A MEAN-VALUE LEMMA AND APPLICATIONS TO HEAT DIFFUSION

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We control the gap between the mean value of a function on a submanifold (or a point), and its mean value on any tube around the submanifold (in fact, we give the exact value of the second derivative of the gap). We then apply this formula to obtain comparison theorems between eigenvalues of the Laplace-Beltrami operator, and, also, to obtain bounds of solutions of the heat equation: these bounds are optimal, and are valid for all values of time. Moreover, we get the asymptotic time-expansion of a heat diffusion process on convex polyhedrons in euclidean spaces of arbitrary dimension, and on domains with smooth boundaries in Riemannian manifolds, and we write explicit bounds for the remainder terms of the above expansions, which hold for all values of time.

Contents

Introduction

Sections 1 and 2 contain the technical background of the paper. Let N be a compact, piecewise-smooth submanifold of the complete, $n-$dimensional Riemannian manifold M. The *tube* of radius r around N is the set $M(r) = \{x \in M : \rho(x) < r\}$, where ρ is the *distance function from* N. Given a function u on M, our aim is to describe, in Theorem 2.8, the second derivative of the *content function* :

$$F(r) = \int_{M(r)} u \, dv_n$$

where $r > 0$, and where dv_n is the volume form on M given by the metric. It turns out that the answer involves the Laplacian of u, as well as the Laplacian of the distance function ρ. Now, if we stay within the injectivity radius of N, i.e. if we stay away from the cut-locus of N in M, both ρ and F will be smooth functions (of $x \in M$ and r respectively); however, the nature of the problems we intend to investigate (which include the piecewise-smooth case), and the kind of answers we want to give to these problems (namely, control solutions of the heat equation for all values of time), forced us to take into account all points of the manifold M, and then consider $F(r)$ as a function on the whole half-line, and not just restricted to the (often too small) injectivity tube around the submanifold N.

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In general, both F and ρ will only be Lipschitz regular, and their Laplacians must therefore be taken in the sense of distributions. Hence, our first preoccupation will be to describe, in Lemma 1.4, the distributional Laplacian of the distance function, and to show that it decomposes in a regular part $\Delta_{reg}\rho$ (an $L^1_{loc}-$ function on M), and a singular part, which is in turn the sum of a positive Radon measure $\Delta_{cut}\rho$, supported on the cut-locus of N, and the Dirac measure $-2\delta_N$, supported on the submanifold N and vanishing when N has codimension greater than 1. In particular, $\Delta\rho$ is a Radon measure itself.

As preparatory steps, and for further use in the applications, we then prove a version of Green's theorem for the tubes $M(r)$ (Proposition 2.4), and we show that F is C^1-smooth almost everywhere on $(0,\infty)$; more precisely, on the set of regular values of ρ (by definition, r is a regular value of the distance function if the level set $\rho^{-1}(r)$ meets the cut-locus of N in a subset of zero $(n-1)-$ dimensional Hausdorff measure).

Section 2 ends with the proof of the main technical lemma, Theorem 2.8:

$$-F''(r) = \int_{M(r)} \Delta u \, dv_n + \rho_*(u\Delta\rho)(r) \tag{0.1}$$

where ρ_* is the operator of push-forward on distributions (in our case: measures), which is dual to the pull-back operator ρ^*. (If $r = \rho(x)$ is smaller than the injectivity radius of N, then $\Delta\rho$ is smooth at x, and gives the trace of the second fundamental form of $\rho^{-1}(r)$ at x; in that case, $\rho_*(u\Delta\rho)(r) = \int_{\rho^{-1}(r)} u\Delta\rho$, the integration being performed with respect to the induced measure on $\rho^{-1}(r)$).

Section 3 deals with the applications of Theorem 2.8 to eigenvalue estimates. Some of the results exposed here are already known, but the proofs we provide are, we believe, new, and we have chosen to include them to show the usefulness of our approach, which gives a simple unified proof of all these results. So let us select an eigenfunction u of the Laplace-Beltrami operator: $\Delta u = \lambda u$, and let $F(r) = \int_{M(r)} u$. Theorem 2.8 becomes the following statement:

$$-F'' = \lambda F + \rho_*(u\Delta\rho) \tag{0.2}$$

If u is harmonic, and if all the geodesic spheres of M around x_0 have constant mean curvature (in particular, if M is a manifold of revolution around x_0, or if M is a symmetric space) then we immediately re-derive the "classical mean-value lemma" (Proposition 3.1), by applying (0.2) in the case where ρ is the distance from x_0. This fact justifies the name "mean-value lemma" we have given to Theorem 2.8.

The basic idea in the use of equation (0.2) is that it is possible to bound from below the distribution $\Delta\rho$ by an explicit radial function on M (that is, a function which depends only on the distance from N), if one assumes in addition a lower bound of the Ricci curvature on M. Then we derive from (0.2) a second order differential inequality in F, which can be studied by standard comparison arguments. We explicitly carry out the idea in the following two cases: when ρ is the distance from a point, and when ρ is the distance from the boundary of a domain.

Let us apply the principle (0.2) when $N = \{x_0\}$ and $\rho = d(x_0, \cdot)$. Let us assume $Ricci \geq (n-1)K$, where K is any real number. Let $B(x_0, r)$ (resp. $\bar{B}(r)$) be any geodesic ball of radius r in M (resp. in the simply connected manifold $\bar{M}$ of constant curvature K). We then obtain, in Theorem 3.4, for any positive solution of u of $\Delta u \geq \lambda u$ on $B(x_0, r)$ (resp. for any positive solution of $\Delta\bar{u} = \bar{\lambda}u$ on $\bar{B}(r)$), the following inequality:

$$\frac{\displaystyle\int_{\partial B(x_0,r)} u}{\displaystyle\int_{B(x_0,r)} u} \leq \frac{\displaystyle\int_{\partial\bar{B}(r)} \bar{u}}{\displaystyle\int_{\bar{B}(r)} \bar{u}}$$

for all $0 < r < R$. Theorem 3.4 reduces to the classical Bishop-Gromov inequality if $u = \bar{u} = 1$. Notice that R is not assumed to be smaller than the injectivity radius of x_0, so that the above inequality extends beyond the cut-locus of x_0.

We observe two consequences of Theorem 3.4: the first (Corollary 3.7), states that if u is a positive superharmonic function on $B(x_0, R)$, then, for $0 < r < R$:

$$u(x_0) \geq \frac{1}{\mathrm{vol}\partial\bar{B}(r)} \int_{\partial B(x_0,R)} u$$

and the second (Theorem 3.8) is a well-known inequality of Cheng's regarding the first eigenvalues of the Dirichlet Laplacian on open balls in M and $\bar{M}$ respectively:

$$\lambda_1(B(R)) \leq \lambda_1(\bar{B}(R))$$

which is proved in [7], by different methods.

In the second part of Section 3, we use equation (0.1) in the case where ρ is the distance function from the boundary of the domain Ω in M. We assume a lower bound $\bar{\eta}$ for the mean curvature of $\partial\Omega$, a lower bound $(n-1)K$ for the Ricci curvature of $\partial\Omega$, and we denote by R the inner radius of Ω (that is, the radius of the biggest ball that fits into Ω). We then consider the "symmetrized" domain $\bar{\Omega}$ corresponding to the data $\bar{\eta}, K, R$: it will be the cylinder of constant curvature K, and width R, having constant mean curvature equal to $\bar{\eta}$ on one, say Γ, of the two connected components of the boundary. We then show, in Theorem 3.10, that:

$$\lambda_1(\Omega) \geq \lambda_1(\bar{\Omega})$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet problem on Ω, and $\lambda_1(\bar{\Omega})$ is the first eigenvalue of the following mixed problem on $\bar{\Omega}$: Dirichlet condition on the component having mean curvature $\bar{\eta}$, Neumann condition on the other. The result extends to any domain with piecewise-smooth boundary satisfying an additional property (see Property (P), before Lemma 3.9), and should be compared with the corresponding result obtained by Kasue [15], by different methods. In the special case $\bar{\eta} = 0$, $K = 0$, Theorem 3.10 reduces to the following well-known inequality, due to Li and Yau (see [17], theorem 11):

$$\lambda_1(\Omega) \geq \frac{\pi^2}{4R^2}$$

Section 4 deals with the most original applications of the mean-value lemma, namely, applications to heat diffusion. We fix a domain Ω (we assume $\partial\Omega$ piecewise-smooth and compact), and we fix a solution $w(t, x)$ of the heat equation on Ω. We then introduce the function, depending only on time:

$$f(t) = \int_\Omega w(t, x)\, dx \tag{0.4}$$

Our main interest is for the solution of the heat equation satisfying Dirichlet boundary conditions, and having unit initial conditions ($u(0, x) = 1$ for all $x \in \Omega$). We denote this particular solution by $u(t, x)$ and call it simply the *temperature* function. Integrating in dx, and assuming $\mathrm{vol}(\Omega) < \infty$, we then obtain the *heat content* function $H(t)$:

$$H(t) = \int_\Omega u(t, x)\, dx$$

The function $H(t)$ has been the object of investigation by a number of authors (see [1],[2],[3]).

Our basic idea is to introduce an auxiliary variable $r \geq 0$, and then work with what we call the *complementary heat content* function:

$$F(t,r)=\int_{\Omega(r)}(1-u(t,x))\,dx$$

where $\Omega(r)$ is the level domain of ρ, the distance function from the boundary: $\Omega(r)=\{x\in\Omega: d(x,\partial\Omega)>r\}$. By the mean-value lemma, applied to $N=\partial\Omega$, we immediately obtain that $F(t,r)$ satisfies a heat equation on the half- line $(0,\infty)$, of the type:

$$(-\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial t})F=-\rho_*((1-u(t,\cdot))\Delta\rho) \tag{0.5}$$

The main advantage of the method is that it reduces the problem to a one-dimensional one, where all computations can be performed explicitly: for example, using Duhamel principle (Lemma 4.3), we can represent $F(t,r)$ in terms of the measure $\rho_*((1-u(t,\cdot))\Delta\rho)$ and in terms of suitable heat kernels on the half-line, which, unlike $k(t,x,y)$, have the advantage of being explicit. We emphasize the fact that all these computations extend beyond the cut-radius and the focal radius of the normal exponential map, and therefore the estimates are valid for arbitrary values of time, and not just for small t's.

The Section is divided into four subsections, corresponding to various geometric situations.

In Section 4A, we assume that $\Delta\rho$ is a *positive measure* : this occurs, for example, if both the mean curvature of $\partial\Omega$ and the Ricci curvature of the domain are non-negative (if the boundary is only piecewise-smooth, we add the condition that the foot of any geodesic which minimizes the distance from $\partial\Omega$ is a regular point of $\partial\Omega$). We then apply (0.5) and show that $F(t,r)=\int_{\Omega(r)}(1-u(t,x))\,dx$ is a sub-solution of the heat equation on the interval $(0,R)$, where R is the inner radius of the domain. The main consequences of this fact are the following:

Theorem 4A.1: If $\Delta\rho\geq 0$, then, for all $t>0$, $r\geq 0$:

$$\int_{\Omega(r)}u(t,x)\,dx\geq \mathrm{vol}(\Omega(r))-\mathrm{vol}(\partial\Omega)\cdot\int_0^t e_R(\tau,r,0)\,d\tau$$

where $e_R(t,r,s)$ is the heat kernel of the interval $(0,R)$, with Neumann condition at 0, and Dirichlet condition at R; moreover, equality holds for a flat cylindrical domain with inner radius R.

Hence: among all domains with piecewise-smooth boundary satisfying $\Delta\rho\geq 0$, with fixed inner radius, and with boundary of fixed volume, flat cylinders hold the maximum complementary heat content.

In particular, we have that, for all $t>0$:

$$\int_\Omega u(t,x)\,dx\geq \mathrm{vol}(\Omega)-\frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}$$

an inequality which continues to be true, by polyhedral approximation, for *any* compact, convex set in $\mathbb{R}^n$ (but see also Theorem 4A.7 for sharp upper and lower bounds of the difference between the left and the right-hand sides of the inequality in terms of the second derivative of the function $r\to\mathrm{vol}(\Omega(r))$).

Let us denote by η the trace of the second fundamental form of $\partial\Omega$, by R_{inj} the injectivity radius of normal exponential map of $\partial\Omega$, and by *scal* the scalar curvature.

Theorem 4A.8: If $\partial\Omega$ is smooth (and $\Delta\rho\geq 0$), then, for all $t>0$:

$$\int_\Omega u(t,x)\,dx\geq \mathrm{vol}(\Omega)-\frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}+\frac{1}{2}\int_{\partial\Omega}\eta(x)\,dv_{n-1}(x)\cdot t+\min\{C,0\}t^{3/2}-g(t)$$

where $C=\frac{1}{3\sqrt{\pi}}\inf_{r\in(0,a)}\int_{\rho^{-1}(r)}(scal_M-Ricci(\nabla\rho,\nabla\rho)-scal_{\rho^{-1}(r)})\,dv_{n-1}$, where $g(t)$ is the exponentially decreasing function $(\int_{\partial\Omega}\eta)\int_0^t\int_a^\infty\frac{1}{\sqrt{\pi\tau}}e^{-r^2/\tau}\,dr\,d\tau$, and where a is a fixed number $0<a<R_{inj}$. In particular, if $\Omega\subseteq\mathbb{R}^3$, then $C=-\dfrac{4\sqrt{\pi}}{3}\chi(\partial\Omega)$, where $\chi(\partial\Omega)$ is the Euler characteristic of $\partial\Omega$.

Section 4A ends with a discussion of the case $\Delta\rho \leq 0$ (an example: the complement of a compact, convex set in $\mathbb{R}^n$), and with the corresponding bounds for the complementary heat content function $F(t) = \int_\Omega (1 - u(t,x))\, dx$ (see Theorem 4A.10).

In section 4.B, we drop the assumption on the positivity of $\Delta\rho$, and we assume that $\partial\Omega$ is smooth. We obtain upper and lower bounds of the heat content $H(t)$ which holds in the general case. However, the bounds are given in terms of the temperature function $u(t,x)$, and therefore are not as explicit as the others in the paper.

In Sections 4C and 4D, we compute the first three coefficients of the asymptotic expansion of the heat content function $H(t)$, as $t \to 0$. We consider two cases: when the boundary is smooth (in 4C), and when the domain is a convex polyhedron in $\mathbb{R}^n$ (in 4D). An important feature of our approach is that we are able to give an explicit bound of the remainder term, so what we get is really both an upper and a lower bound for the heat content $H(t) = \int_\Omega u(t,x)\, dx$, which hold for all values of t, and which are sharp, up and including the term in t, for small values of t.

We can derive both developments from the same expression (see 4.4) of the heat content, which holds for arbitrary piecewise-smooth boundaries, and follows from Duhamel principle:

$$\int_\Omega u(t,x)\, dx = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}} \mathrm{vol}(\partial\Omega)\sqrt{t} + \int_0^t \int_0^\infty e(t-\tau, r, 0)\rho_*((1 - u(\tau,\cdot))\Delta\rho)\, dr\, d\tau$$

where $e(t,r,0) = \dfrac{1}{\sqrt{\pi t}} e^{-r^2/4t}$.

In order to approximate the double integral, one needs to accomplish two tasks:

1. Approximate the temperature function, near the boundary, by an explicit, simpler "model".
2. Control the distribution $\Delta\rho$ near the boundary.

If the boundary is *smooth* , the singular part $\Delta_{cut}\rho$ of $\Delta\rho$ is supported on the cut-locus of the normal exponential map, so its support lies *far* from the boundary, and therefore $\Delta_{cut}\rho$ will contribute only with exponentially decreasing terms; on the other hand, $\Delta_{reg}\rho$ is the trace of the second fundamental form of the level sets of the distance function. An appropriate model for $u(t,x)$ is given by the temperature function on a half-space in $\mathbb{R}^n$ (this is proved in Lemma 4C.2), and so we get:

$$\int_\Omega u(t,x)\, dx = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}} \mathrm{vol}(\partial\Omega)\sqrt{t} + \frac{1}{2}\int_{\partial\Omega} \eta\, dv_{n-1} \cdot t + l(t)$$

The remainder $|l(t)|$ is bounded, for all $t > 0$, by $Ct^{3/2} + h(t)$, where C is a constant which depends on the curvatures of Ω and $\partial\Omega$, and where $h(t)$ is exponentially decreasing as $t \to 0$ (see Theorem 4C.3 for their explicit computation).

Theorem 4C.7 generalizes to manifolds the result of [2], which was obtained for domains in euclidean space, and which was proved by probabilistic methods; in [1] the first five terms of the expansion of $H(t)$ were computed, but no estimate of the remainder was given.

If Ω is a *convex polyhedron* in $\mathbb{R}^n$ (the case examined in section 4D), then the regular part of the Laplacian of the distance function vanishes, i.e. $\Delta\rho$ is "purely singular"; moreover, the cut-locus is a polyhedral set itself, and we can describe $\Delta_{cut}\rho$ in Proposition 4D.3. The appropriate model for $u(t,\cdot)$, near an $(n-2)-$dimensional face of Ω, is shown to be the temperature function on the infinite wedge in $\mathbb{R}^n$ bounded by the two hyperplanes which meet at the given face (this is the most delicate step in the proof). Since we only need to approximate u on the cut-locus, which is contained in the bisecting plane of the wedge, we can, by a symmetry argument, reduce our calculations to the bisectrix of a wedge in the plane, and there use an explicit expression of the temperature function. Then, we obtain Theorem 4D.1:

$$\int_\Omega u(t,x)\, dx = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}} \mathrm{vol}(\partial\Omega)\sqrt{t} + c_2 t + l_1(t)$$

with:

$$c_2 = 4 \sum_E \mathrm{vol}_{n-2}(E) \cdot \int_0^\infty \left(1 - \frac{\tanh(\gamma(E)x)}{\tanh(\pi x)}\right) dx$$

where E runs through the set of all $(n-2)-$dimensional faces of $\partial\Omega$ (the "edges" if $\Omega \subseteq \mathbb{R}^3$), and $\gamma(E)$ is the interior angle of the two $(n-1)-$planes whose intersection is E. The remainder $|l_1(t)|$ is bounded, for all t, by $C_1 t^{3/2} + h_1(t)$, for a constant C_1, and for an exponentially decreasing function $h_1(t)$, both explicited in the proof of the theorem.

Theorem 4D.1 generalizes to arbitrary dimension, in the convex case, the result of [3] obtained for domains in the plane having polygonal boundary.

In fact, if $n = 2$ the constant C_1 is zero, our proof simplifies considerably, and we can extend it to cover the (not necessarily convex) polygonal case in $\mathbb{R}^2$.

A few more remarks are in order. First, we observe that the coefficient c_2 is supported on the $(n-2)-$dimensional skeleton of Ω, and therefore it should be related to some kind of distributional mean curvature of the boundary of the polyhedron; on the other hand, c_2 is not the limit of the integral mean curvatures of a sequence of smooth domains which approximate the polyhedron Ω: in other words, c_2 does not pass to the limit under smooth approximations. This fact can be explained by the observation that, in the polyhedral case, the cut-locus goes to the boundary, and cannot be neglected in the computation of the asymptotic terms of order greater than $t^{1/2}$.

As for the arbitrary, piecewise-smooth case, we conjecture the following fact: let $\gamma(y)$ denote the interior angle of the tangent spaces of the two smooth pieces of $\partial\Omega$ meeting at the singular point y, and assume that $\gamma(y) > 0$ (that is, the intersections are transversal). Then the coefficient of the term in t in the asymptotics of the heat content should be given by:

$$4 \int_{Sk_{n-2}} \int_0^\infty \left(1 - \frac{\tanh(\gamma(y)x)}{\tanh(\pi x)}\right) dx \, dv_{n-2}(y) + \frac{1}{2} \int_{\partial_{reg}\Omega} \eta(y) \, dv_{n-1}(y)$$

where Sk_{n-2} is the union of all pieces of dimension $n-2$ in the cellular decomposition of $\partial\Omega$, and η is the trace of the second fundamental form of the regular part of the boundary.

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1. The cut-locus and the Laplacian of the distance function

Let N be a smooth submanifold of the complete Riemannian manifold M of dimension n. We assume N compact. The properties of the cut-locus stated below are proved in [16] in the case $N = \{x_0\}$. They can be extended to arbitrary codimensions by replacing the unit sphere in the tangent space $T_{x_0}M$ with the unit normal bundle $U(N)$ of N. However, all we say in this section holds if N is assumed, more generally, *piecewise-smooth* ; we refer to Appendix D for the extension, to the piecewise-smooth case, of all the results exposed below under the assumption of smoothness for N.

So let $\pi(\xi)$ be the projection of the unit vector $\xi \in U(N)$ onto its base point, and let $c(\xi)$ be the non-negative real number (possibly ∞), having the property that:

the geodesic $\gamma : [0, r] \to M$ *defined by* $\gamma(t) = \exp_{\pi(\xi)} t\xi$ *minimizes the distance from* N *if and only if* $r \in [0, c(\xi)]$.

The map c is then continuous from $U(N)$ to $[0, \infty]$, the 1-point compactification of $[0, \infty)$. The *cut-locus* $\mathrm{Cut}(N)$ of N is, by definition, the set of all points $\exp_{\pi(\xi)} c(\xi)\xi$, as ξ runs through $U(N)$. $\mathrm{Cut}(N)$ is a *closed set* of *measure zero* in M.

Let, for $r > 0$, and $\xi \in U(N)$: $\Phi(r, \xi) = \exp_{\pi(\xi)} r\xi$. Then Φ gives rise to a diffeomorphism from the open set $U = \{(r, \xi) \in (0, \infty) \times U(N) : 0 < r < c(\xi)\}$ to $\Phi(U) = M \setminus (N \cup \mathrm{Cut}(N))$. The (r, ξ) are called the *normal coordinates* of M.

Let dv_n be the Riemannian volume form on M. We pull it back by the diffeomorphism Φ, and we will write: $\Phi^*(dv_n) = \theta(r, \xi) dr d\xi$ on U, θ being the *density of the Riemannian measure in normal coordinates*, and $d\xi$ being the canonical volume form on $U(N)$.

We denote by $\rho_N(x)$, or simply by $\rho(x)$, the distance of x from N. The function $\rho : M \to [0, \infty)$ is Lipschitz:

$$|\rho(x) - \rho(y)| \leq d(x, y) \qquad x, y \in M$$

as it immediately follows from the triangle inequality. In normal coordinates we have, simply, $\rho(r, \xi) = r$, hence ρ, restricted to the set of its "regular points" $\Phi(U) = M \setminus (N \cup \mathrm{Cut}(N))$ is C^∞ smooth, and, on $\Phi(U)$, we have $\|\nabla \rho\| = 1$ by Gauss' lemma. We let $\Delta_{reg}\rho$ denote the Laplacian of $\rho|_{\Phi(U)}$ with respect to the Riemannian metric. The following formula holds true on U:

$$\Delta_{reg}\rho \circ \Phi = -\frac{1}{\theta} \frac{\partial \theta}{\partial r} \tag{1.1}$$

For the proof, see [12], p.40. Since θ vanishes at the focal points of N, we see that $\Delta_{reg}\rho$ is not bounded. Nevertheless, viewed as a function on M (recall that $M \setminus \Phi(U)$ has measure zero), we have :

$$\Delta_{reg}\rho \in L^1_{loc}(M) \tag{1.2}$$

For the proof of this fact, see Appendix A.

The distance function ρ is not, in general, C^1-smooth all over M, and therefore its Laplacian is not a function in the usual sense, but only a distribution; precisely, we define the *distributional Laplacian of* ρ in the natural way: if $\phi \in C_c^\infty(M)$ is a test-function, then:

$$\langle \Delta \rho, \phi \rangle = \int_M \rho \Delta \phi \, dv_n \tag{1.3}$$

$\langle \cdot, \cdot \rangle$ denoting here the duality between a test-function and a distribution. The following lemma clarifies the structure of $\Delta\rho$.

