Tensor product of Hopf bimodules over a group

Claude Cibils*

Abstract

We describe the monoidal structure of the category of Hopf bimodules of a finite group and we derive a surjective ring map from the Grothendieck ring of the category of Hopf bimodules to the center of the integral group ring. We consider analogous results for the multiplicative structure of the Hochschild cohomology.

1 Introduction

Hopf bimodules are natural representations of a Hopf algebra H, they form an abelian category $\mathcal{B}(H)$ provided with a tensor product, their definition is given below. M. Rosso proved in [15] that $\mathcal{B}(H)$ is equivalent –as a monoidal category– to the category of modules over the Drinfeld double of H, therefore $\mathcal{B}(H)$ is a braided category (see [2, 7, 9, 11]). M. Rosso and S.L. Woronowicz ([14, 17]) has considered those structures in relation to differential calculus on quantum groups; Hopf bimodules over a group algebra provides the classification of quantum structures on path algebras obtained in [4].

The main purpose of this paper is to describe the tensor product of Hopf bimodules over an arbitrary group G with coefficients in a ring k, which are k-projective and finitely generated. Alternatively, if G is finite the result concerns Hopf bimodules without restrictions on their k-structure. Notice that the computation of the monoidal structure of Hopf bimodules is perhaps related with the work of Wassermann concerning the computation of Conne's fusion for $SU(2)$, see [16] and [10].

The quoted structure result is performed by constructing an explicit monoidal category in terms of conjugacy classes and representations of the centralizers provided with a tensor product defined in terms of induced modules.

*Supported by University of Bern and University of Grenoble.

1991 Mathematics Subject Classification : 18D10, 20G05 and 20G10.

We prove that the resulting category is equivalent –as a monoidal category– to the one formed by the Hopf bimodules. As a consequence, we obtain that the Grothendieck ring of Hopf bimodules surjects on the center of the integral group ring.

This results are related with the study of the Hochschild cohomology algebra of a group algebra. Indeed, we recall that the algebra structure of the later is unknown and that a link between the tensor product of Hopf bimodules and the cup product of Hochschild cocycles seems to exist; in the abelian group case, such a relation is clear since both structures are graduations over the group of products of more usual structures (respectively left modules and usual cocycles), see [5]. We consider this point at the last section.

We know from [4] that the category of kG-Hopf bimodules is equivalent –just as an abelian category– to the product of categories of usual modules $\prod_{C\in\mathcal{C}} \operatorname{mod} kZ_C$ where $\mathcal{C}$ is the set of conjugacy classes and Z_C is the centralizer of one of the elements of $C \in \mathcal{C}$. Objects of this product are possibly infinite families of right modules over the centralizers. W.D. Nichols ([13]) has initiated this classification for a finite abelian group and a field of characteristic not dividing the order of G. D. Dijkgraaf, V. Pasquier and P. Roche in [8] considered the case of a finite group with complex coefficients and modules over the Drinfeld double of the group algebra, using Lusztig's results ([12]) for studying its semisimplicity.

We begin this paper with an account of Hopf bimodules and their monoidal structure, including the proof of the additive structure result obtained in [4] presented here in a simplified version. This enables us to obtain the complete multiplicative structure Theorem in section 3.

2 Hopf bimodules

A Hopf bimodule B over a Hopf algebra H is an H-bimodule which is simultaneously an H-bicomodule; moreover, it is required that the structure maps $\delta_L : B \to H \otimes B$ and $\delta_R : B \to B \otimes H$ are H-bimodule maps for the H-bimodule structure of the tensor products obtained through the comultiplication of H. For more details concerning this definition, see for instance [5].

