SOME ULTRABORNOLOGICAL NORMED FUNCTION SPACES

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ABSTRACT. — A variety of normed function spaces, including the space $E_0(X)$ of continuous functions on the compact Hausdorff space X that are locally constant on a dense open subset of X, are shown to be ultrabornological.

1. Introduction. — Our main objective is to prove that various uniformly normed, but not (in general) complete, spaces of continuous functions are ultrabornological. Our particular interest is in the space $E_0(X)$ of continuous functions on the compact Hausdorff space X that are locally constant on a (varying with the function) dense open subset of X.

The path leading us to this point has its origin in a study by one of us [1] of functions that operate on function spaces. The essential features of [1], Lemma 12, were later isolated in the following result [2], Theorem 1: *if a nonconstant continuous function* φ *on an interval* $[a, b]$ *operates from* $C(X)$ *to a linear subspace* E *of* $C(X)$ *in the sense that* $\varphi \circ u \in E$ *whenever* $u \in C(X)$ *and* $u(X) \subset [a, b]$*, then* E*, endowed with the supremum norm, is a barreled normed linear space.* In particular, it follows that $E_0(X)$ is barreled (and point-separating) if X has a dense open locally connected subset, for then any $\varphi \in E_0([a, b])$ operates from $C(X)$ to $E_0(X)$. By a very different argument, it is shown in [7], Theorem 2, that $E_0(X)$ is barreled whenever X is metrizable. This argument can be adapted to apply to *all* X, even those for which $E_0(X)$ is not point-separating. $E_0(X)$ *is* point-separating if X is metrizable ([6], [7]), but M. E. Rudin and W. Rudin have constructed an X, not a singleton, for which $E_0(X)$ reduces to the constant functions [6]. Recently the point-separation results on $E_0(X)$ have been elaborated and extended by J. Hart and K. Kunen [5], who also study a variety of other aspects of the spaces $E_0(X)$ and some of their close relatives.

It turns out that the spaces $E_0(X)$, and many other spaces to which the method of [7] applies, are in fact ultrabornological, which implies that they are barreled. For $X = [a, b]$, this follows from Theorem 2.4 of the beautiful and fundamental paper [4] by

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A. Gilioli, who used a powerful method based on a family of projections $\{P_\lambda\}_{\lambda\in[a,b]}$ to treat a large number of spaces of vector-valued functions, including both some spaces of continuous functions and some spaces of integrable functions. Independently, a similar projection method was developed in [3] by S. Díaz, A. Fernández, M. Florencio and P. J. Paúl, who used their method to treat many additional spaces. We heartily recommend the papers [4] and [3] to those interested in ultrabornological function spaces.

Our approach is more topological than those in [4] and [3]. All three papers rely on some form of localization : if things go wrong, there must be a point at or near which they go wrong. Gilioli's approach is especially well suited to spaces of functions defined on intervals ; in this setting, his method succeeds in pretty much any situation in which ours does, as well as in many to which ours does not apply. The approach of Díaz *et al.* works particularly well on rather general spaces of integrable functions. Our method yields good results for spaces of continuous functions on general compact Hausdorff spaces.

This paper is organized in the following way. In the next section we review various notions from the theory of topological vector spaces, and introduce the spaces $E_0(X)$ formally. The heart of the paper is section 3, in which it is proven that if X is metrizable then $E_0(X)$ is ultrabornological. This proof shows very clearly how our method works. Sections 4 and 5 extend this result to a variety of other spaces, including $E_0(X)$ when X is *not* metrizable.

While our normed spaces will consist of scalar-valued functions, it will be clear that the results and proofs extend to spaces of vector-valued functions with values in a normed space. Similarly, it will be clear that Theorem 4.1 will remain valid if E has any norm that dominates the supremum norm and the F_α have norms dominating the norm inherited from E; in fact, a version of the theorem in which E is a Hausdorff locally convex topological vector space can be proven.