1.4 Lemma. (i) *There exists a positive distribution on* M*, denoted by* $\Delta_{cut}\rho$ *and supported on Cut*(N)*, such that:*

$$\Delta \rho = \begin{cases} \Delta_{reg}\rho + \Delta_{cut}\rho & \text{if} \quad codim(N) \geq 2 \\ \Delta_{reg}\rho + \Delta_{cut}\rho - 2\delta_N & \text{if} \quad codim(N) = 1 \end{cases}$$

where $\langle \delta_N, \phi \rangle = \int_N \phi \, dv_{n-1}$*;*

(ii) $\Delta\rho$ *is a Radon measure (i.e. it can be extended to a continuous linear functional on* $C_c^0(M)$*, endowed with its canonical topology), and if* ϕ *is a Lipschitz, compactly supported function on* M*:*

$$\langle \Delta\rho, \phi\rangle = \int_M (\nabla\rho \cdot \nabla\phi)\, dv_n$$

Proof. We show that the lemma holds with:

$$\langle \Delta_{cut}\rho, \phi\rangle = \int_{\{\xi\in U(N): c(\xi)<\infty\}} \theta(c(\xi),\xi)\phi(\exp_{\pi(\xi)} c(\xi)\xi)\, d\xi \tag{1.5}$$

where $\theta(c(\xi),\xi) = \lim_{r\to\xi_-}\theta(r,\xi)$ (it is a continuous function of ξ).

Now, since ρ is Lipschitz, and since $M\setminus\Phi(U)$ has measure zero, we have, by Green's theorem: $\int_M \rho\Delta\phi = \int_{\Phi(U)} \nabla\rho\cdot\nabla\phi$. Integrating in normal coordinates (in which $\nabla\rho = \dfrac{\partial}{\partial r}$):

$$\langle \Delta\rho, \phi\rangle = \int_{U(N)}\int_0^{c(\xi)} \theta(r,\xi)\frac{\partial(\phi\circ\Phi)}{\partial r}(r,\xi)\, dr\, d\xi$$

Integrating by parts in dr, the inner integral reduces to:

$$\theta(c(\xi),\xi)(\phi\circ\Phi)(c(\xi),\xi) - \theta(0,\xi)\phi(\pi(\xi)) - \int_0^{c(\xi)} \frac{\partial\theta}{\partial r}(r,\xi)(\phi\circ\Phi)(r,\xi)\, dr$$

hence, integrating in $d\xi$, we obtain, thanks to (1.1) and (1.5):

$$\langle \Delta\rho, \phi\rangle = \langle \Delta_{cut}\rho, \phi\rangle + \int_{\Phi(U)} (\Delta_{reg}\rho)\,\phi - \int_{U(N)} \theta(0,\xi)\phi(\pi(\xi))\, d\xi$$

Now $\int_M (\Delta_{reg}\rho)\,\phi = \langle\Delta_{reg}\rho, \phi\rangle$; moreover the last integral is zero if $\mathrm{codim}(N)\geq 2$ (because then $\theta(0,\xi)\equiv 0$), and it equals $2\int_N \phi\, dv_{n-1}$ if $\mathrm{codim}(N) = 1$ (because in that case $\theta(0,\xi)\equiv 1$, and $U(N)$ is locally isometric with $N\times\mathbb{Z}_2$). That $\Delta_{cut}\rho$ is positive, and supported on $\mathrm{Cut}(N)$, is immediate from (1.5). Hence (i) is proved.

Proof of (ii). It follows from (i) that $\Delta\rho$ is a *zero-order distribution* : that is, for any compact set $K\subseteq M$, there is a constant C_K such that, for all ϕ with support contained in K, we have:

$$|\langle\Delta\rho,\phi\rangle| \leq C_K \sup_K|\phi|$$

A classical result (see for example [19]) implies that $\Delta\rho$ is a Radon measure, as asserted. (Note in particular that $\Delta_{cut}\rho$ is a positive Radon measure).

As regards to the last statement, pick a sequence of smooth, compactly supported functions ϕ_n which converge to ϕ in the $C^{0,1}-$topology on the space of Lipschitz, compactly supported functions on M, which is the topology induced by the semi-norms:

$$\|\phi\|_{C^{0,1}(K)} = \sup_{x\in K}|\phi(x)| + \sup_{x\neq y\in K}\frac{|\phi(x)-\phi(y)|}{d(x,y)}$$

for all K compact. Since $\Delta\rho$ is C^0-continuous, it is certainly $C^{0,1}-$ continuous, hence:

$$\begin{aligned}\langle\Delta\rho,\phi\rangle &= \lim_{n\to\infty}\langle\Delta\rho,\phi_n\rangle\\ &= \lim_{n\to\infty}\int_M \nabla\rho\cdot\nabla\phi_n\\ &= \int_M \nabla\rho\cdot\nabla\phi\end{aligned}$$

Proof is complete.

The singular Laplacian of the distance function has been considered by Courtois in [9]: for a second description of $\Delta_{cut}\rho$ (which is essentially the one found in [9]), see Appendix B. For the extension of Lemma 1.4 to the case where N is only piecewise-smooth, see Appendix D.

2. The mean value lemma.

Let N be a compact, piecewise-smooth submanifold of M, and let $\rho : M \to N$ denote the distance function from N. Fix $u \in C^2(M)$, and consider the map $F : (0, \infty) \to \mathbb{R}$ defined by:

$$F(r) = \int_{M(r)} u \, dv_n \tag{2.1}$$

where $M(r)$ is the tube $\{x \in M : \rho(x) < r\}$. The map F is locally Lipschitz, (Proposition 2.6) but generally not even C^1-smooth. The aim of this section is to describe, in Theorem 2.8, the second derivative of F, as a distribution on $(0, \infty)$ (it will turn out in fact, that F'' is a Radon measure). To that end, we first need to prove a version of Green's theorem for the level domains $M(r)$, which are not always regular.

We make use of the Hausdorff measures H_p, for the definition and properties of which we refer to [6] (but see also [11]); let us only remark here that if A is a subset of a Riemannian manifold of dimension p, then the $p-$dimensional Hausdorff measure of A coincides with the Riemannian measure of A, and in particular if V is a domain of M with piecewise-smooth boundary, then $H_{n-1}(\partial V) = \text{vol}_{n-1}(\partial V)$.

We start from:

2.2 Lemma. *Let K be a compact subset of M with $H_{n-1}(K) < \infty$, $n = dim(M)$. Then, for all $0 < \epsilon < \epsilon_0$, there exists an open set $V(\epsilon)$ with piecewise smooth boundary which covers K and satisfies:*

$$vol(\partial V(\epsilon)) \leq C_{n-1} \cdot H_{n-1}(K) + C\epsilon$$

where $C_{n-1} = 2^{n-1}\dfrac{vol(\partial B^{n-1})}{vol(B^{n-1})}$ (B^{n-1} is the unit ball in $\mathbb{R}^{n-1}$) and C is a positive constant which depends only on ϵ_0 and on a lower bound of the Ricci curvature on a neighborhood of K.

Proof. Appendix B.

2.3 Corollary. *Let K be a compact set, and let $\phi \in C^0(M)$. Then:*

$$|\int_K \phi \Delta_{cut}\rho| \leq C_{n-1} H_{n-1}(K \cap Cut(N)) \cdot \|\phi\|_{C^0(K)}$$

in particular, $\Delta_{cut}\rho$ is absolutely continuous with respect to H_{n-1}.

Proof. Appendix B.

Another consequence of Lemma 2.2 is a version of Green's theorem which will suit our needs. Given the domain Ω, we will say that $\partial\Omega$ is *almost regular* if it is the disjoint union of two pieces $\partial_{reg}\Omega$, $\partial_{sing}\Omega$, where $\partial_{reg}\Omega$ is a C^1-smooth submanifold of M, and where $\partial_{sing}\Omega$ is compact, and has zero $H_{n-1}-$measure.

2.4 Proposition. *Let Ω be a domain with almost regular boundary, and let ν denote the unit vector, normal to $\partial_{reg}\Omega$ and pointing inside Ω. Then if $u \in C^2(\Omega)$:*

$$\int_\Omega \Delta u = \int_{\partial_{reg}\Omega} \frac{\partial u}{\partial \nu} \, dv_{n-1}$$

where dv_{n-1} is the induced volume form on $\partial_{reg}\Omega$, or, equivalently:

$$\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dH_{n-1}$$

where H_{n-1} is Hausdorff measure.

Proof. Fix $\epsilon > 0$, and apply the lemma to $K = \partial_{sing}\Omega$. Then we have:

$$\int_\Omega \Delta u\, dv_n = \lim_{\epsilon\to 0} \int_{\Omega\setminus(V(\epsilon)\cap\Omega)} \Delta u\, dv_n$$

The domain $\Omega \setminus (V(\epsilon) \cap \Omega)$ has piecewise smooth boundary given by the disjoint union of $\partial\Omega \cap (V(\epsilon))^c$ and $\partial V(\epsilon) \cap \Omega$. Hence, by the classical version of Green's theorem:

$$\int_{\Omega\setminus(V(\epsilon)\cap\Omega)} \Delta u\, dv_n = \int_{\partial\Omega\cap V(\epsilon)^c} \frac{\partial u}{\partial \nu}\, dv_{n-1} + \int_{\partial V(\epsilon)\cap\Omega} \frac{\partial u}{\partial \nu}\, dv_{n-1}$$

Since $V(\epsilon)$ is contained in a $2\epsilon-$neighborhood of K (see the proof of Lemma 2.2), we deduce that $\int_{\partial_{reg}\Omega\cap V(\epsilon)} \frac{\partial u}{\partial \nu}\, dv_{n-1}$ tends to zero as $\epsilon \to 0$, by Lebesgue bounded convergence theorem. Therefore $\int_{\partial\Omega\cap V(\epsilon)^c} \frac{\partial u}{\partial \nu}\, dv_{n-1}$ converges to $\int_{\partial_{reg}\Omega} \frac{\partial u}{\partial \nu}\, dv_{n-1}$. On the other hand, by Lemma 2.2, $\mathrm{vol}(\partial V(\epsilon)) \to 0$, and therefore $\int_{\partial V(\epsilon)\cap\Omega} \frac{\partial u}{\partial \nu}\, dv_{n-1}$ converges to 0 as $\epsilon \to 0$, since $|\frac{\partial u}{\partial \nu}| \leq \|\nabla u\|$ is bounded. $\square$

Now fix $r > 0$. We say that r is a *regular value* of ρ if:

$$H_{n-1}(\rho^{-1}(r) \cap \mathrm{Cut}(N)) = 0$$

We see immediately that, if r is a regular value of ρ, then $\partial M(r) = \rho^{-1}(r)$ is almost regular, with $\partial_{reg}\Omega = \rho^{-1}(r) \cap \Phi(U)$, and $\partial_{sing}\Omega = \rho^{-1}(r) \cap \mathrm{Cut}(N)$. Since $\mathrm{Cut}(N)$ has zero measure in M, we have, as a consequence of Eilenberg's inequality ([6], Thm 13.3.1), that *the complement of the set of regular values of ρ is of zero Lebesgue measure in* $(0, \infty)$. Therefore, for almost all $r \in (0, \infty)$, $\partial M(r)$ is almost regular; and since $\nabla\rho$ coincides with the unit normal to $\partial_{reg}M(r)$, pointing outside $M(r)$, we have proved the following:

2.5 Corollary. *At all regular values r of ρ, hence almost everywhere on* $(0, \infty)$*:*

$$\int_{M(r)} \Delta u\, dv_n = -\int_{\rho^{-1}(r)} (\nabla u \cdot \nabla\rho)\, dH_{n-1}$$

for all $u \in C^2(M)$.

Next, we show that the map $F(r)$, as in (2.1), is locally Lipschitz. From the co-area formula ([6], Corollary 13.4.6), we see that:

$$F(r) = \int_0^r \int_{\rho^{-1}(s)} u\, dH_{n-1}\, ds$$

Although F is C^∞-smooth for $r < R_{inj}$ (the injectivity radius of N in M, which is positive if N is smooth), there are examples showing that F is not even C^1 past R_{inj} . However:

2.6 Proposition. *The map F is Lipschitz on every bounded interval in $[0,\infty)$, and it is C^1-smooth at all regular values of ρ, where we have:*

$$F'(r) = \int_{\rho^{-1}(r)} u\, dH_{n-1} = \int_{\partial_{reg} M(r)} u\, dv_{n-1}$$

The proof will be an easy consequence of the following lemma, which will be also used later on. We consider the map:

$$V(r) = \int_{\rho^{-1}(r)\cap\Phi(U)} u\, dv_{n-1}$$

Then we have:

2.7 Lemma.
(i) *V(r) is continuous from the right at all $r \in [0,\infty)$;*

(ii)
$$\lim_{s\to r_-} V(s) = V(r) + \int_{\rho^{-1}(r)} u\, \Delta_{cut}\rho$$

the last term denoting the integral of u (with respect to the measure $\Delta_{cut}\rho$) on the measurable set $\rho^{-1}(r)$. In particular, $V(r)$ is continuous at all regular values of ρ.

Proof. We use normal coordinates, and abbreviate $u(\Phi(r,\xi))$ with $u(r,\xi)$. Then it is easy to verify that:

$$V(r) = \int_{U(N)} f_r(\xi)\, d\xi$$

where:

$$f_r(\xi) = \begin{cases} u(r,\xi)\,\theta(r,\xi) & \text{if} \quad c(\xi) > r \\ 0 \quad \text{if} \quad c(\xi) \le r \end{cases}$$

Consider two sequences: $s_n \to r_+$ and $r_n \to r_-$. Then, for all $\xi \in U(N)$, we have, easily:

$\lim\limits_{n\to\infty} f_{s_n}(\xi) = f_r(\xi)$, and:

$\lim\limits_{n\to\infty} f_{r_n}(\xi) = \begin{cases} f_r(\xi) & \text{if} \quad c(\xi) \ne r \\ u(c(\xi),\xi)\,\theta(c(\xi),\xi) & \text{if} \quad c(\xi) = r \end{cases}$

$f_t(\xi)$ being bounded, we can apply Lebesgue bounded convergence theorem, and get immediately (i); as for (ii):

$$\begin{aligned} \lim_{r_n\to r} V(r_n) &= \int_{U(N)} f_r(\xi) d\xi + \int_{\{\xi : c(\xi)=r\}} u(c(\xi),\xi)\theta(c(\xi),\xi)\, d\xi \\ &= V(r) + \int_{\rho^{-1}(r)} u\, \Delta_{cut}\rho \end{aligned}$$

If r is a regular value, then $\rho^{-1}(r)$ has zero $\Delta_{cut}\rho$−measure by Corollary 2.3, hence the right and left limits coincide with $V(r)$. Proof is complete.

Proof of Proposition 2.6. In order to show that F is Lipschitz on each compact interval $[a,b]$, it is enough to show that the map $r \mapsto \int_{\rho^{-1}(r)} u\, dH_{n-1}$ is essentially bounded on $[a,b]$, or that $r \mapsto V(r)$ is bounded on $[a, b]$. But this can be done by giving an upper bound of the Jacobian $\theta(r, \xi)$ on the compact set $[a, b] \times U(N)$. The equality $F'(r) = V(r)$ follows immediately. $\square$

We now come to the computation of F''. Let $\psi \in C^0_c(0, \infty)$. Since ρ is a proper map, the pull back $\psi \circ \rho$ is a continuous, compactly supported map on M. Hence if T is a Radon measure on M, we can consider its *push-forward* $\rho_*(T)$: it will be the Radon measure on $(0, \infty)$ defined by the relation:

$$\langle \rho_*(T), \psi \rangle = \langle T, \psi \circ \rho \rangle$$

In particular, if $T \in L^1(M)$, then $\rho_* T$ is the regular measure given by:

$$\rho_* T(r) = \int_{\rho^{-1}(r)} T \, dv_{n-1}$$

defined almost everywhere on $[0, \infty)$. (This follows immediately from the co-area formula).

We now come to the main theorem of the section.

2.8 Theorem (Mean-value lemma). *Let $\rho : M \to [0, \infty)$ be the function: distance from N, where N is a compact, piecewise-smooth submanifold of M; let $u \in C^2(M)$, and let $M(r) = \{x \in M : \rho(x) < r\}$. If $F(r) = \int_{M(r)} u \, dv_n$, then we have, as Radon measures on $(0, \infty)$:*

$$-F''(r) = \int_{M(r)} \Delta u \, dv_n + \rho_*(u\Delta\rho)(r)$$

where ρ_* denotes push-forward.

Proof. It is enough to verify the equality when both sides are tested against a smooth, compactly supported function on $(0, \infty)$. So let ψ be one such. Then:

$$\begin{aligned}
-\langle F'', \psi \rangle &= -\int_0^\infty F\psi'' \\
&= \int_0^\infty F'\psi' \\
&= \int_0^\infty \left(\int_{\rho^{-1}(r)} u \, dH_{n-1} \right) \psi'(r) \, dr \\
&= \int_0^\infty \int_{\rho^{-1}(r)} u(\psi' \circ \rho) dH_{n-1} \, dr \\
&= \int_M u(\psi' \circ \rho)
\end{aligned}$$

The last equality uses co-area formula. Now, on the set of regular points of ρ, (hence a.e. on M) the map $\psi \circ \rho$ is C^∞ and we have: $\nabla(\psi \circ \rho) = (\psi' \circ \rho)\nabla\rho$. Hence:

$$\begin{aligned}
-\langle F'', \psi \rangle &= \int_M u(\nabla(\psi \circ \rho) \cdot \nabla\rho) \\
&= \int_M \nabla(u(\psi \circ \rho)) \cdot \nabla\rho - \int_M (\psi \circ \rho)(\nabla u \cdot \nabla\rho)
\end{aligned}$$

Since $\psi \circ \rho$ is Lipschitz, we have, by Lemma 1.4(ii), that the first term is equal to $\langle \Delta\rho, u(\psi \circ \rho) \rangle$, and then, by the definition of push-forward, also equal to: $\langle \rho_*(u\Delta\rho), \psi \rangle$. The second term is equal to: $-\int_0^\infty \psi \int_{\rho^{-1}(r)} (\nabla u \cdot \nabla\rho) dH_{n-1} dr$ by the co-area formula, and then, thanks to Corollary 2.5, also equal to: $\int_0^\infty \psi \left(\int_{M(r)} \Delta u \right) dr$. The proof is complete. $\square$

An important particular case is when N is the boundary of a domain Ω in M; for future convenience, we restrict ρ to Ω, and consider $\Delta\rho$ as a distribution on Ω. The mean-value lemma takes the form:

2.8 Theorem (Special case of Mean-value lemma). *Let Ω be a domain with piecewise-smooth boundary, and let $\rho : \Omega \to (0, \infty)$ denote the distance function from the boundary. Let $u \in C^2(\Omega)$, and let $F(r) = \int_{\Omega(r)} u\, dv_n$, where $\Omega(r) = \{x \in \Omega : \rho(x) > r\}$. Then:*

$$-F''(r) = \int_{\Omega(r)} \Delta u\, dv_n - \rho_*(u\Delta\rho)$$

as Radon measures on $(0, \infty)$.

Proof. Repeat the proof of Theorem 2.5, with the indicated changes; and observe that we have $F'(r) = -\int_{\rho^{-1}(r)} u\, dH_{n-1}$ (a.e. on $(0, \infty)$). □

3. Applications to eigenvalue estimates.

The classical mean-value lemma on harmonic manifolds. The scope of this subsection is to show that Theorem 2.8 implies the classical mean-value lemma when M is a manifold such that all geodesic spheres of M have constant mean curvature. By definition, this condition is satisfied (for small spheres around a given point x_0), by a manifold which is *locally harmonic* at x_0 (in the sense of [4], §6.10): that is, there exists $\epsilon = \epsilon(x_0) > 0$ and a smooth map $\bar{\theta} : (0, \epsilon) \to (0, \infty)$ such that $\theta(r, \xi) = \bar{\theta}(r)$ for all $r \in (0, \epsilon), \xi \in S^{n-1}$ (the density of the Riemannian metric, in polar coordinates centered at x_0, depends only on the distance from x_0).

3.1 Proposition. *Assume that M is locally harmonic at x_0. If u is a harmonic map on $B(x_0, \epsilon)$, then:*

$$u(x_0) = \frac{1}{vol(\partial B(x_0, r))} \int_{\partial B(x_0,r)} u \qquad \textit{for all} \quad r < \epsilon$$

Proof. Let $F(r) = \int_{B(x_0,r)} u$, and let $0 < r < \epsilon$. We apply Theorem 2.8 with $\rho =$ distance from x_0. Since $\rho_*(u\Delta_{cut}\rho)$ is supported for r greater than the injectivity radius of x_0, and since, by (1.1) and our assumptions, $\rho_*(u\Delta_{reg}\rho)(r) = -\frac{\bar{\theta}'}{\bar{\theta}}(r)F'(r)$, we see that F satisfies the equation:

$$F''(r) = \frac{\bar{\theta}'}{\bar{\theta}}(r)F'(r)$$

on the interval $0 < r < \epsilon$. Hence, on that interval, the function $\frac{F'}{\bar{\theta}}$ is constant in r. This implies that:

$$\frac{F'(r)}{\theta(r)\,\mathrm{vol}(S^{n-1})} = \lim_{s\to 0} \frac{F'(s)}{\theta(s)\,\mathrm{vol}(S^{n-1})}$$

and the assertion follows by observing that $\theta(r)\,\mathrm{vol}(S^{n-1}) = \mathrm{vol}(\partial B(x_0, r))$. □

Applications when ρ is the distance from a point. Now let M be a manifold on which we make the following curvature assumptions:

(3.2) $$Ricci \geq (n-1)K$$

where K can assume all real values. Let $\theta = \theta(r, \xi)$ denote, as before, the density of the Riemannian measure in normal (polar) coordinates centered at a given point $x_0 \in M$, and let $\bar{\theta}$ be the corresponding density, relative to a given point $\bar{x}_0$, on the *simply connected manifold $\bar{M}_K$ of constant curvature* K (Note that any manifold with Ricci curvature bounded from below is homothetic to some manifold satisfying:

$Ricci \geq (n-1)Kg$ with $K \in \{-1,0,1\}$, hence we could restrict our attention to these cases). By Bishop comparison theorem (see [5]), we have, for all (r,ξ) such that $r < c(\xi)$:

$$\frac{\theta'}{\theta}(r,\xi) \leq \frac{\bar{\theta}'}{\bar{\theta}}(r)$$

and therefore:

$$\Delta_{reg}\rho \geq -\frac{\bar{\theta}'}{\bar{\theta}} \circ \rho \tag{3.3}$$

at all regular points of ρ.

We will be working with the integral of a map on geodesic spheres centered at some point x_0 in M. We point out the fact that, when $r > R_{inj}(x_0)$, $\partial B(x_0,r)$ is no longer a regular submanifold of M; however, we can integrate a function on the "regular part" of it: $\partial_{reg}B(x_0,r) \equiv \partial B(x_0,r) \cap \Phi(U)$. *Hence, in this section, we set* :

$$\int_{\partial B(x_0,r)} u \equiv \int_{\partial_{reg}B(x_0,r)} u \, dv_{n-1}$$

We can now state the main theorem of this subsection.

