A SIMPLE PROOF OF P. CARTER'S THEOREM

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ABSTRACT: We give a simple proof of the following result of P. Carter: Given a twist homeomorphism of an annulus with at most one fixed point in the interior of the annulus, then there exists an essential simple closed curve inside this annulus meeting its image in at most the (possible) interior fixed point.

0. Introduction

Let us first recall the classical

Theorem (Poincaré-Birkhoff). *Let F be a twist homeomorphism of the annulus $A = S^1 \times [0,1]$. If F preserves area then F has at least two fixed points.*

Very soon, Birkhoff, Kérékjártó and others looked for a more topological statement avoiding the area preserving hypothesis. They obtained the following [B], [K]

Theorem. *Let F be a twist homeomorphism of the annulus without fixed point in $intA$. Then there is an essential simple closed curve $c \subset intA$ such that $F(c) \bigcap c = \emptyset$.*

Of course this statement implies that in the classical setting one gets one fixed point. A natural question now is: is it possible to get the second fixed point working along the same lines? It was not before 1982 that the answer came out with the following result of P. Carter [C]

Theorem. *Let F be a twist homeomorphism of the annulus A with at most one fixed point in $intA$. Then there is an essential simple closed curve $c \subset intA$ which meets its image in at most one point (the fixed point of F in $intA$ if it exists).*

In this note we propose to give a simplified proof of this theorem. It is known that one can deduce rapidly the theorem of Birkhoff and Kérékjártó from the Brouwer plane translation theorem [K], [G], and we shall follow the same path, using the recent proof of that theorem given by P. Le Calvez and A. Sauzet [LS] (but as we shall need to start from scratch, in view of the fixed point, this paper is almost self contained). Other proofs of Brouwer's theorem known to me do not seem to lead to such an easy proof as the Brouwer lines they construct may converge on a fixed point (compare with lemma 2 below).

1. Preliminaries

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Definition. A homeomorphism F, isotopic to the identity, of the annulus $A = S^1 \times [0, 1]$ is a **twist homeomorphism** if there is a lift f of F or F^{-1} to the band $B = \mathbb{R} \times [0, 1]$ satisfying $f_1(x, 0) < x$ and $f_1(x, 1) > x$ where $f = (f_1, f_2)$.

To prove P. Carter's theorem we first extend F to a homeomorphism of $S^1 \times [-\delta, 1]$ in such a way that F (or F^{-1} but it is clearly enough to consider one case and we shall do so) admits a lift $f : \mathbb{R} \times [-\delta, 1] \to \mathbb{R} \times [-\delta, 1]$ such that

i) $f(x, t) = (f_1(x, t), t)$ if $-\delta \leq t \leq 0$
ii) $f_1(x, t) < x$ if $-\delta \leq t \leq 0$
iii) $f(x, -\delta) = (x - \alpha, -\delta)$ for some small $\alpha > 0$
iv) $f_1(x, t) > x$ if $t = 1$

Thinking of S^1 as $\mathbb{R}/\mathbb{Z}$, we can also suppose that f has $\mathbb{Z} \times \{\frac{1}{2}\}$ as set of fixed points (in which case F fixes $\{1\} \times \{\frac{1}{2}\}$): if f has no fixed point then the result follows from the proof of the Birkhoff-Kérékjártó theorem alluded to above and in any case can be obtained as a simplification of what follows.

Convention : All int, Fr, Adh or $\overline{}$ below are with respect to $\mathbb{R}^2$.

Definition.

1. A **brick decomposition** of a subset E in some surface is a collection $\{V_i\}_{i \in \mathbb{N}}$ of closed discs such that

i) $\bigcup_{i=0}^{\infty} V_i = E$

ii) Every point of E admits a neighborhood which meets at most three of the V_i's.

iii) If $V_i \bigcap V_j \neq \emptyset$ then $V_i \bigcap V_j$ is a (non degenerated) arc in $FrV_i \bigcap FrV_j$.

2. A brick decomposition is said **generic** (with respect to some homeomorphism f of E) if every arc γ in the family $\{V_i \bigcap V_j\}_{i,j \in \mathbb{N}}$ satisfies: for every arc $\gamma' = V_k \bigcap V_l$ in the same family such that $f(\gamma) \bigcap \gamma' \neq \emptyset$ (resp. $f^{-1}(\gamma) \bigcap \gamma' \neq \emptyset$) then $f(\gamma)$ (resp.$f^{-1}(\gamma)$) meets both $intV_k$ and $intV_l$.

Note that the union of the elements of any finite subcollection of the V_i's is a 2-submanifold with boundary of E.

Definition. A subset X of some space endowed with a homeomorphism f is **free** if $f(X) \bigcap X = \emptyset$. It is **free rel** A if $f(X \setminus A) \bigcap X \setminus A = \emptyset$ where $A \subset X$.

