Dense lattices in dimensions 28 and 29.

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Abstract: We exhibit an integral lattice of determinant $2^n\ 3$ with minimum 8 in dimension $n = 28$ and 29. These lattices are denser than the laminated lattices and yield hence the densest known sphere packings in 28 and 29 dimensions.

The standard reference for lattices and lattice-packings is **[CS]**. In order to keep this paper self-contained we recall briefly the fundamental definitions.

A *lattice* is a discrete cocompact subgroup Λ in the $n-$dimensional Euclidean vector space $\mathbf{E}^n$ (with scalar product denoted by $\langle\ ,\ \rangle$). Two lattices Λ and M are *isomorphic* if there exists a group isomorphism $\varphi : \Lambda \longrightarrow M$ which is an isometry. The *determinant* of a lattice Λ is the square of the volume of a fundamental domain $\mathbf{E}^n/\Lambda$. The *norm* of a lattice element $\lambda \in \Lambda$ is defined as $\langle\lambda, \lambda\rangle$ (and is hence the squared Euclidean length of λ). A matrix G is a Gram matrix of Λ if $G_{i,j} = \langle b_i, b_j\rangle$ where $b_1, \ldots, b_n$ is a basis of Λ. It can easily be shown that det Λ = det G. A lattice Λ is *integral* if all scalar products between lattice elements are integral. An integral lattice is *even* if all its elements have even norm. Its *minimum* is the norm of a shortest non-zero element. The *Voronoï polytope*

$$\mathrm{Vor}_\Lambda = \{x \in \mathbf{E}^n \mid 2\langle x, \lambda\rangle \leq \langle\lambda, \lambda\rangle \ \text{ for all } \lambda \in \Lambda \setminus \{0\}\ \} \subset \mathbf{E}^n$$

of Λ is the set of all points $x \in \mathbf{E}^n$ such that 0 is the closest lattice point. For two lattices Λ and M we denote by $\Lambda \oplus M$ the orthogonal sum lattice $\{(\lambda, \mu)\}_{\lambda\in\Lambda,\mu\in M}$ with the obvious scalar product.

A *sphere-packing* in dimension n is a collection S of points in $\mathbf{E}^n$ and a strictly positive real number ρ such that $\inf_{a\neq b\in S} \ \| a - b \| \leq \rho$ where $\| x \|$ denotes the Euclidean length of x. One takes generally $\rho = \inf_{a\neq b\in S} \ \| a - b \|$. The *density* of a sphere packing S, ρ is defined as

$$\mathrm{limsup}_{t\to\infty} \frac{\text{Volume } \{x \in \mathbf{E}^n,\ \| x \| \leq t \text{ and } \exists s \in S,\ \| x - s \| \leq \rho\}}{\text{Volume } \{x \in \mathbf{E}^n,\ \| x \| \leq t\}} \quad .$$

A sphere-packing is a *lattice-packing* if the set S is a lattice. The density of an $n-$dimensional lattice is the maximal density of an associated sphere-packing and is given by

$$\sqrt{\frac{(\min \Lambda)^n}{4^n \ \det \Lambda}} V_n$$

where $V_n = \dfrac{\pi^{n/2}}{(n/2)!}$ is the volume of the unit-ball in $\mathbf{E}^n$. The *center density* of Λ is by definition

$$\sqrt{\frac{(\min \Lambda)^n}{4^n \ \det \Lambda}}$$

and is proportional to the density (cf. Chapter 1, Formulas 20 and 27 in **[CS]**). The *kissing number* of a lattice Λ is the cardinality of the set $\Lambda_{\min} = \{\lambda \in \Lambda \mid \langle\lambda, \lambda\rangle = \mathrm{minimum}(\Lambda)\}$.

The root lattice A_2 is a lattice which is isomorphic to the even sublattice of $\mathbf{Z}^3$ generated by $(1, -1, 0)$ and $(0, 1, -1)$. There are 6 vectors of norm 2, called *roots*, in A_2. The lattice A_2 is $2-$dimensional and has determinant 3.

Let M be an $n-$dimensional lattice with minimum 4. Denote by $M_{12} = \{\mu \in M \mid \langle\mu, \mu\rangle = 12\}$ the set of norm 12 elements in M. Suppose that there exists $R' = \{\pm r_1, \pm r_2, \pm r_3\} \subset M_{12} \cap \mathrm{Vor}_{2M}$ such that $\frac{1}{\sqrt{6}}R'$ are the roots of A_2 (recall that $\mathrm{Vor}_{2M} = \{x \in \mathbf{E}^n \mid \langle x, \mu\rangle \leq \langle\mu, \mu\rangle \text{ for all } \mu \in M \setminus \{0\}\}$). Let $m_1, \ldots, m_n$ be a basis of M.