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2. Some definitions and terminology. — We recall here some definitions and facts. By an *LCS* we shall understand a Hausdorff locally convex topological vector space. A closed absorbent absolutely convex subset of an LCS E is called a *barrel* in E; E is said to be *barreled* if every barrel in E is a neighborhood of 0 in E. Banach spaces, or more generally LCSs which are of the second category in themselves, provide standard examples of barreled spaces.

Let E be an LCS and let $\{(F_\alpha,\tau_\alpha)\}_{\alpha\in A}$ be an indexed family of ordered pairs (F_α,τ_α) in which for each α, F_α is an LCS and τ_α is a linear mapping of F_α into E. E is the *locally convex inductive limit* of the family $\{(F_\alpha,\tau_\alpha)\}_{\alpha\in A}$ if the sets of the form

$H(\{U_\alpha\}_{\alpha\in A}) := \mathrm{conv}\Big(\bigcup_{\alpha\in A}\tau_\alpha(U_\alpha)\Big)$ ("conv" denotes convex hull), where for each α, U_α is an absolutely convex neighborhood of 0 in F_α, constitute a base at 0 for the topology of E. This amounts to requiring that each τ_α be continuous and that each $H(\{U_\alpha\})$ be a neighborhood of 0 in E, and implies that the union of the $\tau_\alpha(F_\alpha)$ spans E; the locally convex inductive limit topology on E is the finest locally convex topology on E that makes all the τ_α continuous.

The LCS E is *ultrabornological* if it is the locally convex inductive limit of a family $\{(F_\alpha,\tau_\alpha)\}_{\alpha\in A}$ in which each F_α is a Banach space. If in this definition we replace "Banach space" by "normed linear space," then E is said to be *bornological*. Bornological LCSs are precisely those from which every linear mapping into another LCS that takes bounded sets into bounded sets is necessarily continuous; every metrizable LCS is bornological, and every complete bornological LCS is ultrabornological. It is easy to check that an ultrabornological LCS must be barreled.

If an LCS E is the locally convex inductive limit of a family $\{(F_\alpha,\tau_\alpha)\}_{\alpha\in A}$ of LCSs and mappings, and if Λ is a linear mapping from E into some LCS, then Λ is continuous if and only if every mapping $\Lambda\circ\tau_\alpha$ is continuous. We shall use this fact in the proof of Corollary 3.2. In the metrizable case this fact has a converse which we will not need: if E is a metrizable LCS and $\{(F_\alpha,\tau_\alpha)\}$ is a family of LCSs F_α and linear mappings $\tau_\alpha : F_\alpha \to E$ such that any linear mapping Λ from E into an LCS is continuous iff every $\Lambda\circ\tau_\alpha$ is continuous, then E is the locally convex inductive limit of $\{(F_\alpha,\tau_\alpha)\}$.

If X is a compact Hausdorff space, $C(X)$ denotes the uniformly normed Banach algebra of continuous scalar-valued functions on X (the scalars may be real or complex), and $M(X)$ is its dual space, the space of regular scalar-valued Borel measures on X in the total variation norm. If $u\in C(X)$, Ω_u denotes the union of all open subsets of X on which u is constant; thus u is locally constant, but not necessarily constant, on Ω_u. Let $E_0(X) := \{u\in C(X) : \Omega_u \text{ is dense in } X\}$, a self-adjoint subalgebra of $C(X)$. If $a<b$, $u\in C([a,b])$ belongs to $E_0([a,b])$ precisely if $[a,b]$ contains a Cantor set on each of whose complementary intervals u is constant.

3. The basic proof, and $E_0(X)$ for metrizable X. — The basic argument we use is best seen, shorn of technical complications, in the proof that $E_0(X)$ is ultrabornological if X is metrizable. It is similar in spirit to a "sliding hump" argument used in [7]. Here we present this proof and a corollary, and later we will describe the modifications involved in extending it to other situations.

