Alexander Stratifications of Character Varieties

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Abstract

The first cohomology group of one-dimensional representations of a finitely presented group Γ defines a natural stratification of its character variety $\widehat{\Gamma}$. This stratification is related to the Alexander stratification, for which one can obtain explicit defining equations using Fox calculus. We give an elementary exposition of Fox calculus from two points of view: in terms of group cohomology and in terms of finite abelian coverings of CW complexes. Work of Simpson, Arapura and others show that if Γ is the fundamental group of a compact Kähler manifold, then the Alexander strata are finite unions of translated affine tori. This gives obstructions to finitely presented groups which can arise as the fundamental group of a compact Kähler manifold. With this in mind, we give properties of the Alexander strata of general finitely presented groups and apply them to certain examples.

1 Introduction

Let $\mathcal{P}$ be the set of isomorphism classes of fundamental groups of smooth complex projective varieties. By the Lefschetz hyperplane theorem, each group in $\mathcal{P}$ is isomorphic to the fundamental group of some smooth complex projective surface. While it is well known that any finitely presented group is isomorphic to the fundamental group of some closed real 4-dimensional manifold, there are many obstructions for groups to lie in $\mathcal{P}$. For example, it follows from Hodge theory that groups in $\mathcal{P}$ must abelianize to a group with even rank. A more complete discussion of these and other properties and examples of groups contained in $\mathcal{P}$ can be found in the surveys [Ar2], [A-B-C-K-T]. In this paper we will show how Alexander invariants together with known properties of groups in $\mathcal{P}$ can be used as a tool for finding obstructions for a given finitely presented group to lie in $\mathcal{P}$.

The Alexander stratification of character varieties is the finite sequence of nested algebraic subsets $V_i(\Gamma)$ of the character variety $\widehat{\Gamma}$ defined by the Alexander ideals associated to a finite presentation of Γ. This stratification is closely related to the jumping loci $W_i(\Gamma)$ for the cohomology of Γ with respect to one dimensinal representations. The Fox calculus computes defining polynomials for the Alexander strata. We give an exposition of this tool and give general properties of the Alexander strata and jumping loci in section 2.

An algebraic subset $P \subset (\mathbb{C}^*)^r$ is called a *rational plane* if it is a connected subgroup or translation of a connected subgroup by multiplication by a unitary element of $(C^*)^r$ (an element whose components have norm one.) The position of torsion points $\mathrm{Tor}(V)$ for V any algebraic subset of the affine torus $(\mathbb{C}^*)^r$ is described by the following result due to Sarnak [A-S] (cf. Laurant [La]).

Proposition 1 *(Sarnak) If $V \subset (\mathbb{C}^*)^r$ is any algebraic subset, then there exist rational planes $P_1, \ldots, P_k$ in $(\mathbb{C}^*)^r$ such that $P_i \subset V$ for each $i = 1, \ldots, k$ and*

$$Tor(V) = \bigcup_{i=1}^{k} Tor(P_i).$$

One motivation of this paper is the problem of relating properties of a group Γ with the existence of epimorphisms of Γ onto the fundamental group Γ_g of a smooth genus g curve. The existence of such an epimorphism has strong geometric consequences. If X is a smooth complex projective variety and there exists an epimorphism $\pi_1(X) \to \Gamma_g$, where $g \geq 2$, then Beauville [Be] shows that there is a pencil of divisors on X of genus greater than or equal to g. (cf. [Cat].)

For the curve groups Γ_g, the Alexander strata and jumping loci are all rational planes. Furthermore, given an epimorphism $\Gamma \to \Gamma_g$, there are inclusions $V_i(\Gamma_g) \hookrightarrow V_i(\Gamma)$ (see Proposition 5 in section 2.5), giving rational planes contained in $V_i(\Gamma)$. The same is also true for the jumping loci $W_i(\Gamma)$. By a result of Arapura [Ar1] (in the slightly more general setting of compact Kähler manifolds), generalizing work of Simpson in [Sim], for groups in $\mathcal{P}$ the strata are finite unions of rational planes.

Theorem 2 *(Simpson,Arapura) Let X be a smooth complex projective surface and let $\Gamma = \pi_1(X)$. Then each $W_i(\Gamma)$ is a finite union of affine subtori translated by unitary characters.*

For W_1 one can state more about the structure of the rational planes, by the following theorem due to Green and Lazarsfeld [G-L], Beauville [Be], Simpson [Sim] and Arapura [Ar1].

Theorem 3 *(Green-Lazarsfeld, Beauville, Simpson, Arapura) Let X be a smooth complex projective surface and let $\Gamma = \pi_1(X)$. Then there are a finite number of surjective morphisms $f_i : X \to C_i$ onto smooth curves C_i with genus g_i, torsion characters ρ_i and a finite number of unitary characters ρ'_j such that*

$$W_1(\Gamma) = \bigcup_i \rho_i f_i^* W_1(\Gamma_{g_i}) \cup \bigcup_j \rho'_j.$$

For most finitely presented groups, the Alexander stratification can't be decomposed into a finite union of rational planes. Thus, these theorems give strong obstructions for groups that can occur in $\mathcal{P}$. We study some examples in section 3.

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2 Fox Calculus and Alexander Invariants

In this section we give an exposition of the techniques and results surrounding Fox Calculus, Alexander invariants (cf. [Fox]) and their applications.

Fox calculus and Alexander invariants can be summarized as follows. For F any finitely generated free group, let $\mathrm{ab} : F \to F_{\mathrm{ab}}$ be the abelianization map. Then $\mathbb{Z}F_{\mathrm{ab}}$ can be identified with the ring of Laurent poynomials $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, where x_i maps to t_i. Define

$$\mathrm{D}_i : F \to \Lambda$$

by

$$\begin{aligned} \mathrm{D}_i(x_j) &= \delta_{i,j} \\ \mathrm{D}_i(fg) &= \mathrm{D}(f) + \mathrm{ab}(f)\mathrm{D}(g). \end{aligned}$$

We call D_i the *i-th partial Fox derivative.* The map $\mathrm{D} : F \to \Lambda^r$ given by $\mathrm{D} = (\mathrm{D}_1, \ldots, \mathrm{D}_r)$ is called the *Fox derivative.* For a given finitely presented group Γ with generators in a free group F and relations $R_1, \ldots, R_s$, the Alexander invariants are associated with the matrix of partials $M(R_1, \ldots, R_s) = [\mathrm{D}_i(R_j)]$ called the *Alexander matrix.* These matrices can be evaluated on points of the character variety $\widehat{F} = (\mathbb{C}^*)^r$ of F. Let $q : F \to \Gamma$ be the quotient map. Then composition by q defines an inclusion $\widehat{q} : \widehat{\Gamma} \to \widehat{F}$. For $i = 0, 1, 2, \ldots$, define

$$V_i(\Gamma) = \{ \rho \in \widehat{\Gamma} \mid \mathrm{rank} M(R_1, \ldots, R_s)(\widehat{q}(\rho)) \leq r - i \}.$$

These are the subvarieties of $\widehat{\Gamma}$ defined by the ideal of $(r-i) \times (r-i)$ minors of $M(R_1, \ldots, R_s)$ restricted to the image of $\widehat{\Gamma}$ in $\widehat{F}$. The nested sequence

$$\widehat{\Gamma} \supset V_0(\Gamma) \supset V_1(\Gamma) \supset \ldots$$

is called the *Alexander stratifiction* for the group Γ.

In section 2.1, we relate the Alexander stratification to group cohomology. A convenient way to view the Alexander stratification is in terms of a certain coherent sheaf which we define in section 2.2. In section 2.3, we show how Fox calculus can be used to find defining polynomials for the Alexander strata.

Fox calculus gives an effective method for computing the first Betti number of abelian coverings of a finite CW complex. In section 2.4, we give a geometric interpretation of the calculus in terms of CW complexes and regular coverings and derive a formula (see Proposition 4 in 2.4) for the first Betti number of finite abelian coverings in terms of the Alexander stratifications. We discuss properties of the Alexander stratifications in section 2.5.

2.1 Group cohomology and the Alexander stratification.

For a group Γ, let $\mathbb{C}\Gamma$ to be the group ring associated to Γ and let $\widehat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ be the group of characters of Γ. Then this has the natural structure of an algebraic variety with coordinate ring $\mathbb{C}\Gamma_{\mathrm{ab}}$, where Γ_{ab} is the abelianization of Γ. For $f \in \mathbb{C}\Gamma_{\mathrm{ab}}$ and $\rho \in \widehat{\Gamma}$, $f(\rho)$ is defined to be $\rho(f)$. Let $\mathrm{ab} : \Gamma \to \Gamma_{\mathrm{ab}}$ be the abelianization map. If $\alpha : \Gamma \to \Gamma'$ is a homomorphism of groups, let $\widehat{\alpha} : \widehat{\Gamma'} \to \widehat{\Gamma}$ be the corresponding map on character varieties. This is a morphism and the corresponding map on coordinate rings is given by

$$\widehat{\alpha}^* : \mathbb{C}\Gamma'_{\text{ab}} \to \mathbb{C}\Gamma_{\text{ab}},$$

where $\widehat{\alpha}^*(\text{ab}(f))(\rho) = f(\widehat{\alpha}(\rho)) = \widehat{\alpha}(\rho)(f) = \rho(\alpha(f))$, for all $f \in \Gamma$.

Let $C^1(\Gamma, \rho)$ be the set of *crossed homomorphisms* $f : \Gamma \to \mathbb{C}$ satisfying

$$f(g_1 g_2) = f(g_1) + \rho(g_1) f(g_2).$$

Then $C^1(\Gamma, \rho)$ is a vector space over $\mathbb{C}$. Note that for any $f \in C^1(\Gamma, \rho)$, $f(1) = 0$.