3.4 Theorem. *Let M be a manifold satisfying: $Ricci \geq (n-1)K$, and let $\lambda \in \mathbb{R}$, and $R \leq diam(M)$. Let u be a solution of $\Delta u \geq \lambda u$ which is never zero on the open ball $B(x_0,R)$ in M, and let $\bar{u}$ be a solution of $\Delta \bar{u} = \lambda \bar{u}$ on the open ball $B(\bar{x}_0,R) \equiv \bar{B}(R)$ in $\bar{M}_K$ such that $\bar{u}(\bar{x}_0) \neq 0$. Then we have, for all $r \leq R$:*

$$\frac{\displaystyle\int_{\partial B(x_0,r)} u}{\displaystyle\int_{B(x_0,r)} u} \leq \frac{\displaystyle\int_{\partial \bar{B}(r)} \bar{u}}{\displaystyle\int_{\bar{B}(r)} \bar{u}} \tag{i}$$

$$\frac{1}{u(x_0)} \int_{B(x_0,r)} u \leq \frac{1}{\bar{u}(\bar{x}_0)} \int_{\bar{B}(r)} \bar{u} \tag{ii}$$

About the existence of solutions of $\Delta \bar{u} = \lambda \bar{u}$ on the space form $\bar{M}_K$, we have the following:

3.5 Lemma. *Let $\lambda \in \mathbb{R}$, and let $R \leq diam(\bar{M}_K)$. Then there exists a unique radial solution of $\Delta \bar{u} = \lambda \bar{u}$ on the open ball $\bar{B}(R)$, having a preassigned value at its center. Here "radial" means that there exists a function $f : [0,R) \to \mathbb{R}$ such that $\bar{u} = f \circ \rho$.*

Proof. Since $\Delta(f \circ \rho) = -(f'' + \frac{\bar{\theta}'}{\bar{\theta}} f') \circ \rho$, solving the equation $\Delta \bar{u} = \lambda \bar{u}$ on $\bar{B}(R)$ amounts to solve the equation:

$$\bar{\theta} f'' + \bar{\theta}' f' + \lambda \bar{\theta} f = 0$$

on the interval $(0,R)$. The value $r = 0$ is a regular singular point of the equation, and the indicial equation has roots: 0 and $2-n$. The solution corresponding to the zero root will satisfy the requirements (see a textbook on second order differential equations, for example, [8]). $\square$

For the proof of the theorem, we need the following additional lemma:

3.6 Lemma. *Let $W : (a,b) \to \mathbb{R}$ be continuous on a dense subset of (a,b), and suppose that the one-sided limits $\lim_{r\to a_+} W = W(a_+)$ and $\lim_{r\to b_-} W = W(b_-)$ both exist. If $W' \leq 0$ (resp. $W' \geq 0$) in the sense of distributions, then:*

$$W(b_-) \leq W(a_+) \qquad \textit{(resp.} \quad W(b_-) \geq W(a_+))$$

Proof of Lemma. (i) Fix $r < s$ in (a,b), and pick a sequence of smooth, positive functions ψ_n supported inside (a,b) and satisfying: $\lim_{n\to\infty} \psi_n' = \delta_r - \delta_s$ (the sequence is easily constructed). Then, for all n:

$$0 \geq \langle W', \psi_n \rangle = -\int_{-\infty}^{\infty} W \psi_n' \, dr$$

If W is continuous at r and s, we get, by taking the limit as $n \to \infty$: $0 \geq W(s) - W(r)$. We then pass to the limit as $r \to a_+$ and $s \to b_-$. $\square$

Proof of Theorem 3.4. It suffices to consider only the case in which $u > 0$ on $B(x_0, R)$ and $\bar{u}(\bar{x}_0) > 0$. Let $F(r) = \displaystyle\int_{B(x_0,r)} u$, and fix a small $\epsilon > 0$. By the mean-value lemma, we have, as distributions on (ϵ, R):

$$\begin{aligned} -F'' &\geq \lambda F + \int_{\partial B(x_0,r)} u\Delta_{reg}\rho + \rho_*(u\Delta_{cut}\rho) \\ &\geq \lambda F - \frac{\bar{\theta}'}{\bar{\theta}} F' \end{aligned}$$

the inequality following from (3.3), the positivity of $\Delta_{cut}\rho$ and the fact that $F'(r) = \displaystyle\int_{\partial B(x_0,r)} u$ almost everywhere on (ϵ, R), hence as distributions on (ϵ, R) (Proposition 2.6). Therefore F is seen to verify, on (ϵ, R), the differential inequality:

$$(*) \qquad F'' - \frac{\bar{\theta}'}{\bar{\theta}} F' + \lambda F \leq 0$$

On the other hand, the corresponding map $\bar{F}(r) = \int_{\bar{B}(r)} \bar{u}$ satisfies, on (ϵ, R), the equation:

$$(**) \qquad \bar{F}'' - \frac{\bar{\theta}'}{\bar{\theta}} \bar{F}' + \lambda \bar{F} = 0$$

in fact, on $\bar{M}_K$ the cut-locus of any point reduces to a single point or is empty, so that $\Delta_{cut}\rho = 0$; and as $\Delta_{reg}\rho = -\dfrac{\bar{\theta}'}{\bar{\theta}} \circ \rho$, we have (**) by the mean-value lemma. Now let R_0 be the first zero of $\bar{F}$, so that $\bar{F} \geq 0$ on (ϵ, R_0), and let $R_1 = \min\{R_0, R\}$. We multiply (*) by $\bar{F}$, (**) by F and subtract. Then, on (ϵ, R_1), we have the inequality:

$$(F'\,\bar{F} - \bar{F}'\,F)' - \frac{\bar{\theta}'}{\bar{\theta}}(F'\,\bar{F} - \bar{F}'\,F) \leq 0$$

so that, if $W(r) = \dfrac{F'(r)\bar{F}(r) - \bar{F}'(r)F(r)}{\bar{\theta}(r)}$, then $W' \leq 0$ in the sense of distributions on (ϵ, R_1). From Lemma 3.6, and Lemma 2.7(ii):

$$W(\epsilon) \geq W(r_-) \geq W(r)$$

Next, we observe that $\lim_{\epsilon\to 0} W(\epsilon) = 0$: in fact, as $\epsilon \to 0$: $F(\epsilon), \bar{F}(\epsilon) \sim \epsilon^n$, $F'(\epsilon), \bar{F}'(\epsilon) \sim \epsilon^{n-1}$, and $\bar{\theta}(\epsilon) \sim \epsilon^{n-1}$. We can then conclude that $F'\bar{F} - F\bar{F}' \leq 0$ on $(0, R_1)$, which is (i) with R_1 replacing R.

Next, we integrate both sides of $\dfrac{F'}{F} \leq \dfrac{\bar{F}'}{\bar{F}}$ from ϵ to r, and get: $\dfrac{F(r)}{\bar{F}(r)} \leq \dfrac{F(\epsilon)}{\bar{F}(\epsilon)}$; but since $\dfrac{\mathrm{vol}(B(x_0,\epsilon))}{\mathrm{vol}(\bar{B}(\epsilon))} \to 1$ when $\epsilon \to 0$, we see that the limit inequality is:

$$\frac{F(r)}{\bar{F}(r)} \leq \frac{u(x_0)}{\bar{u}(\bar{x}_0)}$$

which is precisely (ii) with R_1 replacing R. It then remains to show that $R_0 \geq R$. Assume not. Then we would have:

$$0 < F(R_0) \leq \frac{u(x_0)}{\bar{u}(\bar{x}_0)} \bar{F}(R_0) = 0$$

Proof is complete. □

We observe that, for $u = 1$, the theorem reduces to the well-known Bishop-Gromov inequality.

3.7 Corollary. *Assume Ricci $\geq (n-1)K$. If u is a positive super-harmonic function on $B(x_0, R)$, then, for all $r \leq R$:*

$$u(x_0) \geq \frac{1}{vol(\partial\bar{B}(r))} \int_{\partial B(x_0,r)} u$$

Proof. Simply take $\lambda = 0$, $\bar{u} = constant = u(x_0)$ in the theorem. Then from (i) and (ii) combined:

$$\frac{1}{\mathrm{vol}(\partial\bar{B}(r))} \int_{\partial B(x_0,r)} u \leq \frac{1}{\mathrm{vol}(\partial\bar{B}(r))} \int_{\partial\bar{B}(r)} \bar{u} = u(x_0)$$

□

Another application of Theorem 3.4 is a new proof of the following result of S.Y.Cheng (see [7]) on the first non-zero eigenvalue of the Laplace-Beltrami operator on geodesic balls. Let us denote by $\lambda_1(\Omega)$ the first non-zero eigenvalue of the Dirichlet problem on Ω.

3.8 Theorem (S. Y. Cheng). *If Ricci $\geq (n-1)K$, then, for all R:*

$$\lambda_1(B(x_0, R)) \leq \lambda_1(\bar{B}(R))$$

where $\bar{B}(R)$ is the ball of radius R in the simply connected manifold of constant sectional curvature K.

Proof. Let u be a positive eigenfunction on M corresponding to $\lambda = \lambda_1(B(x_0, R))$, and let $\bar{u}$ be a positive eigenfunction of the Dirichlet Laplacian on $\bar{B}(R)$ corresponding to $\bar{\lambda} = \lambda_1(\bar{B}(R))$. Then $\bar{u}$ is radial (since $\bar{\lambda}$ is simple): say $\bar{u} = \bar{f} \circ \rho$, with $\bar{f}(R) = 0$, and $\bar{f}'(R) \leq 0$. Next, let $v = f \circ \rho$ be the radial solution of $\Delta v = \lambda v$ on $\bar{B}(R)$, satisfying $v(\bar{x}_0) = u(x_0)$, where $\bar{x}_0$ is the center of $\bar{B}(R)$: then, by Theorem 3.4, $f \geq 0$ on $(0, R)$. We can now prove that $\lambda \leq \bar{\lambda}$. In fact, from the relations:

$$\begin{cases} f'' + \dfrac{\bar{\theta}'}{\bar{\theta}} f' + \lambda f = 0 \\ \bar{f}'' + \dfrac{\bar{\theta}'}{\bar{\theta}} \bar{f}' + \bar{\lambda}\bar{f} = 0 \end{cases}$$

we have, multiplying the first relation by $\bar{f}$, the second by f, and subtracting:

$$(\bar{\lambda} - \lambda) f\bar{f}\bar{\theta} = (\bar{\theta}(\bar{f}f' - f\bar{f}'))'$$

and integrating from $r = 0$ to $r = R$:

$$(\bar{\lambda} - \lambda) \int_0^R f \bar{f} \bar{\theta} \, dr = -\bar{\theta}(R) \bar{f}'(R) f(R) \geq 0$$

which immediately implies $\bar{\lambda} \geq \lambda$, since the integral on the left-hand side is positive. $\square$

Applications when ρ is the distance from the boundary of a domain. In this subsection we give a lower bound for the first eigenvalue of the Dirichlet Laplacian of a relatively compact domain Ω having smooth boundary, or piecewise-smooth boundary satisfying an additional condition (see *property (P)* below). The bound is given in terms of a lower bound of the Ricci curvature of Ω, a lower bound of the mean curvature of $\partial\Omega$, and the *inner radius* of Ω (the radius of the biggest ball that fits into Ω), and has been obtained by Kasue (see [15]), for domains with smooth boundary. We remark, however, that our proof differs, in the smooth case, from the one in [15].

So let Ω as above and denote by $\rho : \Omega \to (0, \infty)$ the distance function from the boundary. Then we have, as distributions on Ω (i.e. as continuous linear maps on $C_c^\infty(\Omega)$):

$$\Delta\rho = \Delta_{reg}\rho + \Delta_{cut}\rho$$

where $\Delta_{cut}\rho$ is positive, and supported on the cut-locus of $\partial\Omega$. Let us write $\partial\Omega = \partial_{reg}\Omega \cup \partial_{sing}\Omega$, where $\partial_{reg}\Omega$ is a smooth submanifold of codimension 1 and $\partial_{sing}\Omega$ is a piecewise smooth submanifold of top codimension ≥ 2.

We will say that Ω *satisfies property (P) if:*

for each $x \in \Omega \setminus Cut(\partial\Omega)$ the foot of the geodesic which minimizes the distance from x to $\partial\Omega$ is a regular point of $\partial\Omega$.

Under the assumption (P), we then have: $\Delta_{reg}\rho = -\dfrac{\theta'}{\theta} \circ \rho$, where θ is the Jacobian of the diffeomorphism (normal chart): $\Phi : U \to \Omega \setminus \mathrm{Cut}(\partial\Omega)$ which sends (r, ξ) to $\exp_{\pi(\xi)} r\,\xi$. Here $U = \{(r, \xi) \in (0, \infty) \times U(\partial_{reg}\Omega) : 0 < r < c(\xi), \Phi(r, \xi) \in \Omega\}$. If ρ is smooth at x, and $\rho(x) = r$, then $\Delta_{reg}\rho(x)$ gives the trace of the second fundamental form of the level submanifold $\rho^{-1}(r)$ at x; the mean curvature is then given by $\dfrac{1}{n-1}\Delta_{reg}\rho(x)$ (our sign convention is that the mean curvature of the unit sphere in euclidean space is positive for the choice of the inward unit normal).

3.9 Lemma. *Let Ω be a relatively compact, open set of M, with piecewise smooth boundary satisfying property (P). Assume that the mean curvature of $\partial_{reg}\Omega$ is bounded below by $\bar{\eta}$, and that Ricci $\geq (n-1)K$ on Ω. Then, as distributions on Ω:*

$$\Delta\rho \geq -\frac{\bar{\theta}'}{\bar{\theta}} \circ \rho$$

where $\bar{\theta}(r) = (s_K'(r) - \bar{\eta} s_K(r))^{n-1}$ and:

$$s_K(r) = \begin{cases} \dfrac{1}{\sqrt{K}} \sin(r\sqrt{K}) & \text{if} \quad K > 0 \\ r \quad \text{if} \quad K = 0 \\ \dfrac{1}{\sqrt{|K|}} \sinh(r\sqrt{|K|}) & \text{if} \quad K < 0 \end{cases}$$

In particular we have the following sufficient conditions for the positivity of the Laplacian of the distance function:

Assume that Ω satisfies property (P), that the mean curvature of $\partial\Omega$ is non-negative at all regular points, and that the Ricci curvature of Ω is non-negative. Then $\Delta\rho \geq 0$.

Proof. The lemma is a consequence of Heintze-Karcher's estimates in [14]; however, we can re-derive it by the procedure followed in [12], p. 41.

Since $\Delta_{cut}\rho$ is positive, it is enough to show that $\Delta_{reg}\rho \geq -\dfrac{\bar\theta'}{\bar\theta} \circ \rho$. Fix a unit normal vector $\xi \in U(\partial_{reg}\Omega)$ pointing inside Ω, and let $b(r,\xi) = \theta(r,\xi)^{\frac{1}{n-1}}$ for $r \in (0, c(\xi))$. The function b satisfies:

$$\begin{cases} b'' + \dfrac{1}{n-1} Ricci(\nabla\rho, \nabla\rho)\, b \leq 0 \\ b(0) = 1 \\ b'(0) = -\eta(\xi) \end{cases}$$

By our assumptions, we can compare b with the solution $\bar b$ of:

$$\begin{cases} b'' + K\, b = 0 \\ b(0) = 1 \\ b'(0) = -\bar\eta \end{cases}$$

and conclude:

$$\begin{cases} \dfrac{b'(r,\xi)}{b(r,\xi)} \leq \dfrac{\bar b'(r)}{\bar b(r)} \\ b(r,\xi) \leq \bar b(r) \end{cases}$$

on $(0, \bar R)$, where $\bar R$ is the first zero of $\bar b$. But the last inequality displayed shows that $\bar R \geq c(\xi)$ for all $\xi \in \partial\Omega$, hence $\bar R \geq R \doteq \max_{\xi\in\partial\Omega} c(\xi)$ (the inner radius of Ω). Now set: $\bar\theta(r) = \bar b(r)^{n-1}$. Then:

$$\frac{\theta'(r,\xi)}{\theta(r,\xi)} \leq \frac{\bar\theta'(r)}{\bar\theta(r)} \qquad \text{for all} \quad (r,\xi) : r < c(\xi)$$

The explicit expression of $\bar\theta$ given by the lemma is easily obtained. As for the proof of the last statement, first observe that, since $\Delta_{cut}\rho$ is positive, it is enough to show that, under the given assumptions, $\Delta_{reg}\rho(x) \geq 0$ at all regular points $x \in \Omega$. By property (P), we can write $x = \exp_{\pi(\xi)} r\xi$ for some $r > 0$, and some $\xi \in U(\partial_{reg}\Omega)$. Then:

$$\Delta_{reg}\rho(x) = -\frac{\theta'(r,\xi)}{\theta(r,\xi)} \geq -\frac{\bar\theta'(r)}{\bar\theta(r)} = (n-1)\frac{\bar\eta}{1 - r\bar\eta}$$

and the last quantity is indeed non-negative. $\square$

To state our comparison theorem, we need to define the model domains to which we will compare our domain Ω. Then let $\bar\Omega \equiv \bar\Omega(K, \bar\eta, R)$ be the cylinder with constant curvature K, and width R, such that the mean curvature is constant, equal to $\bar\eta$, on one of the two connected components of the boundary. Depending on K and $\bar\eta$, Ω will be an annulus in either the space form $\bar M_K$, or the hyperbolic cylinder of constant curvature K. We postpone the explicit description of $\bar\Omega$ after we have proved the following comparison theorem.

3.10 Theorem. (Compare with [15]) *Let Ω be a domain with piecewise smooth boundary satisfying property (P). Assume that the Ricci curvature is bounded below by $(n-1)K$ on Ω, that the mean curvature is bounded below by $\bar\eta$ on $\partial_{reg}\Omega$, and let R denote the inner radius of Ω. Then:*

$$\lambda_1(\Omega) \geq \bar\lambda_1(\bar\Omega)$$

where λ_1 is the first non-zero eigenvalue of the Dirichlet problem on Ω, and where $\bar\lambda_1(\bar\Omega)$ denotes the first non-zero eigenvalue of the following mixed problem on $\bar\Omega(K,\bar\eta,R)$: Dirichlet condition on the component having mean curvature $\bar\eta$, Neumann condition on the other.

Proof. Let $\bar\rho:\bar\Omega\to(0,\infty)$ denote the distance function from Γ, the component of $\partial\bar\Omega$ having constant mean curvature $\bar\eta$. From the explicit expression of $\bar\Omega$, it will be clear that the cut-locus of Γ is either empty, or reduces to a point: hence $\Delta_{cut}\bar\rho=0$; moreover $\Delta_{reg}\bar\rho=-\dfrac{\bar\theta'}{\bar\theta}\circ\bar\rho$ where $\bar\theta$ is as in Lemma 3.9. Let u be a positive eigenfunction corresponding to $\lambda=\lambda_1(\Omega)$, let $\bar u$ be the eigenfunction associated to $\bar\lambda=\bar\lambda_1(\bar\Omega)$ which is positive on $\bar\Omega$ and is normalized so that: $\int_{\bar\Omega}\bar u=\int_\Omega u$, and let $F(r)=\int_{\Omega(r)}u$ and $\bar F(r)=\int_{\bar\Omega(r)}\bar u$.
By the special case of Theorem 3.8 we have: $F''=-\lambda F+\rho_*(u\Delta\rho)$, and: $\bar F''=-\bar\lambda\bar F+\bar\rho_*(\bar u\Delta\bar\rho)$.
By Lemma 3.9, and the fact that $\rho_*(u)(r)=-F'(r)$, and $\bar\rho_*(\bar u)(r)=-\bar F'(r)$, we easily arrive at:

$$F''-\frac{\bar\theta'}{\bar\theta}F'+\lambda F\geq 0 \qquad \text{and :}$$

$$\bar F''-\frac{\bar\theta'}{\bar\theta}\bar F'+\bar\lambda\bar F=0$$

We prove that $\lambda\geq\bar\lambda$. Assume $\lambda<\bar\lambda$. We multiply the first inequality by $\bar F$, the second equation by F, and subtract. We get:

$$\left(\frac{F'\bar F-F\bar F'}{\bar\theta}\right)'\geq\frac{F\bar F(\bar\lambda-\lambda)}{\bar\theta}$$

which is therefore strictly positive on $(0,R)$.
Hence $\dfrac{F'\bar F-F\bar F'}{\bar\theta}>\dfrac{F'(0)\bar F(0)-F(0)\bar F'(0)}{\bar\theta(0)}=0$, which implies: $\dfrac{F'(r)}{F(r)}>\dfrac{\bar F'(r)}{\bar F(r)}$ on $(0,R)$.
By our normalization ($F(0)=\bar F(0)$) we obtain: $F(r)>\bar F(r)$, and, in turn: $F'(r)>\bar F'(r)$ on $(0,R)$. Ultimately we would have:

$$F(0)=-\int_0^R F'(r)\,dr<-\int_0^R\bar F'(r)\,dr=\bar F(0)$$

which is a contradiction. Hence $\lambda\geq\bar\lambda$. $\square$

We now proceed to the explicit costruction of the model cylinder $\bar\Omega=\bar\Omega(K,\bar\eta,R)$.

Case 1: $K>0,\bar\eta\in\mathbb{R}$, or: $K<0,\bar\eta>\sqrt{|K|}$, or: $K=0,\bar\eta>0$.

Consider the unique ball $B(\bar R)$ in the space form $\bar M_K$ having boundary of constant mean curvature equal to $\bar\eta$. Its radius is:

$$\bar R=\begin{cases}\dfrac{1}{\sqrt K}\cot^{-1}(\dfrac{\bar\eta}{\sqrt K}) \qquad \text{if} \quad K>0\\ \dfrac{1}{\bar\eta} \qquad \text{if} \quad K=0\\ \dfrac{1}{\sqrt{|K|}}\coth^{-1}(\dfrac{\bar\eta}{\sqrt{|K|}}) \qquad \text{if} \quad K<0\end{cases}$$

We have already observed that $\bar R\geq R$. We let $\bar\Omega$ denote the interior of $B(\bar R)\setminus B(\bar R-R)$, and $\Gamma=\partial B(\bar R)$.

Case 2: $K<0,\bar\eta<-\sqrt{|K|}$, or $K=0,\bar\eta<0$.

Again in the space form $\bar M_K$, let:

$$\bar{R} = \begin{cases} -\dfrac{1}{\bar{\eta}} & \text{if} \quad K = 0 \\ \dfrac{1}{\sqrt{|K|}} \coth^{-1}\left(\dfrac{-\bar{\eta}}{\sqrt{|K|}}\right) & \text{if} \quad K < 0 \end{cases}$$

and let $\Gamma = \partial B(\bar{R})$. Then let $\bar{\Omega}$ denote the interior of $B(\bar{R} + R) \setminus B(\bar{R})$.

Case 3: $K < 0, -\sqrt{|K|} < \bar{\eta} < \sqrt{|K|}$.

Let $M = \mathbb{R} \times S^{n-1}$ with metric: $g = (dr)^2 + \cosh^2(r\sqrt{|K|})g_{S^{n-1}}$. We fix the minimal submanifold $N = \{0\} \times S^{n-1}$ and we consider normal coordinates (r, ξ) based at N. The Jacobian $\bar{\theta}_N(r) = \cosh^{n-1}(r\sqrt{|K|})$, so that the absolute value of the mean curvature of the hypersurface $\{r\} \times S^{n-1}$ is $\dfrac{1}{n-1}\dfrac{\bar{\theta}'}{\bar{\theta}}(r) = \sqrt{|K|}\tanh(r\sqrt{|K|})$. We then take:

$$\bar{R} = \frac{1}{\sqrt{|K|}} \tanh^{-1}\left(\frac{|\bar{\eta}|}{\sqrt{|K|}}\right)$$

so that the hypersurface $\Gamma \equiv \{\bar{R}\} \times S^{n-1}$ will have constant mean curvature equal to $\pm\bar{\eta}$. If we set:

$$\bar{\Omega} = \begin{cases} \{(r, \xi) \in \mathbb{R} \times S^{n-1} : \bar{R} - R < r < \bar{R}\} & \text{if} \quad \bar{\eta} \geq 0 \\ \{(r, \xi) \in \mathbb{R} \times S^{n-1} : \bar{R} < r < \bar{R} + R\} & \text{if} \quad \bar{\eta} < 0 \end{cases}$$

then it is easily verified that $\bar{\Omega}$ satisfies the requirements.

Case 4: $K = 0, \bar{\eta} = 0$.

We take in this case the cylinder $\mathbb{R} \times S^{n-1}$ with the product metric, and we let:

$$\bar{\Omega} = \{(r, \xi) \in \mathbb{R} \times S^{n-1} : 0 < r < R\}$$

and $\Gamma = \{0\} \times S^{n-1}$. Note that $\bar{\rho}^{-1}(r)$ is a minimal submanifold for all r, and that $\Delta\bar{\rho} = 0$ in this case.

Case 5: $K < 0, \bar{\eta} = \pm\sqrt{|K|}$.

These are the limit cases of Case 1 (if $\bar{\eta} = \sqrt{|K|}$) and Case 2 (if $\bar{\eta} = -\sqrt{|K|}$) as $\bar{R} \to \infty$.