THEOREM 3.1. — *If X is a metrizable compact space, then $E_0(X)$, endowed with the supremum norm, is ultrabornological.*

Proof. — Consider a family S of nonempty open subsets of X such that $\Omega(S) :=$

$\cup\{U : U \in S\}$ is dense in X; $\mathcal{S}$ will denote the set of all such families. For $S \in \mathcal{S}$ let $E_S := \{u \in C(X) : u$ is constant on each $U \in S\}$, and let σ_S denote the inclusion mapping from E_S into $E = E_0(X)$. E_S is a uniformly closed subalgrebra of $C(X)$, so is a Banach space in the supremum norm, and σ_S is trivially linear and continuous. (For certain $S \in \mathcal{S}$, for instance $S = \{X\}$, E_S may reduce to the constant functions; as a general rule, E_S does not contain all the functions in $C(X)$ that are locally constant on $\Omega(S)$.) We shall show that E is the locally convex inductive limit of the family $\{(E_S, \sigma_S)\}_{S\in\mathcal{S}}$. This amounts to proving:

(3.1.1) *If $\{\lambda_S\}_{S\in\mathcal{S}}$ is a family of strictly positive real numbers and if $B_S := \{u \in E_S : ||u|| < \lambda_S\}$, then $H = H(\{B_S\}_{S\in\mathcal{S}})$ is a neighborhood of 0 in E.*

We argue by contradiction. Suppose that for some family $\{\lambda_S\}$, H is *not* a neighborhood of 0 in E. Call a nonempty open subset V of X *bad* for H if for every $\varepsilon > 0$ there is $u \in E$ with closed support supp(u) contained in V such that $||u|| < \varepsilon$ but $u \notin H$. We claim that

(3.1.2) *X must contain a point p such that every open neighborhood of p is bad for H.*

For if (3.1.2) fails then X may be covered by finitely many nonempty open sets $V_1, \ldots, V_n$ none of which is bad for H. Thus for each $j = 1, \ldots, n$ there is $\varepsilon_j > 0$ such that if $u \in E$ satisfies supp(u) $\subset V_j$ and $||u|| < \varepsilon_j$, it follows that $u \in H$. Let $\varepsilon := n^{-1} \inf\{\varepsilon_1, \ldots, \varepsilon_n\}$. E contains nonnegative real-valued functions $g_1, \ldots, g_n$ such that $\sum_{k=1}^{n} g_j \equiv 1$ and supp(g_j) $\subset V_j$, so if $u \in E$ and $||u|| < \varepsilon$ then $ng_ju \in H$ for every j, hence $u = n^{-1} \sum_{j=1}^{n} ng_ju \in H$ by convexity, contrary to our assumption that H is not a neighborhood of 0. Thus (3.1.2) is proven.

Fix p as in (3.1.2). Let $(V_n)_{n=0}^{\infty}$ be a sequence of open neighborhoods of p that shrinks to $\{p\}$ and is such that $V_0 = X$ and $\overline{V}_n$ (the closure of V_n) is contained in V_{n-1} for $n \geq 1$. Each V_n is bad for H, so there is $u_n \in E$ such that supp(u_n) $\subset V_n$, $||u_n|| < 2^{-n}$ and $u_n \notin H$. Let S_0 denote the family of open sets U such that, for some $n \geq 0$, $U \subset V_n \setminus \overline{V}_{n+1}$ and the functions $u_0, \ldots, u_n$ all are constant on U. Then $S_0 \in \mathcal{S}$ and all the u_n belong to E_{S_0}. If n is so large that $2^{-n} < \lambda_{S_0}$, then $u_n \in B_{S_0} \subset H$, giving the desired contradiction. Thus (3.1.1) holds, and the theorem is proved. ■