Definition. Let

$$V_i(\Gamma) = \{ \ \rho \in \widehat{\Gamma} \mid \ \dim C^1(\Gamma, \rho) > i \ \}.$$

This defines a nested sequence

$$\widehat{\Gamma} \supset V_0(\Gamma) \supset V_1(\Gamma) \supset \ldots.$$

In section 2.3, we will show that this stratification is the same as the Alexander stratification defined in the introduction to this section. For the moment we will use the same notation.

Define, for $\rho \in \widehat{\Gamma}$,

$$B_1(\Gamma, \rho) = \{ \ f : \Gamma \to \mathbb{C} \mid f(g) = (\rho(g) - 1)c \text{ for some constant } c \in \mathbb{C} \ \}.$$

Then $B_1(\Gamma, \rho)$ is a subspace of $C^1(\gamma, \rho)$. Define

$$\mathrm{H}^1(\Gamma, \rho) = C^1(\Gamma, \rho) / B^1(\Gamma, \rho).$$

This is the *first cohomology group of* Γ *with respect to the representation* ρ. The *jumping loci* for the first cohomology group of Γ is defined to be

$$W_i(\Gamma) = \{ \ \rho \in \widehat{\Gamma} \mid \ \dim \mathrm{H}^1(\Gamma, \rho) \geq i \ \},$$

for $i = 0, 1, 2, \ldots$, which defines a nested sequence

$$\widehat{\Gamma} = W_0(\Gamma) \supset W_1(\Gamma) \supset \ldots.$$

If $\rho = \widehat{1}$ is the identity character in $\widehat{\Gamma}$, then $\rho(g) = 1$, for all $g \in \Gamma$. Thus, $B^1(\Gamma, \rho) = \{0\}$. Also, $C^1(\Gamma, \widehat{1})$ is the set of all homomorphisms from Γ to $\mathbb{C}$ and is isomorphic to the abelianization of Γ tensored with $\mathbb{C}$. Thus,

$$\dim C^1(\Gamma, \widehat{1}) = \dim \mathrm{H}^1(\Gamma, \widehat{1}) = d,$$

where d is the rank of the abelianization of Γ. If $\rho \neq \hat{1}$, then $\dim B^1(\Gamma, \rho) = 1$, so

$$\dim C^1(\Gamma, \rho) = \dim \mathrm{H}^1(\Gamma, \rho) + 1.$$

We have thus shown the following.

Lemma. 2.1.1 *The jumping loci $W_i(\Gamma)$ and the nested sequence $V_i(\Gamma)$ are related as follows.*

$$\begin{aligned} W_i(\Gamma) &= V_i(\Gamma) && \text{for } i \neq d \\ W_i(\Gamma) &= V_i(\Gamma) \cup \{\hat{1}\} && \text{for } i = d. \end{aligned}$$

This shows that $V_0(\Gamma) = \widehat{\Gamma}$ unless Γ is a perfect group, abelianizing to the trivial group.

Remark. The Alexander strata could also have been defined using the cohomology of local systems. Let X be a topological space homotopy equivalent to a finite CW complex with $\pi_1(X) = \Gamma$. Let $\widetilde{X} \to X$ be the universal cover of X. Then, for each $\rho \in \widehat{\Gamma}$, each $g \in \Gamma$ acts on $\widetilde{X} \times \mathbb{C}$ by its action as covering automorphism on $\widetilde{X}$ and by multiplication by $\rho(g)$ on $\mathbb{C}$. This defines a local system $\mathbb{C}_\rho \to X$ over X. Then $W_i(\Gamma)$ is the jumping loci for the rank of the cohomology group $\mathrm{H}^1(X, \mathbb{C}_\rho)$ with coefficients in the local system $\mathbb{C}_\rho$.

2.2 Coherent sheaves over the character variety.

Let Γ be a finitely presented group and let $C^1(\Gamma, \rho)^\vee$ be the dual space of $C^1(\Gamma, \rho)$. We will construct sheaves $\mathcal{C}^1(\Gamma)$ and $\mathcal{C}^1(\Gamma)^\vee$ over $\widehat{\Gamma}$ whose stalks are $C^1(\Gamma, \rho)$ and $C^1(\Gamma, \rho)^\vee$, respectively.

Let F be a free group with r fixed generators $x_1, \ldots, x_r$. For $i = 1, \ldots, r$ and $\rho \in \widehat{F}$, let $\langle x_i \rangle_\rho \in C^1(F, \rho)$ be the element determined by

$$\langle x_i \rangle_\rho (x_j) = \delta_{i,j}.$$

Then $\langle x_1 \rangle_\rho, \ldots, \langle x_r \rangle_\rho$ is a basis for $C^1(F, \rho)$ as a $\mathbb{C}$-vector space.

Let $C^1(F)$ be the $\mathbb{C}^r$-bundle

$$\begin{array}{ccc} C^1(F) & = & \sqcup_{\rho \in \widehat{\Gamma}} C^1(F, \rho) \\ \downarrow & & \\ \widehat{F} & & \end{array}$$

where the fibers are $C^1(F,\rho)$. Define $\langle x_i\rangle \in \mathrm{H}^0(\widehat{F}, \mathcal{C}^1(F)$ by $\langle x_i\rangle(\rho) = \langle x_i\rangle_\rho$. Then the module of holomorphic sections of $\widehat{F}$ to the total space $C^1(F)$ is a free $\mathbb{C}F_{\mathrm{ab}}$ module with basis $\{\langle x_1\rangle, \dots, \langle x_r\rangle\}$. Let $\mathcal{C}^1(F)$ be the associated sheaf and let $\mathcal{C}^1(F)^\vee$ denote the sheaf associated to the dual bundle. The latter has global sections $\langle x_1\rangle^\vee, \dots, \langle x_r\rangle^\vee$.

Let Γ be presented by:

$$\Gamma = \langle x_1, \dots, x_r \ : \ R_1, \dots, R_s\rangle.$$

We will refer to the presentation as $\wp$. Let F, F' be free groups on r and s generators, respectively, where $x_1, \dots, x_r$ are the generators for F and $y_1, \dots, y_s$ are generators for F'. Let

$$F' \xrightarrow{\psi} F \xrightarrow{q} \Gamma$$

be homomorphisms, where $\psi(y_i) = R_i$, for $i = 1, \dots, s$, and q is the quotient map given by modding F out by the normalization $N(R_1, \dots, R_s)$ of the image of ψ. We will sometimes refer to a presentation $\wp$ as $\wp(F' \xrightarrow{\psi} F \xrightarrow{q} \Gamma)$.

The map ψ determines a map on character groups $\widehat{\psi} : \widehat{F} \to \widehat{F'}$. Let $\widehat{\psi}^*(\mathcal{C}^1(F'))$ be the pullback of $\mathcal{C}^1(F')$ over $\widehat{F}$. The global sections $s \in \mathrm{H}^0(\widehat{F}, \widehat{\psi}^*(\mathcal{C}^1(F')))$ are given by $s(\rho) = s'(\widehat{\psi}(\rho))$ for some $s' \in \mathrm{H}^0(\widehat{F'}, \mathcal{C}^1(F'))$.

In general, for any group homomorphism $\alpha : \Gamma' \to \Gamma$ and any $\rho \in \widehat{\Gamma}$, there is a corresponding linear map

$$\alpha^\diamond_\rho : C^1(\Gamma, \rho) \to C^1(\Gamma', \widehat{\alpha}(\rho))$$

given by $\alpha^\diamond_\rho(f) = f \circ \alpha$ for $f \in C^1(\Gamma, \rho)$. In this instance, we have a linear map

$$\psi^\diamond_\rho : C^1(F, \rho) \to C^1(F', \widehat{\psi}(\rho))$$

defined by composition by ψ.

Take any element $s \in \mathrm{H}^0(\widehat{F'}, \mathcal{C}^1(F'))$. For any $\rho \in \widehat{F}$ define $\psi^\diamond(s)(\rho) = \psi^\diamond_\rho \circ s(\widehat{\psi}(\rho)) = s(\rho) \circ \psi$. This defines a ring homomorphism

$$\psi^\diamond : \mathrm{H}^0(\widehat{F}, C^1(F)) \to \mathrm{H}^0(\widehat{F'}, C^1(F'))$$

which determines a $\mathbb{C}F_{\mathrm{ab}}$-module homomorphism

$$\mathrm{H}^0(\widehat{F}, C^1(F)) \to \mathrm{H}^0(\widehat{F}, \psi^*(C^1(F'))) = \mathrm{H}^0(\widehat{F'}, C^1(F')) \otimes_{\mathbb{C}F'_{\mathrm{ab}}} \mathbb{C}F_{\mathrm{ab}}.$$

Let

$$\Psi^\diamond : \mathcal{C}^1(F) \to \widehat{\psi}^*(\mathcal{C}^1(F'))$$

be the associated homomorphism of sheaves over $\widehat{F}$.

Let $\mathcal{C}^1(\Gamma)$ be the kernel of $\Psi^\diamond$ pulled back to $\widehat{\Gamma}$ under the map $\widehat{q} : \widehat{\Gamma} \to \widehat{F}$. The dual $\mathcal{C}^1(\Gamma)^\vee$ is the cokernel of the dual map $\Psi_\diamond : \widehat{\psi}^*\mathcal{C}^1(F')^\vee \to \mathcal{C}^1(F)^\vee$ pulled back to $\widehat{\Gamma}$. The sheaf $\mathcal{C}^1(\Gamma)^\vee$ is coherent since $\widehat{\psi}^*\mathcal{C}^1(F')^\vee$ and $\mathcal{C}^1(F)^\vee$ are generated freely by global sections.

Lemma. 2.2.1 *The sequence*

$$0 \longrightarrow C^1(\Gamma, \rho) \xrightarrow{q_\rho^\diamond} C^1(F, \widehat{q}(\rho)) \xrightarrow{\psi_\rho^\diamond} C^1(F', \widehat{\psi}\widehat{q}(\rho))$$

is exact.

