (AB_{i_k}A^*) \wedge \operatorname{Tr}B_{j_1} \cdots \operatorname{Tr}B_{j_l} \wedge (A \wedge A^*)^m| \\ &\qquad \leq C \prod_{i_p, j_q} |B_{i_p}||B_{j_q}||(\omega + \sqrt{-1}A \wedge A^*)^n|\end{aligned}$$

where $|B|^2 = \sum |b_{ij}|^2$ for $B = (b_{ij})$.

Proof. — By computation,

$$\begin{aligned}&(AB_{i_1}A^*) \cdots (AB_{i_k}A^*) \wedge \operatorname{Tr}B_{j_1} \cdots \operatorname{Tr}B_{j_l} \wedge (A \wedge A^*)^m \\ &\qquad = \sum_{|I|=|J|=k+m} c_{IJ}(B_{i_1}, \dots, B_{j_l}) a_I \wedge \bar{a}_J\end{aligned} \tag{13.11}$$

where $c_{IJ}(B_{i_1}, \dots, B_{j_l})$ is a polynomial of entries of $B_{i_1}, \dots, B_{j_l}$. Then, by using Lemma 13.2,

$$\begin{aligned}&|(AB_{i_1}A^*) \cdots (AB_{i_k}A^*) \wedge \operatorname{Tr}B_{j_1} \cdots \operatorname{Tr}B_{j_l} \wedge (A \wedge A^*)^m| \\ &\qquad \leq \sum_{|I|=|J|=k+m} |c_{IJ}(B_{i_1}, \dots, B_{j_l})||a_I||a_J| \\ &\qquad \leq C(\prod_{i_p, j_q} |B_{i_p}||B_{j_q}|) \sum_I |a_I|^2 \\ &\qquad \leq C \prod_{i_p, j_q} |B_{i_p}||B_{j_q}||(\omega + \sqrt{-1}A \wedge A^*)^n|.\end{aligned} \tag{13.12}$$

■

Proof of Theorem 13.1. — For simplicity, we assume Sing $X_0 = \{p\}$. (The case $\sharp$ Sing $X_0 > 1$ is similar.) Let g' be the Kähler metric considered in Lemma 13.1, $\|\cdot\|'_Q$ the Quillen metric of $\lambda(\mathcal{O}_X)$ associated to $g'_{X/D}$. By the anomaly formula ([B-G-S 1], Theorem 0.2), we get

$$\log \frac{\|\sigma\|_Q}{\|\sigma\|'_Q} = \pi_*(\widetilde{Td}(TX/D; g_{X/D}, g'_{X/D}))^{(0,0)}. \tag{13.13}$$

Set $g_\varepsilon := (1-\varepsilon)g_X + \varepsilon g'_X$, and $g_{\varepsilon/D}$ for the induced metric on TX/D. Let R_ε and $R_{\varepsilon/D}$ be their curvature forms respectively. By the Bott-Chern formula,

$$\widetilde{Td}(TX/D; g_{X/D}, g'_{X/D})^{(0,0)} = \int_0^1 \frac{d}{d\delta}\big|_{\delta=0} Td(R_{\varepsilon/D} + \delta\theta_\varepsilon), \tag{13.14}$$

where $\theta_\varepsilon = g_\varepsilon^{-1}\frac{d}{d\varepsilon}g_\varepsilon$. By the Chern-Weil theory, there exists a polynomial $f(x_1, \dots, x_{n+1}) \in \mathbb{Q}[x_1, \dots, x_{n+1}]$ such that

$$Td_{n+1}(\Omega) = f(\mathrm{Tr}\,\Omega, \mathrm{Tr}\,\Omega^2, \dots, \mathrm{Tr}\,\Omega^{n+1}) \tag{13.15}$$

where $Td_{n+1}(\Omega)$ stands for the homogeneous part of degree $n+1$ of $Td(\Omega)$. Then, by (13.13–13.15), we have

$$\begin{aligned} &\pi_*(\widetilde{Td}(TX/D; g_{X/D}, g'_{X/D}))^{(0,0)}(t) \\ &= \int_0^1 d\varepsilon \int_{X_t} \frac{d}{d\delta}\big|_{\delta=0} f_{n+1}(\mathrm{Tr}(R_{\varepsilon/D} + \delta\theta_\varepsilon), \dots, \mathrm{Tr}(R_{\varepsilon/D} + \delta\theta_\varepsilon)^{n+1}) \\ &= \int_0^1 d\varepsilon \int_{X_t} \sum_{k=1}^{n+1} k \frac{\partial f_{n+1}}{\partial x_k}(\mathrm{Tr}\,R_{\varepsilon/D}, \dots, \mathrm{Tr}\,R_{\varepsilon/D}^{k-1}\theta_\varepsilon, \dots, \mathrm{Tr}\,R_{\varepsilon/D}^{n+1}) \\ &= \int_0^1 d\varepsilon \int_{X_t} g(\mathrm{Tr}\,R_{\varepsilon/D}, \dots, \mathrm{Tr}\,R_{\varepsilon/D}^{n+1}, \mathrm{Tr}\,\theta_\varepsilon, \mathrm{Tr}\,R_{\varepsilon/D}\theta_\varepsilon, \dots, \mathrm{Tr}\,R_{\varepsilon/D}^{n}\theta_\varepsilon) \end{aligned} \tag{13.16}$$

for some $g(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) \in \mathbb{Q}[x, y]$.