The tensor product of Hopf bimodules B and B' is the bimodule $B \otimes_H B'$ equipped with the bicomodule structure given by the maps

$$B\otimes_H B' \overset{\delta_L\otimes\delta_L}{\to} (H\otimes B)\otimes_H(H\otimes B') \overset{1\otimes\tau\otimes 1}{\to} H\otimes_H H\otimes B\otimes_H B' \overset{m\otimes 1\otimes 1}{\to} H\otimes B\otimes_H B'$$

$B\otimes_H B' \overset{\delta_R\otimes\delta_R}{\to} (B\otimes H)\otimes_H(B'\otimes H) \overset{1\otimes\tau\otimes 1}{\to} B\otimes_H B'\otimes H\otimes_H H \overset{1\otimes 1\otimes m}{\to} B\otimes_H B'\otimes H$

where τ is the flip map and m is the multiplication map of H. There is no difficulty to check that $1_\otimes\tau_\otimes 1$ is a well defined H-bimodule map, both compositions gives H-bimodule maps and define an Hopf bimodule structure on the bimodule $B\otimes_H B'$.

Theorem 2.1 ([4]) *Let k be a ring and G be a group. The category $\mathcal{B}(kG)$ of kG-Hopf bimodules is equivalent to $\prod_{C\in\mathcal{C}}\mathrm{mod}\, kZ_C$.*

Proof : Let $M=(M_C)_{C\in\mathcal{C}}$ be a possibly infinite family of right $kZ_{u(C)}$-modules, where $Z_{u(C)}$ is the centralizer of a chosen element $u(C)$ in each conjugacy class C. Let VM be the kG-bimodule $\prod_{C\in\mathcal{C}} kG\otimes M_C\uparrow_{Z_C}^G$ where the right structure of $M_C\uparrow_{Z_C}^G$ is the induced one –recall that $M_C\uparrow_{Z_C}^G = M_C\otimes_{kZ_C} kG$– and the left structure is the trivial one. The k-module kG is provided with the regular left and right actions and each tensor product is equipped with the resulting diagonal left and right actions.

Concernig the bicomodule structure, it is useful to know that any Hopf bimodule is bigraded by G : we have $B=\bigoplus_{x,y\in G}{}^yB^x$ where ${}^yB^x$ is the isotypic cocomponent of type (y,x), i.e.

$$ {}^yB^x=\{b\in B \mid \delta_L(b)=y\otimes b \text{ and } \delta_R(b)=b\otimes x\}. $$

A proof of this lemma is provided in [4], notice however that if G is finite the result is straightforward since $(kG)^*$ is a split semisimple basic algebra and results about kG-bicomodules are duals of results about $(kG)^*$-bimodules.

The isotypic cocomponents of VM are defined as follows

$$ x\otimes M_C\otimes K = {}^{xK^{-1}u(C)K}(VM)^x $$

where K is a left class of $Z_{u(C)}\backslash G$. It is easely checked that this data provides VM with a Hopf bimodule structure.

The reverse functor W associates to a Hopf bimodule B the collection of isotypic cocomponents $\left\{{}^{u(C)}B^1\right\}_{C\in\mathcal{C}}$. Notice that ${}^{u(C)}B^1$ is a right $kZ_{u(C)}$-module through "conjugation" of actions: $b.z=z^{-1}bz$. Indeed, the right actions takes ${}^{u(C)}B^1$ to ${}^{u(C)z}B^z$ and the left action returns it to ${}^{z^{-1}u(C)z}B^1$. Since $z\in Z_{u(C)}$, the latter is exactly ${}^{u(C)}B^1$ as required.

Remark 2.2 Let $\mathcal{B}_k(kG)$ be the full subcategory of $\mathcal{B}(kG)$ of Hopf bimodules with underlying k-module projective and finitely generated, and let $\times_{C\in\mathcal{C}}\mathrm{mod}_k\, kZ_C$ be the category of finite families of right kZ_C-modules which are k-projective and finitely generated. The functors of the preceeding proof restrict to equivalences between these sub-categories.

