FUNDAMENTAL GROUP OF A CLASS OF RATIONAL CUSPIDAL CURVES

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In [FZ], H. Flenner and M. Zaidenberg have found new examples of rational cuspidal curves in $\mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$; some of them were found earlier in [tD]. They have classified all such curves having at least three singularities, one of them of multiplicity $d-2$, where d is the degree of the curve. They have found also that these curves are projectively rigid.

It is known that there is a close relationship between some analytic invariants of the complement of the curve in the projective plane and the fundamental group of this complement. This fundamental group is also interesting in order to know the coverings of $\mathbb{P}^2$ ramified along the curve. In this paper we are going to compute the fundamental groups of the curves cited above, see Theorem in §1 below. As a consequence we find an infinite class of groups which admit hyperbolic triangle groups as quotients. We find also irreducible curves C_1, C_2 such that the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are non-homeomorphic but $\pi_1(\mathbb{P}^2 \setminus C_1)$ is a non-abelian group isomorphic to $\pi_1(\mathbb{P}^2 \setminus C_2)$, see Corollary 3.

The computation of the group of a curve of degree d with a singular point of multiplicity $d-2$ has been performed by A. Degtyarev in [D]; we present this particular case in order to get an explicite proof and in order to make self-contained our last statement.

Some group calculations have been made using GAP. I thanks Institut Fourier for its kind hospitality and M. Zaidenberg and L. Haddak for helpful discussions.

In a subsequent paper with L. Haddak we will discuss the relationship between the fundamental group of the complement of these curves and the fundamental group of the 3-manifold obtained as the boundary of a regular neighbourhood of the curve in the projective plane.

§1.- Definitions and results

Definition. Let $C \subset \mathbb{P}^2$ be a projective plane curve. Following O. Zariski [Z] we define *the group of the curve* as the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ of its complement, denoted G_C.

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Definition. Let $C \subset \mathbb{P}^2$ be an irreducible projective plane curve. We say that C is a *rational cuspidal curve* if it is rational (i.e. its normalization is isomorphic to $\mathbb{P}^1$) and all its singularities are locally irreducible (i.e. it is homeomorphic to $\mathbb{P}^1$).

Recall the classification of rational cuspidal curves having at least three singularities, one of them of multiplicity $d-2$, where d is the degree of the curve, see [FZ] and also [tD] for some particular cases. For each $(d,a,b) \in \mathbb{Z}^3$ such that $d > 3$, $a \geq b > 0$ and $a+b = d-2$, there is exactly one curve $C_{d,a,b}$, up to projective equivalence, having three singular points P_0, A_0, B_0; the germs $(C, P_0), (C, A_0), (C, B_0)$ have exactly one Puiseux pair, $(d-1, d-2), (2a+1, 2), (2b+1, 2)$ respectively.

Example. The curve $C_{4,1,1}$ is the tricuspidal quartic.

We state the main result of the paper:

Theorem. *Let $C := C_{d,a,b}$ be as above. Then, G_C is the group*

$$G_{d,n} := \langle c_1, c_2 : (c_2 c_1)^{d-1} = c_2^{d-2}, \quad (c_2 c_1)^n c_2 = c_1 (c_2 c_1)^n \rangle,$$

where $n \geq 0$ and $2n+1 = \gcd(2a+1, 2b+1)$. In particular, the group of $C_{d,a,b}$ depends only on (d,n).

We emphasize some consequences of this theorem. Recall first that for $p, q, r \in \mathbb{Z}$, $p, q, r > 1$, the group

$$T_{p,q,r} := |x, y, z : x^p = y^q = z^r = xyz = 1|$$

has a representation as the group of orientation-preserving isometries which preserve the tesselation of the 2-sphere (resp. euclidean plane, resp. hyperbolic plane) by triangles of angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ (resp. $= 1$, resp. < 1)

Corollary 1. *Let $C := C_{d,a,b}$ and n be as above. The fundamental group G_C is abelian if and only if $n = 0$ (i.e. $2a+1$ and $2b+1$ are coprime).*

If $n = 0$, the group G_C is cyclic of order d. If $n > 0$, G_C is a central extension of the triangle group $T_{2,2n+1,d-2}$.