Let U be a neighborhood of p such that $g'\big|_U = \partial\bar{\partial}\|z\|^2$, and we restrict our consideration on U for a while. Consider the following exact sequence:

$$0 \longrightarrow TX/D \longrightarrow TX \longrightarrow N \longrightarrow 0 \tag{13.17}$$

where $N = TX/(TX/D)$ is the normal bundle. Let $A_\varepsilon \in A^{1,0}(\mathrm{End}(TX/D, (TX/D)^\perp))$ be the second fundamental form of (13.7) with respect to g_ε. Then, R_ε is represented as follows:

$$R_\varepsilon = \begin{pmatrix} R_{\varepsilon/D} - A_\varepsilon^* \wedge A_\varepsilon & -\partial A_\varepsilon^* \\ \bar{\partial} A_\varepsilon & R_{N,\varepsilon} - A_\varepsilon \wedge A_\varepsilon^* \end{pmatrix}. \tag{13.18}$$

Put $B_\varepsilon := R_\varepsilon - A^* \wedge A_\varepsilon$, and $E_\varepsilon = R_{N,\varepsilon} - A_\varepsilon \wedge A_\varepsilon^*$ for the $(1,1)$ and $(2,2)$-entries. Since g_ε ($\varepsilon \in [0,1]$) is a smooth family of nondegenerate Kähler metrics, there is $C > 0$ such that, on U,

$$|B_\varepsilon| + |E_\varepsilon| \le |R_\varepsilon| \le C, \quad |\theta_\varepsilon| \le C. \tag{13.19}$$

Note that C does not depend on a choice of unitary frames of TX/D and N. Now substitute $R_{\varepsilon/D} = A^* \wedge A + B_\varepsilon$ to (13.16). Then, by Lemma 13.3 and 13.4, we obtain on U

$$\begin{aligned}(13.20)\qquad &\left|g(\operatorname{Tr} R_{\varepsilon/D}, \dots, \operatorname{Tr} R_{\varepsilon/D}^{n+1}, \operatorname{Tr} \theta_\varepsilon, \operatorname{Tr} R_{\varepsilon/D}\theta_\varepsilon, \dots, \operatorname{Tr} R_{\varepsilon/D}^{n}\theta_\varepsilon)\big|_{X_t}\right| \\ &\qquad\leq C\left|(\omega_\varepsilon + \sqrt{-1}A_\varepsilon \wedge A_\varepsilon^*)^n\big|_{X_t}\right|\end{aligned}$$

where ω_ε is the Kähler form of g_ε. By Lemma 13.1, for any $\delta > 0$, there exists $C(n,\delta) > 0$ such that

$$\begin{aligned}(13.21)\qquad \left|(\omega_\varepsilon + \sqrt{-1}A_\varepsilon \wedge A_\varepsilon^*)^n\big|_{X_t}\right| dv_{\varepsilon.t} &\leq C(n,\delta)(\delta^{-1}\omega_\varepsilon + \sqrt{-1}A_\varepsilon \wedge A_\varepsilon^*)^n\big|_{X_t} \\ &= C(n,\delta)(\delta^{-1}\omega_\varepsilon + \sqrt{-1}R_{N,\varepsilon} - \sqrt{-1}E_\varepsilon)^n\big|_{X_t}.\end{aligned}$$

Let $f \in \mathcal{O}(U)$ be the defining function of π on U, *i.e.*, $\pi(z) = f(z)$. Then, in the same manner as Proposition 1.1 (1.11), we get

$$\begin{aligned}(13.22)\qquad R_{N,\varepsilon} &= -R_{N^*,\varepsilon} = -\bar\partial\partial|df|_\varepsilon^2 \\ &= \frac{|df|_\varepsilon^2\partial\bar\partial|df|_\varepsilon^2 - \partial|df|_\varepsilon^2 \wedge \bar\partial|df|_\varepsilon^2}{|df|_\varepsilon^4} \\ &= \frac{|df|_\varepsilon^2\langle\nabla df, \nabla df\rangle - \partial|df|_\varepsilon^2 \wedge \bar\partial|df|_\varepsilon^2}{|df|_\varepsilon^4} - \frac{\langle\rho_\varepsilon(df), df\rangle_\varepsilon}{|df|_\varepsilon^2}\end{aligned}$$

where $\langle\nabla df, \nabla df\rangle = \sum g^{ij}\langle\nabla_i df, \nabla_j df\rangle dz_i \wedge d\bar z_j$ and ρ_ε is the curvature of $(\Omega_X^1, g_\varepsilon^*)$. Put

$$(13.23)\qquad S_\varepsilon = \frac{|df|_\varepsilon^2\langle\nabla df, \nabla df\rangle - \partial|df|_\varepsilon^2 \wedge \bar\partial|df|_\varepsilon^2}{|df|_\varepsilon^4},$$

$$(13.24)\qquad T_\varepsilon = \frac{\langle\rho_\varepsilon(df), df\rangle_\varepsilon}{|df|_\varepsilon^2},$$

and identify S_ε with the associated Hermitian form. Then, $S_\varepsilon \geq 0$. In fact, for any $\xi \in T^{1,0}X$,

$$\begin{aligned}(13.25)\qquad S_\varepsilon(\xi,\xi) &= |df|_\varepsilon^2|\nabla_\xi df|_\varepsilon^2 - |\partial_\xi|df|_\varepsilon^2|^2/|df|_\varepsilon^4 \\ &= |df|_\varepsilon^2|\nabla_\xi df|_\varepsilon^2 - |\langle\nabla_\xi df, df\rangle_\varepsilon|^2/|df|_\varepsilon^4 \geq 0.\end{aligned}$$

