Group actions on the Dolbeault cohomology of homogeneous manifolds

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1 Introduction

Let M be a compact complex manifold, $\mathcal{O} = \mathcal{O}_M$ its structure sheaf, $\Omega^p = \Omega^p_M$ the sheaf of germs of local holomorphic p-forms on M, and $H^{p,q}(M) = H^q(M, \Omega^p)$ the Dolbeault cohomology spaces. If G is a complex Lie group acting holomorphically on M, then G has a natural representation on $H^{p,q}(M)$. Since M is compact, the spaces $H^{p,q}(M)$ are finite-dimensional and, as one can easily show, the representations of G on $H^{p,q}(M)$ are holomorphic (see e.g. [1], § 4.1). For M Kähler we have the Hodge decomposition

$$H^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M). \quad (1)$$

If G is connected then G acts trivially on $H^r(M, \mathbb{C})$ by the homotopy argument. Thus (1) shows that $H^{p,q}(M)$ is a trivial G-module for all p, q. However, for an arbitrary M this is in general not the case. Namely, F.Lescure [10] constructed recently examples of non-trivial actions on $H^1(M, \mathcal{O})$. In his first example, $\dim M = 4$ and M is acted on by $\mathrm{SL}(2, \mathbb{C})$ in such a way that all orbits have dimension 2. This action induces a non-trivial representation of $\mathrm{SL}(2, \mathbb{C})$ on $H^1(M, \mathcal{O})$. In his second example, M is a homogeneous manifold of the form $M = G/\Gamma$, where G is a connected solvable complex Lie group of dimension 3, Γ a cocompact discrete subgroup in G, and the G-action on $H^1(M, \mathcal{O})$ is again non-trivial. A natural question arising from these two examples is the following one. Assume that G is a connected semisimple complex Lie group or, more generally, a reductive linear algebraic group over $\mathbb{C}$. Let $\Gamma \subset G$ be a cocompact discrete subgroup. Is it then possible that the induced G-action on $H^1(G/\Gamma, \mathcal{O})$ is non-trivial? The answer turns out to be negative. Moreover, we have the following result.

Theorem 1 *Let G be a connected reductive linear algebraic group over $\mathbb{C}$, $\mathfrak{g}$ the Lie algebra of G, $\Gamma \subset G$ a cocompact discrete subgroup. Then there is an isomorphism of G-modules*

$$H^{p,q}(G/\Gamma) \simeq H^q(\Gamma, \mathbb{C}) \otimes \wedge^p(\mathfrak{g}), \tag{2}$$

where G acts trivially on $H^q(\Gamma, \mathbb{C})$ and the action on $\wedge^p(\mathfrak{g})$ is induced by the adjoint representation on $\mathfrak{g}$. In particular, $H^{0,q}(G/\Gamma)$ is a trivial G-module.

As a consequence, we obtain Raghunathan's vanishing theorem for one-dimensional cohomology, see Section 3. We also generalize Theorem 1 (for $p = 0$) to arbitrary compact complex homogeneous manifolds of reductive linear algebraic groups.

Theorem 2 *Let G be as in Theorem 1, $H \subset G$ a closed complex Lie subgroup, and assume that G/H is compact. Let $H^\circ \subset H$ be the connected component of the identity element, P the normalizer of H° in G. There exists a connected reductive algebraic subgroup $G^\star \subset P$, such that $P = G^\star \cdot H^\circ$ and $G^\star \cap H$ is discrete. The G-action on $H^q(G/H, \mathcal{O})$ is trivial and*

$$H^q(G/H, \mathcal{O}) \simeq H^q(\Gamma^\star, \mathbb{C}),$$

where $\Gamma^\star = G^\star \cap H$.

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2 Preliminaries

This section contains some known results, which will be used later on. Let $\mathfrak{g}$ be a real Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We identify $\mathfrak{g}_{\mathbb{C}}$ with its image in $U(\mathfrak{g})$ and denote by $U^+(\mathfrak{g})$ the ideal in $U(\mathfrak{g})$ generated by $\mathfrak{g}_{\mathbb{C}}$. Let $X \mapsto X^t$ be the principal anti-automorphism of $U(\mathfrak{g})$ defined by $X^t = -X$ for $X \in \mathfrak{g}_{\mathbb{C}}$. Let $X \mapsto \bar{X}$ be the complex conjugation in $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$, extended canonically to $U(\mathfrak{g})$. Note that $Z(\mathfrak{g})$ is invariant under these two mappings.

