ARE ALL UNIFORM ALGEBRAS AMNM?

by Stuart J. SIDNEY

ABSTRACT. — A Banach algebra $\mathfrak{a}$ is AMNM if whenever a linear functional ϕ on $\mathfrak{a}$ and a positive number δ satisfy $|\phi(ab) - \phi(a)\phi(b)| \leq \delta||a|| \cdot ||b||$ for all $a, b \in \mathfrak{a}$, there is a multiplicative linear functional ψ on $\mathfrak{a}$ such that $||\phi - \psi|| = o(1)$ as $\delta \to 0$. K. Jarosz [1] asked whether every Banach algebra, or every uniform algebra, is AMNM. B.E. Johnson [2] studied the AMNM property and constructed a commutative semisimple Banach algebra that is *not* AMNM. In this note we construct uniform algebras that are not AMNM.

If $\mathfrak{a}$ is a Banach algebra, ψ is a (possibly zero) multiplicative linear functional on $\mathfrak{a}$, and τ is a bounded linear functional on $\mathfrak{a}$ of norm $||\tau|| = \sigma$, a trivial calculation shows that the linear functional $\phi := \psi + \tau$ is δ-*multiplicative* with $\delta = (3 + \sigma)\sigma$ in the sense that

$$|\phi(ab) - \phi(a)\phi(b)| \leq \delta||a|| \cdot ||b|| \text{ for all } a \in \mathfrak{a}, b \in \mathfrak{a}.$$

Loosely, if the linear functional ϕ on $\mathfrak{a}$ is near a multiplicative linear functional, then it is *approximately multiplicative*. The notion of approximately multiplicative linear functional, or even operator, is discussed in Krzysztof Jarosz' monograph [1], in which it is shown (Proposition 5.5) that if the linear functional ϕ on $\mathfrak{a}$ is δ-multiplicative then $||\phi|| \leq 1 + \delta$, which is a step in the direction of proving that ϕ must be near a multiplicative linear functional. Jarosz poses the problem (Problem 5, page 111): if $\mathfrak{a}$ is a Banach algebra [resp. uniform algebra] and ϕ is a δ-multiplicative linear functional on $\mathfrak{a}$, must there exist a multiplicative linear functional on $\mathfrak{a}$ such that $||\phi - \psi|| = o(1)$ as $\delta \to 0$? In Barry Johnson's definitive study [2] of approximately multiplicative linear functionals, he calls commutative Banach algebras for which the answer to this question is affirmative AMNM (approximately multiplicative is near multiplicative) algebras. Johnson shows that many classical commutative Banach algebras are AMNM, while presenting (Example 9.1) a commutative semisimple Banach algebra that is *not* AMNM. In particular, he shows (Theorem 7.1 *et seq.*) that polydisc algebras are AMNM, but leaves open the question of whether H^∞, or indeed every uniform algebra, is AMNM. Our purpose here is to

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fill part of this gap by producing uniform algebras that are *not* AMNM. This is accomplished by means of the

THEOREM. — *For any two positive numbers $\eta < 1$ and δ, there are a uniform algebra $\mathfrak{a}$ and a δ-multiplicative linear functional ϕ on $\mathfrak{a}$ such that $||\phi - \psi|| > \eta$ for every multiplicative linear functional ψ on $\mathfrak{a}$.*

COROLLARY. — *There is a uniform algebra $\mathfrak{a}$ that is not AMNM.*

Proof of the corollary. — For $k = 1, 2, \ldots$ let $\mathfrak{a}_k$ and ϕ_k be as in the theorem for $\eta_k = 1 - (k+1)^{-1}$ and $\delta_k = k^{-1}$. $\mathfrak{a}_k$ is a uniform algebra on some compact Hausdorff space X_k. Let X be the one-point compactification of the disjoint union of the X_k, and let

$$\mathfrak{a} := \left\{ f \in C(X) : f\big|_{X_k} \in \mathfrak{a}_k \text{ for all } k \right\}.$$