Substituting (13.24) and (13.25) into (13.21), we obtain

$$\begin{aligned}(13.26)\qquad \left|(\omega_\varepsilon + \sqrt{-1}A_\varepsilon \wedge A_\varepsilon^*)^n\big|_{X_t}\right| dv_{\varepsilon.t} &\leq C(n,\delta)(\Omega_\varepsilon(\delta) + \sqrt{-1}S_\varepsilon)^n\big|_{X_t} \\ &\leq C(n,\delta)\sum_{k=0}^{n}\binom{n}{k}\Omega_\varepsilon(\delta)^{n-k}S_\varepsilon^k\big|_{X_t}\end{aligned}$$

where $\Omega_\varepsilon(\delta) = \delta^{-1}\omega_\varepsilon - \sqrt{-1}E_\varepsilon - \sqrt{-1}T_\varepsilon$. Since $\{g_\varepsilon\}$ is a smooth family of nondegenerate Kähler metrics on X, chosing δ sufficiently small, there exist $C_1(\delta), C_2(\delta), C_3(\delta) > 0$ which does not depend on ε such that, on U,

$$(13.27)\qquad 0 < \omega_\varepsilon \leq C_1(\delta)\Omega_\varepsilon(\delta) \leq C_2(\delta)(\delta^{-1}\omega_\varepsilon - \sqrt{-1}T_\varepsilon) \leq C_3(\delta)\omega_\varepsilon.$$

As $S_\varepsilon \geq 0$, substituting (13.27) to (13.26), and using Lemma 13.2, we have

(13.28)

$$\begin{aligned}\big|(\omega_\varepsilon + \sqrt{-1}A_\varepsilon \wedge A_\varepsilon^*)^n\big|_{X_t}\big|dv_{\varepsilon.t} &\leq C(\delta)\sum_{k=0}^{n}\binom{n}{k}(\delta^{-1}\omega_\varepsilon - \sqrt{-1}T_\varepsilon)^{n-k}(\sqrt{-1}S_\varepsilon)^k\big|_{X_t} \\ &\leq C(\delta)(\delta^{-1}\omega_\varepsilon + \sqrt{-1}S_\varepsilon - \sqrt{-1}T_\varepsilon)^n\big|_{X_t} \\ &= C(\delta)(\delta^{-1}\omega_\varepsilon + \sqrt{-1}R_{N,\varepsilon})^n\big|_{X_t}.\end{aligned}$$

Since $\big|g(\operatorname{Tr} R_{\varepsilon/D}, \dots, \operatorname{Tr} R_{\varepsilon/D}^{n+1}, \operatorname{Tr}\theta_\varepsilon, \operatorname{Tr} R_{\varepsilon/D}\theta_\varepsilon, \dots, \operatorname{Tr} R_{\varepsilon/D}^{n}\theta_\varepsilon)\big|_{X_t}\big|$ is uniformly bounded on $X - U$, combining (13.14), (13.16), (13.20) and (13.28), we get

$$(13.29)\qquad \begin{aligned}\big|\pi_*(\widetilde{Td}(TX/D; g_{X/D}, g'_{X/D}))^{(0,0)}(t)\big| &\leq C(\delta)\int_{X_t\cap U}(\delta^{-1}\omega_\varepsilon + \sqrt{-1}R_{N,\varepsilon})^n + C \\ &\leq C(\delta)\int_{X_t}(\delta^{-1}\omega_\varepsilon + \sqrt{-1}R_{N,\varepsilon})^n + C \\ &\leq C(\delta)\{\delta^{-1}[\omega]\big|_{X_t} + 2\pi c_1(N_{X_t})\}^n[X_t] + C \\ &\leq C(\delta),\end{aligned}$$

which, combined with (13.13), yields

$$(13.30)\qquad \big|\log\frac{\|\sigma\|_Q}{\|\sigma\|'_Q}\big| \leq C$$

for some $C > 0$, and therefore, $\log\|\sigma\|_Q/\|\sigma\|'_Q \in L^\infty_{loc}(D)$.

Now we estimate $\pi_*(Td(TX/D, g_{X/D}))^{(1,1)}$. In the same way as (13.20), (13.21) and (13.26), on U, we get

$$(13.31)\qquad -C(\omega_X + \sqrt{-1}R_N)^{n+1} \leq Td(TX/D, g_{X/D})^{(n+1,n+1)} \leq C(\omega_X + \sqrt{-1}R_N)^{n+1}$$

where $R_N = R_{N,0} = \partial\bar{\partial}\log|df|_0^2$. Put

(13.32)

$$|df|_0^2 = |df|^2 + A(df, df), \quad |df|^2 = \sum\big|\frac{\partial f}{\partial z_i}\big|^2, \quad A(df, df) = \sum_{i,j} a_{ij}\frac{\partial f}{\partial z_i}\overline{\frac{\partial f}{\partial z_j}}.$$

Since $\nu : U - \{0\} \ni z \to [\frac{\partial f}{\partial z_0} : \cdots : \frac{\partial f}{\partial z_n}] \in \mathbb{P}^n$ extends to a meromorphic map from $\nu : U \to \mathbb{P}^n$, due to Hironaka, there is a proper modification $p : V \to U$ such that

1) $p^{-1}(0)$ is a divisor of simple normal crossing, and p is an isomorphism between $V - p^{-1}(0)$ and $U - \{0\}$,

2) There exists a holomorphic map $\mu : V \to \mathbb{P}^n$ such that $\mu = \nu \circ p$ on $V - p^{-1}(0)$.