A $U(\mathfrak{g})$-module M is said to be a module with infinitesimal character χ_M if $Xv = \chi_M(X)v$ for all $X \in Z(\mathfrak{g}), v \in M$, where $\chi_M : Z(\mathfrak{g}) \to \mathbb{C}$ is a homomorphism of algebras over $\mathbb{C}$ with a unit. By a trivial infinitesimal character we mean the homomorphism $Z(\mathfrak{g}) \to \mathbb{C}$, whose kernel coincides with $Z(\mathfrak{g}) \cap U^+(\mathfrak{g})$. One example of a $U(\mathfrak{g})$-module with infinitesimal character is given by an irreducible representation $\mathfrak{g} \to \mathfrak{gl}(V)$, where V is a finite-dimensional complex vector space. This representation extends to a representation of $U(\mathfrak{g})$ on V and, by Schur's lemma, $Z(\mathfrak{g})$ acts on V by scalar operators. Another example comes from the theory of unitary representations. Namely, let G be a real Lie group with Lie algebra $\mathfrak{g}$ and let H be a topologically irreducible unitary G-module. Then the subspace of differentiable vectors $H^\infty \subset H$ is a $U(\mathfrak{g})$-module with infinitesimal character (see e.g. [9], §11.3). In this case we write χ_H instead of χ_{H^∞}. Since H is unitary, we have

$$\chi_H(X) = \overline{\chi_H(\bar{X}^t)}. \tag{3}$$

In what follows we are interested in the special case, when G is itself a complex Lie group. Then $\mathfrak{g}$ also has a complex structure, i.e., there is a linear mapping $J : \mathfrak{g} \to \mathfrak{g}$, such that $[JX, Y] = J[X, Y]$ and $J^2 = -\mathrm{Id}$. The complexification $\mathfrak{g}_\mathbb{C}$ decomposes into the sum of two ideals,

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\mathfrak{g}_1 = \{X - iJX \mid X \in \mathfrak{g}\}, \quad \mathfrak{g}_2 = \{X + iJX \mid X \in \mathfrak{g}\}.$$

Fix a maximal compact subgroup $K \subset G$ and let $\mathfrak{k} \subset \mathfrak{g}$ be the corresponding Lie subalgebra. Recall that a connected complex Lie group G has a structure of a reductive linear algebraic group if and only if $\mathfrak{g} = \mathfrak{k} \oplus J\mathfrak{k}$.

Lemma 1 *Let G be a connected complex Lie group, V an irreducible holomorphic finite-dimensional G-module, and H a topologically irreducible unitary G-module. If $\chi_H = \chi_V$, then both infinitesimal characters are trivial. Moreover, if G is linear algebraic and reductive then $V = \mathbb{C}$ and G acts trivially on V.*

Proof We have $U(\mathfrak{g}) = U_1 \otimes U_2$ and $Z(\mathfrak{g}) = Z_1 \otimes Z_2$, where U_i is the universal enveloping algebra of $\mathfrak{g}_i$, Z_i the center of U_i, $i = 1, 2$. The complex conjugation $X \mapsto \bar{X}$ interchanges the subalgebras $Z_1 \otimes 1$ and $1 \otimes Z_2$, whereas the map $X \mapsto X^t$ leaves them stable. Since V is a holomorphic G-module, $(JX)v = iXv$ for $X \in \mathfrak{g}, v \in V$. Therefore χ_V is trivial on $1 \otimes Z_2$. But $\chi_V = \chi_H$, and (3) shows that χ_V is also trivial on $Z_1 \otimes 1$.