The family $\{(E_S, \sigma_S)\}_{S\in\mathcal{S}}$ used in the above proof is essentially forced on us. For suppose $E_0(X)$ is the locally convex inductive limit of *some* family $\{(F_\alpha, \tau_\alpha)\}_{\alpha\in A}$ of Banach spaces and mappings. Let $\{U_k\}$ be a countable base for the topology of X with $U_k \neq \emptyset$. If $\alpha \in A$ is fixed, then a simple category argument shows that for each nonempty open set W there is at least one k for which $U_k \subset W$ and $\tau_\alpha(F_\alpha) \subset \{u \in E_0(X) : U_k \subset \Omega_u\}$; thus there is a set of integers J_α such that $S_\alpha := \{U_k : k \in J_\alpha\} \in \mathcal{S}$ and $\tau_\alpha(F_\alpha) \subset E_{S_\alpha}$. Note that a deceptive candidate for the F_α might be the spaces $F_\Omega := \{u \in C(X) : \Omega_u \supset \Omega\}$ for dense open subsets Ω of X, with τ_Ω the inclusion mapping; however, in general F_Ω is not uniformly closed (though it is if X is locally connected, since then F_Ω

consists of those continuous functions on X that are constant on the (necessarily open) components of Ω).

The case $X = [a, b]$ of Theorem 3.1 also follows from [4], as we mentioned earlier. This case has an interesting corollary, which can also be proven directly by the method of either [7] or the above proof; the corollary is essentially the dual form of the theorem for $X = [a, b]$, and inasmuch as it can be shown that the dual form actually implies the original version in some generality, the theorem for $X = [a, b]$ can in fact be deduced from the corollary.

COROLLARY 3.2. — *Let $a < b$. Suppose that for every Cantor set K contained in $[a, b]$ there is given a measure $\mu_K \in M(K)$ so that the following compatibility condition holds:*

> *Whenever K_1 and K_2 are Cantor sets contained in $[a, b]$ and $u \in C([a, b])$ is constant on each interval of $[a, b] \setminus K_1$ and on each interval of $[a, b] \setminus K_2$, it follows that $\int u d\mu_{K_1} = \int u d\mu_{K_2}$.*

Then there is a unique measure $\mu \in M([a, b])$ such that, for every Cantor set K contained in $[a, b]$ and every $u \in C([a, b])$ that is constant on each interval of $[a, b] \setminus K$,

$$\int u d\mu_K = \int u d\mu.$$

Proof. — By a Cantor set we mean a nonempty totally disconnected metrizable compact space having no isolated points. If K is a Cantor set contained in $[a, b]$, denote by $S(K)$ the set of intervals (or components) that make up $[a, b] \setminus K$. In the notation of the proof of Theorem 3.1, $S(K) \in \mathcal{S}$, and the E_S that do not reduce to the constant functions are just the $E_{S(K)}$. Thus a linear functional on $E_0([a, b])$ is continuous if and only if its restriction to each $E_{S(K)}$ is continuous.

Because $E_0([a, b])$ is the union of the $E_{S(K)}$ and $E_{S(K_1)} + E_{S(K_2)} \subset E_{S(K_1 \cup K_2)}$, the compatibility hypothesis shows that a linear functional f may be defined consistently on $E_0([a, b])$ by the rule

$$f(u) = \int u d\mu_K \text{ if } u \in E_{S(K)}.$$

Trivially the restriction of f to each $E_{S(K)}$ is continuous, so f is continuous on $E_0([a, b])$, whence there is $\mu \in M([a, b])$ such that $\int u d\mu = f(u)$ for every $u \in E_0([a, b])$. This μ is unique because, by the Stone-Weierstrass theorem, $E_0([a, b])$ is uniformly dense in $C([a, b])$. ■

In contrast to the uniqueness of μ, many different $\mu_K \in M(K)$ give the same functional on $E_{S(K)}$, except for trivial cases. Of course, the corollary remains true if $[a, b]$ is replaced by other one-dimensional spaces, such as circles.