Corollary 2. *Let $C := C_{d,a,b}$ and n be as above with $n > 0$.*

(i) *If $(d,a,b) = (4,1,1)$, then the group G_C is a non-abelian finite group of order 12, admitting $T_{2,2,3}$ as a quotient. If $(d,a,b) = (7,4,1)$, then G_C is also non-abelian finite group of order 840, admitting $T_{2,3,5}$ as a quotient. Both are spherical triangle groups.*

(ii) *If $(d,a,b) \neq (4,2,2), (7,4,1)$, then the group of the curve is a non-abelian infinite group, admitting $T_{2,2n+1,d-2}$ as a quotient.*

The statement about $C_{4,1,1}$ appears already in [Z].

Next, we are going to apply the fact that the groups of these curves depend only on (d,n).

Corollary 3. *There exist curves $C_1, C_2 \subset \mathbb{P}^2$ such that G_{C_1} and G_{C_2} are isomorphic and non-abelian, but $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.*

In fact, there is an infinite number of non-equivalent such pairs. Take for instance $C_1 = C_{13,10,1}$ and $C_2 = C_{13,7,4}$, where $n = 1$ in both cases.

We finish this section with two definitions:

Definition. Let X be a smooth projective surface and let D be a compact curve such that all its irreducible components $D_1, \ldots, D_r$ are smooth and rational and the singularities of D are only nodes. We will say that the ordered r-tuple $(D_1, \ldots, D_r)$ is a *symmetric r-string* if the dual graph of D is a linear tree ordered by $D_1, \ldots, D_r$ (i.e. $D_i \cdot D_{i+1} = 1$, $1 \leq i \leq r-1$, and $D_i \cdot D_j = 0$ if $|i-j| > 1$) and $D_1 \cdot D_1 = D_r \cdot D_r = -1$, $D_j \cdot D_j = -2$, $1 < j < r$ (we remark that if $(D_1, \ldots, D_r)$ is an r-symmetric curve this is also the case for $(D_r, \ldots, D_1)$).

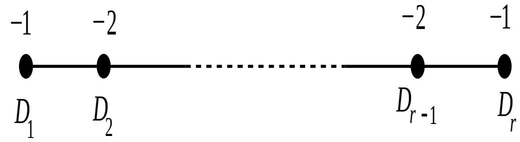

Figure 1.

Definition. Let X be a smooth projective manifold and let $H, K \subset X$ be hypersurfaces. Let $* \in X \setminus (H \cup K)$. A *meridian* of H in the group $\pi_1(X \setminus K, *)$ is the homotopy class of a loop μ defined as follows: take a point $P \in H$ which is smooth in $H \cup K$; take a small disk Δ around P transverse to H and disjoint from K; fix a point $*' \in \partial\Delta$ and let m be the loop based at $*'$ which turns once along $\partial\Delta$ in the positive direction. Choose any path ℓ from $*$ to $*'$ in $X \setminus (H \cup K)$ such that $\ell \cap \Delta = \{*'\}$. Then $\mu := \ell \cdot m \cdot \ell^{-1}$ (we note that two meridians of H are conjugate if H is irreducible).

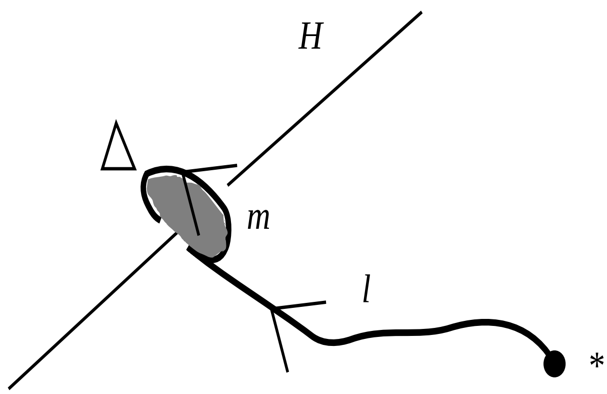

Figure 2.