Lemma 1. There exists a generic brick decomposition of $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$, $\{V_i\}_{i \in \mathbb{N}}$ which satisfies:

(1) It is periodic (i.e. $\tau(V_j) \in \{V_i\}_{i\in\mathbb{N}}$ for all $V_j \in \{V_i\}_{i\in\mathbb{N}}$ where $\tau(x,t) = (x+1,t)$)

(2) Each V_i is free

(3) $V_0 \bigcap \mathbb{R} \times \{-\delta\} = [0, \frac{2}{3}\alpha] \times \{-\delta\}$

ă$V_2 \bigcap \mathbb{R} \times \{-\delta\} = [\frac{2}{3}\alpha, \frac{4}{3}\alpha] \times \{-\delta\}$

ă$V_1 \bigcap \mathbb{R} \times \{-\delta\} = [-\frac{2}{3}\alpha, 0] \times \{-\delta\}$

(therefore $f(V_0) \bigcap V_1 \neq \emptyset \neq f^{-1}(V_0) \bigcap V_2$).

Proof: We shall construct a brick decomposition$\{W_i\}$ of the annulus $S^1 \times [-\delta, 1] \setminus \{1\} \times \{\frac{1}{2}\}$ which will then be lifted to $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$. We first construct the bricks along $S^1 \times \{-\delta\}$ such that there are free under F and satisfy the obvious analog of (3), which is certainly possible if α is small enough, and we complete this decomposition to a brick decomposition covering $S^1 \times [-\delta, 0] \bigcup S^1 \times \{1\}$ consisting of 2-cells of diameter less than β where $\beta = inf_{t\in[-\delta,0],x\in[0,1]}(x - f_1(x,t), f_1(x,1) - x)$ (recall that F may have fixed points on $S^1 \times \{0\} \bigcup S^1 \times \{1\}$). We then complete the brick decomposition successively on each annulus $B_{n+1} \setminus intB_n$ where B_n is the closed ball of radius $\frac{1}{n}$ centered at $\{1\} \times \{\frac{1}{2}\}$ by bricks free under F. The lift of the decomposition so obtained satisfies the lemma except perhaps genericity. To get genericity choose some numbering $\{\gamma_k\}_{k\in\mathbb{N}}$ of the set $\{W_i \bigcap W_j\}_{i,j\in\mathbb{N}}$ and modify slightly γ_0 (if necessary) so that it becomes generic. Modify then γ_1 so slightly that γ_0 is still generic and γ_1 becomes generic. We continue in this way, each γ_i is modified only a finite number of times and we get a brick decomposition $\{W_i\}_{i\in\mathbb{N}}$ whose lift to $\mathbb{R}\times[-\delta,1]\setminus\mathbb{Z}\times\{\frac{1}{2}\}$ is generic and satisfies (1), (2), (3) if our perturbations have been small enough.

Lemma 2. Let $\{V_i\}_{i\in\mathbb{N}}$ be a brick decomposition of $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$ and for $X \subset \mathbb{N}$, let $W_X = int(\bigcup_{i\in X} V_i)$. Then if W_X is connected, unbounded and if $\mathbb{R}^2 \setminus \overline{W}_X$ has no bounded components, $\overline{W}_X$ is a submanifold whose boundary contains no bounded components (i.e. circles).

Proof: $\overline{W}_X$ is certainly a 2-manifold away from $Fix f = \mathbb{Z} \times \{\frac{1}{2}\}$. Let z be a fixed point in $Fr\overline{W}_X$ and let C be a small circle around z. Near C, $\overline{W}_X$ is a manifold and we can suppose C transversal to $Fr\overline{W}_X$ so that $C \bigcap \overline{W}_X$ is some finite number of arcs and $C \bigcap Fr\overline{W}_X$ an even number of points. If C is small enough all these points have to be joined by arcs of $Fr\overline{W}_X$ to z. Since $\overline{W}_X$ is connected and $\mathbb{R}^2 \setminus \overline{W}_X$ has no bounded component, only two such arcs can exist and $\overline{W}_X$ is a 2-submanifold of $\mathbb{R}^2$. The assertion on the boundary now follows as a circle in $\partial\overline{W}_X$ would have to bound a disc included in $\overline{W}_X$ or $\mathbb{R}^2 \setminus \overline{W}_X$.

We will also have to use the following result of Franks [F, Proposition 1.3] which follows easily from Brouwer's lemma that an orientation preserving homeomorphism of the plane with a periodic point must have some compact fixed point set of positive index . To get the statement below from Franks' proposition note that every fixed point of f has index 0 by the Lefchetz fixed point formula (applied to a large finite covering of the annulus A if F has fixed points on ∂A).

Lemma 3. There is no finite family $D_0, \dots, D_n = D_0$ of free open discs in $\mathbb{R} \times [-\delta, 1]$ such that $f(D_i) \bigcap D_{i+1} \neq \emptyset$, $0 \leq i < n$.

2. Proof of P. Carter's theorem

Let us consider a brick decomposition $\{V_i\}_{i \in \mathbb{N}}$ of $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$ as given by Lemma 1.