3 Monoidal structure

We define first the monoidal category $\mathcal{M}(kG)$ mentionned in the introduction for k a commutative ring and G any group. Objects are formal finite sums $\sum_{C\in\mathcal{C}}[M_C]C$ where M_C is a right module over the centralizer $Z_{u(C)}$ of a chosen element $u(C)$ of each conjugacy class C, such that the underlying k-structure of M_C is projective and finitely generated. Morphisms and direct sums are the natural ones.

In order to define the monoidal structure of $\mathcal{M}(kG)$, recall that $M{\uparrow}_Z^G$ is the induced module $M\otimes_{kZ}kG$ of a right kZ-module M where Z is a subgroup of a group G. The tensor product is defined on elementary objects,

$$[M]A\otimes[N]B=\sum_{C\in\mathcal{C}}\left[X_{A,B}^C\right]C$$

where $X_{A,B}^C$ is a sub $kZ_{u(C)}$-module of $\left(M{\uparrow}_{Z_{u(A)}}^G\otimes N{\uparrow}_{Z_{u(B)}}^G\right){\downarrow}_{Z_{u(C)}}$. Let

$$E_{A,B}^C=\{(K,L)\in Z_{u(A)}\backslash G\times Z_{u(B)}\backslash G\ \mid\ K^{-1}u(A)KL^{-1}u(B)L=u(C)\},$$

we define

$$X_{A,B}^C=\bigoplus_{(K,L)\in E_{A,B}^C}M\otimes K\ \otimes_k\ N\otimes L.$$

Notice that $E_{A,B}^C$ is actually preserved by right multiplication by elements of G. We are able to extend this product to the full category since objects are finite formal sums. It is not clear a priori that this product is associative, once the next Theorem will be proved the associativity derives since the tensor product of Hopf bimodules is clearely associative.

Theorem 3.1 *Let k be a ring and G be any group. The categories $\mathcal{B}_k(kG)$ and $\mathcal{M}(kG)$ are equivalent through functors which preserves tensor products.*

Proof : We refer to the proof of the previous Theorem 2.1 for notations and definitions. Let I and I' be Hopf bimodules constructed via the functor V by means of elementary objects $[M]A$ and $[N]B$, and consider their tensor product $I\otimes_{kG}I'=V\left([M]A\right)\otimes_{kG}V\left([N]B\right)$. We need to prove that W of this tensor product is the announced object of $\mathcal{B}_k(kG)$, in other words we have to show that

$${}^{u(C)}(I\otimes_{kG}I')^1=X_{A,B}^C$$

as right $kZ_{u(C)}$-modules, for each $C\in\mathcal{C}$.

First we collect the following easy equality:

$$ {}^{d}I^{c} \otimes_{kG} {}^{xa}I'^{xb} \;=\; {}^{dc}I^{cx} \otimes_{kG} {}^{a}I'^{b} \;\subset\; {}^{dxa}(I \otimes_{kG} I')^{cxb}\,. $$

In particular this means that a normalization can be performed as follows:

$$ I \otimes_{kG} I' \;=\; \bigoplus_{z,y,x \in G} {}^{z}I^{y} \otimes_{k} {}^{x}I'^{1} $$

and the isotypic cocomponent ${}^{v}(I \otimes_{kG} I')^{u}$ is given in this normalization by $\bigoplus_{yx=v} {}^{y}I^{u} \otimes_{k} {}^{x}I'^{1}$. Hence

$$ {}^{u(C)}(I \otimes_{kG} I')^{1} \;=\; \bigoplus_{xy=u(C)} {}^{y}I^{1} \otimes_{k} {}^{x}I'^{1}. $$