Proposition. *The $(n+2)-$dimensional sublattice $\Lambda \subset \frac{1}{\sqrt{2}}M \oplus \frac{1}{\sqrt{6}}M$ generated by*

$$(\sqrt{2}m_1, 0), \ldots, (\sqrt{2}m_n, 0) \text{ and } (\frac{1}{\sqrt{2}}r_1, \frac{1}{\sqrt{6}}r_1), (\frac{1}{\sqrt{2}}r_2, \frac{1}{\sqrt{6}}r_2)$$

has determinant $2^n\ 3$ det M and minimum 8. Moreover, Λ is integral if M is integral.

Proof: Λ contains the lattice generated by $(\sqrt{2}m_1, 0), \ldots, (\sqrt{2}m_n, 0), (0, \sqrt{\frac{2}{3}}r_1), (0, \sqrt{\frac{2}{3}}r_2)$ as a sublattice of index 4. The determinant of this sublattice is easily seen to be $2^{n+4}\ 3$ det M. This shows that det $\Lambda = \dfrac{2^{n+4}\ 3 \text{ det } M}{(2^2)^2}$.

Let $x = (x_1, x_2)$ be a non-zero element of Λ. If x_2 is zero then $x \in (\sqrt{2}M, 0)$ and x has norm at least 8. Let us hence suppose $x_2 \neq 0$. The element x_2 is an element of the even lattice A_2 which does not contain elements of norm 4. The norm of x_2 is hence equal to $2, 6$ or ≥ 8. In the last case we are done. If x_2 is of norm 6 then $x_2 \equiv x_2' \pmod{2A_2}$ where x_2' is a root of A_2. We can hence suppose that x_2 is of norm 2. This implies that $x = (\frac{1}{\sqrt{2}}r + \sqrt{2}\mu, \frac{1}{\sqrt{6}}r)$ for some $r \in R'$ and some $\mu \in M$. We have then

$$\begin{aligned} \langle x, x \rangle &= \tfrac{1}{2}\langle r, r \rangle + 2\langle r, \mu \rangle + 2\langle \mu, \mu \rangle + \tfrac{1}{6}\langle r, r \rangle \\ &= 8 + 2(\langle r, \mu \rangle + \langle \mu, \mu \rangle) \quad . \end{aligned}$$

But $\pm r \in \mathrm{Vor}_{2M}$ implies $|\langle r, \mu \rangle| \leq \langle \mu, \mu \rangle$ and hence $\langle x, x \rangle \geq 8$.

A straightforward computation shows that the integrality of M implies the integrality of Λ. QED.

Set

$$m_1 = \frac{1}{53}(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 74, 22, 23, 24, 25, 26) \in \mathbf{E}^{26}$$

and denote by e_i the $i-$th basis vector of the standard orthonormal basis of $\mathbf{E}^n$. For $2 \leq i \leq 25$ set then $m_i = e_i + a_i e_{26}$ where a_i is an integer such that $i + 26a_i$ is divisible by 53 (this ensures the integrality of the scalar product $\langle m_1, m_i \rangle$) and set finally $m_{26} = 53e_{26}$. The vectors $m_1, .., m_{26}$ generate then an integral unimodular lattice without roots. Indeed, $53m_1, m_2, \ldots, m_{26}$ generate the sublattice of index 53 in $\mathbf{Z}^{26}$ consisting of all elements in $\mathbf{Z}^{26}$ which have integral scalar product with m_1. Adjoining m_1 to this lattice yields a unimodular lattice since m_1 is of index at least 53 in this lattice (the integrality of the resulting lattice is easy). Let r be a (non-zero) vector of minimal length in M. If r belongs to $\mathbf{Z}^{26}$ then r has norm at least 3 since the coordinates $\mu_1, \ldots, \mu_{26}$ of $53m_1$ are all non-zero modulo 53 and satisfy $\mu_i \neq \pm\mu_j$ for $i \neq j$. If the shortest vector r does not belong to $\mathbf{Z}^{26}$ then the coordinates $\rho_1, \ldots, \rho_{26}$ of $26r$ represent all elements of the set $\{\pm 1 \pmod{53}, \pm 2 \pmod{53}, \ldots, \pm 26 \pmod{53}\}$. The norm of r is hence at least $(1^2 + 2^2 + \ldots + 26^2)/53^2 = 26.27.53/(6.53^2) > 2$. Since all norms are integral the minimum of M is at least 3 and it is easy to find elements of norm 3 or 4 in M. This shows that M is a unimodular lattice without roots (elements of norm 1 or 2) in dimension 26. According to Borcherds such a lattice is unique (cf. **[Bo]**). For $i = 1, \ldots, 26$ set now $\tilde{m}_i = m_i + \epsilon_i m_{26}$ where $\epsilon_i = 1$ if m_i has odd norm and $\epsilon_i = 0$ otherwise. The vectors $\tilde{m}_i$ generate the even sublattice M_e of M. Consider

$$\begin{aligned} r_1 &= (1, -1, 1, -1, -1, -1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)\ , \\ r_2 &= (1, -1, 1, 1, -1, -1, 1, 1, 0, 0, 0, -1, -1, 0, 0, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0)\ , \\ r_3 &= r_1 - r_2 \end{aligned}$$

(coordinates are with respect to the orthonormal basis $e_1, \ldots, e_{26}$ of $\mathbf{E}^{26}$).