Proof. Clearly, $q_\rho^\diamond$ is injective. An element $f \in C^1(\widehat{F}, \widehat{q}(\rho))$ lies in the image of $q_\rho^\diamond$ if and only if it vanishes on $N(\psi(F'))$.

Suppose f vanishes on $N(\psi(F'))$. Then $f(\psi(R)) = 0$ for all $R \in F'$, so $\psi_\rho^\diamond(f) = 0$ and $f \in \ker(\psi_\rho^\diamond)$.

If $f \in \ker(\psi_\rho^\diamond)$, then $f(\psi(R)) = 0$ for all $R \in F'$. Take any $a \in F$ and $R \in F'$. Then

$$\begin{aligned} f(a\psi(R)a^{-1}) &= f(a) + \widehat{q}(\rho)(a)f(\psi(R)) + \widehat{q}(\rho)(a)\widehat{q}(\rho)(R)f(a^{-1}) \\ &= f(a) + \widehat{q}(\rho)(a)f(a^{-1}) \\ &= f(aa^{-1}) = f(1) = 0 \end{aligned}$$

Thus, f vanishes on $N(\psi(F'))$ and is in the image of $q_\rho^\diamond$.

■

Corollary 4 *The stalks of* $\mathcal{C}^1(\Gamma)$ *and* $\mathcal{C}^1(\Gamma)^\vee$ *are canonically isomorphic to* $C^1(\Gamma, \rho)$ *and* $C^1(\Gamma, \rho)^\vee$*, respectively.*

Proof. Fix $\rho \in \widehat{\Gamma}$. Then the restriction of $\Psi^\diamond$ to the stalks over $\widehat{q}(\rho)$ is given by

$$\psi_\rho^\diamond : C^1(F, \widehat{q}(\rho)) \to C^1(F', \widehat{\psi}\widehat{q}(\rho)).$$

The claim for $\mathcal{C}^1(F)$ follows from Lemma 2.2.1. The dual case follows from naturality.

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2.3 Fox Calculus and the Alexander stratification.

In this section, we show how to use the Fox calculus to find defining equations for the stratification defined in section 2.1, associated to a finitely presented group. For technical reasons we will concentrate on the stalks $C^1(\Gamma,\rho)^\vee$ of $\mathcal{C}^1(\Gamma)^\vee$ instead of on the stalks $C^1(\Gamma,\rho)$ of $\mathcal{C}^1(\Gamma)$.

For any group Γ, there is an exact bilinear pairing

$$(\mathbb{C}\Gamma)_\rho \times C^1(\Gamma,\rho) \to \mathbb{C},$$

where

$$(\mathbb{C}\Gamma)_\rho = \mathbb{C}\Gamma/\{g_1g_2 - g_1 - \rho(g_1)g_2 \mid g_1, g_2 \in \Gamma\}$$

given by

$$[g,f] = f(g).$$

This determines a $\mathbb{C}$-linear map

$$\Phi_\rho : (\mathbb{C}\Gamma)_\rho \to C^1(\Gamma,\rho)^\vee.$$

Let $\alpha : \Gamma' \to \Gamma$ be a homomorphism of groups. Then, for each $\rho \in \widehat{\Gamma}$, α determines a map $\alpha : (\mathbb{C}\Gamma')_{\widehat{\alpha}(\rho)} \to (\mathbb{C}\Gamma)_\rho$, by simply applying α to $f \in \Gamma'$ and extending linearly. To see this one need only check the following:

$$\begin{aligned}\alpha(g_1g_2) &= \alpha(g_1 - \widehat{\alpha}(\rho)(g_1)g_2)\\ &= \alpha(g_1 - \rho(\alpha(g_1))g_2)\\ &= \alpha(g_1) - \rho(\alpha(g_1))\alpha(g_2).\end{aligned}$$

Lemma. 2.3.1 *For each $\rho \in \widehat{\Gamma}$, we have a commutative diagram*

$$\begin{array}{ccc}(\mathbb{C}\Gamma')_{\widehat{\alpha}(\rho)} & \xrightarrow{\Phi_{\widehat{\alpha}(\rho)}} & C^1(\Gamma',\widehat{\alpha(\rho)})^\vee \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_{\diamond,\rho}} \\ (\mathbb{C}\Gamma)_\rho & \xrightarrow{\Phi_\rho} & C^1(F,\rho)^\vee,\end{array}$$

where $\alpha_{\diamond,\rho}$ is the dual map of $\alpha^\diamond_\rho$.

Proof. For $g \in \mathbb{C}\Gamma'$ and $f \in C^1(\Gamma,\rho)$, the pairing $[,]$ gives

$$[g,\alpha^\diamond_\rho(f)] = \alpha^\diamond_\rho(f)(g) = f(\alpha(g)) = [\alpha(g),f].$$

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Define

$$\Phi : \mathbb{C}F \to \mathrm{H}^0(\widehat{F}, C^1(F)^\vee)$$

by

$$\begin{aligned}\Phi(x_i) &= \langle x_i\rangle^\vee \\ \Phi(g_1 g_2) &= \Phi(g_1) + \mathrm{ab}(g_1)\Phi(g_2) \qquad \text{for } g_1, g_2 \in \mathbb{F}.\end{aligned}$$

Note that for any $\rho \in \widehat{F}$ and $f \in \mathbb{C}F$ with image $f_\rho \in (\mathbb{C}F)_\rho$, we have $\Phi_\rho(f_\rho) = \Phi(f)(\rho)$. Thus, the map Φ is the globalization of the maps Φ_ρ.

Lemma. 2.3.2 *For each $\rho \in \widehat{\Gamma}$ the dimension of $C^1(\Gamma, \rho)$ is given by*

$$r - \dim(\Phi(\wp)_\rho),$$

where $\Phi(\wp)_\rho$ is the subspace of $C^1(F, \rho)$ generated by

$$\Phi(R_1)(\widehat{\alpha}(\rho)), \ldots, \Phi(R_s)(\widehat{\alpha}(\rho)).$$

Proof. The presentation $\wp(F' \xrightarrow{\psi} F \xrightarrow{q} \Gamma)$ of Γ gives the exact sequence

$$C^1(F', \widehat{\psi}\widehat{q}(\rho))^\vee \xrightarrow{\psi_{\diamond,\widehat{q}(\rho)}} C^1(F, \widehat{q}(\rho))^\vee \xrightarrow{q_{\diamond,\rho}} C^1(\Gamma, \rho)^\vee \longrightarrow 0.$$

By Lemma 2.3.1, the diagram

$$\begin{array}{ccc} (\mathbb{C}F')_{\widehat{\psi}\widehat{q}(\rho)} & \xrightarrow{\Phi_{\widehat{\psi}\widehat{q}(\rho)}} & C^1(F', \widehat{\psi}\widehat{q}(\rho))^\vee \\ \downarrow \psi & & \downarrow \psi_{\diamond,\widehat{q}(\rho)} \\ (\mathbb{C}F)_{\widehat{q}(\rho)} & \xrightarrow{\Phi_{\widehat{q}(\rho)}} & C^1(F, \widehat{q}(\rho))^\vee \\ \downarrow q & & \downarrow q_{\diamond,\rho} \\ (\mathbb{C}\Gamma)_\rho & \xrightarrow{\Phi_\rho} & C^1(\Gamma, \rho)^\vee \end{array}$$

commutes for each $\rho \in \widehat{\Gamma}$. Note that, for free groups F, $C^1(F)^\vee$ has a basis of global sections $\langle x_1\rangle^\vee, \ldots, \langle x_r\rangle^\vee$. Thus, the maps $\Phi_{\widehat{\psi}\widehat{q}(\rho)}$ and $\Phi_{\widehat{q}(\rho)}$ are surjective. Also, the maps q and $q_{\diamond,\rho}$ are surjective. Therefore, the dimension of $C^1(\Gamma, \rho)^\vee$ equals the rank of the cokernel of $\psi_{\diamond,\rho}$. The top of the commutative diagram shows that the image of $\psi_{\diamond,\widehat{q}(\rho)}$ equals the image of $\Phi_{\widehat{q}(\rho)} \circ \psi$ which is generated by $(\Phi_{\widehat{q}(\rho)} \circ \psi)(y_1), \ldots, (\Phi_{\widehat{q}(\rho)} \circ \psi)(y_s)$, for each $\rho \in \widehat{F}$. Since $\psi(y_i) = R_i$, for $i = 1, \ldots, s$, the claim follows.

■

We will now write the map Φ in terms of Laurent polynomials. Let F be the free group on r generators $x_1, \ldots, x_r$. Then there is a natural identification of the coordinate ring $\mathbb{C}F_{\text{ab}}$ for $\widehat{F}$ with the ring of Laurent polynomials $\Lambda = \mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$, given by sending x_i to t_i, for $i = 1, \ldots, r$. The ring of sections $H^0(\widehat{F}, \mathcal{C}^1(F)^\vee)$ is canonically isomorphic to Λ^r, where $\langle x_1 \rangle^\vee, \ldots, \langle x_r \rangle^\vee$ map to the generators of Λ^r considered as a module over Λ.

The corresponding map $\mathbb{C}F \to \Lambda^r$ restricted to F is the Fox derivative $D : F \to \Lambda^r$. Let $R_1, \ldots, R_s$ be elements of F and let $M(R_1, \ldots, R_s)$ be the $s \times r$ Alexander matrix with entries in Λ.