Now we have $I = V([M]A) = kG \otimes_{k} M{\uparrow}^{G}_{Z_{u(A)}}$, and by definition of the isotypic cocomponents

$$ z \otimes M \otimes K \;=\; {}^{zK^{-1}u(C)K}I^{z} $$

for each $z \in G$ and $K \in Z_{u(A)}\backslash G$. Consequently the isotypic cocomponents of type ${}^{y}I^{1}$ are of the form ${}^{K^{-1}u(C)K}I^{1}$ for all $K \in Z_{u(A)}\backslash G$ and we derive

$$ {}^{u(C)}(I \otimes_{kG} I')^{1} \;=\; \bigoplus_{(K,L) \in E^{C}_{A,B}} {}^{K^{-1}u(A)K}I^{1} \otimes_{k} {}^{L^{-1}u(B)L}I'^{1} \;= $$

$$ \bigoplus_{(K,L) \in E^{C}_{A,B}} (1 \otimes M \otimes K) \otimes (1 \otimes N \otimes L). $$

The identification with the described k-module $X^{C}_{A,B}$ is now clear. To end the proof it is enough to record that the right action of $Z_{u(C)}$ on this direct sum coincides with the one described on $X^{C}_{A,B}$. Indeed, the action is given by "conjugation" of the actions (c.f. the proof of 2.1) and the left structure on the right induced modules is the trivial one.

We consider now the Grothendieck ring $K_0\mathcal{B}_k(kG)$ of the category of k-projective and finitely generated Hopf bimodules over kG, namely the free abelian group on the set of isomorphism classes of objects of $\mathcal{B}_k(kG)$ divided by the subgroup generated by the relations $Y - Y' - Y''$ associated to each split exact sequence $0 \to Y' \to Y \to Y'' \to 0$ of the category. In case the

category is a Krull-Schmidt one, this quotient is free with basis the set of isomorphism classes of indecomposable objects. If k is a field, the monoidal structure of the category induces a product on the Grothendieck group which provides a ring structure with unit (notice that the Hopf algebra is the unit object for the tensor product of Hopf bimodules).

Corollary 3.2 *Let k be a field and G a group. There is a ring surjection*

$$\phi : K_0\mathcal{B}(kG) \longrightarrow \mathcal{Z}(kG)$$

where $\mathcal{Z}(kG)$ is the center of the group algebra $\mathbf{Z}G$.

Proof : Recall first that $\mathcal{Z}(\mathbf{Z}G)$ is free with basis $\left\{\overline{C}\right\}_{C\in\mathcal{C}}$ where $\overline{C}$ is the sum in $\mathbf{Z}G$ of all the elements of the conjugacy class C. The structure constants of this ring are non negative integers given by $\overline{A}\,\overline{B} = \sum_{C\in\mathcal{C}} e^{C}_{A,B}\overline{C}$. It is elementary to prove that $e^{C}_{A,B}$ is the cardinal of the set $E^{C}_{A,B}$ considered before.

Next we identify $K_0\mathcal{B}(kG)$ and $K_0\mathcal{M}(kG)$ through the functors V and W and we define

$$\phi\left(\sum_{C\in\mathcal{C}}[M_C]\,C\right) = \sum_{C\in\mathcal{C}}\dim_k[M_C]\,\overline{C}.$$

This gives a ring homomorphism since from one side

$$\phi\left([M]A\otimes[N]B\right) = \phi\left(\sum_{C\in\mathcal{C}}\left[X^{C}_{A,B}\right]C\right) = \sum_{C\in\mathcal{C}}\dim_k X^{C}_{A,B}\,\overline{C} =$$

$$\sum_{C\in\mathcal{C}}\dim_k M\dim_k N\,|\,E^{C}_{A,B}\,|\,\overline{C}$$

and from the other side we have

$$\phi\left([M]A\right)\phi\left([N]B\right) = \left(\dim_k M\overline{A}\right)\left(\dim_k N\overline{B}\right) = \dim_k M\dim_k N\sum_{C\in\mathcal{C}}e^{C}_{A,B}\,\overline{C}.$$

Each basis element $\overline{C}$ of $\mathcal{Z}(\mathbf{Z}G)$ is reached through ϕ, using the elementary object having the trivial representation of the centralizer of C as coefficient of C.