Assume now that G is linear algebraic and reductive. Choosing an appropriate G-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$, we get a

Casimir element of the form

$$C = -\sum_j X_j^2 + \sum_j (JX_j)^2,$$

where $\{X_j\}$ is a basis of $\mathfrak{k}$. Let $(\cdot,\cdot)$ denote a K-invariant Hermitian inner product in V. Then

$$0 = \chi_V(C)(v,v) = (Cv,v) = 2\sum_j (X_j v, X_j v) \qquad (v \in V),$$

and so we obtain that V is a trivial $\mathfrak{k}$-module. Since G is connected, G acts trivially on V. □

Lemma 2 *Let G be a connected reductive linear algebraic group, V a non-trivial irreducible holomorphic finite-dimensional G-module, and H a topologically irreducible unitary G-module. Then*

$$H^*(\mathfrak{g}, \mathfrak{k}, H^\infty \otimes V) = \{0\}.$$

Proof A vanishing theorem in [4], Ch. 1, § 4.1, tells us that $H^*(\mathfrak{g}, \mathfrak{k}, H^\infty \otimes V)$ may be non-trivial only if $\chi_{V^*} = \chi_H$, where V^* is the dual to the G-module V. In view of Lemma 1 this is in fact impossible. □

Lemma 3 *Let G and V be as in Lemma 2 and let $\Gamma \subset G$ be a discrete cocompact subgroup. Then $H^*(\Gamma, V) = \{0\}$.*

Proof The cohomology of a discrete group Γ can be expressed in terms of relative Lie algebra cohomology (see [12], [13] and [4], Ch. 7, § 2.7). Namely,

$$H^*(\Gamma, V) \simeq H^*(\mathfrak{g}, \mathfrak{k}, C^\infty(G/\Gamma) \otimes V) = H^*(\mathfrak{g}, \mathfrak{k}, (L^2(G/\Gamma))^\infty \otimes V).$$

Let $\hat{G}$ be the set of equivalence classes of topologically irreducible unitary representations of G. For each $\pi \in \hat{G}$ choose a representative H_π. Then $L^2(G/\Gamma)$ decomposes into a discrete Hilbert direct sum of H_π with some finite multiplicities $m(\pi, \Gamma)$ (see [6], Ch. 1, § 2.3). An argument using the finiteness of dim $H^*(\Gamma, V)$ shows that $H^*(\Gamma, V)$ decomposes into an algebraic direct sum of the corresponding Lie algebra cohomology spaces, namely

$$H^*(\Gamma, V) \simeq \oplus_{\pi \in \hat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, \mathfrak{k}, H_\pi^\infty \otimes V)$$

(see [4], Ch. 7, §3.2). The result follows from Lemma 2. □

3 Parallelizable manifolds

For the proof of Theorem 1 we shall need a spectral sequence, which was constructed by A.Grothendieck in a more general context (see [7]).

Let $(X, \mathcal{O}_X)$ be a reduced complex space, L a group acting on X by biholomorphic automorphisms, and $\mathcal{S}$ a sheaf of $\mathcal{O}_X$-modules. Recall that $\mathcal{S}$ is called a L-sheaf if L acts on $\mathcal{S}$ in such a way, that this action commutes with the projection map $\mathcal{S} \to X$ and is compatible with the natural L-action on $\mathcal{O}_X$. As usual, we denote by $\mathcal{S}(U)$ the space of sections of $\mathcal{S}$ over $U \subset X$ and write $l : \mathcal{S}(U) \to \mathcal{S}(l \cdot U)$ for the isomorphism induced by $l \in L$. In our situation L will be a direct product of the form $L = G \times \Gamma$, where Γ is a discrete group acting on X properly discontinuously and freely. Let $Y = X/\Gamma$ be the quotient considered with the natural complex structure and let $\mathcal{S}^\Gamma$ be the sheaf of invariant elements of $\mathcal{S}$, i.e.,

$$\mathcal{S}^\Gamma(U) = \{s \in \mathcal{S}(\pi^{-1}(U)) \mid \gamma \cdot s = s \quad \text{for all} \quad \gamma \in \Gamma\}$$

for $U \subset Y$. Then G acts on Y in a natural way and $\mathcal{S}^\Gamma$ is a G-sheaf of $\mathcal{O}_Y$-modules.

We now apply Theorem 2.4.1 of [7] to the sequence

$$\mathrm{C} \overset{\Phi}{\to} \mathrm{C}' \overset{\Psi}{\to} \mathrm{C}'',$$

where C is the category of L-sheaves of $\mathcal{O}_X$-modules, C′ the category of L-modules, C″ the category of G-modules, $\Phi(\mathcal{S}) = \mathcal{S}(X)$ the functor of global sections, and $\Psi(V) = V^\Gamma$ the functor of Γ-invariants.