Let Γ be a finitely presented group with presentation $\wp(F' \xrightarrow{\psi} F \xrightarrow{q} \Gamma)$. Let $\widehat{q}^* : \mathbb{C}F_{\text{ab}} \to \mathbb{C}\Gamma_{\text{ab}}$ be the corresponding map on coordinate rings of the character varieties $\widehat{F}$ and $\widehat{\Gamma}$. Then $\widehat{q}^*$ represents $\mathbb{C}\Gamma_{\text{ab}}$ as a quotient ring of $\mathbb{C}F_{\text{ab}}$ with kernel given by the ideal generated by $\text{ab}(R_1) - 1, \ldots, \text{ab}(R_s) - 1$. Thus, q determines an identification of the coordinate ring $\mathbb{C}\Gamma_{\text{ab}}$ of $\widehat{\Gamma}$ with a quotient of Λ. Call this quotient Λ_Γ and let $\widetilde{q} : \Lambda \to \Lambda_\Gamma$ be the quotient map. Let $\mathrm{D}_\Gamma : F \to \Lambda_\Gamma^r$ be the composition $\mathrm{D}_\Gamma = \widetilde{q}^r \circ \mathrm{D}$. Let $\mathrm{D}_{\Gamma,i}$ be the i-th component of D_Γ. Let $M(\wp)$ be the $s \times r$ matrix with entries in Λ_Γ given by $\mathrm{D}_{\Gamma,i}(R_j)$, for $i = 1, \ldots, r$ and $j = 1, \ldots, s$. We call $M(\wp)$ the *Alexander matrix for* Γ *associated to the presentation* $\wp$. Note that $M(\wp)$ can be obtained from $M(R_1, \ldots, R_s)$ by applying the map $\widetilde{q}$ to all the entries.

Lemma. 2.3.3 *The stratification $V_i(\Gamma)$ are the algebraic subsets of $\widehat{\Gamma}$ defined by the ideals of $(r-i) \times (r-i)$ minors of $M(\wp)$ and hence is the same as the Alexander stratification.*

Proof. For each $\rho \in \widehat{\Gamma}$, let $M(\wp)(\rho)$ be the matrix $M(\wp)$ evaluated at ρ. Then the columns of $M(\wp)(\rho)$ are $\mathrm{D}_\Gamma(R_i)(\rho) = \mathrm{D}(R_i)(\widehat{q}(\rho))$, for $i = 1, \ldots, s$. By lemma 2.3.2, we have

$$\dim C^1(\Gamma, \rho) = r - \text{rank}(M(\wp)(\rho)).$$

It follows that $\dim C^1(\Gamma, \rho) > i$ if and only if $\text{rank}(M(\wp)(\rho)) < r - i$. This happens, if and only if ρ satisfies all $(r-i) \times (r-i)$ minors of $M(\wp)$.

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2.4 Abelian coverings of finite CW complexes.

In this section we explain the Fox calculus and Alexander stratification in terms of finite abelian coverings of a finite CW complex. An illustration of the techniques is given at the end of this section.

Let X be a finite CW complex and let $\Gamma = \pi_1(X)$. Suppose Γ has presentation $\wp$ given by $\Gamma = \langle x_1, \dots, x_r : R_1, \dots, R_s \rangle$. Then X is homotopy equivalent to a CW complex with cell decomposition whose tail end is given by

$$\dots \supset \Sigma_2 \supset \Sigma_1 \supset \Sigma_0,$$

where Σ_0 consists of a point P, Σ_1 is a bouquet of r oriented circles S^1 joined at P. Identify F with $\pi_1(\Sigma_1)$ so that each x_i is the positively oriented loop around the i-th circle. Each R_i defines a homotopy class of map from S^1 to Σ_1. The 2-skeleton Σ_2 is the union of s disks attached along their boundaries to Σ_1 by maps in the homotopy class defined by $R_1, \dots, R_s$.

Let $\alpha : \Gamma \to G$ be any epimorphism of Γ to an abelian group G. Let $\widetilde{\alpha} : \mathbb{Z}\Gamma_{\text{ab}} \to \mathbb{Z}G$ be the corresponding map on group rings. Let $q_\alpha : F \to G$ be the map $q_\alpha = \alpha \circ q$ and let $\widetilde{q_\alpha} : \mathbb{Z}F_{\text{ab}} \to \mathbb{Z}G$ be the corresponding map on group rings. Identify $\mathbb{Z}F_{\text{ab}}$ with $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ by $\text{ab}(x_i) = t_i$ and let Λ_Γ and Λ_α be the quotient rings of Λ corresponding to $\widetilde{q}$ and $\widetilde{q_\alpha}$. The monomials in Λ, Λ_Γ and Λ_α are in one to one correspondence with elements of F_{ab}, Γ_{ab} and G, respectively.

Let $\rho_\alpha : X_\alpha \to X$ be the regular unbranched covering determined by α with G acting as group of covering automorphisms. We will now show how Fox calculus can be used to compute the first Betti number of X_α. Choose a basepoint $1P \in \rho_\alpha^{-1}(P)$. For each i-chain $\sigma \in \Sigma_i$ and $g \in G$, let $g\sigma$ be the the component of its preimage which passes through gP. For each generating i-cell in Σ_i, there are exactly G copies of isomorphic cells in its preimage. Thus X_α has a cell decomposition

$$\dots \supset \Sigma_{2,\alpha} \supset \Sigma_{1,\alpha} \supset \Sigma_{0,\alpha},$$

where the i-cells in $\Sigma_{i,\alpha}$ are given by the set $\{g\sigma \ : \ g \in G, \sigma \text{ an } i\text{-cell in } \Sigma_i\}$. With this notation if σ attaches to $\Sigma_{i-1,\alpha}$ according to the homotopy class of mapping $f : \partial\sigma \to \Sigma_{i-1}$, where $\partial\sigma$ is the boundary of σ, then $g\sigma$ attaches to $\Sigma_{i-1,\alpha}$ by the map $f' : \partial g\sigma \to \Sigma_{i-1,\alpha}$ lifting f at the basepoint gP.

Let C_i be the i-chains on X and let $C_{i,\alpha}$ be the i-chains on X_α. Then there is a commutative diagram for the chain complexes for X and X_α:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{2,\alpha} & \xrightarrow{\delta_{2,\alpha}} & C_{1,\alpha} & \xrightarrow{\delta_{1,\alpha}} & C_{0,\alpha} & \xrightarrow{\epsilon} & \mathbb{Z} \\ & & \downarrow{\rho_\alpha} & & \downarrow{\rho_\alpha} & & \downarrow{\rho_\alpha} & & \\ \dots & \longrightarrow & C_2 & \xrightarrow{\delta_2} & C_1 & \xrightarrow{\delta_1} & C_0, & & \end{array}$$

where the map ϵ is the augmentation map

$$\epsilon(\sum_{g\in G}(a_g g)) = \sum_{g\in G} a_g.$$

Here elements of G are are identified with monomials in Λ_α.

Let $\langle x_i\rangle$ be the elements of $C_{1,\alpha}$ given by lifting $x_1,\dots,x_r$, considered as loops on Σ_1, to 1-chains on $\Sigma_{1,\alpha}$ with basepoint $1P$. Then $C_{1,\alpha}$ can be identified with Λ_α^r, with basis $\langle x_1\rangle,\dots,\langle x_r\rangle$ and $C_{0,\alpha}$ can be identified with Λ_α, where each monomial $t\in\Lambda_\alpha$ corresponds to a translation of $1P$ by the corresponding element in G.

The above commutative diagram can be rewritten as

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \Lambda_\alpha^s & \xrightarrow{\delta_{2,\alpha}} & \Lambda_\alpha^r & \xrightarrow{\delta_{1,\alpha}} & \Lambda_\alpha & \xrightarrow{\epsilon} & \mathbb{Z} \\ & & \downarrow{\rho_\alpha} & & \downarrow{\rho_\alpha} & & \downarrow{\rho_\alpha} & & \\ \dots & \longrightarrow & \mathbb{Z}^s & \xrightarrow{\delta_2} & \mathbb{Z}^r & \xrightarrow{\delta_1} & \mathbb{Z}. & & \end{array} \tag{1}$$

For any finite set S, let $|S|$ denote its order. The map ϵ is surjective, so we have the formula

$$b_1(X_\alpha) \;=\; \mathrm{nullity}(\delta_{1,\alpha}) - \mathrm{rank}(\delta_{2,\alpha}) \;=\; (r-1)|G| + 1 - \mathrm{rank}(\delta_{2,\alpha}), \tag{2}$$

where $b_1(X_\alpha)$ is the rank of $\ker\delta_{1,\alpha}/\mathrm{image}\delta_{2,\alpha}$ and is the rank of $\mathrm{H}_1(X_\alpha;\mathbb{Z})$. We will rewrite this formula in terms of the Alexander stratification.

Lemma. 2.4.1 *The map $\delta_{1,\alpha}$ is given by*

$$\delta_{1,\alpha}(\sum_{i=1}^r f_i\langle x_i\rangle) = \sum_{i=1}^r f_i\widetilde{q_\alpha}(t_i-1).$$

Proof. It's enough to notice that the lift of x_i to $C_{1,\alpha}$ at the basepoint $1P$ has end point t_iP.

■

We will now relate the map $\delta_{2,\alpha}$ with the Fox derivative.

Recall that Σ_1 equals a bouquet of r circles $\wedge_r S^1$. Let $\rho : \mathcal{L}_r \to \wedge_r S^1$ be the universal abelian covering. Then $\mathcal{L}_r$ is a lattice on r generators with F_{ab} acting as covering automorphisms. The vertices of the lattice can be identified with F_{ab} and hence with the monomials in Λ. Let $K_\alpha = \ker(\alpha \circ q) \subset F$ and let $\widetilde{K_\alpha}$ be its image in F_{ab}. Then $\Sigma_{1,\alpha} = \mathcal{L}_r/\widetilde{K_\alpha}$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_r & \xrightarrow{\eta_\alpha} & \Sigma_{1,\alpha} \\ \downarrow \rho & & \downarrow \rho_\alpha \\ \wedge_r S^1 & = & \Sigma_1 \end{array}$$

where $\eta_\alpha : \mathcal{L}_r \to \Sigma_{1,\alpha}$ is the quotient map. Let $(\eta_\alpha)_* : C_1(\mathcal{L}_r) \to C_1(\Sigma_{1,\alpha})$ be the induced map on one chains. Then identifying $C_1(\mathcal{L}_r)$ with Λ^r and $C_1(\Sigma_{1,\alpha})$ with Λ_α^r, we have $(\eta_\alpha)_* = \widetilde{q_\alpha}^r$.