We observe that in Case 4 the theorem reduces to the following well-known inequality, due to Li and Yau (see [17], theorem 11):

$$\lambda_1^D(\Omega) \geq \frac{\pi^2}{4R^2}$$

4. Applications to heat diffusion

In this section Ω is an open set with *piecewise-smooth* boundary in a complete Riemannian manifold, and $\rho : \Omega \to \mathbb{R}$ denotes *the distance function from the boundary.* We assume $\partial\Omega$ compact, and we denote by R the inner radius of Ω: $0 < R \leq \infty$.

Let $k(t, x, y)$ denote the heat kernel for the Dirichlet problem on Ω, at time $t > 0$ and at the points $x, y \in \Omega$. Assume that Ω is at constant unit temperature at time $t = 0$, and that its boundary $\partial\Omega$ is kept at temperature zero for all $t > 0$: then the temperature at time t, at the point $x \in \Omega$, is given by:

$$u(t, x) = \int_{\Omega} k(t, x, y)\, dy$$

In the sequel, $u(t,x)$ will be simply referred to as the *temperature function.*

If Ω has finite volume, the total amount of heat inside Ω, at time t, is expressed by the *heat content* function:

$$H(t)=\int_{\Omega}u(t,x)\,dx=\int_{\Omega\times\Omega}k(t,x,y)\,dx\,dy$$

We will study $H(t)$ by viewing it as the value of:

$$H(t,r)=\int_{\Omega(r)}u(t,x)\,dx$$

at $r=0$. Here $\Omega(r)=\{x\in\Omega:\rho(x)>r\}$ are the level domains of ρ. Note that $H(t,r)=0$ for all $t>0$, and all $r\geq R=$ inner radius of Ω. But to study the heat content function $H(t,r)$, we will consider more generally functions $f(t,r)$ of type:

$$f(t,r)=\int_{\Omega(r)}w(t,x)\,dx$$

where $w(t,x)$ is *any* summable solution of the heat equation on Ω. An example is the *complementary heat content* function:

$$F(t,r)=\int_{\Omega(r)}(1-u(t,x))\,dx$$

which, unlike $H(t,r)$, is finite even when $\mathrm{vol}(\Omega)=\infty$. Note that $F(t)\equiv F(t,0)$ is the amount of heat inside Ω, assuming zero initial temperature, and assuming that the boundary of Ω is kept at constant unit temperature. Note also that $F(t,r)=\mathrm{vol}(\Omega(r))-H(t,r)$ for all $t>0$, $r\geq 0$.

It is an immediate consequence of the mean-value lemma (Theorem 2.8, special case) that $f(t,r)$ satisfies, on each of the intervals $(0,a)$, with $0<a\leq\infty$, the following heat equation:

$$(-\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial t})f=-\rho_*(w_t\Delta\rho) \tag{4.1}$$

where $w_t(x)\equiv w(t,x)$. The equality is one between Radon measures. Note the boundary condition $f(t,a)=0$ for all $t>0$, in case $R\leq a<\infty$. Hence $f(t,r)$ can be expressed, via Duhamel principle, in terms of the heat kernel $e_a(t,r,s)$ relative to the *mixed problem* on $(0,a)$: *Neumann* condition at $r=0$, and *Dirichlet* condition at $r=a$. Among all choices of a, the heat kernel $e_R(t,r,s)$ is best suited for geometry, and, as we shall see, its use will produce sharp bounds on the heat content function. Nevertheless, the heat kernel corresponding to $a=\infty$ (that is, the heat kernel of $(0,\infty)$ satisfying Neumann conditions at 0) which will be denoted simply by $e(t,r,s)$, is more explicit: in fact, by the reflection principle, it is given by:

$$e(t,r,s)=\frac{1}{\sqrt{4\pi t}}(e^{-(r-s)^2/4t}+e^{-(r+s)^2/4t})\qquad\text{for all}\quad t>0,\,r,s\geq 0 \tag{4.2}$$

and in particular:

$$e(t,r,0)=\frac{1}{\sqrt{\pi t}}e^{-r^2/4t}$$

We always have: $e_R(t,r,s)\leq e(t,r,s)$.

We start from:

4.3 Lemma (Duhamel principle). *Let $f(t,r) = \int_{\Omega(r)} w(t,x)\,dx$, where $w(t,x)$ is a solution of the heat equation on Ω, and let $e_a(t,r,s)$ denote the heat kernel associated to the mixed problem on $(0,a)$: Neumann at $r=0$, Dirichlet at $r=a$, where $a \in [R,\infty]$. Then, for all $t>0$, $r \in [0,a]$:*

$$f(t,r) = \int_0^a e_a(t,r,s)f(0,s)\,ds - \int_0^t \int_0^a e_a(t-\tau,r,s)\rho_*(w_\tau \Delta\rho)(s)\,ds\,d\tau - \int_0^t \frac{\partial f}{\partial r}(\tau,0)\,e_a(t-\tau,r,0)\,d\tau$$

Proof. The classical proof, which we reproduce here, applies to our case. We only need to give the proof for $R<a<\infty$. The regularity properties of $w(t,x)$ guarantee the convergence of all integrals appearing below; and we can perform freely all the operations indicated in the sequel (differentiation under the integral sign, etc.). Fix $t>0$, and $r \in (0,a)$, and let $\tau \in [\alpha,\beta] \subseteq (0,t)$. Then, by (4.1):

$$\int_0^a e_a(t-\tau,r,s)\rho_*(w_\tau\Delta\rho)(s)\,ds = \int_0^a e_a(t-\tau,r,s)\frac{\partial^2 f}{\partial s^2}(\tau,s)\,ds - \int_0^a e_a(t-\tau,r,s)\frac{\partial f}{\partial \tau}(\tau,s)\,ds$$

The first integral on the right-hand side must be interpreted as the integral of the function $e(t-\tau,r,\cdot)$ with respect to the measure $\dfrac{\partial^2 f}{\partial s^2}(\tau,\cdot)$ on $(0,a)$. Integrating by parts we obtain:

$$\int_0^a e_a(t-\tau,r,s)\frac{\partial^2 f}{\partial s^2}(\tau,s)\,ds = -\frac{\partial f}{\partial r}(\tau,0)e_a(t-\tau,r,0) - \int_0^a \frac{\partial e_a}{\partial s}(t-\tau,r,s)\frac{\partial f}{\partial s}(\tau,s)\,ds$$

and integrating by parts in the second term we obtain, since $f(\tau,a)=0$:

$$\int_0^a e_a(t-\tau,r,s)\frac{\partial^2 f}{\partial s^2}(\tau,s)\,ds = -\frac{\partial f}{\partial r}(\tau,0)e_a(t-\tau,r,0) + \int_0^a \frac{\partial^2 e_a}{\partial s^2}(t-\tau,r,s)f(\tau,s)\,ds$$

we add to the second integral and easily obtain:

$$\int_0^a e_a(t-\tau,r,s)\rho_*(u_\tau\Delta\rho)(s)\,ds = -\frac{\partial f}{\partial r}(\tau,0)e_a(t-\tau,r,0) - \frac{\partial}{\partial \tau}\int_0^a e_a(t-\tau,r,s)f(\tau,s)\,ds$$

We integrate this relation from $\tau=\alpha$ to $\tau=\beta$ and then pass to the limit as $\alpha \to 0$ and $\beta \to t$. $\square$

We point out two applications of Lemma 4.3. In the first, we take $w(t,x) = 1-u(t,x)$, and $a=\infty$. Since $f(0,s)=0$ for all s, and $\dfrac{\partial f}{\partial r}(t,0) = -\mathrm{vol}(\partial\Omega)$ for all t, we obtain the following expression of the complementary heat content function (at $r=0$):

$$F(t) = \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} - \int_0^t\int_0^\infty e(t-\tau,r,0)\rho_*((1-u_\tau)\Delta\rho))\,dr\,d\tau \tag{4.4}$$

and consequently:

$$H(t) = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + \int_0^t\int_0^\infty e(t-\tau,r,0)\rho_*((1-u_\tau)\Delta\rho))\,dr\,d\tau$$

For the second application, we fix $x \in \Omega$, and let $f(t,r) = \int_{\Omega(r)} k(t,x,y)\,dy$. Then $f(t,0) = u(t,x)$; moreover:

$$f(0,r) = \begin{cases} 1 & \text{if} \quad r<\rho(x) \\ 0 & \text{if} \quad r>\rho(x) \end{cases}$$

and $\dfrac{\partial f}{\partial r}(t,0)=0$ for all $t>0$. By Duhamel principle, applied for $a=\infty$:

$$u(t,x)=\int_0^{\rho(x)} e(t,r,0)\,dr-\int_0^t\int_0^\infty e(t-\tau,r,0)\rho_*(k(\tau,x,\cdot)\Delta\rho)(r)\,dr\,d\tau \tag{4.5}$$

The two expressions will be repeatedly used in the section. We will also make use of the following estimates:

$$\int_0^t e(\tau,a,0)\,d\tau\le\frac{4}{\sqrt{\pi}a^2}t^{3/2}e^{-a^2/4t} \tag{4.6}$$

$$\int_a^\infty e(t,r,0)\,dr\le\frac{2}{\sqrt{\pi}a}t^{1/2}e^{-a^2/4t} \tag{4.7}$$

$$\int_0^\infty e(t,r,0)r\,dr=\frac{2}{\sqrt{\pi}}t^{1/2} \tag{4.8}$$

$$\int_0^t e(t-\tau,r,0)(\int_r^\infty e(\tau,s,0)\,ds)\,d\tau=\int_0^t e(\tau,2r,0)\,d\tau \tag{4.9}$$

The proof of (4.6), (4.7) and (4.8) follow easily by suitable integration by parts; (4.9) is proved by observing that both sides admit the same Laplace transform with respect to t, since the Laplace transform, at $p>0$, of $e(t,r,0)$ is $\dfrac{1}{\sqrt{p}}e^{-\sqrt{p}r}$.

4A. Bounds in the case: $\Delta\rho\ge 0$

We assume, in this subsection, that Ω is a domain with *piecewise-smooth boundary* and we also assume that the measure $\Delta\rho$ is *positive* on Ω: $\Delta\rho\ge 0$. We recall that sufficient conditions for this to occur (see Lemma 3.9) are that both the mean curvature of $\partial\Omega$, and the Ricci curvature of Ω, are non-negative; if $\partial\Omega$ is merely piecewise-smooth, we also require that Ω satisfies the following property: the foot of any geodesic segment in Ω which minimizes the distance from the boundary is a regular point of $\partial\Omega$.

We give bounds of the heat content and the temperature function on Ω which will often be sharp for a (flat) cylinder. Hence we first treat that case.

By a *cylinder,* we mean a domain of type:

$$\Omega=N\times(0,2R)$$

where N is a closed Riemannian manifold and the metric is the product metric. Note that: $\partial\Omega=(N\times\{0\})\cup(N\times\{2R\})$, and that R is the inner radius of Ω. The cut-locus of $\partial\Omega$ is the smooth submanifold $\rho^{-1}(R)=N\times\{R\}$, and, for all $r\in[0,R)$, we have: $\rho^{-1}(r)=(N\times\{r\})\cup(N\times\{2R-r\})$, a totally geodesic submanifold. Hence: $\Delta_{reg}\rho=0$, and so $\Delta\rho=\Delta_{cut}\rho=2\delta_{\rho^{-1}(R)}$. Explicitly:

$$\langle\Delta\rho,\phi\rangle=2\int_{\rho^{-1}(R)}\phi(x)\,dv_{n-1}(x)$$

and:

$$\langle \rho_*(u\Delta\rho), \psi\rangle = 2\psi(R)\int_{\rho^{-1}(R)} u(x)\, dv_{n-1}(x)$$

In particular, $\langle \rho_*(u\Delta\rho), \psi\rangle = 0$ whenever $\psi(R) = 0$.

As for the temperature function $u(t, x)$, it is obvious that it depends only on the distance of x from the boundary; more precisely, one has:

$$u(t, x) = \bar{u}(t, \rho(x))$$

where $\bar{u}(t, r)$ is the corresponding temperature function on the interval $(0, 2R)$ (note that $\rho(x) \leq R$ for all $x \in \Omega$).

We now come to the bounds.

4A.1 Theorem. *Let Ω a domain with piecewise-smooth boundary satisfying: $\Delta\rho \geq 0$, and let $u(t, x)$ and $F(t, r)$ denote, respectively, the temperature and complementary heat content functions on Ω. The following inequalities hold for all $t > 0$, $x \in \Omega$, and $r \in [0, R]$, where $0 < R \leq \infty$:*

(i)
$$F(t, r) \leq vol(\partial\Omega) \cdot \int_0^t e_R(\tau, r, 0)\, d\tau$$

Equality holds if Ω is a cylinder.
Let $\bar{u}(t, r)$ be the temperature function on the interval $(0, 2R)$. Then:

(ii)
$$u(t, x) \leq \bar{u}(t, \rho(x))$$

Equality holds if Ω is a cylinder. In particular:

(iii)
$$u(t, x) \leq \int_0^{\rho(x)} e(t, r, 0)\, dr$$

Equality holds if Ω is the semi-infinite cylinder $\partial\Omega \times (0, \infty)$.

Proof. (i) We apply Duhamel principle (Lemma 4.3), to the function $F(t, r) = \int_{\Omega(r)} (1 - u(t, x))\, dx$ as in (4.4), but this time we take $a = R$. We end-up with:

$$F(t, r) = \mathrm{vol}(\partial\Omega) \cdot \int_0^t e_R(t - \tau, r, 0)\, d\tau - \int_0^t \int_0^R e_R(t - \tau, r, s)\rho_*((1 - u_\tau)\Delta\rho)(s)\, ds\, d\tau$$

Inequality (i) now follows from the positivity of $\Delta\rho$ and the fact that $u(\tau, x) \leq 1$ for all τ and x; if Ω is a cylinder the double integral vanishes because $e_R(t - \tau, r, R) = 0$ (recall that $\rho_*((1 - u_\tau)\Delta\rho)$ is supported at $s = R$) and we have the equality.

Proof of (ii): fix $x \in \Omega$ and let $f(t, r) = \int_{\Omega(r)} k(t, x, y)\, dy$ as in (4.5). By Duhamel principle, applied for $a = R$:

$$u(t, x) = f(t, 0) = \int_0^{\rho(x)} e_R(t, r, 0)\, dr - \int_0^t \int_0^R e_R(t - \tau, r, 0)\rho_*(k(\tau, x, \cdot)\Delta\rho)(r)\, dr\, d\tau \tag{4A.2}$$

hence $u(t, x) \leq \int_0^{\rho(x)} e_R(t, r, 0)\, dr$. Equality again holds for a cylinder, and in that case the right-hand side is $\bar{u}(t, \rho(x))$. (ii) is then proved. For the proof of (iii), just recall that $e_R(t, r, s) \leq e(t, r, s)$. $\square$

Remark It is perhaps worth noting two immediate consequences of Theorem 4A.1(i): the first is that, for all $t > 0$, and $r \geq 0$:

$$F(t,r) \leq \mathrm{vol}(\partial\Omega) \cdot \int_0^t \frac{1}{\sqrt{\pi\tau}} e^{-r^2/4\tau}\, d\tau$$

and in particular:

$$F(t) \leq \frac{2}{\sqrt{\pi}} \mathrm{vol}(\partial\Omega)\sqrt{t}$$

(note that the equality holds for the semi-infinite cylinder $\partial\Omega \times (0,\infty)$); and the second is the following maximizing property of cylinders:

Among all domains satisfying $\Delta\rho \geq 0$, with fixed inner radius, and with boundary of fixed volume, flat cylinders hold the maximum (complementary) heat content.

We have corresponding inequalities for the heat content:

4A.3 Theorem. *Let Ω be a domain with piecewise-smooth boundary and finite volume, satisfying: $\Delta\rho \geq 0$, and let $H(t,r) = \int_{\Omega(r)} u(t,x)\, dx$ denote the heat content function on Ω. The following inequalities hold for all $t > 0$, $x \in \Omega$, and $r \in [0,R]$:*

(i) $$H(t,r) \geq vol(\Omega(r)) - vol(\partial\Omega) \cdot \int_0^t e_R(\tau, r, 0)\, d\tau$$

Equality holds if Ω is a cylinder. In particular:

$$H(t) \geq vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega) \cdot \sqrt{t}$$

Let $H_{(0,2R)}(t,r)$ denote the heat content of a segment of length $2R$. Then:

(ii) $$H(t,r) \leq \frac{1}{2}\, vol(\partial\Omega)\, H_{(0,2R)}(t,r)$$

Equality holds if Ω is a cylinder.

Proof. (i) follows immediately from 4A.1(i) since $H(t,r) = \mathrm{vol}(\Omega(r)) - F(t,r)$. Proof of (ii): we have, by 4A.1(iii) and co-area formula:

$$\begin{aligned} H(t,r) &= \int_r^R \int_{\rho^{-1}(s)} u(t,x)\, dx\, ds \\ &\leq \int_r^R \bar{u}(t,s)\, \mathrm{vol}(\rho^{-1}(s))\, ds \end{aligned}$$

Since (by the special case of Theorem 2.8, applied to $u = 1$), $\frac{d}{ds}\mathrm{vol}(\rho^{-1}(s)) = -\rho_*(\Delta\rho) \leq 0$, we see that $\mathrm{vol}(\rho^{-1}(s))$ is a decreasing function of s, hence $\mathrm{vol}(\rho^{-1}(s)) \leq \mathrm{vol}(\partial\Omega)$ for all s and (iii) follows from the symmetry of $\bar{u}(t,s)$ with respect to $s = R$. $\square$

Inequality 4A.1(i), and its counterpart 4A.3(i) hold also for some domains whose boundary is not necessarily piecewise-smooth:

4A.4 Corollary. *Let Ω be an open set in a Riemannian manifold, and assume that there is a sequence of domains $\Omega_n \subseteq \Omega$, all satisfying the assumptions of Theorem 4A.3, and such that, in addition:*
1. $\lim_{n\to\infty} \rho_n(x) = \rho(x)$ for each $x \in \Omega$, where $\rho_n(x) = d(x, \Omega \setminus \Omega_n)$;
2. $vol(\partial\Omega_n)$ converges to a number $\underline{vol}(\partial\Omega)$.
Then, if $H(t,r)$ is the heat content on Ω, we have, for all t, r:

$$H(t,r) \geq vol(\Omega(r)) - \underline{vol}(\partial\Omega) \int_0^t e_R(\tau, r, 0)\, d\tau$$

and:

$$H(t) \geq vol(\Omega) - \underline{vol}(\partial\Omega) \cdot \sqrt{t}$$

In particular, if Ω is a convex, bounded, open subset of $\mathbb{R}^n$, the inequalities hold with $\underline{vol}(\partial\Omega) = vol(\partial\Omega) =$ canonical $(n-1)-$volume of the boundary of Ω (all notions of $(n-1)$-dimensional volume coincide in this case). The corresponding inequality hold for the function $F(t,r)$:

$$F(t,r) \leq \underline{vol}(\partial\Omega) \int_0^t e_R(\tau, r, 0)\, d\tau$$

Proof. Since $\rho_n \leq \rho$, for each fixed r we have: $\Omega_n(r) \subseteq \Omega(r)$; moreover $\mathrm{vol}(\Omega_n(r))$ converges to $\mathrm{vol}(\Omega(r))$ as $n \to \infty$. Since $R_n \leq R$, we also have $e_{R_n}(\tau, r, 0) \leq e_R(\tau, r, 0)$. Now, if $u_n(t,x)$ is the temperature function on Ω_n, then, for all n, t, x: $u(t,x) \geq u_n(t,x)$. Therefore the heat content $H(t,r)$ on Ω, satisfies, by Theorem 4A.3(i):

$$H(t,r) \geq \mathrm{vol}(\Omega_n(r)) - \mathrm{vol}(\partial\Omega_n) \cdot \int_0^t e_R(\tau, r, 0)\, d\tau$$

for all n, and the assertion follows by a passage to the limit. If Ω is a convex, bounded subset of $\mathbb{R}^n$, then Ω is the limit of an increasing sequence of convex polyhedra, the limit being taken with respect to the Hausdorff distance on convex subsets of $\mathbb{R}^n$. Hence $\rho_n \to \rho$ (uniformly) on $\bar{\Omega}$, and $\mathrm{vol}(\partial\Omega_n)$ converges to the canonical volume of the boundary of Ω (see [20], Theorem 12.5). $\square$

Note finally that inequality 4A.1(iii), is obvious for convex subsets of $\mathbb{R}^n$, since a convex set is contained in a half-space.

The next theorem gives an estimate of the "defects":

$$H_2(t) = H(t) - (\mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}) \tag{4A.5}$$

and:

$$F_2(t) = \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} - F(t) \tag{4A.6}$$

Note that, in fact, $H_2(t) = F_2(t)$ for all t. The estimate is given in terms of the measure $\rho_*(\Delta\rho)$, which is just the second derivative of the function $r \to \mathrm{vol}(\Omega(r))$.

4A.7 Theorem. *Let Ω be a domain with piecewise-smooth boundary satisfying $\Delta\rho \geq 0$. Then, at all times $t > 0$:*

$$\int_0^t \int_0^\infty e(\tau, 2r, 0)\rho_*(\Delta\rho)(r)\, dr\, d\tau \leq H_2(t) = F_2(t) \leq \int_0^t \int_0^\infty e(\tau, r, 0)\rho_*(\Delta\rho)(r)\, dr\, d\tau$$

The inequalities are sharp for the semi-infinite cylinder $\partial\Omega \times (0,\infty)$, in which case all three quantities reduce to zero.

Proof. From (4.4), we have:

$$F_2(t) = \int_0^t \int_0^\infty e(t-\tau, r, 0)\rho_*((1-u_\tau)\Delta\rho)\, dr\, d\tau$$

Since $u(\tau, x) \geq 0$ for all τ, x, we immediately obtain the right-hand inequality. On the other hand, by Theorem 4A.1(iii) we have:

$$1 - u(\tau, x) \geq \int_{\rho(x)}^\infty e(\tau, r, 0)\, dr$$

and therefore:

$$\rho_*((1-u_\tau)\Delta\rho)(r) \geq \left(\int_r^\infty e(\tau, s, 0)\, ds\right) \rho_*(\Delta\rho)(r)$$

We now plug the above into (4.4): the left-hand inequality of the theorem will then follow from (4.9). $\square$

Remark. If $\partial\Omega$ is smooth, the left-hand inequality, besides being optimal, has good asymptotic properties as $t \to 0$, in the sense that the difference between $H_2(t)$ and $\int_0^t \int_0^\infty e(\tau, 2r, 0)\rho_*(\Delta\rho)(r)\, dr\, d\tau$ is $O(t^{3/2})$ as $t \to 0$ (see Theorem 4C.3). In fact, if $\partial\Omega$ is smooth, the left-hand inequality can be made more explicit: to that purpose, let R_{inj} be the injectivity radius of $\partial\Omega$, and let η denote the trace of the second fundamental form of the boundary. Then:

4A.8 Theorem. *Let Ω be a domain with smooth boundary, satisfying $\Delta_{reg}\rho \geq 0$. Then, for all $t > 0$:*

$$\int_\Omega u(t,x)\, dx \geq vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\sqrt{t} + \frac{1}{2}(\int_{\partial\Omega} \eta\, dv_{n-1})t + \min\{C, 0\}t^{3/2} - g(t)$$

where $C = \dfrac{1}{3\sqrt{\pi}} \inf_{r\in(0,a)} \int_{\rho^{-1}(r)} (scal_M - Ricci(\nabla\rho, \nabla\rho) - scal_{\rho^{-1}(r)})\, dv_{n-1}$ and where $g(t)$ is the exponentially decreasing function: $g(t) = (\int_{\partial\Omega} \eta) \int_0^t \int_a^\infty \dfrac{1}{\sqrt{\pi\tau}} e^{-r^2/\tau}\, dr\, d\tau$; here a is a fixed number $0 < a < R_{inj}$ and "scal" denotes scalar curvature. In particular, if $\Omega \subseteq \mathbb{R}^3$: $C = -\dfrac{4\sqrt{\pi}}{3}\chi(\partial\Omega)$, where $\chi(\partial\Omega)$ is the Euler characteristic of $\partial\Omega$.

