A remark on complex polynomials of least deviation

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Abstract

We obtain a generalization of two recent results of [Pa] concerning the best approximation problem and a unicity problem for complex polynomials of one variable.

In [Pa] the second author described the polynomials of least deviation from zero on certain compacta in $\mathbb{C}$. This result was applied in [Pa] to obtain a solution of a problem posed by C.-C. Yang [Ya]. Namely, it was proven that, up to a sign, a complex polynomial of a given degree is determined uniquely by the preimage of the two-point subset $\{1, -1\} \subset \mathbb{C}$. In this note we generalize both of these results along the same line of ideas.

Remind that, for a given compact $K \subset \mathbb{C}$, a monic polinomial $p(z) \in \mathbb{C}[z]$ of degree $n > 0$ is called *the n-th polynomial of least deviation* (from zero) if $||p||_K \leq ||q||_K$ for any monic polynomial $q(z) \in \mathbb{C}[z]$ of degree n, where $||p||_K := \max_{z \in K}\{|p(z)|\}$. By the Tonelli–Walsh–Kolmogorov Theorem [1] [To, Wa, Ko] such a polynomial is unique as soon as card $K \geq n$. To formulate our main result we need the following

Definition 1. Let $\Delta = \Delta_{a,r}$ be a closed disc centered at $a \in \mathbb{C}$ of radius r. We say that a compact $K \subset \Delta$ *supports*[2] Δ if Δ is the (unique) disc of smallest radius which contains K.

The next result generalizes Theorem 1 of [Pa].

Theorem 1. *Let $\Delta_r = \Delta_{0,r} \subset \mathbb{C}$ be the disc of radius r centered at the origin, $K \subset \Delta_r$ be a supporting compact of Δ_r, and $p \in \mathbb{C}[z]$ be a monic polynomial of degree n. Set $K_p = p^{-1}(K)$. Then p is the unique n-th polynomial of least deviation on K_p.*

[1] Walsh (1930) proved this (and, in fact, a more general) theorem using the Tonelli approach (1908) to the Chebyshev unicity theorem for real polynomials of least deviation on the interval. The Kolmogorov approach (1948), which deals more generally with Chebyshev systems, is based on his well known criterion for polynomials of best approximation.

[2] See Lemma 2 below for some properties of supporting compacta.

The following corollary describes a class of plane compacta for which a given monic polynomial p serves as the polynomial of least deviation. The maximal such class is called *the Chebyshev cluster of* p.

Corollary (cf. [Pa, Theorem 1]). *Let* $K \subset S_r$ *be a supporting compact of the disc*[3] Δ_r *and* $p \in \mathbb{C}[z]$ *be a monic polynomial of degree* $n > 0$. *Then* p *is the* n*-th polynomial of least deviation on any compact* T *such that* $K_p \subset T \subset p^{-1}(\Delta_r)$.

In the proof of Theorem 1 we use the following averaging projection.

Definition 2. Let $p,\ q \in \mathbb{C}[z]$ and $\deg p = n > 0$. We call *the average of* q *over* p the transform $q \longmapsto \sigma_p(q) = \hat{q} \circ p$, where

$$\hat{q}(z) = \frac{1}{n} \sum_{p^{-1}(z)=\{\xi_1,\dots,\xi_n\}} q(\xi_j)\,, \tag{1}$$

the summation is over all the roots of the polynomial $p(\xi)-z$, and a root of multiplicity m is repeated m times.

Lemma 1. *a)* $\sigma_p : \mathbb{C}[z] \to \mathbb{C}[p]$ *is a linear projection. Moreover, it is a homomorphism of* $\mathbb{C}[p]$*–moduli, i.e.* $\sigma_p(\varphi(p)\cdot q) = \varphi(p)\cdot\sigma_p(q)$ *for any* $q,\ \varphi \in \mathbb{C}[z]$.
b) $\deg \hat{q} = [(\deg q)/n]$. *In particular,* $\deg \hat{q} = 1$ *if* $\deg q = n$. *Furthermore, if both* p *and* q *are monic polynomials of degree* n*, then* $\sigma_p(q) = p + c$*, where* $c \in \mathbb{C}$.

Proof. By definition, the function $\sigma_p(q)$ is constant on each fibre $p^{-1}(z)$ of p, and $\sigma_p(p) = p$. Let $q(z) = \sum_k b_k(z)p^k$, where $b_k \in \mathbb{C}[z]$ and $\deg b_k < n$ for all k, be the p–adic decomposition[4] of q. Then, clearly, $\sigma_p(q) = \sum_k \delta_k p^k$, where $\delta_k = \sigma_p(b_k)$. Here δ_k are constants, since $\sigma_p(q)$ is constant for any polynomial q of degree $m < n$. Indeed, this is enough to check for the monomials $q_m(z) = z^m$, $0 \le m < n$. But the Newton sum of the roots of $p - z$

$$\hat{q}_m(z) = \frac{1}{n} \sum_{p^{-1}(z)=\{\xi_1,\dots,\xi_n\}} \xi_j^m$$

is a polynomial on the coefficients $a_{n-1},\dots,a_{n-m}$ of $p(\xi) = \xi^n + \sum_{i=0}^{n-1} a_i\xi^i$, and therefore, it is constant. The lemma easily follows from these observations. $\square$

[3]For instance, we may take for K either a pair of symmetric points or a triple of points in S_r, as in Lemma 2(iv) below.

[4]We are thankful to D. N. Akhiezer who proposed to use the p–adic decomposition.