Choose $1P' \in \rho^{-1}(P)$. Let $C_1(\mathcal{L}_r)$ be the 1-chains on $\mathcal{L}_r$. Let $\langle x_1\rangle, \dots, \langle x_r\rangle$ be the lifts of $x_1, \dots, x_r$ to $C_1(\mathcal{L}_r)$ at the base point $1P'$. This determines an identification of $C_1(\mathcal{L}_r)$ with Λ^r and determines a choice of homotopy lifting map $\ell : \pi_1(\Sigma_1) \to C_1(\mathcal{L}_r)$.

The action of F_{ab} on Λ_r determines an action of F_{ab} on $C_1(\mathcal{L}_r)$. Let $\mathrm{D} : F \to C_1(\mathcal{L}_r)$ be defined by

$$\begin{aligned} \mathrm{D}(x_i) &= \langle x_i \rangle \\ \mathrm{D}(fg) &= \mathrm{D}(f) + \mathrm{ab}(f)\mathrm{D}(g). \end{aligned}$$

Define $\mathrm{D}_\alpha : F \to C_1(\Sigma_{1,\alpha})$ to be the map $(\eta_\alpha)_* \circ \mathrm{D}$. Under the identification of $C_1(\mathcal{L}_r)$ with Λ^r, the map D is the Fox derivative map defined in the begining of section 2. Let $M(\wp)$ be the matrix (as defined in section 2.3) given by taking the entries of $M(R_1, \dots, R_s)$ and composing on the left by $\widetilde{q}$. Let $M(\wp)_\alpha$ be the matrix obtained from $M(\wp)$ be composing on the left with $\widetilde{\alpha}$. (Here we are ignoring the distinction between $\mathbb{Z}F_{\text{ab}}$, $\mathbb{Z}\Gamma_{\text{ab}}$, $\mathbb{Z}G$ and Λ, Λ_Γ and Λ_α, respectively.)

Lemma. 2.4.2 *The lifting map $\ell : \pi_1(\wedge_r S^1) \to C_1(\mathcal{L}_r)$ is given by the Fox derivative $D : F \to \Lambda^r$ and $D_\alpha = (\widetilde{q_\alpha})^r \circ \ell$.*

Proof. By definition, both maps ℓ and D send x_i to $\langle x_i \rangle$, for $i = 1, \dots, r$. We have left to check products. Let $f, g \in F$, be thought of as loops on $\wedge_r S^1$. Then the lift of f has endpoint $\mathrm{ab}(f)$, where $\mathrm{ab} : F \to F_{\text{ab}}$ is the abelianization map. Therefore, $\ell(fg) = \ell(f) + \text{ab}(f)\ell(g)$. Since these rules are the same as those for the Fox derivative map, the maps must be the same.

■

Corollary 5 *Let Γ be a finitely presented group with presentation $\wp$ given by generators $x_1, \ldots, x_r$ and relations $R_1, \ldots, R_s$. Let $\alpha : \Gamma \to G$ be an epimorphism to an abelian group G. Let $\widetilde{q_\alpha} : \Lambda \to \Lambda_\alpha$ be the the associated quotient map. Suppose the Alexander matrix $M(R_1, \ldots, R_s)$ is given by $[f_{i,j}]$. Then the map $\delta_{2,\alpha}$ is given by $M(\wp)_\alpha = [\widetilde{q_\alpha}(f_{i,j})]$.*

Proof. Let $\sigma_1, \ldots, \sigma_s$ be the s disks generating the 2-cells $C_2(\Sigma_2)$. For each $i = 1, \ldots, s$ and $g \in G$, let $g\sigma_i$ denote the lift of σ_i at gP. By Lemma 2.4.2, the boundary $\partial\sigma_i$ maps to $\text{D}(R_i)$ in $C_1(\mathcal{L}_r)$. Thus, the boundary of $g\sigma_i$ equals $g\text{D}(R_i)$, and for $g_1, \ldots, g_s \in \Lambda_\alpha$,

$$\delta_{\alpha,2}(\sum_{i=1}^{s} g_i\sigma_i) = \sum_{i=1}^{s} g_i\text{D}(R_i).$$

This is the same as the application of $M(\wp)_\alpha$ on the s-tuple $(g_1, \ldots, g_s)$.

■

We now give a formula for the first Betti number $b_1(X_\alpha)$ in terms of the Alexander stratification in the case where G is finite.

Tensor the top row in diagram (1) by $\mathbb{C}$. Then Λ_α is a finite dimensional vector space (isomorphic to $\mathbb{C}G$) and the action of G diagonalizes to get

$$\Lambda_\alpha = \oplus_{\rho\in\widehat{G}}\Lambda_{\alpha,\rho},$$

where $g \in G$ acts on $\Lambda_{\alpha,\rho}$ by multiplication by $\rho(g)$.

The top row of diagram (1) becomes

$$\oplus_{\rho\in\widehat{G}}\Lambda^s_{\alpha,\rho} \xrightarrow{\delta_{\alpha,2}} \oplus_{\rho\in\widehat{G}}\Lambda^r_{\alpha,\rho} \xrightarrow{\delta_{\alpha,1}} \oplus_{\rho\in\widehat{G}}\Lambda_{\alpha,\rho} \xrightarrow{\epsilon} \mathbb{C}.$$

The map $\delta_{\alpha,2}$ considered as a matrix $M(\wp)_\alpha$, as in Lemma 2.4.3, decomposes into blocks

$$M(\wp)_\alpha = \oplus_{\rho\in\widehat{G}}M(\wp)_\alpha(\rho),$$

where, if $M(\wp)_\alpha = [f_{i,j}]$, then $M(\wp)_\alpha(\rho) = [f_{i,j}(\rho)]$. We thus have the following formula for the rank of $M(\wp)_\alpha$:

$$\operatorname{rank}(M(\wp)_\alpha) \;=\; \sum_{\rho\in\widehat{G}} \operatorname{rank}(M(\wp)_\alpha(\rho)). \tag{3}$$

Recall that the Alexander stratification $V_i(\Gamma)$ was defined in the beginning of section 2 to be the zero set in $\widehat{\Gamma}$ of the $(r-i)\times(r-i)$ ideals of $M(\wp)$. For any $\rho \in \widehat{G}$, $M(\wp)_\alpha(\rho) = M(\wp)(\widehat{\alpha}(\rho)) = M(R_1,\ldots,R_s)(\widehat{q_\alpha}(\rho))$, since $\widetilde{\alpha}(f)(\rho) = f(\widehat{\alpha}(\rho))$ and $\widetilde{q_\alpha}(f)(\rho) = f(\widehat{q_\alpha}(\rho))$.

We thus have the following Lemma.

Lemma. 2.4.3 *For $\rho \in \widehat{G}$, $\widehat{\alpha}(\rho) \in V_i(\Gamma)$ if and only if rank$(M(\wp)_\alpha(\rho)) < r-i$.*

For each $i = 0,\ldots,r-1$, let $\chi_{V_i(\Gamma)}$ be the indicator function for $V_i(\Gamma)$. Then, for $\rho \in \widehat{G}$, we have

$$\operatorname{rank}(M(\wp)_\alpha(\rho)) = r - \sum_{i=0}^{r-1} \chi_{V_i(\widehat{\Gamma})}(\widehat{\alpha}(\rho)). \tag{4}$$

Lemma. 2.4.4 *For the special character $\widehat{1}$,*

$$\textit{rank}(M(\wp)_\alpha(\widehat{1})) = r - b_1(X)$$

and rank$(M(\wp)_\alpha(\widehat{1})) = r$ if and only if $\widehat{\Gamma} = \{\widehat{1}\}$ and Γ has no nontrivial abelian quotients.

Proof. The group G acts trivially on $\Lambda_{\alpha,\widehat{1}}$. Thus, in the commutative diagram

$$\begin{array}{ccccc}
\Lambda^s_{\alpha,\widehat{1}} & \xrightarrow{M(\wp)_\alpha(\widehat{1})} & \Lambda^r_{\alpha,\widehat{1}} & \xrightarrow{\delta_\alpha(\widehat{1})} & \Lambda_{\alpha,\widehat{1}} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{C})^s & \xrightarrow{\delta_2} & (\mathbb{C})^r & \xrightarrow{\delta_1} & \mathbb{C}
\end{array}$$

the vertical arrows are isomorphisms.