Proof. Since $\Delta_{cut}\rho \geq 0$, we have, by the left-hand inequality in Theorem 4A.7, setting for brevity $\phi(r) = \int_{\rho^{-1}(r)} \Delta_{reg}\rho\, dv_{n-1}$:

$$\begin{aligned}
H(t) - \mathrm{vol}(\Omega) + \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} &\geq \int_0^t \int_0^a e(\tau, 2r, 0)\phi(r)\, dr\, d\tau \\
&= \phi(0) \int_0^t \int_0^a e(\tau, 2r, 0)\, dr\, d\tau + \int_0^t \int_0^a e(\tau, 2r, 0)(\phi(r) - \phi(0))\, dr\, d\tau \\
\geq \phi(0) \int_0^t \int_0^\infty e(\tau, 2r, 0)\, dr\, d\tau &- \phi(0) \int_0^t \int_a^\infty e(\tau, 2r, 0)\, dr\, d\tau + \inf_{(0,a)} \phi' \cdot \int_0^t \int_0^a e(\tau, 2r, 0) r\, dr\, d\tau
\end{aligned}$$

If $\inf_{(0,a)} \phi' \leq 0$, we can minorize the last term by $\dfrac{1}{3\sqrt{\pi}} \inf_{(0,a)} \phi' \cdot t^{3/2}$ (just replace a in the upper limit of integration by ∞); if $\inf_{(0,a)} \phi' \geq 0$, we can minorize it by 0. The theorem now follows from the fact that, for $0 < r < a$:

$$\phi'(r) = \int_{\rho^{-1}(r)} (scal_M - Ricci(\nabla\rho, \nabla\rho) - scal_{\rho^{-1}(r)})\, dv_{n-1}$$

In fact, if $\rho(x) < R_{inj}$, then $\Delta\rho(x) = \Delta_{reg}\rho(x)$ and if $r < R_{inj}$, then: $\frac{d}{dr}\int_{\rho^{-1}(r)} f = \int_{\rho^{-1}(r)}(\nabla f \cdot \nabla\rho - f\Delta\rho)$. Taking $f = \Delta\rho$ and applying Bochner formula, we obtain:

$$\frac{d}{dr}\int_{\rho^{-1}(r)} \Delta\rho = \int_{\rho^{-1}(r)} (\|Dd\rho\|^2 - (\Delta\rho)^2 + Ricci(\nabla\rho \cdot \nabla\rho))$$

By Gauss' formula: $scal_{\rho^{-1}(r)} = scal_M - 2Ricci(\nabla\rho \cdot \nabla\rho) + (\Delta\rho)^2 - \|Dd\rho\|^2$. We substitute and get the stated expression of $\phi'(r)$. $\square$

We now give an example where $\Delta\rho$ is a *negative distribution* . Let Ω be the complement of a compact, convex set in $\mathbb{R}^n$. Then $\Delta\rho$ is a negative, regular distribution: in fact this is true if Ω is the complement of a convex polyhedron (the cut-locus of $\partial\Omega$ is empty, so that $\Delta_{cut}\rho = 0$; and $\Delta_{reg}\rho \leq 0$ because the mean curvature of $\partial\Omega(r)$ is non-positive) and the assertion follows by polyhedral approximation. Moreover, the distribution $\rho_*(\Delta\rho)$ is a *polynomial function* of degree $n-2$ in $r \in [0, \infty)$. In fact (see [7], Section 9.13) if K is compact and convex, and if $K_+(r) \equiv K + B(r)$, where $B(r)$ is the open ball of radius r centered at the origin, then $r \to \mathrm{vol}(K_+(r))$ is a polynomial function on $[0, \infty)$:

$$\mathrm{vol}(K_+(r)) = \mathrm{vol}(K) + \mathrm{vol}(\partial K)r + \alpha_2(K)r^2 + \cdots + \alpha_n(K)r^n$$

and the non-negative numbers $\alpha_2(K), \ldots, \alpha_n(K)$ are the so-called *cross-sectional measures* of K ; in particular, $\alpha_2(K)$ is half of the *total mean curvature* $m(K)$ of ∂K (if ∂K is C^2-smooth, then $\alpha_2(K)$ is half the integral of the sum of the principal curvatures of ∂K). For example, if $n = 2$: $\alpha_2(K) = \pi$, $\alpha_i(K) = 0, i \geq 3$; and, if $n = 3$: $\alpha_3(K) = \frac{4\pi}{3}$, $\alpha_i(K) = 0, i \geq 4$.

Now, if Ω is the complement of the compact, convex set K, and $\rho : \Omega \to \mathbb{R}$ is the distance from $\partial\Omega = \partial K$, then:

$$\mathrm{vol}(\rho^{-1}(r)) = \frac{d}{dr}\mathrm{vol}(K_+(r))$$

and since: $\rho_*(\Delta\rho)(r) = -\frac{d}{dr}\mathrm{vol}(\rho^{-1}(r))$, we see that:

$$\rho_*(\Delta\rho)(r) = -m(K) - 6\alpha_3(K)r - \cdots - n(n-1)\alpha_n(K)r^{n-2} \tag{4A.9}$$

Note that, since $\alpha_k(K) \geq 0$, we have: $\rho_*(\Delta\rho)(r) \leq -m(K)$, for all r. Now, since $\Delta\rho$ is negative, we have the validity of Theorem 4A.1 with all inequalities reversed; in particular, for all $t > 0$:

$$0 \leq 1 - u(t,x) \leq \int_{\rho(x)}^{\infty} e(t,r,0)\, dr$$

and Theorem 4A.7 becomes the following statement:

4A.10 Theorem. *Let Ω be the complement of the compact, convex subset K of $\mathbb{R}^n$. Then, at all times $t > 0$:*

$$0 \leq F(t) - \frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} \leq -\int_0^t \int_0^{\infty} e(\tau, 2r, 0)\rho_*(\Delta\rho)(r)\, dr\, d\tau$$

substituting the expression (4A.9) of $\rho_*(\Delta\rho)$, we get:

$$\frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} \le F(t) \le \frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} + \frac{m(K)}{2}t + O(t^{3/2})$$

where $O(t^{3/2})$ is a polynomial in $t^{1/2}$, explicitly computable from (4A.9). In particular, if $n = 2$:

$$\frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} \le F(t) \le \frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} + \pi t$$

and if $n = 3$:

$$\frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} \le F(t) \le \frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} + \frac{m(K)}{2}t + \frac{8\sqrt{\pi}}{3}t^{3/2}$$

Finally, let us observe that, if Ω is a compact, convex set, then $\rho_*(\Delta\rho)$ is positive and no longer regular; for example, if Ω is a convex polyhedron in $\mathbb{R}^n$, then $\rho_*(\Delta\rho)(r)$ is the sum of a (piecewise-polynomial) discontinuous function and the Dirac term $\mathrm{vol}(\rho^{-1}(R))\delta_R$, where R is the inner radius of Ω. Moreover, the value at zero of its regular part does not coincide with the total mean curvature of Ω. For example, if $n = 2$:

$$\rho_*(\Delta\rho)(0) = 2\sum_P \cot(\gamma(P)/2)$$

where the sum is taken over all vertices P of Ω, and $\gamma(P)$ is the interior angle at P. Note that, unlike the total mean curvature itself, which is always 2π, the functional $\rho_*(\Delta\rho)(0)$ is not bounded above on the set of all convex polygons in the plane. It is, however, *bounded below by* 2π in the sense that, for any convex polygon in the plane:

$$2\sum_P \cot(\gamma(P)/2) \ge 2\pi$$

For the proof, just observe that $\cot(\gamma(P)/2) = \tan(\gamma_{ext}(P)/2)$, where $\gamma_{ext}(P)$ is the exterior angle at the vertex P. Since $\tan(\gamma_{ext}(P)/2) \ge \gamma_{ext}(P)/2$, we get the assertion by summing over all vertices of Ω.

This observation leads to the following inequalities:

4A.11 Theorem. *Let Ω be a compact, convex subset of $\mathbb{R}^2$. Then, for all $t > 0$:*

$$H(t) \ge vol(\Omega) - \frac{2}{\sqrt{\pi}}vol(\partial\Omega)\sqrt{t} + \pi t - \pi\int_0^t\int_{2R}^\infty e(\tau, r, 0)\,dr\,d\tau$$

Proof. Let $\Omega_n \subseteq \Omega, n \in \mathbb{N}$ be a sequence of convex polygons converging to Ω in the Hausdorff metric. Then, for all n, and for all $t > 0$:

$$H(t) \ge H_n(t)$$

where $H_n(t)$ is the heat content of Ω_n. It is then enough to prove the inequality when Ω is a convex polygon; the general case will follow by a passage to the limit. By Theorem 4A.7:

$$H(t) \ge \mathrm{vol}(\Omega) - \frac{2}{\pi}\mathrm{vol}(\partial\Omega)\sqrt{t} + \int_0^t\int_0^R e(\tau, 2r, 0)\rho_*(\Delta\rho)(r)\,dr\,d\tau \tag{4A.12}$$

Now, for each r, $\Omega(r)$ is a convex polygon; hence the distribution $\rho_*(\Delta\rho)$ is the sum of the step-function $r \to 2\sum_{P(r)} \cot(\gamma(P(r))/2)$ (where the sum is taken over all vertices $P(r)$ of the convex polygon $\Omega(r)$), and the singular part $\mathrm{vol}(\rho^{-1}(R))\,\delta_R$, which is always *non-negative* . By the observation preceeding the theorem, we have, as distributions on $(0,R)$:

$$\rho_*(\Delta\rho)(r) \geq 2\pi$$

and the assertion follows by inserting in the inequality 4A.12.

4B. Bounds in the general case.

In this subsection, Ω will be a domain with smooth boundary. We *no longer* assume $\Delta\rho \geq 0$.

We give upper and lower bounds for the heat content. The bounds come from the fact that $\Delta_{cut}\rho$ is always a positive measure. Let R_{inj} denote, as usual, the injectivity radius of $\partial\Omega$.

4B.1 Theorem. *Let Ω be a domain with smooth boundary, and let:*

$$H_-(t) = vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\cdot\sqrt{t} + \int_0^t\int_0^\infty e(t-\tau,r,0)\left(\int_{\rho^{-1}(r)}(1-u_\tau)\Delta_{reg}\rho\right)dr\,d\tau$$

Then, for all $t>0$:

$$0 \leq H(t) - H_-(t) \leq \frac{4}{\sqrt{\pi}R_{inj}^2}\left(\int_\Omega \Delta_{cut}\rho\right)t^{3/2}e^{-R_{inj}^2/4t}$$

Proof. We look at the expression 4.4 of $H(t)$. Since $\Delta_{cut}\rho \geq 0$:

$$\rho_*((1-u_\tau)\Delta\rho)(r) \geq \int_{\rho^{-1}(r)}(1-u_\tau)\Delta_{reg}\rho$$

and we immediately have the inequality. Now, since $\Delta_{cut}\rho$ is supported on the cut-locus, we see that $\rho_*((1-u_\tau)\Delta_{cut}\rho)$ is supported for $r \geq R_{inj}$. Hence:

$$\begin{aligned} H(t) - H_-(t) &= \int_0^t\int_{R_{inj}}^\infty e(t-\tau,r,0)\rho_*((1-u_\tau)\Delta_{cut}\rho)(r)\,dr\,d\tau \\ &\leq \int_0^t e(t-\tau,R_{inj},0)\,d\tau\cdot\int_{\Omega(R_{inj})}\Delta_{cut}\rho \end{aligned}$$

and the assertion follows from (4.6). $\square$

4B.2 Theorem. *Let Ω be a domain with smooth boundary, and let $H_+(t,r)$ be the solution, satisfying Neumann boundary conditions, of the following initial-value problem on the half-line:*

$$\begin{cases} (-\dfrac{\partial^2}{\partial r^2} + \dfrac{\partial}{\partial t})H_+ = -\displaystyle\int_{\rho^{-1}(r)} u_t\Delta_{reg}\rho \\ H_+(0,r) = vol(\Omega(r)) \qquad \text{for all} \quad r \geq 0 \end{cases}$$

Then, for all $t>0$:

$$0 \leq H_+(t,0) - H(t) \leq \frac{4}{\sqrt{\pi}R_{inj}^2}\left(\int_\Omega \Delta_{cut}\rho\right)t^{3/2}e^{-R_{inj}^2/4t}$$

Proof. We apply Duhamel principle to $H(t,r) = \int_{\Omega(r)} u(t,x)\,dx$ and to $H_+(t,r)$. Then:

$$H_+(t,r) - H(t,r) = \int_0^t \int_0^\infty e(t-\tau, r, s)\rho_*(u_\tau \Delta_{cut}\rho)(s)\, ds\, d\tau$$

and we can estimate $H_+(t,0) - H(t)$ as we estimated $H(t) - H_-(t)$ in the proof of Theorem 4B.1. $\square$

4C. Asymptotics of the heat content: smooth boundaries

We assume, in this subsection, that Ω is an open set with *smooth boundary* . We *no longer assume* $\Delta\rho \geq 0$.

We start by estimating the temperature function $u(t,x)$ near the boundary $\partial\Omega$. Fix a number a so that: $0 < 2a < R_{inj}$, and let $|\Delta\rho|$ denote the measure $|\Delta_{reg}\rho| + \Delta_{cut}\rho$. We will make use of the constant:

$$C(\Omega, a) = \sup_{\{x,y\in\Omega: d(x,y)>a; t>0\}} k(t,x,y)$$

where $k(t,x,y)$ is the heat kernel of Ω (with Dirichlet boundary conditions). $C(\Omega,a)$ is obviously finite, and can be easily estimated if Ω is a domain in euclidean space. If M is any manifold, we give an upper bound of $C(\Omega,a)$ in the lemma below.

4C.1 Lemma. *Assume that the sectional curvature of M is bounded above by σ, and that a is selected so that: $a \leq \inf\{Inj(M), \frac{\pi}{\sqrt{\sigma}}\}$ where $Inj(M)$ is a lower bound of the injectivity radius of M (we take $\frac{\pi}{\sqrt{\sigma}} = \infty$ if $\sigma \leq 0$). Then:*

$$C(\Omega, a) \leq \frac{1}{vol(B_\sigma(a))}$$

where $B_\sigma(a)$ is the ball of radius a in the space form, M_σ, of constant curvature σ.

Proof. Let $\bar{k}(t,\bar{x},\bar{y})$ denote the heat kernel, with Neumann boundary conditions, of a ball $B(a)$ of center $\bar{x}$ and radius a in M_σ. Since $\bar{k}$ is a radial function, we will write it simply as $\bar{k}(t,r)$, where $r = d(\bar{x},\bar{y})$. Now, since the Dirichlet heat kernel $k(t,x,y)$ is less than or equal to the heat kernel of the manifold M at (t,x,y), we can use an estimate of Courtois' (see [10]), and conclude that, for all $x,y\in\Omega$ such that $d(x,y) > a$, and for all $t > 0$:

$$k(t,x,y) \leq \bar{k}(t,a)$$

The lemma will be proved once we show that:

$$\sup_{t>0} \bar{k}(t,a) = \lim_{t\to\infty} \bar{k}(t,a)$$

because the right-hand side is exactly $\frac{1}{\mathrm{vol}(B_\sigma(a))}$. It is enough to show that $\frac{\partial\bar{k}}{\partial t}(t,a) \geq 0$ for all t. Now it is well-known that $\bar{k}(t,r)$ attains its minimum for $r = a$, hence there exists $\epsilon > 0$ such that $\frac{\partial\bar{k}}{\partial r}(t,r) \leq 0$ for $r \in (a-\epsilon, a)$. Then fix $r \in (a-\epsilon, a)$. By Green's theorem:

$$\begin{aligned}\int_{B(a)\setminus B(r)} \frac{\partial\bar{k}}{\partial t} &= -\int_{B(a)\setminus B(r)} \Delta\bar{k} \\ &= -\,\mathrm{vol}(\partial B(r))\frac{\partial\bar{k}}{\partial r}(t,r) \geq 0\end{aligned}$$

The assertion now follows from the fact that $\bar{k}$ is a radial function. $\square$

4C.2 Lemma. *Let a be a number such that $0 < 2a < R_{inj}$ = the injectivity radius of $\partial\Omega$. If $\rho(x) < a$ then, for all $t > 0$:*

$$|1 - u(t,x) - \int_{\rho(x)}^{\infty} e(t,r,0)\,dr| \leq C_1 t^{1/2} + C_2 t^{3/2} e^{-a^2/t}$$

Where:

$$C_1 = \frac{2}{\sqrt{\pi}} \sup_{\{y:\rho(y)<2a\}} |\Delta_{reg}\rho(y)|$$

$$C_2 = \frac{1}{\sqrt{\pi}a^2}\, C(\Omega,a) \int_{\Omega} |\Delta\rho|$$

Proof. By (4.5):

$$1 - u(t,x) - \int_{\rho(x)}^{\infty} e(t,r,0)\,dr = \int_0^t \int_0^{\infty} e(t-\tau,r,0)\rho_*(k(\tau,x,\cdot)\Delta\rho)\,dr\,d\tau$$

So, we need to estimate the double integral in the right-hand side. Split the inner integral at $r = 2a$. Since $\Delta_{cut}\rho$ is supported for $\rho > 2a$:

$$\begin{aligned} &|\int_0^{2a} e(t-\tau,r,0)\rho_*(k(\tau,x,\cdot)\Delta\rho)\,dr| \leq \\ &\qquad \leq \frac{(t-\tau)^{-1/2}}{\sqrt{\pi}} \cdot \int_{\{y:\rho(y)<2a\}} k(\tau,x,y)\,|\Delta_{reg}\rho(y)|\,dy \\ &\qquad \leq \frac{(t-\tau)^{-1/2}}{\sqrt{\pi}} \sup_{\{y:\rho(y)<2a\}} |\Delta_{reg}\rho(y)| \end{aligned}$$

On the other hand, $\rho(y) > 2a$ together with $\rho(x) < a$ imply $d(x,y) > a$. Hence:

$$\begin{aligned} &|\int_{2a}^{\infty} e(t-\tau,r,0)\rho_*(k(\tau,x,\cdot)\Delta\rho)\,dr| \\ &\qquad \leq C(\Omega,a)\,e(t-\tau,2a,0) \int_{2a}^{\infty} \rho_*(|\Delta\rho|) \end{aligned}$$

Now add up, integrate from $\tau = 0$ to $\tau = t$, and use the inequality 4.6. □

Let η denote the trace of the second fundamental form of $\partial\Omega$.

4C.3 Theorem. (Compare with [1],[2]) *Assume that Ω is a domain with smooth boundary in any Riemannian manifold, and let $H(t) = \int_{\Omega} u(t,x)\,dx$ denote the heat content function. Then, for all $t > 0$:*

$$H(t) = vol(\Omega) - \frac{2}{\sqrt{\pi}} vol(\partial\Omega)\sqrt{t} + \frac{1}{2}\left(\int_{\partial\Omega} \eta\,dv_{n-1}\right) t + l(t)$$

where $|l(t)| \leq Ct^{3/2} + C_3 t^{3/2} e^{-a^2/4t} + (C_4 t^{5/2} + C_5 t^{7/2}) e^{-a^2/t}$
The constants are as follows: fix a so that $0 < 2a < R_{inj}$. Then:

$$\begin{aligned} C = \frac{4}{3\sqrt{\pi}} &\sup_{\{y:\rho(y)<2a\}} |\Delta_{reg}\rho(y)| \cdot \sup_{(0,a)} \int_{\rho^{-1}(r)} |\Delta_{reg}\rho| \\ &+ \frac{1}{3\sqrt{\pi}} \sup_{(0,a)} |\int_{\rho^{-1}(r)} (scal_M - Ricci(\nabla\rho,\nabla\rho) - scal_{\rho^{-1}(r)})\,dv_{n-1}| \end{aligned}$$

$$C_3 = \frac{5}{\sqrt{\pi}a^2}\int_\Omega |\Delta\rho|$$

$$C_4 = \frac{1}{2\sqrt{\pi}a^3}\int_{\partial\Omega} |\eta|$$

$$C_5 = \frac{1}{\sqrt{\pi}a^4}C(\Omega, a)\int_\Omega |\Delta\rho| \cdot \sup_{(0,a)}\int_{\rho^{-1}(r)} |\Delta_{reg}\rho|$$

Proof. We let $\psi(t,x) = 1 - u(t,x) - \int_{\rho(x)}^\infty e(t,r,0)\,dr$. Then, by (4.4), and (4.9):

$$\begin{aligned}
H(t) - (\mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}) &= \int_0^t\int_0^\infty e(t-\tau,r,0)\rho_*(\psi(\tau,\cdot)\Delta\rho)\,dr\,d\tau \\
&+ \int_0^t\int_0^\infty e(t-\tau,r,0)\left(\int_r^\infty e(\tau,s,0)\,ds\right)\rho_*(\Delta\rho)\,dr\,d\tau \\
&= \int_0^t\int_0^\infty e(t-\tau,r,0)\rho_*(\psi(\tau,\cdot)\Delta\rho)\,dr\,d\tau \\
&+ \int_0^t\int_0^\infty e(\tau,2r,0)\rho_*(\Delta\rho)(r)\,dr\,d\tau
\end{aligned}$$

We cut the inner integrals at $r = a$ so that $H(t) - (\mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t})$ will be the sum of the four pieces A, B, C, D where:

$$A = \int_0^t\int_0^a e(t-\tau,r,0)\rho_*(\psi(\tau,\cdot)\Delta\rho)\,dr\,d\tau$$

$$B = \int_0^t\int_a^\infty e(t-\tau,r,0)\rho_*(\psi(\tau,\cdot)\Delta\rho)\,dr\,d\tau$$

$$C = \int_0^t\int_0^a e(\tau,2r,0)\rho_*(\Delta\rho)(r)\,dr\,d\tau$$

$$D = \int_0^t\int_a^\infty e(\tau,2r,0)\rho_*(\Delta\rho)(r)\,dr\,d\tau$$

Control of $|A|$. By Lemma 4C.2: $|\psi(\tau,x)| \le C_1\tau^{1/2} + C_2\tau^{3/2}e^{-a^2/\tau}$. Then, since $\int_0^a e(t-\tau,r,0)\,dr \le 1$, and $\Delta_{cut}\rho$ is supported for $\rho > a$:

$$\begin{aligned}
|A| &\le \int_0^t\int_0^a e(t-\tau,r,0)\cdot(C_1\tau^{1/2} + C_2\tau^{3/2}e^{-a^2/\tau})\cdot\int_{\rho^{-1}(r)}|\Delta_{reg}\rho|\,dr\,d\tau \\
&\le C_1\cdot\sup_{(0,a)}\int_{\rho^{-1}(r)}|\Delta_{reg}\rho|\cdot\int_0^t\tau^{1/2}\,d\tau + C_2\cdot\sup_{(0,a)}(\int_{\rho^{-1}(r)}|\Delta_{reg}\rho|)\cdot\int_0^t\tau^{3/2}e^{-a^2/\tau}\,d\tau \\
&\le \frac{2}{3}C_1\cdot\sup_{(0,a)}\int_{\rho^{-1}(r)}|\Delta_{reg}\rho|\cdot t^{3/2} + \frac{C_2}{a^2}\cdot\sup_{(0,a)}(\int_{\rho^{-1}(r)}|\Delta_{reg}\rho|)\cdot t^{7/2}e^{-a^2/t}
\end{aligned}$$

Control of $|B|$. Use $|\psi(\tau,x)| \le 1$. Then:

$$
\begin{aligned}
|B| &\leq \int_0^t \int_a^\infty e(t-\tau, r, 0)\rho_*(|\Delta\rho|)\, dr\, d\tau \\
&\leq \int_0^t e(t-\tau, a, 0)\left(\int_a^\infty \rho_*(|\Delta\rho|)\, dr\right) d\tau \\
&\leq \int_\Omega |\Delta\rho| \cdot \int_0^t e(t-\tau, a, 0)\, d\tau
\end{aligned}
$$

Now use 4.6, and obtain:

$$
|B| \leq \frac{4}{\sqrt{\pi}a^2} \int_\Omega |\Delta\rho| \cdot t^{3/2} e^{-a^2/4t}
$$

Control of $|C|$. Since $2a < R_{inj}$:

$$
C = \int_0^t \int_0^a e(\tau, 2r, 0) \left(\int_{\rho^{-1}(r)} \Delta_{reg}\rho\, dr \right) d\tau
$$

Set $\phi(r) = \int_{\rho^{-1}(r)} \Delta_{reg}\rho$. Then:

$$
\begin{aligned}
C &= \phi(0) \int_0^t \int_0^a e(\tau, 2r, 0)\, dr\, d\tau + \int_0^t \int_0^a e(\tau, 2r, 0)(\phi(r) - \phi(0))\, dr\, d\tau \\
&= \frac{\phi(0)}{2} t - \phi(0) \int_0^t \int_a^\infty e(\tau, 2r, 0)\, dr\, d\tau + \int_0^t \int_0^a e(\tau, 2r, 0)(\phi(r) - \phi(0))\, dr\, d\tau
\end{aligned}
$$

Therefore, by 4.7 and 4.8, since $\phi(0) = \int_{\partial\Omega} \eta$:

$$
|C - \frac{1}{2} \int_{\partial\Omega} \eta \cdot t| \leq \frac{|\phi(0)|}{2\sqrt{\pi}a^3} t^{5/2} e^{-a^2/t} + \frac{1}{3\sqrt{\pi}} \sup_{(0,a)} |\phi'| \cdot t^{3/2}
$$

The following expression of ϕ' holds (see the proof of Theorem 4A.8):

$$
\phi'(r) = \sup_{(0,a)} \int_{\rho^{-1}(r)} (scal_M - Ricci(\nabla\rho, \nabla\rho) - scal_{\rho^{-1}(r)})\, dv_{n-1}
$$

Control of $|D|$. By 4.6:

$$
\begin{aligned}
|D| &\leq \int_0^t e(\tau, 2a, 0) \int_a^\infty \rho_*(|\Delta\rho|)\, dr\, d\tau \\
&\leq \frac{1}{\sqrt{\pi}a^2} \int_\Omega |\Delta\rho| \cdot t^{3/2} e^{-a^2/t} \\
&\leq \frac{1}{\sqrt{\pi}a^2} \int_\Omega |\Delta\rho| \cdot t^{3/2} e^{-a^2/4t}
\end{aligned}
$$

The theorem follows.