§2.- Construction of $C_{d,a,b}$

In this section we are going to construct a birational map of $\mathbb{P}^2$ which will give a lot of information about $C := C_{d,a,b}$.

Conventions. Let X, Y be smooth projective surfaces and let $\sigma \colon Y \to X$ be the blow-up of a point $P \in X$.

(a) We will denote also by P the exceptional curve of σ; we recall that P is a smooth rational curve with $(P \cdot P)_Y = -1$. Recall that by Castelnuovo's criterion, the contraction of a smooth rational curve of self-intersection -1 is a blow-down.

(b) Let C be an irreducible curve in X. Let m be the multiplicity of C at P. We denote also by C the proper transform of C by σ, i.e. the closure in Y of $\sigma^{-1}(C \setminus \{P\})$. We recall that $(C \cdot C)_Y = (C \cdot C)_X - m^2$ and $(C \cdot P)_Y = m$. In general, if $C_1, C_2 \subset X$ are irreducible curves with multiplicities m_1 and m_2 at P, respectively, then $(C_1 \cdot C_2)_Y = (C_1 \cdot C_2)_X - m_1 m_2$.

(c) If Q is any point of X different from P, then we denote also by Q its unique preimage by σ. Recall that the restriction of σ to $Y \backslash P$ is an analytic isomorphism onto $X \setminus \{P\}$.

Construction of the curve.

Step 0. Consider the blow-up $\sigma_0 \colon X_0 \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the point P_0 which is the cusp of multiplicity $d-2$ of C. Denote by P_1 the unique intersection point of C and P_0 in X_0. It is clear that C is smooth at P_1; the curves C and P_0 are tangent at P_1 and have contact of order $d-2$ there. We observe that X_0 is a relatively minimal rational ruled surface; the curve P_0 is the unique section with self-intersection -1.

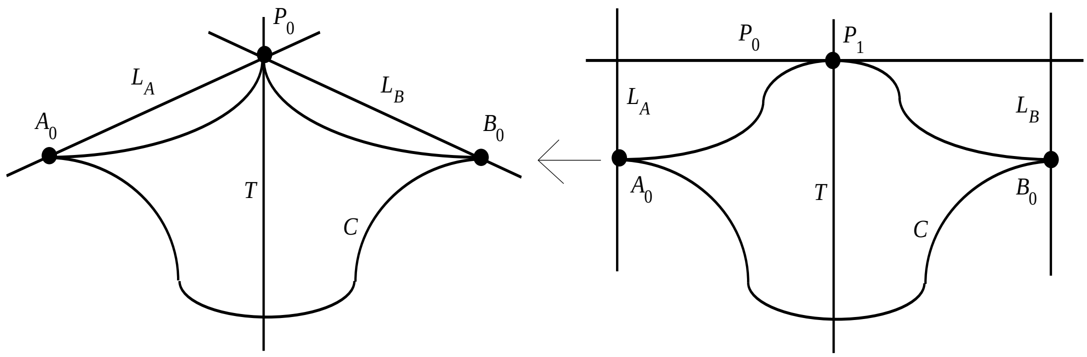

Figure 3.

Let T be the tangent line of C at P_0; let L_A (resp. L_B) be the line detemrined by P_0 and A_0 (resp. B_0). Then, $T \subset X_0$ is the fiber of the rulling passing through P_1, $L_A \subset X_0$ is the fiber passing through A_0 and $L_B \subset X_0$ is the fiber passing through B_0.

Step 1. Consider the point A_0. We recall that C has a singular point at A_0 with only one Puiseux pair $(2a+1, 2)$. Recall the construction of the infinitely near points of C at A_0.

Let $\alpha_1 : X_1 \to X_0$ be the blow-up of $A_0 \in X_0$. By our convention, we denote the exceptional curve A_0. Let A_1 be the unique intersection point of C and A_0 in X_1.

We may construct a sequence of blow-ups $\alpha_i \colon X_i \to X_{i-1}$ such that the center of α_i is the unique intersection point A_{i-1} of C (i.e the strict transform of C by the map $\alpha_i \circ \cdots \circ \alpha_{i-1}$) and A_{i-2} (i.e. the exceptional curve of α_{i-1}). Denote $\tilde{\alpha}_i := \alpha_1 \circ \cdots \circ \alpha_i$.