Remark 3.3 A suitable version of the preceding surjection probably still exists from the Grothendieck ring of Hopf bimodules over a finite dimensional Hopf algebra H to the center of H.

4 Hochschild cohomology

The behavior of the cup product for the Hochschild cohomology classes of a finite group is probably similar to the behavior of tensor product of Hopf bimodules as described in Theorem 3.1. Indeed, we have at first that the underground additive structures agrees through analogous descriptions with respect to centralizers of conjugacy classes, since it is well known that

$$HH^*(kG, kG) = \bigoplus_{C \in \mathcal{C}} H^*(Z_C, k)$$

where HH^* denotes Hochschild cohomology and H^* usual group cohomology see for instance [3] or [1] and compare with Theorem 2.1. Moreover in [5] we have obtained that for a finite abelian group the similarity holds since there is an algebra isomorphism between $HH^*(kG, kG)$ and $kG \otimes H^*(G, k)$, while for Hopf bimodules the category $\mathcal{B}_k(kG)$ is equivalent to $[\mathrm{mod}\, kG]G$: for abelian finite groups both multiplicative structures –cup product in Hochschild cohomology and tensor product of Hopf bimodules– are obtained by grading over G simpler multiplicative structures (cup product in usual cohomology and tensor product of usual representations).

In order to state a precise conjecture for the Hochschild cohomology algebra of a finite group, it is necessary to understand differently the modules $X^C_{A,B}$ of the preceding section, I am indebted to J. Thévenaz for pointing out the following

Lemma 4.1 *Let G be a finite group and let k be a ring. Let A and B be conjugacy classes and let a and b be chosen elements in A and B. Let M and N be right kZ_a and kZ_b-modules, and let C be a conjugacy class with chosen element c. The kZ_C-module*

$$X^C_{A,B} = \bigoplus_{(K,L) \in E^C_{A,B}} (M \otimes K) \otimes (N \otimes L) \subset \left(M{\uparrow}^G_{Z_a} \otimes N \uparrow^G_{Z_b} \right) {\downarrow}_{Z_C}$$

is isomorphic to

$$\bigoplus_{E^C_{A,B}/Z_c} \left(M^K {\downarrow}_{Z_a^K \cap Z_b^L} \otimes N^L {\downarrow}_{Z_a^K \cap Z_b^L} \right) {\uparrow}^{Z_c}_{Z_a^K \cap Z_b^L}$$

Recall that $E^C_{A,B} = \{(K, L) \in Z_a \backslash G \times Z_b \backslash G \mid K^{-1} a K L^{-1} b L = c\}$ and that Z_C acts on the right on this set. The latter direct sum is obtained by chosing one element in each orbit of $E^C_{A,B}$ under this action. The module

M^K denotes the same underlying k-module than M, equipped with the right action of $Z_a^K = K^{-1} Z_a K = Z_{K^{-1}aK}$ defined by $(K^{-1}aK)\, M = am$.

Proof : The module $X_{A,B}^C$ decomposes along the orbits of the action of Z_C on $E_{A,B}^C$. Each resulting piece corresponds to the criteria of being and induced module –see for instance [6]– since its direct summands are transitively permuted by Z_C and the stabilizer of (K, L) is $Z_a^K \cap Z_b^L$.

We construct now an algebra based on the usual cohomology algebras of the centralizers of conjugacy classes in a very similar way we have defined the category $\mathcal{M}(kG)$.