4D. Asymptotics of the heat content on a convex polyhedron

In this section Ω is a convex, bounded, open set in $\mathbb{R}^n$ with a polyhedral boundary. $\bar\Omega$ is then a *polytope* in the sense that it is the intersection of a finite family of closed half-spaces:

$$\bar\Omega = \underset{i\in I}{\cap}\mathcal{H}_i \qquad I=\{1,\dots,m\}$$

where $\mathcal{H}_i=\{x\in\mathbb{R}^n:\rho_{\pi_i}(x)\geq 0\}$ and where ρ_{π_i} denotes the distance, taken with sign, from the oriented affine hyperplane π_i of $\mathbb{R}^n$. Note that ρ_{π_i} is an affine map. The *faces* of $\bar\Omega$ are the (possibly empty) subsets of $\partial\Omega$ defined by:

$$\mathcal{F}_i=\pi_i\cap\bar\Omega \qquad i\in I$$

Each $\mathcal{F}_i$ is a polytope in π_i; the support hyperplanes (in π_i) defining it are: $\pi_i\cap\pi_j, j\neq i$ (with the obvious orientation), and the faces of $\mathcal{F}_i$ (in π_i) are: $(\pi_i\cap\pi_j)\cap\mathcal{F}_i=\mathcal{F}_i\cap\mathcal{F}_j$ with $j\in I, j\neq i$. In turn, each $\mathcal{F}_i\cap\mathcal{F}_j$, with $j\neq i$, is a polytope in the $(n-2)-$dimensional euclidean space $\pi_i\cap\pi_j$, and so on. By $\mathrm{vol}_d(P)$ we denote the Lebesgue measure of the polytope P in $\mathbb{R}^d$, and by γ_{ij} we denote the interior angle at $\mathcal{F}_i\cap\mathcal{F}_j$: it is the unique angle between 0 and π such that $\cos(\gamma_{ij})=-\nu_i\cdot\nu_j$, where ν_i and ν_j are the respective unit normal vectors of π_i and π_j, positively oriented. Note that, if $\mathcal{F}_i$ and $\mathcal{F}_j$ are incident faces, then $0<\gamma_{ij}<\pi$. The aim of this section is to prove the following:

4D.1 Theorem. *Let Ω, $\mathcal{F}_i$, and γ_{ij} be as above, and let $H(t)=\int_\Omega u(t,x)\,dx$ denote the heat content function on Ω. Then, for all $t>0$:*

$$H(t)=vol(\Omega)-\frac{2}{\sqrt{\pi}}vol(\partial\Omega)\cdot t^{1/2}+2\sum_{i\neq j}vol_{n-2}(\mathcal{F}_i\cap\mathcal{F}_j)\cdot c_{ij}\cdot t+E(t)$$

where $c_{ij}=\displaystyle\int_0^\infty\left(1-\frac{\tanh(\gamma_{ij}x)}{\tanh(\pi x)}\right)dx$ and:

$$|E(t)|\leq Ct^{3/2}+g(t)$$

where C is a positive constant, and $g(t)$ is an exponentially decreasing function as $t\to 0$. Both C and $g(t)$ will be explicited at the end of the proof.

The proof proceeds in the following way: we first describe the cut-locus of $\partial\Omega$, show that it is a polyhedral set, and give a convenient expression of $\Delta_{cut}\rho$ as integration on the cut-locus. We then give the proof in four steps. Finally, we examine the special case $n=2$, and extend our proof to cover the (not necessarily convex) polygonal domains in the plane.

Let, as usual, $\rho:\Omega\to\mathbb{R}$ denote the distance from $\partial\Omega$. We observe the following fact, which follows easily from the convexity of Ω: for all $x\in\Omega$:

$$\rho(x)=\min_{i=1,\dots,m}\rho_{\pi_i}(x)$$

Since there are no focal points of $\partial\Omega$, the cut-locus of $\partial\Omega$ is the closure of the set of points of Ω which can be joined to $\partial\Omega$ by at least two minimizing line segments. Therefore:

$$\mathrm{Cut}(\partial\Omega)=\underset{i\neq j}{\cup}\mathrm{Cut}_{ij}$$

where:

$$\mathrm{Cut}_{ij}=\{x\in\bar\Omega:\rho(x)=\rho_{\pi_i}(x)=\rho_{\pi_j}(x)\}$$

The next proposition shows that each Cut_{ij} is in fact a polytope in the *bisecting plane* π_{ij} of π_i and π_j, where:

$$\pi_{ij} = \{x \in \mathbb{R}^n : \rho_{\pi_i}(x) = \rho_{\pi_j}(x)\}$$

so that $\mathrm{Cut}(\partial\Omega)$ is indeed a polyhedral set.

4D.2 Proposition. *For $i \in I$, let $\bar{\mathcal{R}}_i = \{x \in \bar{\Omega} : \rho(x) = \rho_{\pi_i}(x)\}$. Then $\bar{\Omega} = \bigcup_{i \in I} \bar{\mathcal{R}}_i$, and:*

(i) $\bar{\mathcal{R}}_i$ is a polytope in $\mathbb{R}^n$; its faces are $\mathcal{F}_i$, and all $\mathrm{Cut}_{ij} \equiv \bar{\mathcal{R}}_i \cap \bar{\mathcal{R}}_j$ with $j \in I$, $j \neq i$;

(ii) For each $i \neq j$, Cut_{ij} is a polytope in π_{ij} (possibly empty, or degenerate); its faces are $\mathcal{F}_i \cap \mathcal{F}_j$, and all $\mathrm{Cut}_{ijk} = \bar{\mathcal{R}}_i \cap \bar{\mathcal{R}}_j \cap \bar{\mathcal{R}}_k$ with $k \in I$, $k \neq i$, $k \neq j$.

Proof. Fix $i \in I$, and, for each $j \in I$ with $i \neq j$, let π_{ij} denote the bisecting plane of π_i and π_j. Then $\bar{\mathcal{R}}_i$ lies entirely in one of the two sides (half-spaces) determined by the hyperplane π_{ij}. Moreover, $\bar{\mathcal{R}}_i$ and $\bar{\mathcal{R}}_j$ lies on opposite sides with respect to π_{ij}. Denote by $\mathcal{H}_{ij}$ (with the indices in that order) the *closed half-space determined by π_{ij} and containing $\bar{\mathcal{R}}_i$.* By what we just said:

$$\bar{\mathcal{R}}_i \subseteq \bigcap_{j \in I} \mathcal{H}_{ij}$$

where we agree to set $\mathcal{H}_{ii} = \mathcal{H}_i$.

On the other hand, let $x \in \bigcap_{j \in I} \mathcal{H}_{ij}$. Then $x \in \bar{\Omega}$, hence $x \in \bar{\mathcal{R}}_k$ for some k, and therefore $x \in \mathcal{H}_{ki}$. But $x \in \mathcal{H}_{ik}$ by hypothesis: hence $x \in \mathcal{H}_{ik} \cap \mathcal{H}_{ki} = \pi_{ik}$. Then: $\rho_{\pi_i}(x) = \rho_{\pi_k}(x) = \rho(x)$, i.e. $x \in \bar{\mathcal{R}}_i$. We conclude:

$$\bar{\mathcal{R}}_i = \bigcap_{j \in I} \mathcal{H}_{ij}$$

that is, $\bar{\mathcal{R}}_i$ is a polytope, as asserted. The other assertions follow rather easily from the above representation of $\bar{\mathcal{R}}_i$. $\square$

We now give a convenient description of $\rho_*(u\Delta\rho)$ as integration on the cut-locus.

4D.3 Proposition. *Let $\phi \in C^0_c(\Omega)$, and $\psi \in C^0_c(0, \infty)$. Then:*

$$\langle \Delta\rho, \phi \rangle = \sum_{i \neq j} \cos(\frac{\gamma_{ij}}{2}) \int_{\mathrm{Cut}_{ij}} \phi(x)\, dx;$$

$$\langle \rho_*(u\Delta\rho), \psi \rangle = \sum_{i \neq j} \cos(\frac{\gamma_{ij}}{2}) \int_{\mathrm{Cut}_{ij}} u(x)\, \psi(\rho(x))\, dx.$$

dx denoting Lebesgue measure on the hyperplane π_{ij} of $\mathbb{R}^n$

Proof. We first assume ϕ smooth; the assertion will follow by a density argument. Then:

$$\langle \Delta\rho, \phi \rangle = \int_\Omega \nabla\rho \cdot \nabla\phi$$

Since $\bar{\Omega} = \bigcup_{i \in I} \bar{\mathcal{R}}_i$, and since $\mathrm{Cut}(\partial\Omega)$ has measure zero:

$$\int_\Omega \nabla\rho \cdot \nabla\phi = \sum_{i \in I} \int_{\bar{\mathcal{R}}_i} \nabla\rho \cdot \nabla\phi$$

The restriction of ρ to $\bar{\mathcal{R}}_i$ is ρ_{π_i}, which is an affine map; hence, by Green theorem applied to $\mathcal{R}_i$, and Proposition 4D.2:

$$\int_{\bar{\mathcal{R}}_i} \nabla\rho \cdot \nabla\phi = -\sum_{j \in I, j \neq i} \int_{\mathrm{Cut}_{ij}} \phi \frac{\partial \rho_{\pi_i}}{\partial \nu_{ij}}$$

where ν_{ij} denotes the unit normal to π_{ij}, oriented to the side of $\mathcal{R}_i$. Hence $\dfrac{\partial\rho_{\pi_i}}{\partial\nu_{ij}} = \nu_i\cdot\nu_{ij} = -\cos(\gamma_{ij}/2)$, and the first formula follows by summing over I. The second formula follows from the first by the definition of ρ_*. $\square$

By 4.4 and Proposition 4D.3:

$$H(t)-\mathrm{vol}(\Omega)+\frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}=\sum_{i\neq j}\cos(\frac{\gamma_{ij}}{2})\int_0^t\int_{\mathrm{Cut}_{ij}}e(t-\tau,\rho(x),0)(1-u(\tau,x))\,dx\,d\tau \tag{4D.4}$$

so we need to determine the coefficient c_2 of t in the asymptotic expansion of the right-hand side of 4D.4 as $t\to 0$. To do that, we restrict to a suitable $\epsilon-$neighborhood of $\partial\Omega$. Let us fix some notation on the incidence relations of the $\mathcal{F}_i's$:

$$I_2=\{(i,j)\in I\times I: i\neq j, \mathcal{F}_i\cap\mathcal{F}_j\neq\emptyset\}$$

$$I_3=\{(i,j,k)\in I\times I\times I: i\neq j\neq k\neq i, \mathcal{F}_i\cap\mathcal{F}_j\cap\mathcal{F}_k\neq\emptyset\}$$

Then:

4D.5 Lemma. *Let* $\epsilon=\inf_{(i,j,k)\notin I_3} dist(Cut_{ij},\mathcal{F}_k)$. *Then* $\epsilon>0$, *and:*

(i) *If* $x\in Cut_{ij}$ *and* $\rho(x)<\epsilon$, *then* $(i,j)\in I_2$;

(ii) *If* $x\in Cut_{ijk}$ *and* $\rho(x)<\epsilon$, *then* $(i,j,k)\in I_3$.

Proof. Recall that Cut_{ij} and $\mathcal{F}_k$ are closed subsets of $\mathbb{R}^n$. To show that $\epsilon>0$, it is then enough to show that, if $(i,j,k)\notin I_3$, then $\mathrm{Cut}_{ij}\cap\mathcal{F}_k=\emptyset$. But this is clear, since $\mathrm{Cut}_{ij}\cap\mathcal{F}_k\subseteq\mathcal{F}_i\cap\mathcal{F}_j\cap\mathcal{F}_k$.
Proof of (ii): if $x\in\mathrm{Cut}_{ijk}$, then $\rho(x)=d(x,z)$ for some $z\in\mathcal{F}_k$. If $(i,j,k)\notin I_3$, we have $d(x,z)\geq\epsilon$ by our definition of ϵ, and (ii) is proved.
Proof of (i): let $(i,j)\notin I_2$; the restriction of ρ to Cut_{ij} is just ρ_{π_i}: an affine map. Hence $\rho|_{\mathrm{Cut}_{ij}}$ attains its absolute minimum on the boundary of Cut_{ij}: this implies, since $\partial\mathrm{Cut}_{ij}\cap\partial\Omega=\emptyset$, that there exists, by Proposition 4D.2, an index k, $k\neq i, k\neq j$, and a point $y\in\mathrm{Cut}_{ijk}$ such that $\rho(x)\geq\rho(y)$ for all $x\in\mathrm{Cut}_{ij}$. Since $(i,j)\notin I_2$, a fortiori $(i,j,k)\notin I_3$, hence $\rho(y)\geq\epsilon$ by (ii). $\square$

The proof of the theorem is in four steps, which we outline below. Set, for brevity:

$$Z_{ij}(u;\tau)=\int_{\mathrm{Cut}_{ij}}e(t-\tau,\rho(x),0)(1-u(\tau,x))\,dx$$

Step 1. If $\mathcal{F}_i\cap\mathcal{F}_j=\emptyset$, then Cut_{ij} is at distance $\geq\epsilon$ from $\partial\Omega$. Hence each pair $(i,j)\notin I_2$ contributes to the sum in (4D.4) with an exponentially decreasing term. Precisely, we will show that:

$$|H(t)-\mathrm{vol}(\Omega)+\frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}-\sum_{(i,j)\in I_2}\cos(\gamma_{ij}/2)\int_0^t Z_{ij}(u;\tau)\,d\tau|\leq\frac{4}{\sqrt{\pi}\epsilon^2}\mathrm{vol}(\partial\Omega)t^{3/2}e^{-\epsilon^2/4t} \tag{4D.6}$$

In Steps 2-4, we assume that $(i,j)\in I_2$ (that is, $\mathcal{F}_i$ and $\mathcal{F}_j$ are incident faces).

Step 2. It is the most delicate estimate. We show that, in order to compute the term in t in the expansion of the heat content, we can replace the temperature function u on Cut_{ij} by the *temperature function* u_{ij}, *relative to the infinite open wedge* W_{ij} *in* $\mathbb{R}^n$ *bounded by the oriented hyperplanes* π_i *and* π_j. Precisely:

(4D.7) $$|\int_0^t Z_{ij}(u;\tau)\,d\tau - \int_0^t Z_{ij}(u_{ij};\tau)\,d\tau| \le C_1(i,j)t^{3/2} + C_2(i,j)t^2 e^{-\epsilon^2/4nt}$$

for some positive constant $C_1(i,j), C_2(i,j)$. If $\dim(\Omega) = 2$ then $C_1(i,j) = 0$.

Step 3. We observe that, when restricted to π_{ij} (the bisecting plane of the wedge W_{ij}), the temperature $u_{ij}(t,x)$ depends only on $\rho_{ij}(x)$ = distance of x from $\pi_i \cap \pi_j$, so that it can be written $u_{ij}(t,x) = \tilde{u}_{ij}(t,\rho_{ij}(x))$, for a function $\tilde{u}_{ij}$ of t and $r \ge 0$. Hence we show that:

(4D.8) $$|\int_0^t Z_{ij}(u_{ij};\tau)\,d\tau - \mathrm{vol}_{n-2}(\mathcal{F}_i \cap \mathcal{F}_j)\cdot c_{ij}(t)| \le C_3(i,j)t^{3/2} + C_4(i,j)t^2 e^{-\epsilon^2/4t}$$

where $C_3(i,j), C_4(i,j)$ are positive constants, and:

$$c_{ij}(t) = \int_0^t \int_0^\infty e(t-\tau, r\sin(\gamma_{ij}/2), 0)(1 - \tilde{u}_{ij}(\tau,r))\,dr\,d\tau$$

Step 4. It is the explicit computation:

(4D.9) $$c_{ij}(t) = \frac{2}{\cos(\gamma_{ij}/2)}\left(\int_0^\infty (1 - \frac{\tanh(\gamma_{ij}x)}{\tanh(\pi x)})\,dx\right)\cdot t$$

The theorem will follow from 4D.6-9. See the end of the proof for the explicit expressions of C and $g(t)$.

We now give the proofs of Steps 1-4. We make use, several times, of the following, easily established, facts:

1. If S is a $p-$dimensional affine subspace of $\mathbb{R}^n$, then:

$$\mathrm{vol}_p(S \cap \Omega) \le \mathrm{vol}(B^p(diam(\Omega)))$$

where $B^p(a)$ is the ball of radius a in $\mathbb{R}^p$. Note that $\mathrm{vol}(B^0(a)) = 1$. We set $\mathrm{vol}(B^p(a)) = 0$ if $p < 0$.

2. If $\rho_\pi : \mathbb{R}^n \to \mathbb{R}$ is the distance function from the oriented hyperplane π, with unit normal $\nu = \nabla\rho$, then the gradient of the restriction of ρ_π to the affine subspace S of $\mathbb{R}^n$ is the orthogonal projection of ν onto S.

Proof of 4D.6. From Lemma 4D.5, we see that if $x \in \mathrm{Cut}_{ij}$, and $(i,j) \notin I_2$, then $\rho(x) \ge \epsilon$. Hence, if $(i,j) \notin I_2$:

$$Z_{ij}(u;\tau) \le e(t-\tau,\epsilon,0)\,\mathrm{vol}_{n-1}(\mathrm{Cut}_{ij})$$

and since $\langle \rho_*(\Delta\rho), 1\rangle = \mathrm{vol}(\partial\Omega)$, we easily get, by 4.6 and Proposition 4D.3 applied to $u = \psi = 1$:

$$\sum_{(i,j)\notin I_2} \cos(\frac{\gamma_{ij}}{2}) \int_0^t \int_{\mathrm{Cut}_{ij}} e(t-\tau,\rho(x),0)(1-u(\tau,x))\,dx\,d\tau \le \frac{4}{\sqrt{\pi}\epsilon^2}\mathrm{vol}(\partial\Omega)t^{3/2}\,e^{-\epsilon^2/4t}$$

This proves 4D.6.

Proof of 4D.7. We now fix $(i,j) \in I_2$ and use the notation: $I_{ij} = \{k \in I : (i,j,k) \in I_3\}$. Recall that:

$$Z_{ij}(u;\tau) = \int_{\mathrm{Cut}_{ij}} e(t-\tau,\rho(x),0)(1-u(\tau,x))\,dx$$

Since $\mathcal{F}_i$ and $\mathcal{F}_i$ are incident, so are the hyperplanes π_i and π_j. We denote by W_{ij} the infinite open wedge in $\mathbb{R}^n$ given by the intersection of the two half-spaces determined by π_i and π_j:

$$W_{ij} = \{x : \rho_{\pi_i}(x) > 0\} \cap \{x : \rho_{\pi_j(x)} > 0\}$$

Note that $\Omega \subseteq W_{ij}$. We then let $u_{ij} : (0, \infty) \times W_{ij} \to \mathbb{R}$ denote the solution of:

$$\begin{cases} (\Delta + \dfrac{\partial}{\partial t})u_{ij} = 0 \\ u_{ij}(t, x) = 0 \qquad t > 0 \quad x \in \partial W_{ij} \\ u_{ij}(0, x) = 1 \qquad x \in W_{ij} \end{cases}$$

4D.10 Lemma. *For all $t > 0$ and $x \in \Omega$:*

$$0 \leq u_{ij}(t, x) - u(t, x) \leq 2ne^{-d(x, A_{ij})^2/4nt}$$

where $A_{ij} = \underset{k \neq i, k \neq j}{\cup} \mathcal{F}_k$.

Proof. Let $v_{ij}(t, x)$ be the solution of the following initial-boundary-value problem in $\mathbb{R}^n$:

$$\begin{cases} (\Delta + \dfrac{\partial}{\partial t})v_{ij} = 0 \\ v_{ij}(0, x) = 0 \qquad x \in A_{ij}^c \\ v_{ij}(t, x) = 1 \qquad t > 0 \quad x \in A_{ij} \end{cases}$$

The following facts are easy to verify: $u_{ij} - u$ and v_{ij} are both solutions of the heat equation on Ω; they have the same initial conditions on Ω, and moreover, since $v_{ij} \geq 0$, and $0 \leq u_{ij} - u \leq 1$: $(u_{ij} - u)|_{\partial\Omega} \leq v_{ij}|_{\partial\Omega}$ for all $t > 0$. Therefore, for all t, $x \in \Omega$:

$$0 \leq u_{ij}(t, x) - u(t, x) \leq v_{ij}(t, x)$$

by standard arguments. By Levy's maximal inequality:

$$v_{ij}(t, x) \leq 2 \int_{\|y\| \geq d(x, A_{ij})} \frac{1}{(4\pi t)^{n/2}} e^{-\|y\|^2/4t} dy$$

We estimate the integral $I_n = \int_{\|y\| \geq b} \dfrac{1}{(4\pi t)^{n/2}} e^{-\|y\|^2/4t} dy$ in the following way. Using polar coordinates, we have: $I_2 = e^{-b^2/4t}$, from which it follows that:

$$\int_b^\infty \frac{1}{\sqrt{\pi t}} e^{-r^2/4t}\, dr \leq e^{-b^2/4t}$$

Now, since $\|y\| \geq b$ forces $|y_i| \geq \dfrac{b}{\sqrt{n}}$ for at least one coordinate $i \in \{1, \ldots, n\}$, we obtain $I_n \leq ne^{-b^2/4nt}$. The lemma follows. $\square$

Remark. If $\dim(\Omega) = 2$, then 4D.7 *is an immediate consequence of Lemma* 4D.8: in fact, in that case $d(x, A_{ij}) \geq \epsilon$ for all $x \in Cut_{ij}$, by our definition of ϵ, and therefore the quantity:

$$\text{(4D.8')} \qquad 2n \int_0^t \int_{\mathrm{Cut}_{ij}} e(t - \tau, \rho(x), 0) e^{-d(x, A_{ij})^2/4n\tau}\, dx\, d\tau$$

will be *exponentially decreasing* as $t \to 0$, and $C_1(i, j) = 0$. If $\dim(\Omega) > 2$, then A_{ij} will intersect Cut_{ij} in the set $\underset{k \neq i, k \neq j}{\cup} (\mathcal{F}_i \cap \mathcal{F}_j \cap \mathcal{F}_k)$ which is not empty, in general. Therefore we must proceed with the proof and show that 4D.8' is indeed $0(t^{3/2})$, as $t \to 0$.