Then the unique intersection point of A_{i-1} and C in X_i is denoted by A_i.

From the Puiseux pair of C at A_0 we deduce that the germ $(C, A_i) \subset X_i$ is of multiplicity 2 if $i < a$ and it is a smooth germ if $i \geq a$. We know also that C and A_{a-1} are tangent at A_a in X_a and the contact order is 2.

Consider now the curve $\tilde{\alpha}_a^{-1}(L_A) \subset X_a$; its irreducible components are L_A, $A_0, \ldots, A_{a-2}, A_{a-1}$. It is easy to show that $(L_A, A_0, \ldots, A_{a-2}, A_{a-1})$ is a symmetric $(a+1)$-string.

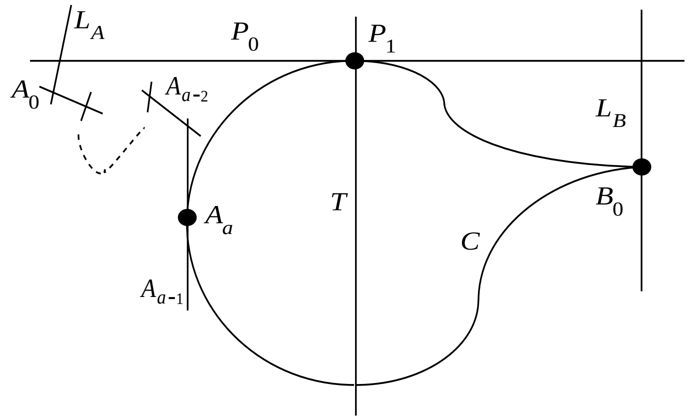

Figure 4.

By Castelnuovo's criterion we may contract $L_A, A_0, \ldots, A_{a-2}$ and we get a map $\alpha\colon X_a \to Y_0$ such that:

- The surface Y_0 is a relatively minimal rational ruled surface.
- The curve P_0 is a section with self-intersection $a-1$.
- The curve T is always the fiber through P_1 and L_B is the fiber through B_0.
- The curve A_{a-1} is the fiber through A_a and it is tangent to C at this point with contact order 2.

Step 2. Consider now the point B_0. We proceed as in step 1: we make blow-ups $\beta_i\colon Y_i \to Y_{i-1}$, $i \geq 1$, and we get the infinitely near points B_i of C at B_0; recall that $B_i \in Y_i$ is the center of the blow-up β_{i+1}, $i \geq 0$. Denote $\tilde{\beta}_i := \beta_1 \circ \cdots \circ \beta_i$.

As above, the germ $(C, B_i) \subset Y_i$ is of multiplicity 2 if $i < b$ and is a smooth germ if $i \geq b$. We know also that C and B_{b-1} are tangent at B_b in Y_b and the contact order is 2.

Consider further the curve $\tilde{\beta}_b^{-1}(L_B) \subset Y_b$; its irreducible components are L_B, $B_0, \ldots, B_{b-2}, B_{b-1}$. As before, $(L_B, B_0, \ldots, B_{b-2}, B_{b-1})$ is a symmetric $(b+1)$-string.

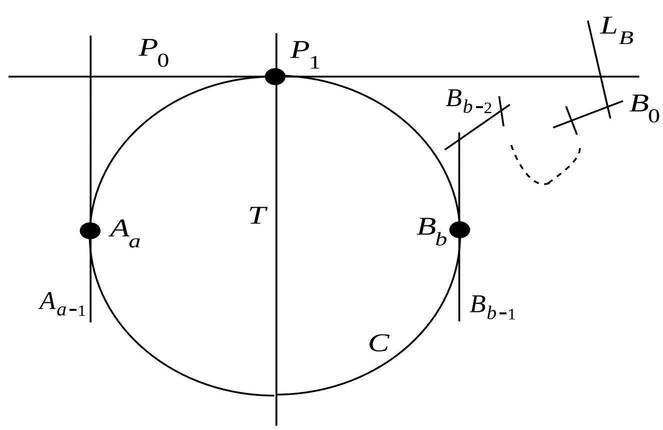

Figure 5.