We thus have

$$\begin{aligned}
\operatorname{rank}(M(\wp)_\alpha(\widehat{1})) &= \operatorname{rank}(\delta_2) \\
&= r - b_1(X).
\end{aligned}$$

■

Proposition 6 *Let Γ be a finitely presented group and let $\alpha : \Gamma \to G$ be an epimorphism where G is a finite abelian group. Let $\widehat{\Gamma} \supset V_0(\Gamma) \supset V_1(\Gamma) \supset \ldots$ be the Alexander stratification for Γ. Let $\widehat{\alpha} : \widehat{G} \hookrightarrow \widehat{\Gamma}$ be the inclusion map induced by α. Then*

$$b_1(X_\alpha) = b_1(X) + \sum_{i=1}^{r-1} |V_i(\Gamma) \cap \widehat{\alpha}(\widehat{G} \setminus \widehat{1})|.$$

Proof. Starting with formula (2) and Lemma 2.4.3, we have

$$\begin{aligned} b_1(X_\alpha) &= (r-1)|G| + 1 - \operatorname{rank}(M(\wp)_\alpha) \\ &= r - \operatorname{rank}(M(\wp)_\alpha(\widehat{1})) + \sum_{\rho \in \widehat{G} \setminus \widehat{1}} (r-1) - \operatorname{rank}(M(\wp)_\alpha(\rho)). \end{aligned}$$

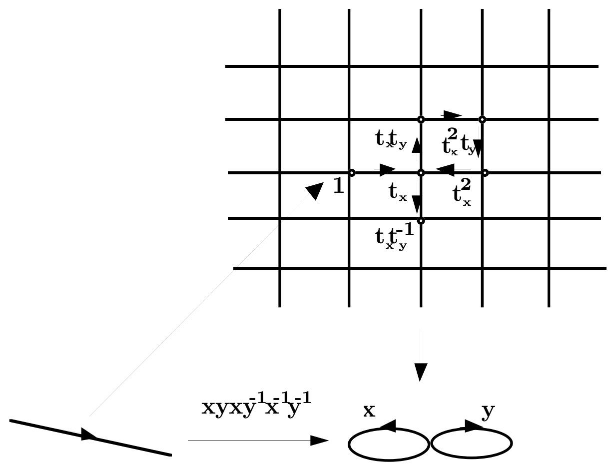
By Lemma 2.4.5, the left hand summand equals $b_1(X)$ and by (4) the right hand side can be written in terms of the indicator functions which leads us to the conclusion we are after:

$$\begin{aligned} b_1(X_\alpha) &= b_1(X) + \sum_{\rho \in \widehat{G} \setminus \widehat{1}} \sum_{i=1}^{r-1} \chi_{V_i(\widehat{\Gamma})}(\widehat{\alpha}(\rho)) \\ &= b_1(X) + \sum_{i=1}^{r-1} |V_i(\Gamma \cap (\widehat{\alpha}(\widehat{G} \setminus \widehat{1}))|. \end{aligned}$$

■

Example. We illustrate the exposition in this section by using the well known case of the trefoil knot in the three sphere S^3:

One presentation of the fundamental group of the complement is $\Gamma = \langle x, y : xyxy^{-1}x^{-1}y^{-1} \rangle$. Then Σ_1 is a bouquet of two circles and $F = \pi_1(\Sigma_1)$ has two generators x, y one for each positive loop around the circles. The maximal abelian covering of Σ_1 is the lattice $\mathcal{L}_2$. Now take the relation $R = xyxy^{-1}x^{-1}y^{-1} \in F$. The lift of R at the origin of the lattice is drawn in the figure below.



Note that the order in which the path segments are taken does not matter in computing the 1-chain. One can verify that $\mathrm{D}(R)$ is the 1-chain defined by

$$(1 - t_x + t_x t_y)\langle x\rangle + (-t_x t_y^{-1} + t_x - t_x^2)\langle y\rangle.$$

Thus, the Alexander matrix for the relation R is

$$M(R) = \begin{bmatrix} 1 - t_x + t_x t_y \\ -t_x t_y^{-1} + t_x - t_x^2 \end{bmatrix}.$$

We will now show how to find $\mathrm{D}_\Gamma(R)$ and $M(\wp)$. Note that $\Gamma_{\mathrm{ab}} \cong \mathbb{Z}$ and x and y both map to the same generator which we'll call t. Thus, Λ_Γ is the quotient of $\Lambda = \mathbb{C}[t_x^{\pm 1}, t_y^{\pm 1}]$ by the ideal generated by $t_x - t_y$ and we can identify Λ_Γ with $\mathbb{C}[t^{\pm 1}]$ by mapping both t_x and t_y to t. The image of $\mathrm{D}_\Gamma(R)$ is $(1 - t + t^2, -1 + t - t^2)$ and the Alexander matrix for this presentation $\wp$ of Γ is

$$M(\wp) = \begin{bmatrix} 1 - t + t^2 \\ -1 + t - t^2 \end{bmatrix}.$$

The Alexander stratification of Γ is thus given by

$$\begin{aligned} V_0(\Gamma) &= \widehat{\Gamma} = \mathbb{C}^*; \\ V_1(\Gamma) &= V(1 - t + t^2); \end{aligned}$$

$$V_i(\Gamma) \;=\; \emptyset \qquad \text{for } i \geq 2.$$

Note that the torsion points on $V_1(\Gamma)$ are the two primitive 6th roots of unity $\exp(\pm 2\pi/6)$.

Now let $\alpha : \Gamma \to G$ be any epimorphism to an abelian group. Then since $\Gamma_{\text{ab}} \cong \mathbb{Z}$, G must be a cyclic group of order n for some n. This means the image of $\widehat{\alpha} : \widehat{G} \to \mathbb{C}^*$ is the set of n-th roots of unity in $\mathbb{C}^*$. Let X_α be the n-cyclic unbranched covering of the complement of the trefoil corresponding to the map α. Then by Proposition 4, we have

$$b_1(X_\alpha) = \begin{cases} 3 & \text{if } 6|n \\ 1 & \text{otherwise.} \end{cases}$$

2.5 Properties of Alexander Stratifications

Let Γ be a finitely presented group with presentation $\wp$ given by $\Gamma = \langle x_1, \ldots, x_r : R_1, \ldots, R_s \rangle$. Hereafter in this paper, we will denote $\wp$ by the pair $\wp(F, \mathcal{R})$, where F is the free group on generators $x_1, \ldots, x_r$ and $\mathcal{R}$ is the set of elements $\{R_1, \ldots, R_s\}$ in F. Let $\mathrm{D} : F \to \Lambda^r$ be the Fox derivative, where Λ be the ring of complex Laurent polynomials in r variables.

Let $\mathcal{D}(\mathcal{R}) = \langle \mathrm{D}(R_1), \ldots \mathrm{D}(R_s) \rangle$ be the ideal generated by Fox derivatives of $R_1, \ldots, R_s$ in Λ^r. For each $\rho \in \widehat{F}$, let $\mathcal{D}(\mathcal{R})(\rho)$ be the subspace of $\mathbb{C}^r$ spanned by $\mathrm{D}(R_1)(\rho), \ldots, \mathrm{D}(R_s)(\rho)$. Then we have

$$V_i(\Gamma) = \{\rho \in \widehat{\Gamma} \mid \dim \mathcal{D}(\mathcal{R})(\rho) < r - i\}.$$

Let $\alpha : \Gamma \to \Gamma'$ be a group homomorphism where Γ and Γ' have presentations $\wp(F, \mathcal{R})$ and $\wp(F', \mathcal{R}')$, respectively. Then α extends (not necessarily uniquely) to a homomorphism $\overline{\alpha} : F \to F'$. Let $\widetilde{\alpha} : \Lambda \to \Lambda'$ be the map on the ring of Laurent polynomials induced by the map $\overline{\alpha}_{\text{ab}} : F_{\text{ab}} \to F'_{\text{ab}}$.

Lemma. 2.5.1 *For $(f_1, \ldots, f_r) \in \Lambda^r$, let*

$$\psi_\alpha(f_1, \ldots, f_r) = \sum_{i=1}^{r} \widetilde{\alpha}(f_i) D(\overline{\alpha}(x_i)).$$

Then the diagram

$$\begin{array}{ccc} \Lambda^r & \xrightarrow{\psi_\alpha} & (\Lambda')^r \\ \uparrow D & & \uparrow D \\ F & \xrightarrow{\overline{\alpha}} & F' \end{array}$$

commutes.

Proof. For each $\rho \in \Gamma'$,

$$\begin{aligned} \psi_\alpha(f_1, \ldots, f_r)(\rho) &= \sum_{i=1}^{r} \widetilde{\alpha}(f_i)(\rho) \\ &= \sum_{i=1}^{r} f_i(\widehat{\alpha}(\rho))\mathrm{D}(\overline{\alpha}(x_i))(\rho). \end{aligned}$$

■

Let $A_{\overline{\alpha}}$ be the matrix with entries in $\Lambda_{\Gamma'}$ having columns $\mathrm{D}(\overline{\alpha}(x_1)), \ldots, \mathrm{D}(\overline{\alpha}(x_r))$. We have thus shown

$$\psi_\alpha(\mathcal{D}(\mathcal{R}))(\rho) = A_{\overline{\alpha}}(\rho)(\mathcal{D}(\mathcal{R})(\widehat{\alpha}(\rho))). \qquad (5)$$

Proposition 7 *Suppose $\alpha : \Gamma \to \Gamma'$ is an epimorphism. Then*

$$\widehat{\alpha}(V_i(\Gamma')) \subset V_i(\Gamma).$$

Proof. For some free group $F = \langle x_1, \ldots, x_r \rangle$, we can find relations $\mathcal{R} \subset \mathcal{R}' \subset F$ such that Γ andΓ' are presented by $\wp(F, \mathcal{R})$ and $\wp(F, \mathcal{R}')$. Taking $\overline{\alpha} : F \to F$ to be the identity map, $A_{\overline{\alpha}} = A_{\overline{\alpha}}(\rho)$ is the identity matrix for every ρ.

Thus (5) gives the inequality

$$\dim(\mathcal{D}(\mathcal{R})(\widehat{\alpha}(\rho))) = \dim(\psi_\alpha(\mathrm{D}(\mathcal{R}))(\rho)) \leq \dim(\mathcal{D}(\mathcal{R}')(\rho)).$$

If $\rho \in V_i(\Gamma')$ then $\dim \mathcal{D}(\mathcal{R}')(\rho) < r - i$, so $\dim \mathcal{D}(\mathcal{R})(\widehat{\alpha}(\rho)) < r - i$ and $\widehat{\alpha}(\rho) \in V_i(\Gamma)$.

■

For any group G, define $V_i(G) = \widehat{G}$ for $i < 0$.