Let $W_1 = int(\bigcup \{V_i \big| intV_i \bigcap f(intV_0) \neq \emptyset\}$

and $W_n = int(\bigcup \{V_i \big| intV_i \bigcap f(W_{n-1}) \neq \emptyset\}$, $n > 1$.

Then $W_+ = \bigcup_{n \geq 1} W_n$ is a connected unbounded set. It is unbounded because by (3) of lemma 1, $W_+ \supset]-\infty, 0[\times\{-\delta\}$. It is connected because each W_n is connected and $V_1 \subset W_1$ by (3) so that if V_k meets $f(V_1 \bigcap V_0)$ then by transversality $V_k \subset W_1 \bigcap W_2$ and so $W_1 \bigcap W_2 \neq \emptyset$ therefore $f(W_1) \bigcap f(W_2) \neq \emptyset$ and by the same reasoning $W_2 \bigcap W_3 \neq \emptyset$ and more generally $W_n \bigcap W_{n+1} \neq \emptyset$, $n \geq 1$.

By construction (and genericity) $W_{n+1} \supset f(\overline{W}_n)$ so that $f(W_+) \subset W_+$ and if $x \in FrW_+$ then $f(x) \in W_+$ except if $x \in Fixf$ so that FrW_+ is free rel $Fixf$.

As a consequence of Lemma 3, $intV_0 \bigcap W_+ = \emptyset$ so let C be the component of $\mathbb{R} \times [-\delta, 1] \setminus \overline{W}_+$ which contains $intV_0$.

Now set $W_{-1} = int(\bigcup \{V_i \big| intV_i \bigcap f^{-1}(intV_0) \neq \emptyset\})$

and $W_{-n} = int(\bigcup \{V_i \big| intV_i \bigcap f^{-1}(W_{-n+1}) \neq \emptyset\})$, $n > 1$.

Then, as above, $W_- = \bigcup_{n \geq 1} W_{-n}$ is a connected unbounded open set.

Let Γ_+ be $FrW_+ \bigcap FrC = Adh\widehat{W}_+ \bigcap AdhC$, where $\widehat{W}_+$ is the union of W_+ and of all bounded components of $\mathbb{R}^2 \setminus \overline{W}_+$. By lemma 3, $W_+ \bigcap W_- = \emptyset$ so that $W_- \bigcup intV_0 \subset C$ and C is unbounded. By lemma 2, FrW_+ is a non compact 1-submanifold without boundary of $\mathbb{R}^2$ so that, Γ_+ being connected since W_+ and C are so, Γ_+ is a half line beginning at $(0, -\delta)$ properly embedded in $\mathbb{R} \times [-\delta, 1]$. In fact Γ_+ does not meet $\mathbb{R} \times \{1\}$ since F is a twist homeomorphism and Γ_+ is free rel $Fixf$. Note also that Γ_+ is composed of sides of the brick decomposition.

Now we imitate the argument in [G, § 5] : Γ_+ separates $\mathbb{R} \times [-\delta, 1]$ into two open connected sets R_1 and R_2 so that (say) $f(\Gamma_+) \subset R_2$ modulo $Fixf$. Let R be the connected component of $\bigcap_{n \in \mathbb{Z}} \tau^n(R_1)$ which contains $\mathbb{R} \times \{1\}$. Lemma 2 applies to $int(\bigcup_{n \in \mathbb{Z}} \tau^n(R_2))$ filled in by the bounded components of its complement since $\bigcup_{n \in \mathbb{Z}} \tau^n(R_2)$ is connected. Therefore $L = FrR \bigcap Fr(\bigcup_{n \in \mathbb{Z}} \tau^n(R_2))$ is a periodic properly embedded line in $\mathbb{R} \times]-\delta, 1[$. Now we show that L is free rel $Fixf$: indeed, outside of any given neigborhood N of $Fixf$, $\bigcap_{n \in \mathbb{Z}} \tau^n(R_1) = \bigcap_{|n| \leq m} \tau^n(R_1)$ for some m, due to the finiteness $mod \tau$ of the brick decomposition outside N. So that if $x \in L$, $x \notin Fixf$, then $x \in \tau^n(\Gamma_+)$ for some n and $f(x) \in \tau^n(f(\Gamma_+)) \subset \tau^n(R_2)$.

This line L projects down in $S^1 \times [-\delta, 0[$ to an essential simple closed curve c such that $c \bigcap F(c) \subset FixF = \{1\} \times \{\frac{1}{2}\}$.

Clearly such a curve cannot meet $S^1 \times [-\delta, 0]$ (look at a point of c realizing $d(c, S^1 \times \{-\delta\})$ and use that every $S^1 \times \{t\}$ is preserved by F for $-\delta \leq t \leq 0$ to contradict that $F(c) \bigcap c$ is at most the fixed point $\{1\} \times \{\frac{1}{2}\}$). This concludes the proof of P. Carter's theorem.

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