4. Other function spaces on metrizable X. — Isolating the features of $E_0(X)$ which were crucial to the proof of Theorem 3.1, we obtain

THEOREM 4.1. — *Let X be a first countable compact Hausdorff space, let E be a linear subspace of $C(X)$ endowed with the supremum norm, and let $\{F_\alpha\}_{\alpha\in A}$ be a family of linear subspaces of E, also endowed with the supremum norm. Suppose the following two conditions hold:*

(4.1.1) To each finite open cover $\{V_1,\ldots,V_n\}$ of X we may associate a positive real number C such that whenever $u \in E$ there are $u_1,\ldots,u_n$ in E satisfying $\operatorname{supp}(u_j) \subset V_j$ and $\|u_j\| \le C\|u\|$ for each $j \in \{1,\ldots,n\}$, and $\sum_{j=1}^{n} u_j = u$.

(4.1.2) Whenever $(V_n)_{n=0}^{\infty}$ is a sequence of open subsets of X such that $\overline{V}_n \subset V_{n-1}$ for $n \ge 1$ and $\bigcap_{n=0}^{\infty} V_n$ reduces to a single point, and $u_n \in E$ satisfies $\operatorname{supp}(u_n) \subset V_n$ for every $n \ge 0$, it follows that there is $\beta \in A$ such that $u_n \in F_\beta$ for infinitely many integers $n \ge 0$.

Then E is the locally convex inductive limit of the family $\{(F_\alpha,\tau_\alpha)\}_{\alpha\in A}$ where $\tau_\alpha : F_\alpha \to E$ is the inclusion mapping. In particular, if in addition all the F_α are uniformly closed, then E is ultrabornological.

This theorem applies in particular to metrizable X. (4.1.1) is often satisfied because E is a subalgebra of $C(X)$ that contains partitions of unity such as $\{g_1,\ldots,g_n\}$ in the proof of Theorem 3.1 (but see example 4.3). Note that if $\bigcap_{u\in E} u^{-1}(\{0\}) = \emptyset$ – in particular if E contains the constant functions – then (4.1.1) implies that E separates the points of X.

We now exhibit several examples.

EXAMPLE 4.2. *Sets of constancy of full measure.* — If X is a compact Hausdorff space and $\mu \in M(X)$ is a positive nonatomic measure, set $E^\mu := \{u \in C(X) : \mu(X \setminus \Omega_u) = 0\}$ and $E_0^\mu(X) := E^\mu \cap E_0(X)$.

LEMMA. — *E^μ separates the points of X, and $E_0^\mu(X)$ separates the same pairs of points of X as does $E_0(X)$.*

Proof. — Let p and q be distinct points of X, and let $u \in C(X)$ [resp. $E_0(X)$] be such that $u(X) \subset (0,1)$ and $u(p) < u(q)$. Let

$$L := \{u(p), u(q)\} \cup \{t \in [0,1] : \mu(u^{-1}(\{t\})) > 0\},$$

a finite or countable subset of $(0,1)$. We now construct a Cantor set $K = \bigcap_{n=0}^{\infty} K_n$ in $[0,1]$ as follows. Take $K_0 = [0,1]$ and $K_1 = [0,\alpha] \cup [\beta,1]$ where $u(p) < \alpha < \beta < u(g)$

and $\{\alpha, \beta\} \cap L = \emptyset$. We will arrange that each K_n be the union of 2^n pairwise disjoint non degenerate closed intervals of length $< 2^{-n+1}$ whose endpoints are not in L, and that $K_0 \supset K_1 \supset K_2 \supset \cdots$. To accomplish this, if $n \geq 1$ and $K_0, \ldots, K_n$ have been constructed, build K_{n+1} by removing from each of the 2^n intervals $[a, b]$ comprising K_n an open subinterval (c, d) chosen so that $a < c < (a+b)/2 < d < b$, $\{c, d\} \cap L = \emptyset$, and $\mu(u^{-1}((c, d))) \geq 2^{-1}\mu(u^{-1}([a, b]))$, possible because by induction $\mu(u^{-1}(\{a, b\})) = 0$. The Cantor set so constructed satisfies $\mu(u^{-1}(K)) = 0$. Now take a nondecreasing continuous real-valued function φ on $[0, 1]$ such that φ is strictly increasing on K and is constant on each interval of $[0, 1] \setminus K$. Then $v := \gamma \circ u$ belongs to E^μ [resp. $E_0^\mu(X)$] and $v(p) < v(q)$. The lemma is proved.