This gives us the following result (cf. [7], Ch. 5, § 5.3, Cor. 3):

(*) *for any L-sheaf of $\mathcal{O}_X$-modules $\mathcal{S}$ there exists a spectral sequence of G-modules converging to $H^*(Y, \mathcal{S}^\Gamma)$, whose second term is given by*

$$E_2^{p,q} = H^p(\Gamma, H^q(X, \mathcal{S})),$$

with the G-action arising from the induced G-action on the coefficient group.

Proof of Theorem 1 Let G be a connected reductive linear algebraic group, $\Gamma \subset G$ a cocompact discrete subgroup. We apply (*) to $X = G$ with G acting on the left and Γ on the right. For $\mathcal{S} := \Omega^p_G$ and $q > 0$ we have $H^q(G, \mathcal{S}) = \{0\}$, because G is a Stein manifold. Therefore the spectral sequence degenerates and $E_2 = E_\infty$. Since Γ is discrete, $\mathcal{S}^\Gamma = \Omega^p_M$, where

$M = Y = G/\Gamma$. Therefore we obtain an isomorphism of G-modules

$$H^r(M, \Omega^p_M) \simeq H^r(\Gamma, \Omega^p(G)), \tag{4}$$

where Γ acts on $\Omega^p(G)$ by the right translations. The G-action on the group cohomology of Γ arises from the left G-action on $\Omega^p(G)$.

Each holomorphic form on G is a linear combination of right invariant holomorphic forms with holomorphic coefficients. As a $(G \times G)$-module,

$$\Omega^p(G) \simeq \mathcal{O}(G) \otimes \wedge^p(\mathfrak{g}^*) \simeq \mathcal{O}(G) \otimes \wedge^p(\mathfrak{g}),$$

where $G \times G$ acts on $\mathcal{O}(G)$ by the left and right translations and the action on $\wedge^p(\mathfrak{g})$ is adjoint on the first factor and trivial on the second one. Thus

$$H^r(\Gamma, \Omega^p(G)) \simeq H^r(\Gamma, \mathcal{O}(G)) \otimes \wedge^p(\mathfrak{g}), \tag{5}$$

where Γ acts on $\mathcal{O}(G)$ by the right translations and G by the left ones. Now, $\mathcal{O}(G)$ is a completion of the algebraic direct sum

$$\mathcal{O}_{\,reg}\,(G) = \oplus_V\,(V^* \otimes V),$$

where V ranges over irreducible holomorphic G-modules, each summand $V^* \otimes V$ is $(G \times G)$-stable, and the left (resp. right) translations act on the first (resp. second) factor. Since $H^r(\Gamma, \mathcal{O}(G))$ is finite-dimensional, we have

$$H^r(\Gamma, \mathcal{O}(G)) = \oplus_V\, H^r(\Gamma, V^* \otimes V) = \oplus_V\, V^* \otimes H^r(\Gamma, V) = H^r(\Gamma, \mathbb{C}),$$

where the last equality follows from Lemma 3. Substituting this in (5) and using (4), we obtain (2). □

Corollary 1 (M.S.Raghunathan [14]) *Let G be a connected semisimple complex Lie group having no epimorphism onto* PSL$(2, \mathbb{C})$. *Let $\Gamma \subset G$ be a cocompact discrete subgroup. Then $H^1(G/\Gamma, \Omega^p) = \{0\}$.*

Proof If G is a connected semisimple real Lie group with finite center, no compact factor and no factor of rank one, then a theorem of Y.Matsushima yields $H^1(\Gamma, \mathbb{C}) = \{0\}$ for any discrete cocompact subgroup $\Gamma \subset G$ (see [11] and [4], Ch. 7, §4.4). In particular, $H^1(\Gamma, \mathbb{C}) = \{0\}$ when G is a complex semisimple Lie group without three-dimensional factors. Therefore the vanishing of $H^1(G/\Gamma, \Omega^p)$ follows from (2). □

Let $\{E_k, d_k\}$ $(k = 1, 2, \ldots)$ be the Hodge-Frölicher spectral sequence of a complex manifold M. Recall that $E_1^{p,q} = H^{p,q}(M)$ and that

$$H^r(M, \mathbb{C}) = \bigoplus_{p+q=r} E_\infty^{p,q}.$$

In the setting of Theorem 1 one can use (2) in order to understand the Hodge-Frölicher spectral sequence of $M = G/\Gamma$. The following corollary, generalizing a result of J.Winkelmann, is an example.