For $x \in \pi_{ij}$, let $\rho_{ij}(x)$ stand for the *distance of x from the hyperplane* $\pi_i \cap \pi_j$ of π_{ij}. Observe that, if $x \in \mathrm{Cut}_{ij}$, then $\rho(x) = \rho_{ij}(x)\sin(\gamma_{ij}/2)$. Hence, by co-area formula, applied to $\rho_{ij} : \mathrm{Cut}_{ij} \to \mathbb{R}$:

$$\text{(4D.11)} \quad \int_{\mathrm{Cut}_{ij}} e(t-\tau, \rho(x), 0) e^{-d(x,A_{ij})^2/4n\tau}\, dx = \int_0^\infty e(t-\tau, r\sin(\gamma_{ij}/2), 0)\Big(\int_{\rho_{ij}^{-1}(r)\cap \mathrm{Cut}_{ij}} e^{-d(x,A_{ij})^2/4n\tau}\, dx\Big)\, dr$$

Next, since: $d(x, A_{ij}) = \min_{k\neq i, k\neq j} d(x, \mathcal{F}_k)$:

$$\text{(4D.12)} \quad \int_{\rho_{ij}^{-1}(r)\cap \mathrm{Cut}_{ij}} e^{-d(x,A_{ij})^2/4n\tau}\, dx \leq \sum_{k\neq i, k\neq j} \int_{\rho_{ij}^{-1}(r)\cap \mathrm{Cut}_{ij}} e^{-d(x,\mathcal{F}_k)^2/4n\tau}\, dx$$

For a fixed r, $\rho_{ij}^{-1}(r) \cap \mathrm{Cut}_{ij}$ is contained in an $(n-2)-$ hyperplane section of Ω; hence, by our definition of ϵ, we see that each term of the above sum involving an index $k \notin I_{ij}$ (that is, an index such that $(i,j,k) \notin I_3$) is majorized by:

$$\text{(4D.13)} \quad e^{-\epsilon^2/4n\tau} \cdot \mathrm{vol}(B^{n-2}(diam(\Omega)))$$

Hence it remains to examine the integrals of type:

$$\int_{\rho_{ij}^{-1}(r)\cap \mathrm{Cut}_{ij}} e^{-d(x,\mathcal{F}_k)^2/4n\tau}\, dx$$

where $k \in I_{ij}$.

First, note that $d(x, \mathcal{F}_k) \geq \rho_{\pi_k}(x)$. Now fix $r \geq 0$, and consider the $(n-2)-$dim polyhedron $Q_{ij} = \rho_{ij}^{-1}(r) \cap \mathrm{Cut}_{ij}$ which lies in a hyperplane parallel to $\pi_i \cap \pi_j$. The function ρ_{π_k}, when restricted to Q_{ij}, has gradient:

$$P_{ijk} = \text{orthogonal projection of } \nabla\rho_{\pi_k} = \nu_k \text{ onto } \pi_i \cap \pi_j$$

and $|P_{ijk}| > 0$ since, by assumption, $\mathcal{F}_i \cap \mathcal{F}_j \cap \mathcal{F}_k \neq \emptyset$, and so π_k is incident $\pi_i \cap \pi_j$. By co-area formula, applied to $\rho_{\pi_k} : Q_{ij} \to \mathbb{R}$:

$$\text{(4D.14)} \quad \begin{aligned} \int_{Q_{ij}} e^{-d(x,\mathcal{F}_k)^2/4n\tau}\, dx &\leq \int_{Q_{ij}} e^{-\rho_{\pi_k}(x)^2/4n\tau}\, dx \\ &= \frac{1}{|P_{ijk}|}\int_0^\infty e^{-s^2/4n\tau} \cdot \mathrm{vol}_{n-3}(\rho_{\pi_k}^{-1}(s) \cap Q_{ij})\, ds \\ &\leq \frac{\mathrm{vol}(B^{n-3}(\mathrm{diam}(\Omega)))}{|P_{ijk}|} \cdot \sqrt{\pi} \cdot \int_0^\infty e(\tau, s/\sqrt{n}, 0)\, ds \cdot \tau^{1/2} \\ &= \sqrt{n\pi}\frac{\mathrm{vol}(B^{n-3}(\mathrm{diam}(\Omega)))}{|P_{ijk}|}\tau^{1/2} \end{aligned}$$

Summing over $k \neq i, k \neq j$, and taking into account 4D.11-14, we obtain:

$$\begin{aligned} \int_{\mathrm{Cut}_{ij}} e(t-\tau, \rho(x), 0) e^{-d(x,A_{ij})^2/4n\tau}\, dx &\leq \\ &\leq \frac{(m-2)}{\sin(\gamma_{ij}/2)} \mathrm{vol}(B^{n-2}(\mathrm{diam}(\Omega))) \cdot e^{-\epsilon^2/4n\tau} + \\ &\quad + \left(\frac{\sqrt{n\pi}\mathrm{vol}(B^{n-3}(\mathrm{diam}(\Omega)))}{\sin(\gamma_{ij}/2)} \cdot \sum_{k\in I_{ij}} \frac{1}{|P_{ijk}|}\right) \cdot \tau^{1/2} \end{aligned}$$

Integrating the above inequality from $\tau = 0$ to $\tau = t$, and multiplying by $2n$, we obtain 4D.7 with:

$$C_1(i,j) = \frac{4n\sqrt{n\pi}}{3\sin(\gamma_{ij}/2)}\mathrm{vol}(B^{n-3}(\mathrm{diam}(\Omega)))\sum_{k\in I_{ij}}\frac{1}{|P_{ijk}|}$$

and:

$$C_2(i,j) = \frac{8n^2(m-2)}{\epsilon^2\sin(\gamma_{ij}/2)}\mathrm{vol}(B^{n-2}(\mathrm{diam}(\Omega)))$$

Proof of 4D.8. We have already observed that, when restricted to the bisecting plane π_{ij} of π_i and π_j, the function $u_{ij}(\tau,x)$ depends only on the distance $\rho_{ij}(x)$ of x from $\pi_i\cap\pi_j$; so let us set $u_{ij}(\tau,x) = \tilde{u}_{ij}(\tau,\rho_{ij}(x))$ for a function $\tilde{u}_{ij}$ of τ and $r\geq 0$. By co-area formula, applied to the map $\rho_{ij}: Cut_{ij}\to\mathbb{R}$, we have:

$$\begin{aligned}
(4\mathrm{D}.15)\quad & \int_{\mathrm{Cut}_{ij}} e(t-\tau,\rho(x),0)(1-u_{ij}(\tau,x))\,dx \\
&= \int_0^\infty e(t-\tau, r\sin(\gamma_{ij}/2),0)\left(\int_{\rho_{ij}^{-1}(r)\cap\mathrm{Cut}_{ij}}(1-u_{ij}(\tau,x))\,dx\right)dr \\
&= \int_0^\infty e(t-\tau, r\sin(\gamma_{ij}/2),0)(1-\tilde{u}_{ij}(\tau,r))\cdot\mathrm{vol}_{n-2}(\rho_{ij}^{-1}(r)\cap\mathrm{Cut}_{ij})\,dr
\end{aligned}$$

4D.16 Lemma. *For* $0<r<\dfrac{\epsilon}{\sin(\gamma_{ij}/2)}$*:*

$$|vol_{n-2}(\rho_{ij}^{-1}(r)\cap Cut_{ij}) - vol_{n-2}(\mathcal{F}_i\cap\mathcal{F}_j)|\leq M_{ij}r$$

where $M_{ij} = vol(B^{n-3}(diam(\Omega)))|\sum_{k\in I_{ij}}\cot\gamma_{ijk}|$ *and where* γ_{ijk} *is the angle between the faces* $\mathcal{F}_i\cap\mathcal{F}_j$ *and* Cut_{ijk} *of the polyhedron* Cut_{ij}*.*

Proof. See Appendix C.

Taking into account 4D.15, Lemma 4D.16 and the expression of $c_{ij}(t)$ as defined in Step 3, and writing for brevity $V_{ij}(r) = \mathrm{vol}_{n-2}(\rho_{ij}^{-1}(r)\cap\mathrm{Cut}_{ij})$, we have that:

$$\begin{aligned}
&|\int_0^t\int_{\mathrm{Cut}_{ij}} e(t-\tau,\rho(x),0)(1-u_{ij}(\tau,x))\,dx - \mathrm{vol}_{n-2}(\mathcal{F}_i\cap\mathcal{F}_j)\cdot c_{ij}(t)|\leq \\
&\leq \int_0^t\int_0^\infty e(t-\tau,r\sin(\gamma_{ij}/2),0)(1-\tilde{u}_{ij}(\tau,r))|V_{ij}(r)-V_{ij}(0)|\,dr \\
&\leq M_{ij}\int_0^t\int_0^{\epsilon/\sin(\gamma_{ij}/2)} e(t-\tau,r\sin(\gamma_{ij}/2),0)r\,dr + \\
&+\mathrm{vol}(B^{n-2}(\mathrm{diam}(\Omega)))\int_0^t\int_{\epsilon/\sin(\gamma_{ij}/2)}^\infty e(t-\tau,r\sin(\gamma_{ij}/2),0)\,dr \\
&\leq \frac{4M_{ij}}{3\sqrt{\pi}\sin^2(\gamma_{ij}/2)}t^{3/2} + \frac{4}{\epsilon^2\sin(\gamma_{ij}/2)}\mathrm{vol}(B^{n-2}(\mathrm{diam}(\Omega)))t^2e^{-\epsilon^2/4t}
\end{aligned}$$

the last inequality following by a change of variable, by 4.8, and by the fact that $\int_\epsilon^\infty e(t-\tau,s,0)\,ds\leq e^{-\epsilon^2/4(t-\tau)}$ (see the proof of Lemma 4D.10). Hence 4D.8 holds with:

$$C_3(i,j) = \frac{4}{3\sqrt{\pi}\sin^2(\gamma_{ij}/2)}\mathrm{vol}(B^{n-3}(\mathrm{diam}(\Omega)))|\sum_{k\in I_{ij}}\cot\gamma_{ijk}|$$

$$C_4(i,j) = \frac{4}{\epsilon^2\sin(\gamma_{ij}/2)}\mathrm{vol}(B^{n-2}(\mathrm{diam}(\Omega)))$$

Remark. If $\dim(\Omega) = 2$, then $V_{ij}(r) - V_{ij}(0) = 0$ for $0 < r < \epsilon$, and therefore we see that in that case $C_3(i,j) = 0$.

Proof of 4D.9. Explicit computation of $c_{ij}(t)$. We make use of the Laplace transform with respect to time, and our notation is the following: if f is a function of t then its Laplace transform at $s > 0$ will be written with capital letters:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

By well-known convolution properties, we then have, for the Laplace transform $C_{ij}(s)$ of $c_{ij}(t)$ (as in Step 3):

$$C_{ij}(s) = s^{-1/2}\int_0^\infty e^{-\sqrt{s}r\sin(\gamma_{ij}/2)}(\frac{1}{s} - \tilde{U}_{ij}(s,r))\,dr$$

We will write down an explicit expression of $\frac{1}{s} - \tilde{U}_{ij}(s,r)$. First observe that W_{ij} is isometric with $W(\gamma_{ij}) \times \mathbb{R}^{n-2}$ (with the product metric), if we denote by $W(\gamma_{ij})$ the *open wedge in* $\mathbb{R}^2$ *with interior angle* γ_{ij}. We adopt cylindrical coordinates $x = (r,\alpha,y)$ where (r,α) are polar coordinates in $W(\gamma_{ij})$ (the angle α being counted from the bisectrix of γ_{ij}), and where $y \in \mathbb{R}^{n-2}$. In these coordinates the temperature function $u_{ij}(t,x)$ is independent from y, hence it can be written, by a slight abuse of language, as $u_{ij}(t,r,\alpha)$. Note that $\tilde{u}_{ij}(t,r) = u_{ij}(t,r,0)$.

The following lemma was suggested by the expression of the Green function of an open wedge in $\mathbb{R}^2$ as a Kontorovich-Lebedev transform (which we learned from [3]).

4D.17 Lemma. *Let $W(\gamma)$ be the open wedge in $\mathbb{R}^2$ with interior angle γ, and let (r,α) be polar coordinates with $\alpha \in (-\gamma/2, \gamma/2)$ being counted from the bisectrix of γ. Let $u(t,r,\alpha)$ be the solution of:*

$$\begin{cases} (\Delta + \frac{\partial}{\partial t})u = 0 \\ u(0,r,\alpha) = 1 \qquad r > 0, \quad \alpha \in (-\gamma/2,\gamma/2) \\ u(t,r,\pm\gamma/2) = 0 \qquad t > 0, r > 0 \end{cases}$$

and let $U(s,r,\alpha) = \int_0^\infty u(t,r,\alpha)e^{-st}\,dt$. Then:

$$U(s,r,\alpha) = \frac{1}{s} - \frac{2}{\pi s}\int_0^\infty K_{ix}(\sqrt{s}r)\frac{\cosh(\frac{\pi x}{2})\cosh(\alpha x)}{\cosh(\frac{\gamma x}{2})}\,dx$$

*where K_{ix} is the modified Bessel function of imaginary argument (*see [13], 8.407.1)

Proof. In polar coordinates $-\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2}$. Hence the function $U(s,r,\alpha)$ must satisfy, on the open wedge $W(\gamma)$, the boundary-value problem:

$$\begin{cases} \Delta U = 1 - sU \\ U(s,r,\pm\gamma/2) = 0 \qquad \text{for all} \quad s > 0, r > 0 \end{cases}$$

Now U is a solution of $\Delta U = 1 - sU$ by [13] (formula 8.491.6), and satisfies the given boundary conditions by [13] (formula 6.794.2). The lemma follows. □

We now come to the computation of $C_{ij}(s)$. From Lemma 4D.17:

$$\tilde{U}_{ij}(s,r) = U_{ij}(s,r,0) = \frac{1}{s} - \frac{2}{\pi s}\int_0^\infty K_{ix}(\sqrt{s}r)\frac{\cosh(\frac{\pi x}{2})}{\cosh(\frac{\gamma_{ij}x}{2})}\,dx$$

hence:

$$C_{ij}(s) = \frac{2}{\pi s^{3/2}}\int_0^\infty \frac{\cosh(\frac{\pi x}{2})}{\cosh(\frac{\gamma_{ij}x}{2})}\left(\int_0^\infty e^{-\sqrt{s}r\sin(\gamma_{ij}/2)}K_{ix}(\sqrt{s}r)\,dr\right)\,dx$$

The inner integral, after the substitution $z = \sqrt{s}r$, will become (see [13], formula 6.611.3):

$$\frac{\pi}{2\cos(\gamma_{ij}/2)s^{1/2}}\left(\frac{\cosh(\frac{\gamma_{ij}x}{2})}{\cosh(\frac{\pi x}{2})} - \frac{\sinh(\frac{\gamma_{ij}x}{2})}{\sinh(\frac{\pi x}{2})}\right)$$

substituting, and changing $\frac{x}{2}$ to x, we then obtain:

$$C_{ij}(s) = \frac{2}{\cos(\gamma_{ij}/2)}\cdot\int_0^\infty\left(1 - \frac{\tanh(\gamma_{ij}x)}{\tanh(\pi x)}\right)\,dx\cdot\frac{1}{s^2}$$

Taking inverse Laplace transform, we obtain (4D.9).

The remainder term. From 4D.6-9, and the expression of the constants $C.(i,j)$, we have:

$$|H(t) - \mathrm{vol}(\Omega) + \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} - 2\sum_{(i,j)\in I_2}\mathrm{vol}_{n-2}(\mathcal{F}_i\cap\mathcal{F}_j)\cdot c_{ij}\cdot t| \le Ct^{3/2} + g(t)$$

where:

$$C = \frac{4n\sqrt{n\pi}}{3}\mathrm{vol}(B^{n-3}(\mathrm{diam}(\Omega)))\sum_{(i,j,k)\in I_3}\frac{1}{|P_{ijk}|}\cot(\gamma_{ij}/2) + \\ + \frac{4}{3\sqrt{\pi}}\mathrm{vol}(B^{n-3}(\mathrm{diam}(\Omega)))|\sum_{(i,j,k)\in I_3}\cot(\gamma_{ijk})|\frac{\cos(\gamma_{ij}/2)}{\sin^2(\gamma_{ij}/2)}|$$

with P_{ijk} and γ_{ijk} as in the proof of 4D.7 and 4D.8, respectively; and:

$$g(t) = \alpha_1 t^{3/2}e^{-\epsilon^2/4t} + \alpha_2 t^2 e^{-\epsilon^2/4t} + \alpha_3 t^2 e^{-\epsilon^2/4nt}$$

with:

$$\alpha_1 = \frac{4}{\sqrt{\pi}\epsilon^2}\mathrm{vol}(\partial\Omega)$$

$$\alpha_2 = \frac{4}{\epsilon^2}\mathrm{vol}(B^{n-2}(\mathrm{diam}(\Omega)))\sum_{(i,j)\in I_2}\cot(\gamma_{ij}/2)$$

$$\alpha_3 = \frac{8n^2(m-2)}{\epsilon^2}\mathrm{vol}(B^{n-2}(\mathrm{diam}(\Omega)))\sum_{(i,j)\in I_2}\cot(\gamma_{ij}/2)$$

If $\dim(\Omega) = 2$, then $C = 0$ and, looking back at the proofs of Steps 1-4, $g(t)$ can be reduced to the following form:

$$g(t) = \frac{4}{\sqrt{\pi}\epsilon^2}\mathrm{vol}(\partial\Omega)t^{3/2}e^{-\epsilon^2/4t} + \frac{12}{\epsilon^2}\sum_{(i,j)\in I_2}\cot(\gamma_{ij}/2)t^2e^{-\epsilon^2/4t}$$

With this, the proof of Theorem 4D.1 is complete.

Heat content asymptotics of a polygonal domain in the plane. (Compare with [3]).

Now let Ω be a (not necessarily convex) polygonal domain in $\mathbb{R}^2$. We show that the coefficient of the term of order t in the asymptotic expansion of the heat content is $4\sum_P c_\gamma$, with:

$$c_\gamma = \int_0^\infty \left(1 - \frac{\tanh(\gamma x)}{\tanh(\pi x)}\right) dx$$

and where γ denotes the interior angle at the vertex P of Ω ($0 < \gamma \leq 2\pi$).

Take a sufficiently small, positive number ϵ, and, on each sector $B(P,\epsilon)\cap\Omega$, approximate the temperature function $u(t,x)$ by $u_P(t,x)$ (the temperature on the infinite open wedge with vertex in P and angle γ): then, proceeding as in Lemma 4D.10, the error in the approximation will be bounded by an exponentially decreasing function ot t, as $t \to 0$.

There are two cases to examine: when the vertex P is *convex* ($0 < \gamma < \pi$), and when it is *concave* ($\pi < \gamma \leq 2\pi$). The contribution to the asymptotics of the heat content when the vertex is convex is $4c_\gamma t$, as we proved in Theorem 4D.1 applied to $n = 2$; it then remains to determine the contribution of concave vertices. Now, near a concave vertex, we have $\Delta_{cut}\rho = 0$, and the level curves of the distance function are C^1 curves given by the union of the two segments parallel to the two sides meeting at P, and an arc of circle of angle $\gamma - \pi$. Precisely, in polar coordinates (r,α) centered at P, with the angle α being counted from the bisectrix of γ, we have, for $0 < r < \epsilon$:

$$\Delta_{reg}\rho(r,\alpha) = \begin{cases} -\dfrac{1}{r} & \text{if} \quad -\dfrac{\gamma-\pi}{2} < \alpha < \dfrac{\gamma-\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

hence the vertex contribution of P is, up to exponentially decreasing terms, given by:

$$\int_0^t\int_0^\infty e(t-\tau,r,0)\rho_*((1-u_P(\tau,\cdot))\Delta_{reg}\rho)\,dr\,d\tau$$
$$= -\int_0^t\int_0^\infty e(t-\tau,r,0)\int_{-(\gamma-\pi)/2}^{(\gamma-\pi)/2}(1-u_P(\tau,r,\alpha))\,d\alpha\,dr\,d\tau$$

Its Laplace transform with respect to time t, at $s > 0$, is, thanks to Lemma 4D.17:

$$-\frac{2}{\pi s^{3/2}}\int_0^\infty e^{-\sqrt{s}r}\int_{-(\gamma-\pi)/2}^{(\gamma-\pi)/2}\int_0^\infty K_{ix}(\sqrt{s}r)\frac{\cosh(\frac{\pi x}{2})\cosh(\alpha x)}{\cosh(\frac{\gamma x}{2})}\,dx\,d\alpha\,dr$$

which can be evaluated using which can be evaluated again by [13], formula 6.611.3. One finds its value to be $\frac{4}{s^2}c_\gamma$. Taking inverse Laplace transform, we obtain, also in this case, the vertex contribution $4c_\gamma t$.

The remainder term of the asymptotic expansion of the heat content will be an exponentially decreasing function of t, as $t \to 0$, which depends on ϵ, on $\mathrm{vol}(\partial\Omega)$, and on the angles γ; it can be easily computed by the same methods used in Theorem 4D.1. We omit the details.

Appendix A.

Lemma. *On any Riemannian manifold M, if ρ is the distance function to a submanifold N, one has that $\Delta_{reg}\rho \in L^1_{loc}(M)$.*

Proof. . We have to show that, if $K \subseteq M$ is compact, then $\int_K |\Delta_{reg}\rho|$ is finite. Let R be large enough so that $K \subseteq \rho^{-1}[0, R)$. Then, integrating in normal coordinates:

$$\int_K |\Delta_{reg}\rho| \leq \int_{U(N)} \int_0^{\min\{c(\xi),R\}} |\theta'(r,\xi)|\, dr\, d\xi$$

Hence it is enough to show that $|\theta'|$ is bounded on $\{(r,\xi) : 0 < r < \min\{c(\xi), R\}, \xi \in U(N)\}$. Let us consider the map $\Phi : (0,\infty) \times U(N) \to M$ defined by $\Phi(r,\xi) = \exp_{\pi(\xi)} r\xi$. As Φ is everywhere C^∞, its Jacobian determinant $\theta(r,\xi) = \dfrac{\Phi^*(dv_n)}{dr \wedge d\xi}$ (where dv_n and $d\xi$ are the canonical volume forms of M and $U(N)$, respectively) is also everywhere C^∞. Now:

$$\lim_{r\to 0} \theta'(r,\xi) = \begin{cases} 0 & \text{if} \quad \dim(N) \leq n-3 \\ 1 & \text{if} \quad \dim(N) = n-2 \\ -\displaystyle\sum_{1\leq i\leq n-1} \eta_i(\xi) \qquad \text{if} \quad \dim(N) = n-1 \end{cases}$$

where $\eta_i(\xi)$ is the i-th principal curvature of N at the unit normal vector ξ. From these facts, we deduce that $|\theta'(r,\xi)|$ is indeed locally bounded on $(0,\infty) \times U(N)$, and then that $\int_K |\Delta_{reg}\rho|\, dv_n$ is finite, as asserted.

In addition, the comparison theorems of Rauch and R.L. Bishop may be used to produce upper and lower bounds of $\theta'(r,\xi)$ in terms of lower and upper bounds of the sectional (or Ricci) curvatures of M.