Theorem 4.1 now applies to show that *if X is metrizable then E^μ and $E_0^\mu(X)$, endowed with the supremum norm, are ultrabornological*. For the family $\{F_\alpha\}_{\alpha \in A}$ we take, as in the proof of Theorem 3.1, the family $\{E_S\}_{S \in \mathcal{S}}$ where this time a member S of $\mathcal{S}$ is a family of nonempty open subsets of X such that $\mu(X \setminus \Omega(S)) = 0$ and in addition, in the case of $E_0^\mu(X)$, $\Omega(S)$ is dense in X. ■

EXAMPLE 4.3. *Piecewise affine functions.* — Most of the spaces in this example can also be handled using Theorem 2.4 of [4], as can E^μ and $E_0^\mu(X)$ when $X = [a, b]$.

Fix real numbers $a < b$ and a family $\mathcal{C}$ of nowhere dense closed subsets of $[a, b]$ that satisfies: $\mathcal{C}$ is closed under finite unions; the union of all the members of $\mathcal{C}$ is dense in $[a, b]$; and whenever (K_j) is a sequence in $\mathcal{C}$ for which $\overline{\bigcup_j K_j} = \left(\bigcup_j K_j\right) \cup \{p\}$ for some point $p \in [a, b]$, it follows that $\overline{\bigcup_j K_j} \in \mathcal{C}$. Possible choices for $\mathcal{C}$ include: all nowhere dense closed subsets of $[a, b]$; all at most countable closed subsets of $[a, b]$; all Cantor sets in $[a, b]$; or all nowhere dense closed subsets of $[a, b]$ that are null for a fixed positive nonatomic $\mu \in M([a, b])$. Then the space $E_A^{\mathcal{C}}$ of all $u \in C([a, b])$ for which there exists $K \in \mathcal{C}$ such that u is affine on each interval of $[a, b] \setminus K$, endowed with the supremum norm, is ultrabornological. For the family $\{F_\alpha\}$ we take the family $\{E_K\}_{K \in \mathcal{C}}$ where E_K consists of those $u \in C([a, b])$ that are affine on each interval in $[a, b] \setminus K$.

Note that $E_A^{\mathcal{C}}$ is not closed under multiplication, so condition (4.1.1) requires a direct verification, which in turn requires the density in $[a, b]$ of the union of the sets in $\mathcal{C}$. ■

EXAMPLE 4.4. *Holomorphic functions.* — Let X be a compact subset of $\mathbf{C}^d$ with dense interior, for some positive integer d. If $u \in C(X)$, W_u will denote the largest open subset of X on which u is holomorphic (thus the scalar field is $\mathbf{C}$). $E_H(X)$ consists of those u for which W_u is dense in X, and if $\mu \in M(X)$ is a positive nonatomic measure, $E_H^\mu(X) := E^\mu \cap E_H(X)$. Both $E_H(X)$ and $E_H^\mu(X)$ are modules over their subspace $E_0^\mu(X)$, making a "partition of unity" argument for (4.1.1) easy. If for the family $\{F_\alpha\}$ we take $\{E_S\}_{S \in \mathcal{S}}$ where a member S of $\mathcal{S}$ is a dense open subset of X (and $\mu(X \setminus S) = 0$ in the case of $E^\mu_H(X)$), and if $E_S := \{u \in C(X) : S \subset W_u\}$, it follows readily that $E_H(X)$ *and* $E^\mu_H(X)$ *are ultrabornological.* ■