Corollary 2 (cf. [16], Part B, §§10,11) *Let* $G = \mathrm{SL}(2, \mathbb{C})$, $\Gamma \subset G$ *a cocompact discrete subgroup, and* $M = G/\Gamma$. *Then*

$$E_2^{0,q} = E_2^{3,q} = H^q(\Gamma, \mathbb{C}), \qquad E_2^{p,q} = \{0\} \quad (p \neq 0, 3),$$

and, consequently, $d_2 = 0$. *Moreover,* $d_k = 0$ *for all* $k \geq 2$. *Let* $l = \dim\ H^1(\Gamma, \mathbb{C})$. *Then* $\dim\ H^2(\Gamma, \mathbb{C}) = l$, $\dim\ H^3(\Gamma, \mathbb{C}) = 1$, *and* $H^q(\Gamma, \mathbb{C}) = \{0\}$ *for* $q > 3$. *The manifold* M *has the following Betti numbers:*

$$b_0 = b_6 = 1, \quad b_1 = b_2 = b_4 = b_5 = l, \quad b_3 = 2.$$

Proof The spaces E_k are bigraded G-modules, the differentials d_k commute with G-action. Since $\wedge^2(\mathfrak{g}) \simeq \mathfrak{g}$, there are only two types of irreducible G-modules occuring in E_1. Namely, $E_1^{1,q}$, $E_1^{2,q}$ are multiples of the adjoint G-module and $E_1^{0,q}$, $E_1^{3,q}$ are trivial G-modules. Since all other terms of E_1 are zero, d_1 equals 0 on $E_1^{0,q}$ and $E_1^{3,q}$. Since the G-action on E_∞ is trivial, d_1 defines an isomorphism between $E_1^{1,q}$ and $E_1^{2,q}$. From this we get the above expression for $E_2^{p,q}$, and it follows that $d_2 = 0$. Furthermore, we observe that $d_k = 0$ for $k \geq 4$.

Let $l_i = \dim\ H^i(\Gamma, \mathbb{C})$. It is an immediate consequence of Theorem 1 that $l_i = 0$ for $i > 3$. In the spectral sequence of the regular covering

$$G \to G/\Gamma = M$$

the second term is equal to $H^*(\Gamma, \mathbb{C}) \otimes H^*(K, \mathbb{C})$, where $K = \mathrm{SU}(2)$. Since $H^i(K, \mathbb{C}) = \{0\}$ for $i \neq 0, 3$, it follows that this spectral sequence degenerates. Thus $H^*(M, \mathbb{C}) = H^*(\Gamma, \mathbb{C}) \otimes H^*(K, \mathbb{C})$ as vector spaces and, consequently, $b_i = l_i$ for $i = 0, 1, 2$, $b_3 = l_3 + 1$ and $b_i = l_{i-3}$ for $i = 4, 5, 6$. Since $b_i = b_{6-i}$ by the Poincaré duality, this yields $l_2 = l_1$, $l_3 = l_0$, and we obtain the above values of b_i.

We still have to prove that $d_3 = 0$ in the Hodge-Frölicher spectral sequence. This reduces to $d_3^{0,2} = 0$ and $d_3^{0,3} = 0$. Assuming $d_3^{0,2} \neq 0$, we get $b_2 = \dim\ E_\infty^{0,2} < \dim\ E_3^{0,2} = \dim\ E_2^{0,2} = l$, contradictory to what we have seen above. Similarly, if $d_3^{0,3} \neq 0$ then $b_4 = \dim\ E_\infty^{3,1} < \dim\ E_3^{3,1} = \dim\ E_2^{3,1} = l$, again a contradiction. □

4 Homogeneous manifolds of general type

We employ the notation introduced in Section 1. In particular, G is a connected reductive linear algebraic group, $H \subset G$ is a closed complex Lie subgroup such that G/H is compact, and P is the normalizer of H° in G. The proof of Theorem 2 is based on the following well-known facts:

(i) *P is a parabolic subgroup in G* (see [15], [3]);

(ii) *H contains the unipotent radical $U = U_P$ of P* (see [8]);

(iii) *$H^p(G/P, \mathcal{O}) = \{0\}$ for $p > 0$* (cf. [5] or [2]).