We verify that $p^* R_N$ extends to a smooth $(1,1)$-form on V. As

$$(13.33)\qquad \begin{aligned}\sqrt{-1}R_N &= \sqrt{-1}\partial\bar{\partial}\log|df|^2 + \sqrt{-1}\partial\bar{\partial}\log\Big(1 + \frac{A(df, df)}{|df|^2}\Big) \\ &= \nu^*\omega_{\mathbb{P}^n} + \sqrt{-1}\partial\bar{\partial}\log\Big(1 + \frac{A(df, df)}{|df|^2}\Big)\end{aligned}$$

where $\omega_{\mathbb{P}^n}$ is the Fubini-Study form as before, it is sufficient to show that $p^*(A(df, df)/|df|^2)$ extends to a smooth function on V which vanishes along $p^{-1}(0)$. Put $O_i := \{[z_0 : \cdots : z_n] \in \mathbb{P}^n; z_i \neq 0\}$,

$$\xi_i^j := \frac{z_j}{z_i}, \quad V_i := \mu^{-1}(O_i). \tag{13.34}$$

Then, by computation, on each V_k, we obtain

$$p^*\Big(\frac{A(df, df)}{|df|^2}\Big) = \sum_{i,j} p^* a_{ij} \mu^* \xi_k^i \overline{\mu^* \xi_k^j} / 1 + \sum_{i \neq k} |\mu^* \xi_k^i|^2. \tag{13.35}$$

As $p^* a_{ij}, \mu^* \xi_k^i \in C^\infty(V_k)$ and $a_{ij}(0) = 0$, it is clear that $p^*(A(df, df)/|df|^2) \in C^\infty(V_k)$ and vanishes along $p^{-1}(0) \cap V_k$, which, combined with (13.31) and (13.33) yields

$$-C \cdot dv_V \leq p^* Td(R_{X/D}, g_{X/D})^{(n+1,n+1)} \leq C \cdot dv_V \tag{13.36}$$

where dv_V is a volume form on V. Set $q := \pi \circ \pi : V \to D$. By the assumption, for any $x \in V$, there is a local coordinates $(w_0, \ldots, w_n)$ such that

(13.37)

$$dv_V = \Big(\frac{\sqrt{-1}}{2}\Big)^{n+1} dw_0 \wedge d\bar{w}_0 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n, \ q(w) = w_0^{k_0} \cdots w_l^{k_l}, \ (k_i \geq 1, \ l \leq n).$$

Set

$$\tau := \frac{1}{l} \sum_{i=1}^{l} \frac{1}{k_i} (-1)^{i-1} w_i dw_0 \wedge \cdots \wedge dw_{i-1} \wedge dw_{i+1} \wedge \cdots \wedge dw_n. \tag{13.38}$$

Since $q^*(dt/t) \wedge \tau = dw_0 \wedge \cdots \wedge dw_n$, by (13.36), we have

(13.39)

$$\begin{aligned} |\pi^*(Td(TX/D; g_{X/D}))^{(1,1)}(t)| &\leq C(|q_*(dw_0 \wedge d\bar{w}_0 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n)| + 1) \\ &= C\Big(\Big|\frac{dt \wedge d\bar{t}}{|t|^2} \int_{q^{-1}(t)} \tau \wedge \bar{\tau}\Big| + 1\Big). \end{aligned}$$

By a theorem of Barlet ([Ba]), there exist $a_0 \in \mathbb{C}$, $\alpha \in \mathbb{Q}_+$, $\beta \geq 0$ such that

$$\int_{q^{-1}(t)} \tau \wedge \bar{\tau} = a_0 + O(|t|^\alpha (\log|t|)^\beta). \tag{13.40}$$

As dv_V is integrable on V, we find $a_0 = 0$, which, combined with (13.39) and (13.40), yields

$$|\pi^*(Td(TX/D; g_{X/D}))^{(1,1)}(t)| \leq C\{|t|^{-2+\alpha}(\log|t|)^\beta + 1\}. \tag{13.41}$$

Therefore, there exists $r > 1$ such that $\pi^*(Td(TX/D; g_{X/D}))^{(1,1)} \in L^r_{loc}(D)$. Using the curvature formula ([B-G-S 1], Theorem 0.1), we get

$$\Big|\frac{i}{2\pi}\bar{\partial}\partial(\log\|\sigma\|_Q - \log\|\sigma\|'_Q)\Big| \leq C\{|t|^{-2+\alpha}(\log|t|)^\beta + 1\} \tag{13.42}$$

which, combined with (13.30) and elliptic regularity theory, yields

$$\log\|\sigma\|_Q - \log\|\sigma\|'_Q \in L^r_{loc,2}(D) \subset C^\alpha(D) \tag{13.43}$$

for some $\alpha > 0$. Clearly, the theorem follows from Theorem 11.2, (13.42), (13.43) and the curvature formula of Bismut, Gillet and Soulé. ■

Remark 13.1. — We remark that the latter part of Main Theorem is proved in the same way as Theorem 13.1.

14. Examples

We treat some examples in this section.