Theorem 1. *The vectors $\sqrt{2}\tilde{m}_1, \ldots, \sqrt{2}\tilde{m}_{26}, (\frac{1}{\sqrt{2}}r_1 + e_{27} - e_{28}), (\frac{1}{\sqrt{2}}r_2 + e_{27} - e_{29})$ generate an integral lattice Λ with minimum 8 and determinant $2^{28}\ 3$. The kissing number of Λ is equal to 112 458.*

Proof. Since M is unimodular and has minimum 3, the even sub-lattice M_e of M has determinant 4 and minimum 4. The vectors $\pm r_1, \pm r_2, \pm r_3$ generate $\sqrt{6}A_2$. On checks by computer that $R' = \{\pm r_1, \pm r_2, \pm r_3\} \subset$

Vor_{2M_e}. The subset $R' = \{\pm r_1, \pm r_2, \pm r_3\}$ satisfies hence the conditions of the Proposition. This shows that Λ is integral, has minimum 8 and determinant $2^{26}3\ 4 = 2^{28}\ 3$. The kissing number of Λ can be established by use of a computer. QED

Consider now the vector

$$m_1 = \frac{1}{61}(1,2,3,5,6,7,53,9,10,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29)$$

and define $m_2, \ldots, m_{26}, m_{27} = qe_{27}$ in the analogous way as for the 26−dimensional lattice above (for instance $m_4 = e_4 + \alpha e_{27}$ where α is an integer such that $5 + 29\alpha \equiv 0 \pmod{61}$). It can again be shown that the lattice M generated by $m_1, \ldots, m_{27}$ is a unimodular lattice without roots. Choose a basis $\tilde{m}_1, \ldots, \tilde{m}_{27}$ of the even sublattice M_e of M and consider the vectors

$$\begin{aligned} r_1 &= (1,-1,-1,1,-1,-1,1,-1,-1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\ , \\ r_2 &= (1,-1,-1,1,1,-1,1,-1,0,0,0,0,-1,1,0,0,0,0,-1,0,0,-1,0,0,0,0,0)\ , \\ r_3 &= r_1 - r_2 \quad . \end{aligned}$$

Theorem 2. *The vectors* $\sqrt{2}\tilde{m}_1, \ldots, \sqrt{2}\tilde{m}_{27}, \frac{1}{\sqrt{2}} r_1 + e_{28} - e_{29}, \frac{1}{\sqrt{2}} r_2 + e_{28} - e_{30}$ *generate an integral lattice* Λ *with minimum 8 and determinant* $2^{29}\ 3$. *The kissing number of* Λ *is equal to* 109 884.

The proof is as for Theorem 1.

Corollary. *In dimension 28 and 29 there exist sphere packings with center density* $\frac{1}{\sqrt{3}}$. *These packings are denser than the packings given by laminated lattices which have center density* $\frac{1}{2}$ *(cf. Chapter 6, Table 6.1 of* **[CS]**).

Remarks. (1) The Proposition can be generalized to other root lattices. One needs $R' \subset M_{12} \cap \mathrm{Vor}_{2M}$ such that $\frac{1}{\sqrt{6}} R'$ is a root system. Moreover R' has to satisfy the following condition: for each element x which belongs to the lattice generated by R' one has $\frac{1}{6}\langle x, x\rangle + 2\langle \overline{x}, \overline{x}\rangle \geq 8$ where $\langle \overline{x}, \overline{x}\rangle$ is the norm of a shortest element in the class of $x \pmod{2M}$.

The construction of the Proposition is in fact very close to laminating.

(2) One may ask if the lattice obtained by the Proposition is perfect if M is perfect. A sufficient condition is that the set R' is contained in the set of vertices of Vor_{2M}. The proof is straightforward.

(3) There are probably many 28−dimensional lattices with the same densities as the lattice given in Theorem 1. Indeed the set

$$\begin{aligned} r_1 &= (1,-1,1,-1,-1,-1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0)\ , \\ r_2 &= (1,1,1,-1,-1,-1,0,1,1,0,0,-1,-1,0,0,0,-1,0,-1,0,0,0,0,0,0,0)\ , \\ r_3 &= r_1 - r_2 \end{aligned}$$

satisfies the same requirements as the set r_1, r_2 and r_3 used in Theorem 1. It yields hence another 28−dimensional lattice with the same density. This lattice is non-isomorphic to the lattice of Theorem 1 since its kissing number is 112 394.

The same remark is probably true for 29−dimensional lattices.

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