The examples treated in this and the preceding section are all normed linear spaces of the first category, except for the trivial cases when X is finite in Theorem 3.1 or Example 4.2. For if $\{U_k\}$ consists of the nonempty non-singleton members of a countable base for the topology of X (or, in Example 4.4, of the interior of X) then E is the union of its proper closed linear subspaces F_k where F_k consists of those $u \in E$ that are: constant on U_k if E is as in Theorem 3.1 or Example 4.2; affine on U_k if E is as in Example 4.3; or holomorphic on U_k if E is as in Example 4.4. On the other hand, being barreled, E is *not* the union of an *increasing* sequence of its proper closed linear subspaces.

5. The non-metrizable situation. — Two new obstacles arise routinely when we attempt to study spaces such as $E_0(X)$ when X is not metrizable. First, the function space E may fail to separate the points of X. This can often be overcome by passing from X to $\widetilde{X}$, the quotient space obtained from X by collapsing to a point each equivalence class for the equivalence relation $x \sim y$ if $u(x) = u(y)$ for all $u \in E$; if $\rho \in X \to \widetilde{X}$ is the quotient map and, for $L \subset E$, $\widetilde{L} := \{\tilde{u} \in C(\widetilde{X}) : \tilde{u} \circ \rho \in L\}$, then $\tilde{u} \mapsto \tilde{u} \circ \rho$ is an isomorphism of $\widetilde{E}$ onto E, and under appropriate circumstances (4.1.1) will hold for $\widetilde{E}$ on $\widetilde{X}$. Second, X (or after the above identification procedure, $\widetilde{X}$) may not be first countable. Then to get an appropriate version of (4.1.2) it often suffices to replace the requirement that $\bigcap_{n=0}^{\infty} V_n$ reduce to a single point, by the more modest requirement that all the u_n be constant on this set.

Rather than attempt to formulate a general result that will deal with these obstacles, we shall show through examples of interest to us how the methods outlined in the previous paragraph work in practice. As in the metrizable setting, $E_0(X)$ will provide the prototype.

THEOREM 5.1. — *If X is a compact Hausdorff space then $E_0(X)$, endowed with the supremum norm, is ultrabornological.*

Proof. — Let $E = E_0(X)$. We follow the proof of Theorem 3.1. Let $\mathcal{S}$, E_S, σ_S, λ_S, B_S, H be as in that proof. We use the identification procedure to pass to $\widetilde{X}$ and $\widetilde{E}$ (N.B.: the new sets of constancy need not be open in $\widetilde{X}$), and now (3.1.2) becomes true: there is a point $\tilde{p} \in \widetilde{X}$ such that every open neighborhood of $\tilde{p}$ is "bad" for $\widetilde{H} = H(\{\widetilde{B}_S\})$. We now take extra care in selecting the sequences $(\widetilde{V}_n)_{n=0}^{\infty}$ of open neighborhoods of $\tilde{p}$ and $(\tilde{u}_n)_{n=0}^{\infty}$ of functions in $\widetilde{E}$. Let $\widetilde{V}_0 = \widetilde{X}$, and let $\tilde{u}_0 \in \widetilde{E}$ satisfy $||\tilde{u}_0|| < 1$ and $\tilde{u}_0 \notin \widetilde{H}$. If $n \geq 1$ and $\widetilde{V}_0, \ldots, \widetilde{V}_{n-1}$ and $\tilde{u}_0, \ldots, \tilde{u}_{n-1}$ have been selected, let $\widetilde{V}_n$ be an open neighborhood of $\tilde{p}$ so small that $(\widetilde{V}_n)^- \subset \widetilde{V}_{n-1}$ and $|\tilde{u}_j - \tilde{u}_j(\tilde{p})| < 2^{-n}$ on $\widetilde{V}_n$ for $j =$