Define $\phi'_k : \mathfrak{a} \to \mathbf{C}$ by $\phi'_k(f) = \phi_k(f\big|_{X_k})$. It is immediate that ϕ'_k is δ_k-multiplicative and that $||\phi'_k - \psi|| > \eta_k$ for every multiplicative linear functional ψ on $\mathfrak{a}$, since ψ must be either identically zero, or evaluation at the point at infinity, or a multiplicative linear functional on some $\mathfrak{a}_j$. ■

Proof of the theorem. — The idea is simple. Imagine a fictitious "almost analytic" disc centered at some point p of the spectrum $\mathfrak{m}$ of $\mathfrak{a}$. For a *true* analytic disc, we would have an algebra homomorphism $f \mapsto \sum_{k=0}^{\infty} \tilde{f}_k \omega^k$ from $\mathfrak{a}$ into the algebra of power series in ω with radius of convergence at least 1, where the $\tilde{f}_k$ are complex numbers and $\tilde{f}_0 = \hat{f}(p)$. For our almost analytic disc, we simply arrange that our algebra $\mathfrak{a}$ can generate appropriate numbers $\tilde{f}_k$ for $k \le n$ but not for $k > n$, where n is some fixed integer, and we consider the functional $f \mapsto \sum_{k=0}^{n} \tilde{f}_k \lambda^k$ for some $\lambda \in \mathbf{C}$. If $\eta < |\lambda| < 1$ and if n is large enough, this functional has the right properties, assuming the functionals $f \mapsto \tilde{f}_k$ are linear and of norm 1. We now give the details.

First, let

$\Delta := \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disc,

$\overline{\Delta} := \{z \in \mathbf{C} : |z| \le 1\}$ the closed unit disc, and

$T := \{z \in \mathbf{C} : |z| = 1\} = \overline{\Delta} \setminus \Delta = \partial\Delta$ the boundary of Δ.

$\widehat{A} := \{u \in C(\overline{\Delta}) : u \text{ is analytic on } \Delta\}$ is the disc algebra on $\overline{\Delta}$,

$A := \{u\big|_T : u \in \widehat{A}\}$ is the disc algebra on T.

Let (z, ω) denote the variable in $\mathbf{C}^2$. For $(j,k) \in \mathbf{Z}^2$ let $h_{jk} \in C(T^2)$ be $h_{jk}(z,\omega) = z^j \omega^k$.

Fix a positive integer n and consider

$$S := \{(j,k) \in \mathbf{Z}^2 : k \geq n+1, \text{ or } j \geq 0 \text{ and } k \geq 0\},$$

a sub-semigroup of $\mathbf{Z}^2$. The algebra $\mathfrak{a}$ we want is the uniformly closed linear span in $C(T^2)$ of $\{h_{jk} : (j,k) \in S\}$. If $f \in \mathfrak{a}$, for each $z \in T$ the function $\omega \mapsto f(z,\omega) : T \to \mathbf{C}$, as a uniform limit of polynomials, belongs to A, so extends to a function F^z in $\widehat{A}$ on $\overline{\Delta}$; that is, $F^z \in C(\overline{\Delta})$, $F^z(\omega) = f(z,\omega)$ if $\omega \in T$, and F^z is analytic in Δ, so has the form $F^z(\omega) = \sum_{k=0}^{\infty} F_k(z)\omega^k$ where $F_k(z) = (2\pi)^{-1}\int_0^{2\pi} f(z,e^{it})e^{-ikt}\,dt = (k!)^{-1}\frac{d^kF^z}{d\omega^k}(0)$. Thus $F_k \in C(T)$; further, if $k \leq n$ then $F_k \in A$, so extends from a function on T to a function $\widehat{F}_k$ in $\widehat{A}$ on $\overline{\Delta}$. We write $f \sim (F_k)$ to indicate that (F_k) is the sequence in $C(T)$ associated to $f \in \mathfrak{a}$ in this way. For each k the mapping $f \mapsto F_k : \mathfrak{a} \to C(T)$ is linear and of norm 1. If also $g \in \mathfrak{a}$ and $g \sim (G_k)$ and $fg \sim (H_k)$, clearly $H_k = \sum_{j=0}^{k} F_jG_{k-j}$.