Proposition 8 *Suppose $\alpha : \Gamma \hookrightarrow \Gamma'$ be an endomorphism. Then there is an $s \in \mathbb{Z}^+$ such that*

$$\widehat{\alpha}(V_i(\Gamma')) \subset V_{i-s}(\Gamma)$$

for all $i = 0, 1, 2, \ldots$.

Proof. Suppose Γ has presentation $\wp(F,\mathcal{R})$, where $F = \langle x_1,\ldots,x_r\rangle$. Then we can find $r' > r$ so that for $F' = \langle x_1,\ldots,x_r,x_{r+1},\ldots,x_{r'}\rangle$ and relations $\mathcal{R}' = \mathcal{R}\cup\mathcal{S} \subset F'$ so that Γ' has presentation $\wp(F',\mathcal{R}')$. In this case no element of $\mathcal{S}$ is a consequence of $\mathcal{R}$ and no word in $\langle x_1,\ldots,x_r\rangle$ which is not a consequence of $\mathcal{R}$ is a consequence of $\mathcal{R}\cup\mathcal{S}$.

Let $\overline{\alpha} : F \to F'$ be the inclusion map. Then $\overline{\alpha}$ induces the map α. For any $\rho \in \widehat{\Gamma'}$, $A_\alpha(\rho) : \mathbb{C}^r \to \mathbb{C}^{r'}$ is the inclusion in the first r coordinates and

$$\begin{aligned} A_\alpha(\rho)(D(\mathcal{R}))(\widehat{\alpha}(\rho)) &= \mathcal{D}(\mathcal{R})(\rho) \\ &\subset \mathcal{D}(\mathcal{R})(\rho) + \mathcal{D}_{\mathcal{S}}(\rho) \\ &= \mathcal{D}(\mathcal{R}')(\rho). \end{aligned}$$

Thus,

$$\dim\mathcal{D}(\mathcal{R})(\widehat{\alpha}(\rho)) \le \dim\mathcal{D}(\mathcal{R}')(\rho) \qquad \text{for all } \rho\in\widehat{\Gamma'}$$

and we have

$$\widehat{\alpha}(V_i(\Gamma')) \subset V_{i-s}(\Gamma).$$

where $s = r' - r$.

■

Note that Propositions 5 and 6 also apply to the jumping loci $W_i(\Gamma)$.

Remark. If $\alpha : F \to F'$ is an endomorphism on free groups, then $\widehat{\alpha} : \widehat{F'} \to \widehat{F}$ is surjective, since any homomorphism $\alpha(F) \to \mathbb{C}^*$ extends to a homomorphism $F' \to \mathbb{C}^*$. Suppose Γ and Γ' are presented by $\wp(F,\mathcal{R})$ and $\wp(F',\mathcal{R}')$, respectively and α induces an endomorphism $\Gamma \to \Gamma'$. Then the restriction of $\widehat{\alpha}$ to $\widehat{\Gamma'} \subset \widehat{F'}$, need not be surjective. For example, consider the groups

$$\begin{aligned} \Gamma &= \langle x,y\rangle \\ \Gamma' &= \langle x,y,z \mid zxz^{-1} = y\rangle. \end{aligned}$$

Then the inclusion map $\alpha : \langle x,y\rangle \hookrightarrow \langle x,y,z\rangle$ induces an endomorphism Γ into Γ', but $\widehat{\alpha}$ is not surjective, since, for example $(1,-1)\in\widehat{\Gamma}$ is not the image of any element of $\widehat{\Gamma'}$.

3 Examples

An algebraic subset $P \subset (\mathbb{C}^*)^r$ of the affine torus, is called a *rational plane* if it is a connected subgroup or a translation of a connected subgroup by a unitary character (character whose components have norm one) in $(\mathbb{C}^*)^r$. Any rational plane $P \subset (\mathbb{C}^*)^r$ is the zero set of a *binomial ideal* in the ring of Laurent polynomials Λ. These are ideals generated by elements of the form $cm - 1$, where m is a monomial in Λ and c is a complex constant (in this case of norm one).

By Proposition 1, the torsion points of any algebraic subset $V \subset (\mathbb{C}^*)^r$ is the set of torsion points on some finite union of rational planes contained in V. As in the introduction to this paper, let $\mathcal{P}$ denote the set of groups isomorphic to the fundamental group of a smooth complex projective variety. We know, from Theorem 2, that the Alexander strata are finite unions of rational planes.

In this section, we will study particular examples of finitely presented groups and study the rational planes contained in their Alexander stratifications. For the case of one relator groups, discussed in section 3.4., we obtain a new obstruction on groups in $\mathcal{P}$.

3.1 Free groups and curve groups

Free groups and curve groups both have very simple Alexander stratifications; and all their strata are rational planes, as we show in this section. Thus, if Γ is a finitely presented group, by Proposition 5, epimorphisms $\Gamma \to \Gamma'$, where Γ' is either a free group or a curve group give rational planes sitting inside the strata $V_i(\Gamma)$.

Since the free group has no relations, if F_r is free of rank r, then

$$V_i(F_r) = \begin{cases} \widehat{F_r} & \text{if } i < r; \\ \emptyset & \text{if } i \geq r. \end{cases} \tag{6}$$

Thus, each nontrivial stratum is isomorphic to the r-dimensional affine torus $(\mathbb{C}^*)^r$. For the jumping loci for the group cohomology (or cohomology of local systems), we have

$$W_i(F_r) = \begin{cases} \widehat{F_r} & \text{if } i < r; \\ \{\hat{1}\} & \text{if } i = r; \\ \emptyset & \text{if } i > r. \end{cases} \tag{7}$$

If $\Gamma_g = \pi_1(C_g)$ is the fundamental group of a smooth complex projective curve (or Riemann surface) of genus g, then Γ_g has presentation $\wp(F_{2g}, \{R_g\})$, where $R_g = [x_1, x_{g+1}] \dots [x_g, x_{2g}]$. The Fox derivative of R_g is given by

$$\mathrm{D}(R_g) = \sum_{i=1}^{g}(t_i - 1)\langle x_i\rangle + \sum_{i=g+1}^{2g}(1 - t_i)\langle x_i\rangle.$$

Thus, we have

$$\dim \mathcal{D}(\{R_g\})(\rho) = \begin{cases} 1 & \text{if } \rho \neq \widehat{1}; \\ 0 & \text{if } \rho = \widehat{1}. \end{cases}$$

This implies

$$V_i(\Gamma_g) = \begin{cases} \widehat{\Gamma_g} \cong (\mathbb{C}^*)^{2g} & \text{if } i < 2g-1; \\ \{\widehat{1}\} & \text{if } i = 2g-1; \\ \emptyset & \text{if } i > 2g-1. \end{cases}$$

and for the jumping loci

$$W_i(\Gamma_g) = \begin{cases} \widehat{\Gamma_g} \cong (\mathbb{C}^*)^{2g} & \text{if } i < 2g-1; \\ \{\widehat{1}\} & \text{if } 2g-1 \leq i \leq 2g; \\ \emptyset & \text{if } i > 2g. \end{cases}$$

3.2 Free products

Suppose Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$, where Γ_1 and Γ_2 are finitely presented groups Γ_1 and Γ with presentations $\wp(F_1, \mathcal{R}_1)$ and $\wp(F_2, \mathcal{R}_2)$, respectively. Suppose $\mathcal{R}_1 = \{R_1, \dots, R_{s_1}\}$ and $\mathcal{R}_2 = \{S_1, \dots, S_{s_2}\}$. Then, setting $F = F_1 * F_2$, Γ has the finite presentation $\wp(F, \mathcal{R})$ where $\mathcal{R} = \{R_1, \dots, R_{s_1}, S_1, \dots, S_{s_2}\}$.

The character group $\widehat{F}$ splits into the cross product $\widehat{F} = \widehat{F_1} \times \widehat{F_2}$. Thus, each $\rho \in \widehat{\Gamma}$ can be written as $\rho = (\rho_1, \rho_2)$, where $\rho_1 \in \widehat{F_1}$ and $\rho_2 \in \widehat{F_2}$. The vector space $\mathcal{D}(\mathcal{R})(\rho)$ splits into a direct sum $\mathcal{D}(\mathcal{R})(\rho) = \mathcal{D}(\mathcal{R}_1)(\rho_1) \oplus \mathcal{D}(\mathcal{R}_2)(\rho_2)$ so we have

$$\dim \mathcal{D}(\mathcal{R})(\rho) = \dim \mathcal{D}(\mathcal{R}_1)(\rho_1) + \dim \mathcal{D}(\mathcal{R}_2)(\rho_2).$$

We have thus shown

Proposition 9 *If* $\Gamma = \Gamma_1 * \Gamma_2$*, then*

$$V_i(\Gamma) = \sum_{i_1+i_2=i} V_{i_1}(\Gamma_1) \oplus V_{i_2}(\Gamma_2).$$

Remark. It has been shown (see [Ar2]) that the free product of two nontrivial groups can't lie in $\mathcal{P}$. It would be interesting to see if Fox calculus could be applied to the problem of which amalgamated products of nontrivial groups can lie in $\mathcal{P}$.

3.3 Abelian products

In this section we deal with groups Γ which are finite abelian products of finitely presented groups.

Lemma. 3.3.1 *Let* Γ *be the free abelian product of free groups* $F_1 \times \ldots \times F_k$ *of ranks* $r_1, \ldots, r_k$*, respectively. Let* $q_i : \Gamma \to F_i$ *be the projections. Let* $r = r_1 + \ldots + r_k$ *and let* $m = \max\{r_1, \ldots, r_k\}$*. Then*

$$V_i(\Gamma) = \begin{cases} \bigcup_{i<r_j} \widehat{q}_j(\widehat{F_j}) & \text{if } i < m; \\ \{\widehat{1}\} & \text{if } m \le i < r; \\ \emptyset & \text{if } i \ge r. \end{cases}$$

Proof. From (6), we know that

$$V_i(F_j) = \begin{cases} \widehat{F_j} & \text{for } i < r_j; \\ \emptyset & \text{for } i \ge r_j. \end{cases}$$

By Proposition 5, the surjective maps $q_j : \Gamma \to F_j$ give inclusions

$$\widehat{q}_j(\widehat{F_j}) \subset V_i(\Gamma)$$

for all j such that $i < r_j$. This gives the inclusion

$$\bigcup_{i<r_j} \widehat{q}_j(\widehat{F_j}) \subset V_i(\Gamma)$$

for all $i < m$.