THEOREM 14.1. — *Let $\pi : X^{n+1} \to D$ be a smoothing of IHS. If (π, X, D) is a family of hypersurfaces in $\mathbb{P}^{n+1}$, then it admits a Morsification. Here, by a family of hypersurfaces, we mean the following commutative diagram:*

$$\begin{array}{ccc} i : X & \hookrightarrow & \mathbb{P}^{n+1} \times D \\ \pi \searrow & \circlearrowleft & \swarrow \pi_2 \\ & D & \end{array} \tag{14.1}$$

where i is the inclusion, pro_2 is the projection to the second factor, and id is the identity map. In particular, Main Theorem holds for degenerating family of projective hypersurfaces.

Let H be the hyperplane bundle of $\mathbb{P}^n$, and F the defining equation of X:

$$X = \{(x, t) \in \mathbb{P}^{n+1} \times D; F(x, t) = 0\}, \tag{14.2}$$

where $F(x, t) \in \mathbb{C}\{t\}[x_0, \dots, x_{n+1}]$ and $F(\cdot, t)$ is a homogeneous polynomial of degree $d = \deg X_t$ for any $t \in D$. As $\operatorname{Sing} X_0$ is discrete, we may assume that $\operatorname{Sing} X_0 \subset \mathbb{P}^{n+1} - H_\infty$, and put

$$F(z, t) := F(z_0, \dots, z_n, 1, t), \quad F(z) = F(z, 0), \quad \delta F(z) = \frac{d}{dt}\Big|_{t=0} F(z, t). \tag{14.3}$$

LEMMA 14.1. — *For any $p \in \operatorname{Sing} X_0$, δF is an unit element of $\mathcal{O}_{X,p}$.*

Proof. — Since X is non-singular, $dF(z, t) \neq 0$ at p, which, combined with $d_z F = \sum_{i=0}^{n} \frac{\partial F}{\partial z_i} dz_i = 0$ at p, yields $\frac{d}{dt}\big|_{t=0} F(p, t) \neq 0$, i.e., $\delta F(p) \neq 0$. ∎

Fix an identification of $H^0(\mathbb{P}^{n+1}, dH)$ with the set of all homogeneous polynomials of degree d. Then, for $G \in H^0(\mathbb{P}^{n+1}, dH)$, $G(z) = G(z_0, \dots, z_n, 1)$ is a polynomial of degree $\leq d$. Set

$$Y_G := \{(z, t, \varepsilon) \in \mathbb{P}^{n+1} \times D^2; F(z, t) + \varepsilon G(z) = 0\}, \quad \Pi : Y_G \ni (z, t, \varepsilon) \to (t, \varepsilon) \in D^2 \tag{14.4}$$

where z is the inhomogeneous coordinates.

LEMMA 14.2. — *If $|\varepsilon| \ll 1$, then Y_G is non-singular and X is reduced in Y_G.*

Proof. — Put $f_G(z, t, \varepsilon) := F(z, t) + \varepsilon G(z)$. Then,

$$df_G = dF + \varepsilon dG + G d\varepsilon. \tag{14.5}$$

Since $dF \neq 0$ on $X = \{(z, t, \varepsilon) \in Y_G; \varepsilon = 0\}$, we find $df_G \neq 0$ on a neighborhood of X. Therefore, for sufficiently small ε, Y_G is non-singular. As

$$df_G \wedge d\varepsilon = (dF + \varepsilon dG) \wedge d\varepsilon \neq 0 \tag{14.6}$$

for sufficiently small ε, $\pi^*\varepsilon$ is chosen to be a coordinates of Y_G, and therefore X is reduced since $X = \{\pi^*\varepsilon = 0\}$. ■

Set

$$Z_p := \{G \in H^0(\mathbb{P}^{n+1}, dH); (\Pi, Y_G, D^2) \text{ is not a Morsification }\}. \tag{14.7}$$

LEMMA 14.3. — *Z_p is an analytic set of $H^0(\mathbb{P}^{n+1}, dH)$.*

Proof. — By translation, if necessary, we may assume that $p = 0$ in the inhomogeneous coordinates. Set

$$\begin{aligned} \Sigma_G &:= \{(z, t, \varepsilon); z \in \operatorname{Sing} Y_{G,(t,\varepsilon)}\} \\ &= \{(z, t, \varepsilon); \frac{\partial f_G}{\partial z_0} = \cdots = \frac{\partial f_G}{\partial z_n} = f_G = 0\} \end{aligned} \tag{14.8}$$

and

$$\begin{aligned} S_G &:= \{(z, t, \varepsilon) \in \Sigma_G; (z, t, \varepsilon) \text{ is not } A_1\} \\ &= \{(z, t, \varepsilon); \det\left(\frac{\partial^2 f_G}{\partial z_i \partial z_j}\right) = \frac{\partial f_G}{\partial z_0} = \cdots = \frac{\partial f_G}{\partial z_n} = f_G = 0\}. \end{aligned} \tag{14.9}$$

If S_G is discrete, then $S_G \cap B_\delta(p) = p$ for some small ball, and (Π, Y_G, D^2) is a Morsification. Therefore,

$$Z_p = \{G \in H^0(\mathbb{P}^{n+1}, dH); \dim S_G > 0\}, \tag{14.10}$$

and it is an analytic set. ■

Proof of Theorem 14.1. — Since Z_p is analytic, it is enough to show that $Z_p \neq H^0(\mathbb{P}^{n+1}, dH)$ for any $p \in \operatorname{Sing} X_0$. Let $(w_0, \ldots, w_n)$be an inhomogeneous coordinates centered at p, and put

$$G(z) = \frac{1}{2}(w_0^2 + \cdots + w_n^2). \tag{14.11}$$

Suppose $G \in Z_p$. Then, as $\det \operatorname{Hess}_w f_G = 0$ on Σ_G, by Nullstellensatz, there exist $k \geq 0$, $h_0, \ldots, h_n, h \in \mathbb{C}\{z_0, \ldots, z_n, \varepsilon, t\}$ such that

$$(\det \operatorname{Hess}_w f_G)^k = \sum_{i=0}^{n} \frac{\partial f_G}{\partial w_i} h_i + f_G h. \tag{14.12}$$