We observe, in particular, that if N is a $p-$dimensional submanifold of $\mathbb{R}^n$, then:

$$\theta(r,\xi) = r^{n-p-1} \prod_{i=1}^{p} (1 - r\eta_i(\xi))$$

Appendix B

The scope of this appendix is to prove Lemma 2.2, Corollary 2.3, and also to give an alternative description of the singular Laplacian of the distance function. We refer to [6] (§13.2) for the definition of Hausdorff measures we use here.

Proof of Lemma 2.2. By the definition of Hausdorff measure and our assumptions, we can find, for each $\epsilon > 0$, a finite or countable covering of K by sets $E_i(\epsilon), i = 1, 2, \ldots$, each of diameter not exceeding ϵ, satisfying:

$$\text{(B.1)} \qquad \sum_i (\operatorname{diam}(E_i(\epsilon)))^{n-1} \leq \frac{2^{n-1}}{\operatorname{vol}(B^{n-1})} \cdot H_{n-1}(K) + \epsilon$$

For each i and each ϵ, pick a point $x \in E_i(\epsilon) \cap K$; then the open ball $B_i(\epsilon)$ with center x and radius $\delta \operatorname{diam}(E_i(\epsilon))$, where $1 < \delta < 2$, contains $E_i(\epsilon)$. K being compact, there is $k(\epsilon)$ such that $K \subseteq \cup_{i=1}^{k(\epsilon)} B_i(\epsilon) \doteq V(\epsilon)$. Note that $V(\epsilon)$ covers K and is contained in a $2\epsilon-$neighborhood of K, and that $\partial V(\epsilon)$ is piecewise smooth. Fix an open neighborhood W of K, and let $\epsilon_0 > 0$ be a number such that $V(\epsilon) \subseteq W$ when $\epsilon < \epsilon_0$.

Claim. Assume that Ricci $\geq -(n-1)\alpha^2 g$ on W. Then there exists a positive constant C_1 depending only on α, ϵ_0, such that:

$$\text{(B.2)} \qquad \frac{\mathrm{vol}(\partial B_i(\epsilon))}{(\mathrm{radius}(B_i(\epsilon)))^{n-1}} \leq \mathrm{vol}(\partial B^{n-1}) + C_1\,\epsilon \qquad \forall i = 1, \dots, k(\epsilon) \qquad \forall \epsilon < \epsilon_0$$

Proof of claim. Using Bishop comparison theorem one argues that, if $B(x,r)$ is *any* ball contained in W, then:

$$\mathrm{vol}(\partial B(x,r)) \leq \mathrm{vol}(\partial B_{-\alpha^2}(r))$$

where $B_{-\alpha^2}(r)$ is the ball of radius r in the simply connected manifold of constant sectional curvature $\sigma = -\alpha^2$. Hence it is enough to prove the claim in that case. Now a classical formula states that: $\mathrm{vol}(\partial B_{-\alpha^2}(r)) = \mathrm{vol}(\partial B^{n-1})(\frac{1}{\alpha}\sinh(\alpha r))^{n-1}$. Write: $\sinh(\alpha r) = \alpha r(1 + \psi_\alpha(r) r)$ with $\psi_\alpha(r)$ smooth and positive for $r \geq 0$. Then:

$$\frac{\mathrm{vol}(\partial B_{-\alpha^2}(r))}{r^{n-1}} \leq \mathrm{vol}(\partial B^{n-1}) + C_1\, r$$

with: $C_1 = \dfrac{1}{2}\mathrm{vol}(\partial B^{n-1}) \sup\limits_{0 \leq r \leq \epsilon_0} \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \psi_\alpha(r)^i r^{i-1} \right)$

Now:

$$\begin{aligned} \mathrm{vol}(\partial V(\epsilon)) &\leq \sum_{i=1}^{k(\epsilon)} \mathrm{vol}(\partial B_i(\epsilon)) \\ &\leq \delta^{n-1} \sum_{i=1}^{k(\epsilon)} \frac{\mathrm{vol}(\partial B_i(\epsilon))}{(\mathrm{radius}(B_i(\epsilon)))^{n-1}} \cdot (\mathrm{diam}(E_i(\epsilon)))^{n-1} \end{aligned}$$

and we get the assertion by B.1, B.2 and the fact that δ was arbitrary. Proof is complete. $\square$

Before giving the proof of Corollary 2.3, we give the following *alternative description of* $\Delta_{cut}\rho$ (see also [9], Lemma 3.3.5). Assume that $\mathrm{supp}\,\phi \subseteq \rho^{-1}[0,R)$. Let $\{W(\epsilon), \epsilon > 0\}$ be any family of open sets with piecewise smooth boundary which cover $\mathrm{Cut}(N) \cap \rho^{-1}[0,R)$ and shrink to zero volume: $\lim_{\epsilon\to 0} \mathrm{vol}\, W(\epsilon) = 0$. Then:

$$(*) \qquad \langle \Delta_{cut}\rho, \phi \rangle = \lim_{\epsilon \to 0} \int_{\partial W(\epsilon)} \phi \frac{\partial \rho}{\partial \nu(\epsilon)}$$

where $\nu(\epsilon)$ is the unit normal to $\partial W(\epsilon)$, pointing inside $W(\epsilon)$.
For the proof, observe first that, if ϵ is small, $\rho^{-1}[0,\epsilon)$ has smooth boundary and covers N. Then let $V(\epsilon)$ be the interior of the set $W(\epsilon)^c \cap \rho^{-1}(\epsilon, \infty)$. Clearly $V(\epsilon)$ has piecewise smooth boundary and satisfies:

$V(\epsilon) \subseteq \Phi(U) \cap \rho^{-1}[0,R)$, and:

$\lim_{\epsilon\to 0} \mathrm{vol}(V(\epsilon)) = \mathrm{vol}(M(R))$

Therefore:

$$
\begin{aligned}
\langle \Delta\rho, \phi\rangle &= \int_{\rho^{-1}[0,R)} \nabla\rho\cdot\nabla\phi \\
&= \lim_{\epsilon\to 0}\int_{V(\epsilon)} \nabla\rho\cdot\nabla\phi \\
&= \lim_{\epsilon\to 0}\int_{V(\epsilon)} \phi\,\Delta_{reg}\rho - \int_{\partial V(\epsilon)} \phi\,\frac{\partial\rho}{\partial n(\epsilon)}
\end{aligned}
$$

where $n(\epsilon)$ denotes the inward unit normal to $\partial V(\epsilon)$. Now, since $\partial M(\epsilon) = \rho^{-1}(\epsilon)$, we can write:

$$
\int_{\partial V(\epsilon)} \phi\frac{\partial\rho}{\partial n(\epsilon)} = -\int_{\rho^{-1}(\epsilon)}\phi - \int_{\partial W(\epsilon)} \phi\frac{\partial\rho}{\partial\nu(\epsilon)}
$$

we then substitute in the above expression and pass to the limit as $\epsilon\to 0$: note that $\lim_{\epsilon\to 0}\displaystyle\int_{V(\epsilon)}\phi\,\Delta_{reg}\rho = \int_M \phi\,\Delta_{reg}\rho$ by Lebesgue's dominated convergence theorem, hence the limit in (*) exists and is indipendent from the sequence $W(\epsilon)$.

Proof of Corollary 2.3. Fix $\phi\in C^0_c(M)$ with $supp\,\phi\subseteq K$, and consider the sequence of $V(\epsilon)$ given by Lemma 2.2, when applied to $K\cap \mathrm{Cut}(N)$. By the description (*) of $\Delta_{cut}\rho$:

$$
\int_K \phi\Delta_{cut}\rho = \lim_{\epsilon\to 0}\int_{\partial V(\epsilon)} \phi\frac{\partial\rho}{\partial\nu(\epsilon)}
$$

The Corollary follows immediately. $\square$

Appendix C

In this appendix we prove Lemma 4D.16, which is in fact a consequence of the following more general:

Lemma. *Let: P = polytope in $\mathbb{R}^d$ with faces $\mathcal{F}_k$, $k=1,\dots,N$; π = hyperplane not intersecting the interior set of P; $\gamma_k = \mathrm{angle}(\nu,\mathcal{F}_k)$, where ν is the unit normal to π, oriented toward P; $\delta = \inf\, dist(\mathcal{F}_k,\pi)$, where the infimum is taken over all indices k such that $\mathcal{F}_k$ is not incident π; $\rho_\pi:\mathbb{R}^d\to\mathbb{R}$: distance from π. Then the map:*

$$
V(r) = vol_{d-1}(P\cap\rho_\pi^{-1}(r))
$$

is differentiable on $(0,\delta)$ and in fact, for $0<r<\delta$:

$$
V'(r) = -\sum_{k=1}^m \cot\gamma_k\cdot vol_{d-2}(\rho_\pi^{-1}(r)\cap\mathcal{F}_k)
$$

where $\mathcal{F}_1,\dots,\mathcal{F}_m$ are the faces incident with π.

Proof. For $0<r<\delta$ the hyperplane $\rho_\pi^{-1}(r)$ will intersect ∂P only in the faces $\mathcal{F}_1,\dots,\mathcal{F}_m$ and the section $\rho_\pi^{-1}(r)\cap P$ will be bounded by the $(d-2)-$dimensional faces $\mathcal{F}_1\cap\rho_\pi^{-1}(r),\dots\mathcal{F}_m\cap\rho_\pi^{-1}(r)$. Consider the strip:

$$
P(r,h) = P\cap\{r\le\rho_\pi\le r+h\}
$$

where h is small and positive. Since ρ_π is an affine function, we have, by Green's theorem:

$$(*) \qquad 0=\int_{P(r,h)}\Delta\rho_\pi = \mathrm{vol}_{d-1}(\rho_\pi^{-1}(r)\cap P) - \mathrm{vol}_{d-1}(\rho_\pi^{-1}(r+h)\cap P) + \sum_{k=1}^{m}(\nabla\rho_\pi\cdot\nu_k)\,Q_k(h)$$

where $Q_k(h)=\mathrm{vol}_{d-1}(\mathcal{F}_k\cap\{r\le\rho_\pi\le r+h\})$. Then:

$$V'(r)_+ = -\sum_{k=1}^{m}\cos\gamma_k(\lim_{h\to 0}\frac{1}{h}Q_k(h))$$

If $\tilde\rho_\pi$ denotes the restriction of ρ_π to $\mathcal{F}_k$, we have, by co-area formula:

$$Q_k(h)=\int_r^{r+h}(\int_{\rho_\pi^{-1}(s)\cap\mathcal{F}_k}\frac{1}{|\nabla\tilde\rho_\pi|}\,dx)\,ds$$

But $\nabla\tilde\rho_\pi$ is the orthogonal projection of $\nabla\rho_\pi$ onto $\mathcal{F}_k$, and therefore $|\nabla\tilde\rho_\pi|=\sin\gamma_k$. We now pass to the limit as $h\to 0_+$ and insert in the expression $*$. The case $h\to 0_-$ is treated similarly. $\square$

Proof of Lemma 4D.16. We let $P=\mathrm{Cut}_{ij}$, and $\pi=\pi_i\cap\pi_j$ in the Lemma. Then $d=n-1$, $\rho_\pi=\rho_{ij}$ and $V(r)=\mathrm{vol}_{n-2}(\rho_{ij}^{-1}(r)\cap Cut_{ij})$. The faces of P incident π are then all polyhedrons Cut_{ijk} with $(i,j,k)\in I_3$. Moreover, if $x\in\mathrm{Cut}_{ijk}$, and $(i,j,k)\notin I_3$, then:

$$\rho_{ij}(x)=\frac{\rho(x)}{\sin(\gamma_{ij}/2)}\ge\frac{\epsilon}{\sin(\gamma_{ij}/2)}$$

by our definition of ϵ (see Lemma 4D.5(ii)). Hence $\delta\ge\dfrac{\epsilon}{\sin(\gamma_{ij}/2)}$ and Lemma 4D.16 follows easily.

Appendix D

The scope of this appendix is to show that, in any Riemannian manifold, the cut-locus of a piecewise-smooth submanifold is a set of zero measure in the manifold.

Let N be a compact subset of a complete Riemannian manifold M. We say that N is a *piecewise-smooth submanifold of M* if N is the disjoint union of a finite family $\mathcal{I}$ of smooth, open submanifolds N_i of dimension $0\le n_i\le n-1$. Let $\rho: M\to\mathbb{R}$ be the distance function from N. Then ρ is Lipschitz.
For each $i\in\mathcal{I}$, let $\mathcal{R}_i$ denote the *maximal open subset of the set of all $x\in M$ for which there is a unique geodesic from x to N minimizing the distance from N, and the foot of this geodesic belongs to N_i.*

Then let:

$$\mathcal{R}=\cup_i\mathcal{R}_i$$

It is clear that, when restricted to $\mathcal{R}_i$, ρ coincides with the smooth function $\rho_{N_i}=$ distance from N_i; moreover ρ is $C^\infty-$smooth on $\mathcal{R}$, and the *regular Laplacian of ρ,* defined on $\mathcal{R}$ by $\Delta_{reg}\rho=\Delta(\rho|_{\mathcal{R}})$, satisfies:

$$\Delta_{reg}\rho|_{\mathcal{R}_i}\circ\Phi_i=-\frac{\theta'_{N_i}}{\theta_{N_i}}$$

where Φ_i is the normal chart relative to N_i, sending $(\xi,r)\in U(N_i)\times(0,\infty)$ to $\exp_{\pi(\xi)}r\xi\in M$, and θ_{N_i} is its Jacobian.

We now come to the main theorem of this appendix:

D.1 Theorem. *The complement of the open set $\mathcal{R}$ of regular points of ρ is of zero measure in M.*

First, we define a surrogate of the "unit normal bundle" of N, N being a compact subset of M.

Let $Un(M)$ be the unit tangent bundle of M, and let $\pi : Un(M) \to M$ be its canonical projection. The cut-radius map:

$$c : \pi^{-1}(N) \to [0, \infty]$$

is defined in the usual way (see §1; no property of continuity is needed at this point). We set:

$$\mathcal{U}(N) = \{\xi \in \pi^{-1}(N) : c(\xi) > 0\}$$

hence $\mathcal{U}(N)$ consists of all unit vectors which are based at points of N, and for which the corresponding geodesic minimizes the distance from N on a segment of positive length. $\mathcal{U}(N)$ does indeed coincide with $U(N)$ when N happens to be a smooth submanifold of M. The normal chart:

$$\Phi : \mathcal{U}(N) \times (0, \infty) \to M$$

where $\Phi(\xi, r) = \exp_{\pi(\xi)} r\xi$ is easily seen to be surjective on $M \setminus N$ and continuous.

Now assume that N is a piecewise-smooth submanifold; then $U(N_i)$ is an open, smooth submanifold of $Un(M)$ of dimension $n-1$, having piecewise-smooth boundary. Set, for each i in the index set $\mathcal{I}$:

$$\mathcal{U}(N_i) = \{\xi \in U(N_i) : c(\xi) > 0\}$$

Note that $\mathcal{U}(N) = \cup_i \mathcal{U}(N_i)$ since, if $\pi(\xi) \in N_i$ and $c(\xi) > 0$, then ξ must be normal to N_i. Now set:

$$\mathcal{U}_{reg}(N) = \cup_i \mathcal{U}_i$$

where $\mathcal{U}_i$ is the *largest open subset of* $U(N_i)$ *contained in* $\mathcal{U}(N_i)$. It follows that $\mathcal{U}_{reg}(N)$ is a smooth, open submanifold of $Un(M)$ of dimension $n-1$; it reduces to $U(N)$ if N is smooth.

We will prove Theorem D.1 by applying the classical proof with $\mathcal{U}_{reg}(N)$ replacing $U(N)$. We first show that $\mathcal{U}(N) \setminus \mathcal{U}_{reg}(N)$ is, for our purposes, a negligible set.

D.2 Proposition. *We have:*

$$\mathcal{U}(N) = \mathcal{U}_{reg}(N) \cup \mathcal{U}_{sing}(N) \quad \textit{(disjoint union)}$$

and $\mathcal{U}_{sing}(N)$ is contained in a $(n-2)-$dimensional submanifold of $Un(M)$.

Proof. We show that, in fact, $\mathcal{U}_{sing}(N) \subseteq \cup_j(\partial U(N_j))$. Let $\xi \in \mathcal{U}_{sing}(N)$, say $\xi \in \mathcal{U}(N_i) \setminus \mathcal{U}_{reg}(N)$. If $\pi(\xi) \in \partial N_i$ we are done, since then $\xi \in \partial U(N_i)$. Hence assume $\pi(\xi) \in N_i$: our aim is to show that then $\xi \in \partial U(N_j)$ for some $j \neq i$. Fix r so that $0 < r < c(\xi)$, and let $x = \Phi(\xi, r)$. The assumption $r < c(\xi)$ implies that x can't be a focal point of N_i along the geodesic $t \to \Phi(\xi, t)$. Hence the normal map $\Phi = \Phi_i : U(N_i) \times (0, \infty) \to M$ is *locally 1-1 near the regular point* (ξ, r). The assumption $\xi \in \mathcal{U}(N_i) \setminus \mathcal{U}_i$ implies the existence of a sequence of vectors $\xi_n \in U(N_i) \setminus \{\xi\}$ such that $\xi_n \to \xi$ as $n \to \infty$, and $c(\xi_n) = 0$, i.e. $\rho(\Phi(\xi_n, t)) < t$ for all $t > 0$. Let $x_n = \Phi(\xi_n, r)$; for each n, there exists $\xi'_n \in \mathcal{U}(N)$ ($\xi'_n \neq \xi_n$), and $r_n < r$ such that $x_n = \Phi(\xi'_n, r_n)$. *We claim that, for n large, $\xi'_n \notin \overline{\mathcal{U}(N_i)}$.*In fact, assume that there exists a subsequence $\{\xi'_{n_k}\} \subseteq \overline{\mathcal{U}(N_i)}$. It must accumulate to a vector $\xi' \in \overline{\mathcal{U}(N_i)}$. Correspondingly, r_{n_k} accumulates to a number $s \leq r$. Now since $x_n \to x$, we see that $\Phi(\xi'_n, r_n) \to x$, so that $\Phi(\xi', s) = \Phi(\xi, r)$ with $s \leq r$. Since, by assumption, r is the minimum distance of x from N, we have necessarily $s = r$, i.e. $r_{n_k} \to r$. Now if $\xi' \neq \xi$, we would have two distinct minimizing geodesics from N to x, and this is impossible since otherwise the geodesic $t \to \Phi(\xi, t)$ would not minimize distance past r. On the other hand, if $\xi' = \xi$ both (ξ'_{n_k}, r_{n_k}) and (ξ_{n_k}, r) converge to (ξ, r), and this is incompatible with the fact that Φ is locally 1-1 near (ξ, r), since $\Phi(\xi'_{n_k}, r_{n_k}) = \Phi(\xi_{n_k}, r)$. The claim is then proved.

Hence, for n large, $\xi'_n \in \cup_{j \neq i} \overline{\mathcal{U}(N_j)}$, a compact set. Pick any accumulation point ξ' of $\{\xi'_n\}$ and assume $\xi' \in \overline{\mathcal{U}(N_j)}$. Reasoning as before, we see that $\xi' \neq \xi$ is impossible, and so $\xi' = \xi$, i.e. $\xi \in \overline{\mathcal{U}(N_j)} \subseteq \overline{U(N_j)}$, with $j \neq i$. If $\xi \in U(N_j)$, then $\pi(\xi) \in N_j$; but also $\pi(\xi) \in N_i$ and $j \neq i$: impossible. Hence, necessarily $\xi \in \partial U(N_j)$. $\square$

D.3 Proposition. *Let $\xi \in \mathcal{U}_{reg}(N)$. If $\Phi(\xi, a) = \exp_{\pi(\xi)} a\xi$ is the cut-point along the geodesic $t \to \Phi(\xi, t)$, then $\Phi(\xi, s) \in \mathcal{R}$ for all $0 < s < a$. Moreover, we have one (or both) of the following alternatives:*
(i) if $\xi \in \mathcal{U}_i$, then $\Phi(\xi, a)$ is the first focal point of N_i along $t \to \Phi(\xi, t)$;
(ii) there are at least two minimizing geodesics from N to $\Phi(\xi, a)$.

Proof. The proof is classical, and, with the obvious changes, it is equal to the proof of Theorem 4.2 in [16].

D.4 Proposition. *Let $c : \mathcal{U}_{reg}(N) \to [0, \infty]$ be the cut-radius map. Then c is continuous.*

Proof. Imitate the proof of Theorem 4.3 in [16].

D.5 Proposition. $M \setminus \mathcal{R} = \Phi(graph(c)) \cup N \cup \Phi(\mathcal{F})$
with: $\mathcal{F} = \{(\xi, r) \in \mathcal{U}_{sing}(N) \times (0, \infty) : 0 < r \leq c(\xi)\}$

Proof. Since Φ is surjective, if $x \in M \setminus \mathcal{R}$, and $x \notin N$, then $x = \Phi(\xi, r)$, for some $\xi \in \mathcal{U}(N)$, $0 < r \leq c(\xi)$. If $\xi \in \mathcal{U}_{sing}(N)$, then $x \in \Phi(\mathcal{F})$. On the other hand, if $\xi \in \mathcal{U}_{reg}(N)$, then $r = c(\xi)$, otherwise $x \in \mathcal{R}$, by Proposition D.3. Hence in that case $x \in \Phi(graph(c))$. $\square$

Proof of Theorem D.1. Since $c : \mathcal{U}_{reg}(N) \to (0, \infty)$ is continuous, $graph(c)$ has zero measure in $\mathcal{U}_{reg} \times (0, \infty)$ by Fubini's theorem, hence $\Phi(graph(c))$ has zero measure in M; similarly, since $\mathcal{U}_{sing}(N)$ is contained in an $(n-2)-$ dimensional manifold, the set $\mathcal{F}$ is contained in an $(n-1)-$dimensional submanifold of $Un(M) \times (0, \infty)$ hence also $\Phi(\mathcal{F})$ has zero measure in M. Theorem D.1 then follows from Proposition D.5. $\square$

We let Cut(N) be the closure of $\Phi(graph(c))$ in M. Then Cut(N) is a subset of $M \setminus \mathcal{R}$, and as such it has measure zero. As for $\Phi(\mathcal{F})$, this set consists of all points $\Phi(\xi, r)$, $0 < r \leq c(\xi)$, with ξ in the overlap of two different pieces $\overline{\mathcal{U}(N_i)}$ and $\overline{\mathcal{U}(N_j)}$ of the "unit normal bundle" $\mathcal{U}(N)$. If $c(\xi) < r$, then ρ is C^1 at $\Phi(\xi, r)$, but not C^2. The reader is invited to draw a picture of the situation when N is, for example, a triangle in the plane.

For a piecewise-smooth submanifold, integration in normal coordinates is the following formula:

$$\int_M f = \sum_i \int_{\mathcal{R}_i} f = \sum_i \int_{\mathcal{U}_i} \int_0^{c(\xi)} f(\Phi(r, \xi)) \theta_{N_i}(r, \xi) \, dr \, d\xi$$

and Lemma 1.4 becomes the following:

D.6 Lemma. *Let N be a piecewise-smooth submanifold of M, and let ρ be the distance function from N. Let $\Delta\rho$ be the distributional Laplacian of ρ. Then:*

$$\Delta\rho = \Delta_{reg}\rho + \Delta_{cut}\rho - 2T$$

where:

$$\Delta_{reg}\rho|_{\mathcal{R}_i} \circ \Phi_i = -\frac{\theta'_{N_i}}{\theta_{N_i}}; \qquad \langle T, \phi \rangle = \sum_{\{N_i : codim(N_i) = 1\}} \int_{N_i} \phi \, dv_{n-1}$$

and where $\Delta_{cut}\rho$ is the positive Radon measure defined by:

$$\langle \Delta_{cut}\rho, \phi \rangle = \sum_i \int_{\mathcal{U}_i} \theta_{N_i}(\xi, c(\xi)) \cdot \phi(exp_{\pi(\xi)} c(\xi)\xi) \, d\xi$$

for all $\phi \in C^0_c(M)$.

Proof. Proceed as in the smooth case, with N_i replacing N, and $\mathcal{U}_i$ replacing $U(N)$, and then sum over the index set $\mathcal{I}$. The Proposition follows easily. $\square$

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