We proceed as before: contracting L_B, $B_0, \ldots, B_{b-2}$ we get a map $\beta\colon Y_b \to Z_1$ such that:

- Z_1 is a ruled surface.

- The curve P_0 is a section with self-intersection $a+b-1=d-3$.
- The curve T is the fiber through P_1 and A_{a-1} is the fiber through A_a.
- The curve B_{b-1} is the fiber through B_b and it is tangent to C at this point with contact order 2.

Notice that C is smooth in Z_1.

Step 3. We are going to proceed in a similar way near P_1.

We make blow-ups $\gamma_i\colon Z_i \to Z_{i-1}$, $i \geq 2$, and we get the infinitely near points P_i of C at P_0; we recall that $P_i \in Z_i$ is the center of the blow-up γ_{i+1}, $i \geq 1$. Denote $\tilde{\gamma}_i := \gamma_2 \circ \cdots \circ \gamma_i$.

From the Puiseux pair of C at P_0 we deduce that the germ $(C, P_i) \subset Z_i$ is smooth if $i \geq 1$. It is easily seen that if we regard C and P_0 as curves in Z_i, $1 \leq i \leq d-2$, they intersect at P_i with contact order $d-1-i$. In the same way, C and P_0 do not intersect in Z_{d-1}.

Consider now the curve $\tilde{\gamma}_{d-1}^{-1}(T) \subset Z_{d-1}$; its irreducible components are T, $P_1, \ldots, P_{d-3}, P_{d-2}$. As before, $(T, P_1, \ldots, P_{d-3}, P_{d-2})$ is a symmetric $(d-1)$-string.

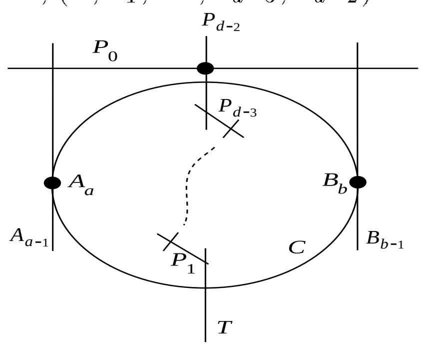

Figure 6.

Contracting $T, P_1, \ldots, P_{d-3}$ we get a map $\gamma\colon Z_{d-1} \to \mathbb{F}$ such that:

- $\mathbb{F}$ is a relatively minimal rational ruled surface.
- The curve P_0 is a section with self-intersection -1.
- The curve B_{b-1} is the fiber through B_b and A_{a-1} is the fiber through A_a.
- The curve P_{d-2} is a fiber which intersects transversally C at two points.

We remark also that C is smooth in $\mathbb{F}$ and does not intersect P_0.

Step 4. Contracting P_0 we get a map $\tilde{\sigma}\colon \mathbb{F} \to \widetilde{\mathbb{P}}^2_1$, where the image $\tilde{C}$ of C in $\widetilde{\mathbb{P}}^2$ is an irreducible conic, A_{a-1}, B_{b-1} and P_{d-2} are straight lines through P_0. The first two lines are tangent to C and the last one is transversal.

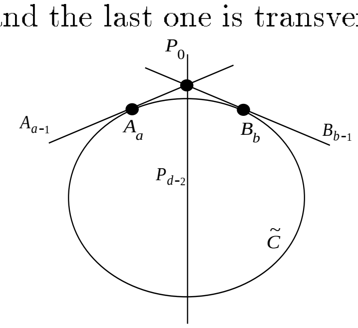

Figure 7.