Remark. An easy alternative proof of the lemma can be obtained by applying the formula of logarithmic residue. Due to this formula, we have

$$\sigma_p(q)(z) = \frac{1}{2\pi i n}\int_{|u|=R}\frac{q(u)p'(u)du}{p(u)-p(z)} = \sum_{k=0}^{\infty}\frac{p^k(z)}{2\pi i n}\int_{|u|=R}\frac{q(u)p'(u)du}{p^{k+1}(u)}\,,$$

where $R = R(z)$ is sufficiently large and where the members of the series are all zero for $k > (\deg q)/n$.

Proof of Theorem 1. Let $q \in \mathbb{C}[z]$ be a monic polynomial of the same degree n as p. Since the compact $K_p = p^{-1}(K)$ is saturated by the fibres of p, it follows easily from Definition 2 that

$$||\sigma_p(q)||_{K_p} \le ||q||_{K_p}\,. \tag{2}$$

By Lemma 1, $\sigma_p(q) = p + c$, where $c \in \mathbb{C}$. Since K supports the disc Δ_r we have

$$r = ||p||_{K_p} \le ||p+c||_{K_p} = ||\sigma_p(q)||_{K_p}\,. \tag{3}$$

From (2) and (3) we obtain

$$||p||_{K_p} \le ||q||_{K_p}\,. \tag{4}$$

This proves that p is, indeed, a polynomial of least deviation on the compact K_p. Note (see e.g. [Pa]) that the geometric preimage of two distinct values a_1, a_2 of the polynomial p of degree n contains at least $n+1$ distinct points. So, the uniqueness of such a polynomial follows from the Tonelli–Walsh–Kolmogorov Theorem cited above. □

Remark. In our particular case we can provide an easy alternative proof of the uniqueness. Assume that in (4) the equality holds, which implies the equality signs also in (2) and (3). This is true only if $c = 0$, and therefore $\sigma_p(q) = p$, i.e. $\hat{q}(z) = z$. Choose two arbitrary distinct points $a_1, a_2 \in K \cap S_r$, where $S_r = \partial\Delta_r$. Then

$$\hat{q}(a_i) = 1/n \sum_{p^{-1}(a_i)=\{\xi_{i1},\ldots,\xi_{in}\}} q(\xi_{ij}) = a_i,\; i = 1,2\,. \tag{5}$$

Hence

$$|1/n \sum_{p^{-1}(a_i)=\{\xi_{i1},\ldots,\xi_{in}\}} q(\xi_{ij})| = r = ||q||_{K_p}\,. \tag{6}$$

Since $\xi_{ij} \in K_p$, we have $|q(\xi_{ij})| \le r$ for all $j = 1,\ldots,n$. This holds only if $q(\xi_{ij}) = a_i = p(\xi_{ij})$, $i = 1,2,\; j = 1,\ldots,n$. Thus, the polynomial $p - q$ of degree at most $n-1$ vanishes in at least $n+1$ distinct points[5] ξ_{ij}, $i = 1,2$, $j = 1,\ldots,n$. This proves that $p = q$. □

[5]See the remark at the end of the proof of Theorem 1.

The next lemma is a simple exercise.

Lemma 2. Let $S = \partial\Delta$ denote the boundary circle of the disc Δ. The following conditions are equivalent:
(i) A compact $K \subset \Delta$ supports Δ.
(ii) The compact $K \cap S \subset S$ supports Δ.
(iii) $K \cap S \subset S$ is not contained in an open half–circle (or, what is the same, in an open half–plane with the boundary line passing through the center of Δ).
(iv) $K \cap S \subset S$ contains either a pair of symmetric points of S, or a triple of points $b,\, c,\, d \in S$ such that the center a of Δ is an inner point of the triangle bcd.

The next theorem shows that, given a plane compact saturated by fibres of a degree n polynomial, there exists a unique such saturation, so that the fibres are uniquely determined by the compact itself. It follows from Theorem 1 in the same way as Corollary 1 follows from Theorem 1 in [Pa]. For the reader's convenience we repeat the arguments.

Theorem 2. *Let a compact $K \subset \mathbb{C}$ contains at least two points. Suppose that $p,\, q \in \mathbb{C}[z]$ are two polynomials of the same degree n such that $p^{-1}(K) = q^{-1}(K)$. Then $p = \alpha(q)$, where $\alpha(z)$ is a rotation of $\mathbb{C}$ which preserves K.*

Proof. Let z_0 be the center of the unique disc Δ supported by K. Replacing K, p, q resp. by $K - z_0$, $p - z_0$, $q - z_0$ we may assume that $z_0 = 0$, so that $\Delta = \Delta_r$ for some $r > 0$. Set $\tilde{p} = (1/a_n)p$, $\tilde{q} = (1/b_n)q$, where $p(z) = a_n z^n + \ldots$, $q(z) = b_n z^n + \ldots$. Then we have

$$K_p := p^{-1}(K) = q^{-1}(K) = \tilde{p}^{-1}((1/a_n)K) = \tilde{q}^{-1}((1/b_n)K)\,.$$

Since the compact $(1/a_n)K$ resp. $(1/b_n)K$ supports the disc $\Delta_{r/|a_n|}$ resp. $\Delta_{r/|b_n|}$, by Theorem 1, $\tilde{p}$ resp. $\tilde{q}$ is the unique n-th monic polynomial of least deviation on the compact

$$K_p = ((1/a_n)K)_{\tilde{p}} = ((1/b_n)K)_{\tilde{q}}\,.$$

It follows that $|a_n| = |b_n|$ and $\tilde{p} = \tilde{q}$, so that $p = e^{i\varphi} q$ for some $\varphi \in \mathbb{R}$. Since $K_p = p^{-1}(K) = q^{-1}(K)$, we have

$$K = p(K_p) = e^{i\varphi} q(K_p) = e^{i\varphi} q(K_q) = e^{i\varphi} K\,.$$

This shows that K is stable under the rotation $z \longmapsto e^{i\varphi} z$. □

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