Let $x_{i,1}, \dots, x_{i,r_i}$ be the generators for F_i, for $i = 1, \dots, k$. Let $F = F_1 * \dots * F_k$. For $i, j = 1, \dots, k$, $i < j$, $\ell = 1, \dots, r_i$ and $m = 1, \dots, r_j$, let $R_{i,\ell,j,m} = [x_{i,\ell}, x_{j,\mathfrak{m}}]$. Let

$$\mathcal{R} = \{\ R_{i,\ell,j,m}\ :\ i \neq j\ \}.$$

Then $\wp(F, \mathcal{R})$ is a presentation for Γ. Let Λ be the Laurent polynomials in the generators $t_{i,\ell}$, $i = 1, \dots, k$, $\ell = 1, \dots, r_i$ and associate this to the ring of functions on $\widehat{F} = \widehat{\Gamma}$ by sending $x_{i,\ell}$ to $t_{i,\ell}$.

Note that, since $F_{\text{ab}} = \Gamma_{\text{ab}}$, the Fox derivatives D and D_Γ are the same. We have

$$\text{D}(R_{i,\ell,j,m}) = (1 - t_{j,m})\langle x_{i,\ell}\rangle + (t_{i,\ell} - 1)\langle x_{j,m}\rangle.$$

It immediately follows that $M(\widehat{1})$ is the zero matrix, so $\widehat{1} \in V_i(\Gamma)$ for $i < r$ and $\widehat{1} \notin V_i(\Gamma)$ for $i \geq r$.

Now consider $\rho \in \widehat{F} = \widehat{\Gamma}$ with $\rho \neq \widehat{1}$. We will show that if $\rho \in \widehat{q_i}(F_i)$ then $\rho \in V_n(\Gamma)$ for $n < r_i$ and $\rho \notin V_n(\Gamma)$ for $n \geq r_i$. If $\rho \notin \widehat{q_i}(F_i)$ for any i, then we'll show that $\rho \in V_0(\Gamma) \setminus V_1(\Gamma)$.

Let $\rho_{i,\ell}$, $i = 1, \dots, k$ and $\ell = 1, \dots, r_i$, be the component of ρ corresponding to the generator $t_{i,\ell}$ in Λ. For each $i = 1, \dots, k$, let $s_i = r_1 + \dots + \widehat{r_i} + \dots + r_k$.

Take $\rho \in \widehat{q_i}(F_i)$. We know from Proposition 5 that $\rho \in V_n(\Gamma)$ for $n < r_i$. Also, $\rho_{j,m} = 1$, for all $j = 1, \dots, \hat{i}, \dots, k$. Assume $\rho \neq \widehat{1}$. Then $\rho_{i,\ell} \neq 1$ for some ℓ. Consider the $s_i \times s_i$ minor of $M(\wp)(\rho)$ with rows corresponding to the generators $\langle x_{j,m}\rangle$ where $j = 1, \dots, \hat{i}, \dots, k$ and columns corresponding to generators $R_{i,\ell,j,m}$, where $j = 1, \dots, \hat{i}, \dots, k$ and $m = 1, \dots, r_j$. This is the $s_i \times s_i$ matrix

$$(1 - \rho_{i,\ell})I_{s_i}$$

where I_{s_i} is the $s_i \times s_i$ identity matrix. Thus, rank $M(\rho) \geq s_i$. This means that $\rho \notin V_n(\Gamma)$ for $n \geq (r - s_i) = r_i$.

Now take $\rho \notin \widehat{q_i}(F_i)$ for any i. Then, for some i and j with $i \neq j$, and some ℓ and m, we have $\rho_{i,\ell} \neq 1$ and $\rho_{j,m} \neq 1$. Consider the minor of $M(\wp)(\rho)$ with rows corresponding to all generators except $x_{i,\ell}$, and rows corresponding to relations $R_{i,\ell,j',m'}$, where $j' = 1, \dots, \hat{i}, \dots, k$ and $m' = 1, \dots, r_j$, and $R_{i,\ell',j,m}$, where $\ell' = 1, \dots, \hat{\ell}, \dots, r_i$. This is the $r - 1 \times r - 1$ matrix

$$\begin{bmatrix} (1 - \rho_{i,\ell})I_{s_i} & 0 \\ 0 & (1 - \rho_{j,m})I_{r_i - 1} \end{bmatrix}$$

which has rank $r-1$. Since $\rho \neq \hat{1}$ this is the maximum possible rank. Thus, $\rho \in V_0(\Gamma) \setminus V_1(\Gamma)$.

∎

Corollary 10 *Let Γ be the abelian product of finitely presented groups*

$$\Gamma = \Gamma_1 \times \ldots \times \Gamma_k$$

with $r_1, \ldots, r_k$ generators, respectively. Let

$$P = F_1 \times \ldots \times F_k$$

where each F_j is the free group of rank r_j. Then

$$V_i(\Gamma) \subset V_i(P)$$

for each i and, in particular,

$$V_i(\Gamma) \subset \{\hat{1}\}$$

if $\max\{r_1, \ldots, r_k\} \leq i$.

3.4 One relator groups

In [Ar2], Arapura gives properties that a one relator group with more than two generators must satisfy in order to lie in $\mathcal{P}$.

Proposition 11 *(Arapura, Green-Lazarsfeld, Gromov) If*

$$\Gamma = \langle\, x_1, x_2, \ldots, x_n \mid R \,\rangle$$

lies in $\mathcal{P}$, with $n > 2$, then

(1) *n is even;*

(2) *each x_i occurs at least once in the word R and R lies in the commutator subgroup $[\Gamma, \Gamma]$ of Γ;*

(3) *Γ surjects onto Γ_g with $g = n/2$.*

Let F be the free group on generators $x_1, \ldots, x_{2g}$ and let R_g be the word in F given by

$$R_g = [x_1, x_{g+1}][x_2, x_{g+2}] \ldots [x_g, x_{2g}].$$

Then $\Gamma_g = \pi_1(C_g)$ has presentation $\wp(F, \{R_g\})$.

Many examples of one relator groups which satisfy the conditions stated in Proposition 9 but which cannot be a group in $\mathcal{P}$ can be constructed using the following proposition.

Proposition 12 *Let*

$$\Gamma = \langle x_1, \ldots, x_{2g} \mid R(x_1, \ldots, x_n) \rangle$$

be a one relator group where

$$R = u_1 R_g^{\epsilon_1} u_1^{-1} u_2 R_g^{\epsilon_2} u_2^{-1} \ldots u_k R_g^{\epsilon_k} u_k^{-1}$$

for some words $u_1, \ldots, u_k$ in F, $\epsilon_i = \pm 1$. Then

$$\tau = \sum_{i=1}^{k} \epsilon_i \widetilde{q}(ab(u_i))$$

must be a constant times the power of a binomial in Λ_Γ.

Proof. The Fox derivative of R is given by $\mathrm{D}(R) = \tau \mathrm{D}(R_g)$. Thus

$$V_i(\Gamma) = \begin{cases} (\mathbb{C}^*)^{2g} & \text{if } i = 1, \ldots, 2g-2 \\ V(\tau) \cup \{\widehat{1}\} & \text{if } i = 2g-1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Since, by Theorem 2, these must all be finite unions of rational planes in particular $V(\tau)$ must be. But $V(\tau)$ is a hypersurface, so $V(\tau)$ is a rational plane only if τ is a constant multiple of a power of a binomial element in Λ_Γ.

■

Example. Consider the group

$$\Gamma = \langle\, x_1, \ldots, x_{2g} \mid R \,\rangle.$$

where

$$R = x_1 R_g x_1^{-1} x_2 R_g x_2^{-1} \ldots x_{2g} R_g x_{2g}^{-1}.$$

Then $\mathrm{D}_\Gamma(R) = (t_1 + \ldots + t_{2g}) \mathrm{D}_\Gamma(R_g)$. If $g > 1$, $\tau = t_1 + \ldots + t_{2g}$ is not a power of a binomial, so Γ does not satisfy the condition of Proposition 10 and is not isomorphic to the fundamental group of any compact Kähler manifold.

By Proposition 1, we know that the torsion points on $V_i(\Gamma)$ must lie on rational planes. Let us consider, in particular, the torsion points on $V(t_1 + t_2 + t_3 + t_4)$. If $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$ satisfies

$$\begin{cases} t_i^{n_i} = 1 & \text{for some } n_i \in \mathbb{Z}_+ \\ t_1 + t_2 + t_3 + t_4 = 0, & \end{cases}$$

then for some permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$, we have $t_{i_1} + t_{i_2} = 0$ and $t_{i_3} + t_{i_4} = 0$. Thus,

$$\mathrm{Tor}(V(t_1 + t_2 + t_3 + t_4)) = \mathrm{Tor}(P_1 \cup P_2 \cup P_3),$$

where $P_1, P_2, P_3 \subset V(t_1 + t_2 + t_3 + t_4)$ are defined by

$$\begin{aligned} P_1 &= V(t_1 + t_2) \cap V(t_3 + t_4) \\ P_2 &= V(t_1 + t_3) \cap V(t_2 + t_4) \\ P_3 &= V(t_1 + t_4) \cap V(t_2 + t_3). \end{aligned}$$

These rational planes have codimension one in $V(t_1 + t_2 + t_3 + t_4)$.

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