Fix $\lambda \in \Delta$ and define $\phi : \mathfrak{a} \to \mathbf{C}$ by

$$\phi(f) := \sum_{k=0}^{n} \widehat{F}_k(0)\lambda^k$$

where $f \sim (F_k)$. ϕ is a continuous linear functional on $\mathfrak{a}$. We shall prove that:

(1) ϕ *is* $\dfrac{n|\lambda|^{n+1}}{1-|\lambda|}$*-multiplicative on* $\mathfrak{a}$*;*

(2) If ψ *is any multiplicative linear functional on* $\mathfrak{a}$ *then* $||\phi - \psi|| \geq |\lambda|$.

The theorem follows by first taking λ so that $|\lambda| > \eta$, then taking n so large that $\frac{n|\lambda|^{n+1}}{1-|\lambda|} \leq \delta$.

To prove (1), let $f \in \mathfrak{a}$ and $g \in \mathfrak{a}$ with $f \sim (F_k)$, $g \sim (G_k)$ and $fg \sim (H_k)$. Then

$$\phi(fg) = \sum_{k=0}^{n} \widehat{H}_k(0)\lambda^k = \sum_{k=0}^{n}\left[\sum_{j=0}^{k}\widehat{F}_j(0)\widehat{G}_{k-j}(0)\right]\lambda^k$$

and

$$\phi(f)\phi(g) = \left[\sum_{j=0}^{n}\widehat{F}_j(0)\lambda^j\right]\left[\sum_{r=0}^{n}\widehat{G}_r(0)\lambda^r\right] = \sum_{k=0}^{2n}\left[\sum_{j=(k-n)\vee 0}^{n\wedge k}\widehat{F}_j(0)\widehat{G}_{k-j}(0)\right]\lambda^k,$$

so

$$\begin{aligned}|\phi(fg) - \phi(f)\phi(g)| &= \left|\sum_{k=n+1}^{2n}\left[\sum_{j=k-n}^{n}\widehat{F}_j(0)\widehat{G}_{k-j}(0)\right]\lambda^k\right| \\ &\leq \sum_{k=n+1}^{2n}(2n-k+1)||f||\cdot||g||\cdot|\lambda|^k \\ &\leq n||f||\cdot||g||\cdot\sum_{k=n+1}^{2n}|\lambda|^k \\ &\leq \frac{n|\lambda|^{n+1}}{1-|\lambda|}||f||\cdot||g||.\end{aligned}$$

To prove (2), we must identify the nonzero multiplicative linear functionals on $\mathfrak{a}$. Let $\mathfrak{m}$ be the set of these. If $\psi \in \mathfrak{m}$ let $\theta(\psi) := (\alpha, \beta) \in \mathbf{C}^2$ where $\alpha = \psi(h_{1,0})$ and $\beta = \psi(h_{0,1})$. $|\alpha| \leq ||h_{1,0}|| = 1$ and $|\beta| \leq ||h_{0,1}|| = 1$, so $\theta(\psi) \in \overline{\Delta}^2$. For all $j > 0$, $\psi(h_{-j,n+1})\alpha^j = \beta^{n+1}$. If $\beta \neq 0$ then $\alpha \neq 0$ and for all $j > 0$, $1 = ||h_{-j,n+1}|| \geq |\psi(h_{-j,n+1})| = |\beta|^{n+1}/|\alpha|^j$, hence $|\alpha| \geq 1$ and $\alpha \in T$. Thus $\theta(\psi) \in X$ where $X := (T \times \overline{\Delta}) \cup (\overline{\Delta} \times \{0\}) \subset \mathbf{C}^2$. Conversely, suppose $(\alpha, \beta) \in X$. If $(\alpha, \beta) \in T \times \overline{\Delta}$ define $\psi(f) = F^{\alpha}(\beta)$; if $(\alpha, \beta) \in \overline{\Delta} \times \{0\}$ define $\psi(f) = \widehat{F}_0(\alpha)$. In either case $\psi \in \mathfrak{m}$ and $\theta(\psi) = (\alpha, \beta)$. Thus θ maps $\mathfrak{m}$ *onto* X. To verify that θ is *injective*, it suffices to verify that if $\psi \in \mathfrak{m}$ then the values $\psi(h_{jk})$ for $(j, k) \in S$ are computable from $\theta(\psi) = (\alpha, \beta)$. If $j \geq 0$ and $k \geq 0$ then $\psi(h_{jk}) = \psi(h_{1,0})^j \psi(h_{0,1})^k = \alpha^j \beta^k$. Suppose $j < 0$ and $k \geq n + 1$. If $\beta = 0$ then $\psi(h_{jk})^2 = \psi(h_{2j,k})\psi(h_{0,1})^k = 0$ so $\psi(h_{jk}) = 0$, while if $\beta \neq 0$ then $|\alpha| = 1$ and $\psi(h_{jk})\psi(h_{1,0})^{-j} = \psi(h_{0,1})^k$ or $\psi(h_{jk})\alpha^{-j} = \beta^k$, giving $\psi(h_{jk}) = \alpha^j \beta^k$.