Note by the way that this construction shows the existence of the curve $C = C_{d,a,b}$. The composition of all these maps gives a birational map $\mathbb{P}^2 \dashrightarrow \widetilde{\mathbb{P}}^2$, which induces an analytic isomorphism

$$\mathbb{P}^2 \setminus (C \cup T \cup L_A \cup L_B) \to \widetilde{\mathbb{P}}^2 \setminus (\tilde{C} \cup P_{d-2} \cup A_{a-1} \cup B_{b-1}).$$

§3.- Proof of the theorem

The group of the curve $\tilde{C} \cup P_{d-2} \cup A_{a-1} \cup B_{b-1}$ is easly computed by looking at the real picture. We will use Zariski-Van Kampen method by projecting from P_0. It is better to explain the computations in the situation just before step 4. We remark that the contraction of P_0 induces an analytic isomorphism

$$\widetilde{\mathbb{P}}^2 \setminus (\tilde{C} \cup P_{d-2} \cup A_{a-1} \cup B_{b-1}) \to \mathbb{F} \setminus (C \cup P_{d-2} \cup A_{a-1} \cup B_{b-1} \cup P_0).$$

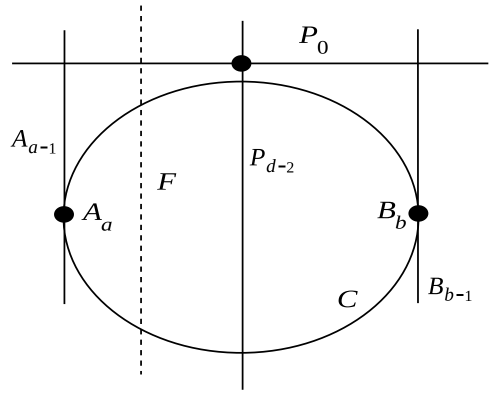

Figure 8.

Denote $p\colon \mathbb{F} \to \mathbb{P}^1$ the projection of the ruled surface $\mathbb{F}$ such that $p(A_{a-1}) = 0$, $p(P_{d-2}) = 1$ and $p(B_{b-1}) = \infty$. The restriction

$$p_|\colon \mathbb{F} \setminus (C \cup P_{d-2} \cup A_{a-1} \cup B_{b-1} \cup P_0) \to \mathbb{C} \setminus \{0, 1\}$$

is a locally trivial fibration. Then, Zariski-Van Kampen is nothing else but the homotopy exact sequence of $p_|$.

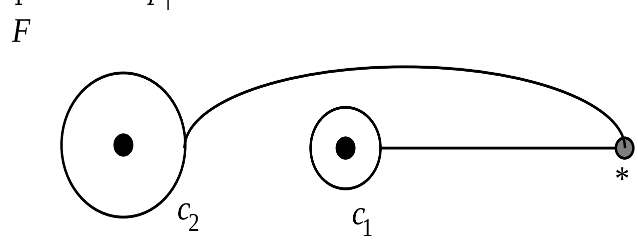

Figure 9.

Step 1: the fiber. Fix $F := p_|^{-1}(\frac{1}{2})$ a generic fiber; it is isomorphic to the punctured sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$; we suppose that ∞ is the intersection with P_0, 1 is the *upper* intersection with C and 0 is the *lower* one. Choose a base point $* \in F$ to be a very big positive real number. We define a meridian c_1 as follows: take the shortest path from $*$ to $3/2$ along the real line; turn once along the circle of radius $1/2$ and center 1 in the positive sense and come back to $*$. Define a meridian c_2 as follows: take the shortest path from $*$ to $3/2$ along the real line; turn one-half along the circle of radius $1/2$ and center 1 in the half-plane $\operatorname{Im} z \geq 0$ reaching $1/2$; turn once along the circle of radius $1/2$ and center 0 in the positive sense and return to $*$ by the same way that you arrived to $1/2$. We denote e a path which turns once along the great circle through $*$ in the clockwise direction. These three loops generate $\pi_1(F, *)$ and we get the first relation:

$$\mathcal{F}: \ c_2 c_1 e = 1.$$

Step 2: the base. Fix a tubular neighbourhood V of P_0 in $\mathbb{F}$; let M be its boundary (it is homeomorphic to the 3-sphere because of self-intersection -1 of P_0). We may choose a $q \colon M \to P_0$ via the natural identification of $\mathbb{P}^1$ with the section P_0.