Since $f_G = F + \varepsilon G$, by (14.11) and (14.12), we get

$$\begin{aligned} \det(\varepsilon I + \operatorname{Hess}_w F)^k &= (\det \operatorname{Hess}_w f_G)^k \\ &= \sum_{i=0}^{n} \frac{\partial f_G}{\partial w_i} h_i + f_G h. \end{aligned} \tag{14.13}$$

Comparing the coefficient of $\varepsilon^{(n+1)k}$, we obtain

$$1 = \sum_{i=0}^{n} \frac{\partial F}{\partial z_i} q_i + \sum_{i=0}^{n} z_i r_i + q F + r G \tag{14.14}$$

where $q_i, r, q, r \in \mathbb{C}\{z_0, \dots, z_n, t\}$, and obtain a contradiction by setting $t = 0$, since $1 \notin \mathfrak{M}_0$. Therefore $G \notin Z_p$, *i.e.*, $Z_p \neq H^0(\mathbb{P}^{n+1}, dH)$. ∎

QUESTION 14.1. — Let (π, X, D) be a smoothing of IHS. Is it true that if (π, X, D) is locally projective, then it admits a Morsification?

THEOREM 14.2. — *Let (A^{n+1}, H) be a polarized abelian variety, ω a flat Kähler form of A such that $c_1(H) = [\omega]$. Let $\mathcal{C}_A(n, d)$ be the Chow variety of n-cycles of A whose degree is d, $\pi : X \to \mathcal{C}_A(n, d)$ the universal family. Suppose D is a divisor of $\mathcal{C}_A(n, d)$ such that π is smooth on $\mathcal{C}_A(n, d) - D$. Set $TX/\mathcal{C}_A(n, d)$ for the relative tangent bundle, $\omega_{TX/\mathcal{C}_A(n,d)}$ for the Hermitian metric of $TX/\mathcal{C}_A(n, d)$ induced by ω, $K_{TX/\mathcal{C}_A(n,d)}$ for the relative canonical bundle, and $\det \omega^{-1}_{TX/\mathcal{C}_A(n,d)}$ for the Hermitian metric induced by $\omega_{TX/\mathcal{C}_A(n,d)}$. Let $\lambda(mK_{X/\mathcal{C}_A(n,d)})$ be the determinant of the direct images, and $\|\cdot\|_Q$ the Quillen metric associated to $\omega_{TX/\mathcal{C}_A(n,d)}$ and $\det \omega^{-1}_{TX/\mathcal{C}_A(n,d)}$. Then, for any $m \in \mathbb{Z}$, $(\lambda(mK_{X/\mathcal{C}_A(n,d)}), \|\cdot\|_Q)$ is flat on $\mathcal{C}_A(n, d) - D$, i.e.,*

$$c_1(\lambda(mK_{X/\mathcal{C}_A(n,d)}), \|\cdot\|_Q) = 0.$$

Proof. — By the same computation as Proposition 1.1, we get

$$\begin{aligned} \left[Td(TX/\mathcal{C}_A(n, d))\, \mathrm{ch}(mK_{X/\mathcal{C}_A(n,d)}) \right]^{(n+1,n+1)} &= C(n, m)\nu^* \omega_{\mathbb{P}^n}^{n+1} \\ &= 0 \end{aligned} \tag{14.15}$$

where $\nu : X_t \to \mathbb{P}^n$ is the Gauss map as before. Now the assertion is clear from the curvature formula ([B-G-S 1], Theorem 0.1). ∎

QUESTION 14.2. — Is it true that, if $D = \cup_{i \in I} D_i$, then

$$c_1(\lambda(mK_{X/\mathcal{C}_A(n,d)}), \|\cdot\|_Q) = \sum_{i \in I} a_i(m, n) \delta_{D_i} \tag{14.16}$$

for some $a_i(m, n) \in \mathbb{Q}$, in the sense of current?

Appendix. Proof of Theorem 6.2 and (4.58)

In this appendix, we give a proof of Theorem 6.2 for $(n-1)$-forms and of (4.58). Let $F(z) \in \mathbb{C}[z_0, \dots, z_n]$ be a homogeneous polynomial such that $V(F) := \{z \in \mathbb{P}^n; F(z) = 0\}$ is a smooth hypersurface. Set $d = \deg F$, and

$$(A.1) \qquad X_t = \{z \in \mathbb{C}^{n+1}; F(z) = t^d\}, \quad g_t = \partial\bar{\partial}\|z\|^2\big|_{X_t},$$

$$(A.2) \qquad N := X_0 \cap S^{2n-1}(1) = \{z \in \mathbb{C}^{n+1}; F(z) = 0, \|z\| = 1\}.$$