Let $\mathbf{M}(kG) = \bigoplus_{C \in \mathcal{C}} H^* \left(Z_{u(C)},\, k \right) C$, that is $\mathbf{M}(kG)$ is the k-module of formal linear combinations of conjugacy classes with coefficients in the respective cohomology group (recall that u is a choice of an element in each conjugacy class). Let mA and nB be elementary elements in $\mathbf{M}(kG)$, with $m \in H^*(Z_a, k)$ and $n \in H^*(Z_b, k)$. The product $mA \cdot nB = \sum_{C \in \mathcal{C}} x_{\mathrm{A,B}}^{\mathrm{C}} C$ is defined by

$$x_{\mathrm{A,B}}^{\mathrm{C}} = \sum_{E_{A,B}^C / Z_c} \mathrm{Ind}_{Z_a^K \cap Z_b^L}^{Z_c} \left(\mathrm{Res}_{Z_a^K \cap Z_b^L}^{Z_a^K} m^K \smile \mathrm{Res}_{Z_a^L \cap Z_b^L}^{Z_a^L} n^L \right)$$

where m^K and n^L denotes elements corresponding to m and n in $H^*(Z_a^K, k)$ and $H^*(Z_b^L, k)$ obtained through conjugation, Ind and Res are the induction and restriction maps defined in cohomology, and $\smile$ is the cup product in the corresponding cohomology algebra.

Conjecture : The Hochschild cohomology algebra $HH^*(kG, kG)$ is isomorphic to $\mathbf{M}(kG)$.

References

[1] Benson, D.J.: Representations and cohomology. I:Basic representation theory of finite groups and associative algebras. Cambridge University Press. 1991

[2] Bruguières A., Théorie tannakienne non commutative. Comm.in Algebra **22**, 5817–5860 (1994)

[3] Cartan, H., Eilenberg, S.:Homological algebra. Oxford University Press. 1956

[4] Cibils C., Rosso M.: Algèbres des chemins quantiques. Publication interne, Genève (1993) et prépublication de l'IRMA 047, Strasbourg (1993). To appear in Advances in Maths.

[5] Cibils, C., Solotar, A.: Hochschild cohomology algebra and Hopf bimodules of an abelian group. Prepublication.

[6] Curtis, C.W., Reiner, I.: Methods of representation theory, Pure & applied mathematics, Wiley-interscience, New York, 1981

[7] Deligne, P. : Catégories tannakiennes. Progress in Mathematics. The Grothendieck Festschrift. Vol II, pp 110–195. Birkhäuser 1990

[8] Dijkgraaf, D. Pasquier,V. Roche, P.: Quasi Hopf algebras, group cohomology and orbifold models. Nuclear Physics B (Proc. Suppl.) **18B** 60–72, (1990)

[9] Drinfeld, V.G.: Quantum groups. Proceedings of the International Congress of Mathematicians. Vol. 1, pp. 798–820. Berkeley: Academic Press 1986

[10] Jones, V.: Fusion en algèbres de von Neumann et groupes de lacets [d'après A. Wassermann]. Séminaire Bourbaki **800** 1995

[11] Kassel, C.: Quantum groups. Grad. Texts in Math. **155** 1995

[12] Luzstig, G. Leading coefficients of character values of Hecke algebras. Arcata Conference on Representations of Finite Groups. Proc. of Symp. in Pure Math. **47** A.M.S. 1987

[13] Nichols, W.D.: Bialgebras of type one. Commun. Algebra **6**, 1521–1552 (1978)

[14] Rosso, M.: Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif. Duke Math. J. **61**, 11–40 (1978)

[15] Rosso, M.: Groupes quantiques et algèbres de battages quantiques, C.R. Acad. Sci. Paris **320**, 145–148 (1995)

[16] Wassermann, A.: Fusion for von Neumann algebras and loop groups, to appear.

[17] Woronowicz, S.L.: Differential calculus on compact matrix pseudogroups (quantum groups). Commun. Math. Phys. **122**, 125–170 (1989)

Section de Mathématiques, Université de Genève, C.P. 240, CH-1211 Genève 24.

Institut Fourier, Laboratoire de Mathématiques, URA 188 du CNRS, BP 74, 38402 St Martin d'Hères Cedex (France).

`cibils@uni2c3.unige.ch`

July 26, 1995