We shall also need an elementary lemma.

Lemma 4 *Let T be an algebraic torus and let $Z \subset T$ be a closed complex Lie subgroup. There exists a subtorus $A \subset T$ such that $T = A \cdot Z$ and $A \cap Z$ is discrete.*

Proof We identify T with $(\mathbb{C}^*)^r$ and denote by $\pi : \mathbb{C}^r \to (\mathbb{C}^*)^r$ the universal covering map, given by $\pi(z_1, \ldots, z_r) = (\exp\, 2\pi i z_1, \ldots, \exp\, 2\pi i z_r)$. The connected component of $\pi^{-1}(Z)$ is a complex subspace in $\mathbb{C}^r$. Denote this subspace by W and let $l := \dim\, W = \dim\, Z$. There exist $r - l$ vectors $v_1, \ldots, v_{r-l} \in \mathbb{Z}^r$, such that

$$\mathbb{C}^r = \mathbb{C}v_1 \oplus \ldots \oplus \mathbb{C}v_{r-l} \oplus W.$$

Then

$$A := \pi(\mathbb{C}v_1 \oplus \ldots \oplus \mathbb{C}v_{r-l})$$

is a subtorus in T. Clearly, $T = A \cdot Z$ and $A \cap Z$ is discrete. □

Proof of Theorem 2 We start by proving the existence of a connected reductive algebraic subgroup $G^\star \subset P$ such that $P = G^\star \cdot H^\circ$ and $G^\star \cap H$ is discrete. Let L be a reductive Levi subgroup of P and write

$$L = T \cdot \prod_{\iota \in I} S_\iota,$$

where T is an algebraic torus and S_ι, $\iota \in I$, are simple factors. It follows from (ii) that H° is of the form

$$H^\circ = Z \cdot (\prod_{\iota \in J} S_\iota) \cdot U,$$

where $J \subset I$ and Z is a connected closed complex Lie subgroup in T. By Lemma 4 we can find a subtorus $A \subset T$ such that $T = A \cdot Z$ and $A \cap Z$ is discrete. Letting

$$G^\star := A \cdot \prod_{\iota \in I-J} S_\iota,$$

we get a subgroup with the desired properties.

Consider the Tits fibration

$$X := G/H \to G/P =: Y$$

with typical fiber $F := P/H$. In the corresponding Leray spectral sequence for $\mathcal{O}_X$ we have

$$E_2^{p,q} = H^p(Y, \mathcal{R}^q),$$

where $\mathcal{R}^q$ is the q-th direct image of $\mathcal{O}_X$. This is a locally free sheaf on Y associated to the homogeneous vector bundle defined by the holomorphic representation

$$P \longrightarrow \mathrm{GL}(H^q(F, \mathcal{O}_F)).$$

The fiber F can be written in the Klein form

$$F = G^\star/\Gamma^\star, \quad \Gamma^\star = G^\star \cap H,$$

and the representation of $G^\star$ on $H^q(F, \mathcal{O}_F)$ is trivial by Theorem 1. Since $P = G^\star \cdot H^\circ$ and H° is normal in P, the same is true for the representation of P on $H^q(F, \mathcal{O}_F)$. Thus $\mathcal{R}^q$ is isomorphic to some power of the structure sheaf $\mathcal{O}_Y$ and, consequently, $E_2^{p,q} = \{0\}$ for $p > 0$ by (iii). Hence

$$H^q(X, \mathcal{O}_X) \simeq E_\infty^{0,q} \simeq E_2^{0,q} = H^0(Y, \mathcal{R}^q),$$

where all isomorphisms are isomorphisms of G-modules. Since P acts trivially on $H^q(F, \mathcal{O}_F)$, the induced G-action on $H^0(Y, \mathcal{R}^q)$ is also trivial. Therefore G acts trivially on $H^q(X, \mathcal{O}_X)$ and

$$H^q(X, \mathcal{O}_X) \simeq H^0(Y, \mathcal{O}_Y) \otimes H^q(F, \mathcal{O}_F) \simeq H^q(\Gamma^\star, \mathbb{C})$$

by Theorem 1. □

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