Clearly, $X_0 = C(N)$, and (X_1, g_1) is a AC manifold with tangent cone X_0. Fix a diffeomorphism:

$$(A.3) \qquad \Psi : X_{0,(A,\infty)} \cong X_1 - K \quad (K \subset\subset X_1),$$

$$(A.4) \qquad \Psi^* g_1 = g_0 + O(r^{-\delta}) \quad (\delta > 0),$$

where $X_{t,(a,b)} := \{z \in X_t; a < \|z\| < b\}$. Define $T_\varepsilon \in GL(n+1, \mathbb{C})$ by $T_\varepsilon z = \varepsilon z$, and put

$$(A.5) \qquad \Psi_t := T_t \circ \Psi \circ T_{t^{-1}} : X_{0,(A|t|,1)} \hookrightarrow X_{t,(A'|t|,2)}$$

which gives an into-diffeomorphism. By (A.4), we have

$$(A.6) \qquad \Psi_t^* g_t = g_0 + O(|t|^\delta \|z\|^{-\delta}).$$

Let $\hat{X}_t$ and $\hat{\wedge}_t^{n-1}$ be the same as in Theorem 6.2. Then, by long but straightforward computations, we can show the following two propositions.

Proposition A.1. — *There exists $C = C(X_0, X_1) > 0$ such that, for any $f \in \Gamma(\hat{X}_t, \hat{\wedge}_t^{n-1})$ with $\operatorname{supp} f \subset \hat{X}_{t,(R|t|,1)}$ $(R \gg 1)$,*

$$1 - CR^{-\delta} \le \|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2 / \|\hat{d}_0 \Psi_t^* f\|^2 + \|\hat{\delta}_0 \Psi_t^* f\|^2 \le 1 + CR^{-\delta}$$

where $\hat{d}_t := d_{X_t} \otimes 1 + 1 \otimes d_{X_t}$, $\hat{\delta}_t := \delta_{X_t} \otimes 1 + 1 \otimes \delta_{X_t}$.

Proposition A.2. — *Suppose $\nu > n^2 + n - 1$. Then, there exists $C = C(n, \nu) > 0$ such that, for any $f \in \Gamma(\hat{X}_0, \hat{\wedge}_0^{n-1})$ with $\operatorname{supp} f \subset \hat{X}_{0,(0,1)}$,*

$$\|f\|^2_{2(1+\frac{1}{\nu})} \le C(\|\hat{d}_0 f\|^2 + \|\hat{\delta}_0 f\|^2).$$

Combining Proposition A.1 and A.2, we obtain the following.

Proposition A.3. — *Suppose $\nu > n^2 + n - 1$ and $R \gg 1$. Then, there exists a constant $C = C(X_0, X_1, \nu, n)$ such that, for any $f \in \Gamma(\hat{X}_t, \hat{\wedge}_t^{n-1})$ with $\operatorname{supp} f \subset \hat{X}_{t,(R|t|,1)}$,*

$$\|f\|^2_{2(1+\frac{1}{\nu})} \leq C(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2).$$

Lemma A.1. — *There exists $C > 0$ such that, for any $f \in \Gamma(\hat{X}_t, \hat{\wedge}_t^{n-1})$ with $\operatorname{supp} f \subset \hat{X}_{t,(R|t|,1)}$,*

$$\int_{X_t} \Big(\frac{1}{(\|x\| \log \|x\|)^2} + \frac{1}{(\|y\| \log \|y\|)^2}\Big)|f|^2 dv_{\hat{X}_t} \leq C(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2).$$

Proof. — Let $\{\phi_i\}$ be a orthonormal basis of $L^2(\hat{X}_{t,(0,\frac{1}{2})}, \hat{\wedge}_t^{n-1})$. Put

$$f(x,y) := \sum_{i,j} a_{ij}\phi_i(x) \otimes \phi_j(y). \tag{A.7}$$

Then, by Lemma 4.1, we get

$$\begin{aligned}\int_{\hat{X}_t} \frac{1}{(\|x\| \log \|x\|)^2}|f(x,y)|^2 dv_{\hat{X}_t} &= \sum_j \int_{X_t} \frac{1}{(\|x\| \log \|x\|)^2}|a_{kj}\phi_j(x)|^2 dv_{X_t} \\ &\leq C\sum_j(\|d\sum_k a_{kj}\phi_j(x)\|^2 + \|\delta\sum_k a_{kj}\phi_j(x)\|^2).\end{aligned} \tag{A.8}$$

Similarly, we obtain

$$\int_{\hat{X}_t} \frac{1}{(\|y\| \log \|y\|)^2}|f(x,y)|^2 dv_{\hat{X}_t} \leq C\sum_j(\|d\sum_k a_{kj}\phi_j(y)\|^2 + \|\delta\sum_k a_{kj}\phi_j(y)\|^2). \tag{A.9}$$

The assertion follows from (A.8) and (A.9). ∎

Proposition A.4. — *There exists $C > 0$ independent of $t \in D$ such that, for any $f \in \Gamma(\hat{X}_t, \hat{\wedge}_t^{n-1})$ with $\operatorname{supp} f \subset \hat{X}_{t,(0,\frac{1}{2})}$,*

$$\|f\|_{\frac{4n}{2n-1}} \leq C \log\frac{1}{|t|}(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2).$$

Proof. — Since the Sobolev inequality holds uniformly in t ([L-T]), by using Weitenböck formula, we get

$$\begin{aligned}\|f\|_{\frac{4n}{2n-1}} &\leq C\|\nabla_{\hat{X}_t} f\|^2 \\ &= C(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2 + (R_t f, f)) \\ &\leq C\Big\{\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2 + \int_{X_t}\Big(\frac{1}{\|x\|^2} + \frac{1}{\|y\|^2}\Big)|f|^2 dv_{\hat{X}_t}\Big\}\end{aligned} \tag{A.10}$$

where R_t is a 0-th order operator arising from the curvature of X_t. As $\big|\log\|x\|\big| \ge C\big|\log|t|\big|$ for some $t \in D$ and $x \in X_{t,(0,\frac{1}{2})}$, combining Lemma A.1 and (A.10), we obtain the desired estimate. ■