$0, \ldots, n-1$, then let $\tilde{u}_n \in \widetilde{E}$ satisfy $\mathrm{supp}(\tilde{u}_n) \subset \widetilde{V}_n$, $\|\tilde{u}_n\| < 2^{-n}$, and $\tilde{u}_n \notin \widetilde{H}$. Let $V_n := \rho^{-1}(\widetilde{V}_n)$ and $u_n := \tilde{u}_n \circ \rho \in E$. Then the V_n and the u_n satisfy the same conditions as in the proof of Theorem 3.1, except that $K := \bigcap_{n=0}^{\infty} V_n = \bigcap_{n=0}^{\infty} \overline{V}_n$ may not reduce to a point. However, all the u_n are constant on K, so if K has nonempty interior we simply make this interior an additional member of the family $S_0 \in \mathcal{S}$. ■

EXAMPLE 5.2. *Sets of constancy of full measure.* — μ, E^μ and $E_0^\mu(X)$ are as in Example 4.2, but X is no longer required to be metrizable. As we follow the models of Theorem 5.1 and Example 4.2 two difficulties may arise, and the second one may not be surmountable in the case of $E_0^\mu(X)$.

First, the boundary of V_n, $\partial V_n = \overline{V}_n \setminus V_n$, may not be μ-null. With extra care in the construction of the $\widetilde{V}_n$, this problem can be avoided. Specifically, once $\widetilde{V}_0, \ldots, \widetilde{V}_{n-1}$ and $\tilde{u}_0, \ldots, \tilde{u}_{n-1}$ have been selected, choose an open neighborhood V_n' of $\tilde{p}$ so that $\overline{V}_n' \subset \widetilde{V}_{n-1}$ and $|\tilde{u}_j - \tilde{u}_j(\tilde{p})| < 2^{-n}$ for $j = 0, \ldots, n-1$, then take a continuous function $h : \widetilde{X} \to [0,1]$ such that $h(\tilde{p}) = 0$ and $h \equiv 1$ on $\widetilde{X} \setminus V_n'$, choose $t \in (0,1)$ so that $\mu(\rho^{-1}(h^{-1}(\{t\}))) = 0$, and let $\widetilde{V}_n := h^{-1}([0,t))$.

The second difficulty is that if $K = \bigcap_{n=0}^{\infty} V_n$, it may be that $\mu(\partial K) \neq 0$. This can be addressed to some extent in the construction of the $\widetilde{V}_n$ by arranging that $\lim_{n\to\infty} \mu(\rho^{-1}(\widetilde{V}_n)) = \mu((\rho^{-1}(\{\tilde{p}\}))$, that is, $\mu(K) = \mu(\rho^{-1}(\{\tilde{p}\}))$. Because E^μ separates the points of X (lemma in Example 4.2), this gives $\mu(K) = 0$ in this case; thus E^μ *is always ultrabornological*. However, the proof breaks down for $E_0^\mu(X)$ if, for some $\tilde{p} \in \widetilde{X}$, the boundary of $\rho^{-1}(\{\tilde{p}\})$ is *not* μ-null. Thus at this writing we can only assert that $E_0^\mu(X)$ *is ultrabornological provided the positive nonatomic measure* μ *is null on the boundary in* X *of every equivalence class for the equivalence relation* $x \sim y$ *if* $u(x) = u(y)$ *for every* $u \in E_0(X)$. ■

EXAMPLE 5.3. *Countable-valued functions.* — Let Y be an uncountable set, and let $E_C(Y)$ denote the space of bounded functions on Y that have finite or countable range. Then $E_C(Y)$ *is ultrabornological*. In fact, if X denotes the Stone-Čech compactification of the discrete space Y, then $E_C(Y)$ becomes identified with a uniformly dense self-adjoint subalgebra E of $C(X)$. Let $\mathcal{S}$ denote the family of all partitions of Y into at most countably many disjoint nonempty subsets, and for $S \in \mathcal{S}$ let $E_S := \{u \in C(X) : u$ is constant on each $U \in S\}$, as in the proof of Theorem 3.1. The proof continues without difficulty. ■

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