Fix $q(*)$ as base point in P_0; we choose three meridians $\tilde{x}, \tilde{t}, \tilde{y}$ in P_0 around $0, 1, \infty$ respectively, in the simplest way such that $\tilde{x}\tilde{y}\tilde{t} = 1$. We lift these loops in the natural way to loops in M based on $*$ and we denote them x, t, y; we get meridians around $A_{a-1}, P_{d-2}, B_{b-1}$, respectively. We remark that e is a positive fiber of q. Applying the definition of the Euler class we get four relations:

$$\mathcal{B}_1: \ xyt = e, \quad \mathcal{B}_2: \ [x, e] = 1, \quad \mathcal{B}_3: \ [y, e] = 1, \quad \mathcal{B}_4: \ [t, e] = 1,$$

where $[g, h] := ghg^{-1}h^{-1}$. It is easily seen that $\mathcal{B}_4$ is a consequence of the other relations.

Step 3: the monodromy. It is just here where we use the real picture of the curve. The singularities of the projection explain us the local behaviour of the monodromy and the real part explains the global behaviour. More precisely, the braid monodromy can be constructed from the real picture.

When we turn around A_{a-1}, we get the braid:

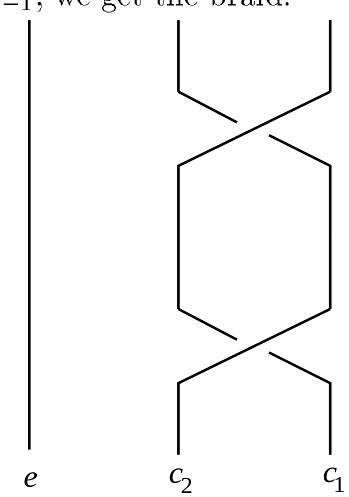

Figure 10.

Induced relations are:

$$\mathcal{X}_1: x^{-1} c_1 x = c_2, \quad \mathcal{X}_2: \ [x, c_2 c_1] = 1.$$

It is easily seen that $\mathcal{X}_2$ is a consequence of $\mathcal{F}$ and $\mathcal{B}_2$. When we turn around P_{d-2} we get the trivial braid.

This implies the relations:

$$\mathcal{T}_1: \ [t, c_1] = 1, \quad \mathcal{T}_2: \ [t, c_2] = 1.$$

As before, $\mathcal{T}_2$ is a consequence of $\mathcal{F}$, $\mathcal{B}_4$ and $\mathcal{T}_1$. When we turn around B_{b-1}, we get:

$$\mathcal{Y}_1 : y^{-1} c_1 y = c_2, \quad \mathcal{Y}_2 : \ [y, c_2 c_1] = 1.$$

As above, we do not need $\mathcal{Y}_2$. It is easily seen that we can also forget $\mathcal{T}_1$. Thus, we have:

$$\pi_1(\mathbb{P}^2 \setminus (C \cup T \cup L_A \cup L_B), *) = |c_1, c_2, x, y, t, e : \mathcal{F}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{X}_1, \mathcal{Y}_1| .$$

Step 4: finding meridians. It is well-known, see [Z] and [F] lemma 4.18, that there exists an epimorphism

$$\pi_1(\mathbb{P}^2 \setminus (C \cup T \cup L_A \cup L_B), *) \to \pi_1(\mathbb{P}^2 \setminus C);$$

its kernel is the normal subgroup generated by a meridian of T, a meridian of L_A and a meridian of L_B. We must express them in terms of the above presentation. The key point is to use the inverse path in §2 and the next lemma (its proof is straightforward), see [F] lemma 7.17:

Lemma. *Let X be a surface and $D \subset X$ a curve. Let $G := \pi_1(X \setminus D, *)$, where $* \in X \setminus D$.*

Suppose that P is an ordinary double point of D. Fix a base point $*' \in X \setminus D$ close to P. Choose two meridians of D: take a path ℓ from $*$ to $*'$ in $X \setminus D$; take m_1 as the positive boundary starting at $*'$ of a small disk transverse to one of the branches of D at P; take m_2 in the same way for the other branch. Define $\mu_i := \ell \cdot m_i \cdot \ell^{-1}$, $= 1, 2$ (note that $[\mu_1, \mu_2] = 1$).