Proof of Theorem 6.2 for $(n-1)$-forms. — By the argumemt of partition of unity, it is sufficient to show the following: for $\nu > n^2+n-1$, and any $f \in \Gamma(\hat{X}_t, \hat{\wedge}_t^{n-1})$ with $\operatorname{supp} f \subset \hat{X}_{t,(0,\frac{1}{2})}$,

$$\|f\|^2_{2(1+\frac{1}{\nu})} \le C(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2). \tag{A.11}$$

Let ρ_ε be the same cut-off function as (4.12), and put $\rho_t(z) := \rho_{R|t|}(\|z\|)$ where z is the coordinate of Theorem 6.2. By Hölder's inequality, Lemma A.1 and Proposition A.4, there exist $\alpha = \alpha(n,\nu) > 0$ and $C > 0$ such that

$$\begin{aligned}
&\|(1-\rho_t(x))\rho_t(y)f\|_{2(1+\frac{1}{\nu})}\\
&\quad\le \{\operatorname{vol}(\operatorname{supp}(1-\rho_t))\operatorname{vol}(\operatorname{supp}\rho_t)\}^\alpha\|(1-\rho_t(x))(\rho_t(y)f\|_{\frac{4n}{2n-1}}\\
&\quad\le C|t|^{2n\alpha}\Big(\log\frac{1}{|t|}\Big)(\|\hat{d}_t\{(1-\rho_t(x))(\rho_t(y)f\}\|^2\\
&\qquad\qquad + \|\hat{\delta}_t\{(1-\rho_t(x))(\rho_t(y)f\}\|^2)\\
&\quad\le C|t|^{2n\alpha}\Big(\log\frac{1}{|t|}\Big)(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2).
\end{aligned} \tag{A.12}$$

Similarly, we obtain

$$\|\rho_t(x)(1-\rho_t(y))f\|_{2(1+\frac{1}{\nu})} \le C|t|^{2n\alpha}\Big(\log\frac{1}{|t|}\Big)(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2), \tag{A.13}$$

$$\|(1-\rho_t(x))(1-\rho_t(y))f\|_{2(1+\frac{1}{\nu})} \le C|t|^{4n\alpha}\Big(\log\frac{1}{|t|}\Big)(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2). \tag{A.14}$$

By Proposition A.3 and Lemma A.1, we have

(A.15)

$$\begin{aligned}
&\|\rho_t(x)\rho_t(y)f\|^2_{2(1+\frac{1}{\nu})}\\
&\quad\le C(\|\hat{d}_t\{\rho_t(x)\rho_t(y)f\}\|^2 + \|\hat{\delta}_t\{\rho_t(x)\rho_t(y)f\}\|^2)\\
&\quad\le C\Big\{\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2 + \int_{X_t}\Big(\frac{1}{(\|x\|\log\|x\|)^2} + \frac{1}{(\|y\|\log\|y\|)^2}\Big)|f|^2 dv_{\hat{X}_t}\Big\}\\
&\quad\le C(\|\hat{d}_t f\|^2 + \|\hat{\delta}_t f\|^2).
\end{aligned}$$

Clearly, (A.11) follows from (A.12-15). ■

Proof of (4.58). — In the sequel, we consider the case that $F(z) = z_0^2 + \cdots + z_n^2$. By the argument of partition of unity and Kählerness, it is enough to show that, for any $f \in A^{0,n}(X_t \cap \mathbb{B}(1))$,

$$\|\frac{1}{r}f\|_{2,t} \le C\|\partial f\|^2_{2,t} \tag{A.16}$$

where $r = \|z\|$, and C is a constant independent of t and f. Since the inequality (A.16) has the scaling invariance, it reduces to the following inequality: for any $f \in A_0^{n,0}(X_1)$,

$$(A.17) \qquad \|\frac{1}{r}f\|_2 \leq C\|\partial f\|_2.$$

As X_1 is a AC manifold with tangent cone X_0, and the Hardy inequality holds for $(0, n)$-forms on X_0 (cf. [Y 1], Theorem 3.1 and Corollary 8.1), in the same way as Proposition A.1 (See also [Y 1], § 3.), we get

$$(A.18) \qquad \|\frac{1}{r}f\|_2 \leq C(\|\partial f\|_2 + \|f\|_{K,2})$$

where K is a compact set. To eliminate the term $\|f\|_{K,2}$ from (A.18), in view of [Y 1], Theorem 3.2, it is sufficient to show

$$(A.19) \qquad \mathcal{H}_n(X_1) := \{f \in \Omega^n_{X_1}; \int_{X_1} \frac{1}{r^2}|f|^2 dv_{X_1} < \infty\} = 0.$$

By computing the growth order of an element of $\mathcal{H}_n(X_1)$, we get

$$(A.20) \qquad \mathcal{H}_n(X_1) \subset H^0(Y, \Omega^n_Y(H))$$

where Y is the projective compactification of X_1, and H is the hyperplane bundle. Since Y is a hyperquadric, we find $H^0(Y, \Omega^n_Y(H)) = 0$ and obtain the assertion. ∎

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