Thus θ is a bijection (homeomorphism, if $\mathfrak{m}$ has its Gelfand topology) of $\mathfrak{m}$ with X. In fact, we may view the functions in $\mathfrak{a}$ as extended from T^2 to X by first extending to $T \times \overline{\Delta}$ by analyticity in ω, then extending to $\overline{\Delta} \times \{0\}$ by analyticity in z.

Now let ψ be a multiplicative linear functional on $\mathfrak{a}$. If $\psi \equiv 0$ then $||\phi - \psi|| \geq |\phi(1) - \psi(1)| = |1 - 0| = 1$. If $\psi \in \mathfrak{m}$, then either $\theta(\psi) \in T \times \overline{\Delta}$ and $||\phi - \psi|| \geq |\phi(h_{1,0}) - \psi(h_{1,0})| = |0 - \alpha| = 1$, or $\theta(\psi) \in \overline{\Delta} \times \{0\}$ and $||\phi - \psi|| \geq |\phi(h_{0,1}) - \psi(h_{0,1})| = |\lambda - 0| = |\lambda|$. Thus (2) is proven, and so is the theorem. ■

A few remarks are in order. First, the $\mathfrak{a}$ constructed in the proof of the theorem actually has several alternative descriptions. For instance, $\mathfrak{a}$ consists precisely of those $f \in C(T^2)$ which extend to $F \in C(T \times \overline{\Delta})$ in such a way that $\omega \mapsto F(z, \omega)$ is an analytic function of $\omega \in \Delta$ for each $z \in T$, and that for $k = 0, \ldots, n$ the function $z \mapsto \frac{\partial^k F}{\partial \omega^k}(z, 0)$ on T belongs to the disc algebra A. Also, $\mathfrak{a}$ is the uniform closure in $C(T^2)$ of $(A \otimes A) + (C(T) \otimes I)$ where I is the principal ideal in A generated by ω^{n+1}; for an analysis of many uniform algebras given by this type of tensor product construction see [3].

The $\mathfrak{a}$ constructed in the proof of the theorem is actually generated, as a unital Banach algebra, by three elements: $h_{1,0}$, $h_{0,1}$ and $\sum_{j=1}^{\infty} (j!)^{-1} h_{-j,n+1}$ (a simple exercise, or see [3], section 5). It is unclear whether there are examples with fewer generators. This $\mathfrak{a}$ is antisymmetric.

If the $\mathfrak{a}_k$ used in the proof of the corollary are those constructed in the proof of the theorem, the resulting $\mathfrak{a}$ has four generators but is not antisymmetric. Well-chosen point identifications convert it to an antisymmetric example, but with no obvious control over the number of generators.

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