Let $\sigma\colon Y \to X$ be the blowing at the point P; identify $G = \pi_1(Y \setminus \sigma^{-1}(D))$ and recall that P is an irreducible component of $\sigma^{-1}(D)$.

Then $\mu := \mu_1 \mu_2$ is a meridian of P.

If we apply this lemma $d-2$ times to the inverse path of Step 3 in §2, we find that $c_2^{d-2} t$ is a meridian of T. Applying the lemma b times in Step 2, we find that $e^b y$ is a meridian of L_B. Finally, $e^a x$ is a meridian of L_A. Denote:

$$\mathcal{Z}_1 : \ c_2^{d-2} t = 1, \quad \mathcal{Z}_2 : \ e^b y = 1, \quad \mathcal{Z}_3 : \ e^a x = 1.$$

Adding these relations to the previous ones, we get a presentation of $\pi_1(\mathbb{P}^2 \setminus C)$. As the first simplification, we can forget $\mathcal{B}_2$ and $\mathcal{B}_3$. Using relations $\mathcal{Z}_1$, $\mathcal{Z}_2$ and $\mathcal{Z}_3$ we can eliminate generators x, y, t and we get:

$$\pi_1(\mathbb{P}^2 \setminus C) = \left| c_1, c_2, e : c_2 c_1 e = 1, c_2^{d-2} e^{d-1} = 1, e^a c_1 e^{-a} = c_2 = e^b c_1 e^{-b} \right| .$$

If we drop e, we find:

$$\pi_1(\mathbb{P}^2 \setminus C) = \left| c_1, c_2 : c_2^{d-2} = (c_2 c_1)^{d-1}, (c_2 c_1)^a c_2 = c_1 (c_2 c_1)^a, (c_2 c_1)^b c_2 = c_1 (c_2 c_1)^b \right| .$$

Finally, if $n \geq 0$ and $2n + 1 = \gcd(2a + 1, 2b + 1)$, we see that the last two relations are equivalent to $(c_2 c_1)^n c_2 = c_1 (c_2 c_1)^n$, and the theorem follows. $\square$

§4.- Proof of Corollaries 1,2,3

Proof of Corollary 1. If $n = 0$, then $c_1 = c_2$ and clearly the group in the question is abelian and cyclic of order d.

If $n > 0$, take $x_1 = (c_2 c_1)^n c_2$ and $y_1 = c_2 c_1$. We get

$$\pi_1(\mathbb{P}^2 \setminus C) = \left| x_1, y_1 : \quad x_1^2 = y_1^{2n+1}, \quad (y_1^{-n} x_1)^{d-2} = y_1^{d-1} \right|.$$

Let $u, v \in \mathbb{Z}$ be such that

$$2a + 1 = u(2n + 1) \text{ and } 2b + 1 = v(2n + 1).$$

We have

$$2(d - 1) = (2a + 1) + (2b + 1) = (u + v)(2n + 1).$$

Thus, $u + v$ is even. Let $w \in \mathbb{Z}$ be such that $u + v = 2w$; then $d - 1 = w(2n + 1)$. Hence, if $\mu = x_1^2$, we obtain

$$(y_1^{-n} x_1)^{d-2} = \mu^w.$$

Note that μ is central. Let G be the quotient of our group by the relation $w = 1$.

If we put $= x_1$, $y = y_1^{-n}$ and $z = (xy)^{-1}$, we find that G is $T_{2,2n+1,d-2}$. □

Proof of Corollary 2. It is easily seen that there are only two cases where the group is a spherical one. The first case was already calculated by Zariski. The second one was found to be finite using GAP.

There is no case where the group is euclidean. It is known that the hyperbolic triangle groups are infinite [CM], so we are done. □

Proof of Corollary 3. It is straightforward from the remark that (d, n) determines the group G_C and a, b determine the topological type of the singularities.

It is also possible to find not only pairs of curves but n-tuples of curves which agree pairwise with the statement of the Corollary, for any $n \in \mathbb{N}$. □

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