A SIMPLE PROOF OF THE RIGIDITY AND THE MINIMAL ENTROPY THEOREMS

by Gérard BESSON, Gilles COURTOIS and Sylvestre GALLOT

SUMMARY

1. Introduction

Let (Y, g) be a compact connected n-dimensional riemannian manifold; let $(\widetilde{Y}, \tilde{g})$ be its universal cover endowed with the pulled-back metric; if $y \in \widetilde{Y}$, we define

$$h(g) = \lim_{R \to +\infty} \frac{1}{R} \operatorname{Log}\big(\operatorname{vol}(B(y, R))\big)$$

where $B(y, R)$ denotes the ball of radius R around y in $\widetilde{Y}$. It is a classical fact that this limit exists and does not depend on y. The invariant $h(g)$ is called the volume-entropy of the metric g but, for the sake of simplicity, we shall use the term entropy (see for example [B-C-G 1] for the relation with the topological entropy of the geodesic flow of g). In the article [B-C-G 1] we prove a theorem involving $h(g)$ which amounts to saying that the functional $g \mapsto h(g)$ (suitably normalized) is minimal for locally symmetric metrics of negative curvature. More precisely, let (X, g_0) be a compact connected n-dimensional riemannian manifold, where the metric g_0 is assumed to be locally symmetric with negative curvature, let us furthermore assume that Y and X are topologically related in the following sense: there exists a continuous map $f : Y \to X$ of non-zero degree, then one has

1.1. Main theorem ([B-C-G 1]). — *With the above notations one has*

i) $h^n(g)\operatorname{vol}(Y, g) \geq h^n(g_0)\operatorname{vol}(X, g_0)|\deg f|$.

ii) The equality case, namely $h(g) = h(g_0)$ *and* $\operatorname{vol}(Y, g) = |\deg f| \operatorname{vol}(X, g_0)$*, occurs if and only if* f *is homotopic to a riemannian covering (i.e. a locally isometric covering).*

Notice that, in this theorem, there is no assumption on the sign of the curvature of (Y, g).

This result has a series of corollaries described in [B-C-G 1], chapter 9. Among them are rigidity type results both of riemannian nature and of dynamical nature. The rigidity results follow from the equality case.

The purpose of this article is to present a quick and simple proof in a particular case, namely when (Y, g) has negative curvature and f is a homotopy equivalence, and to describe briefly how this quick proof extends to an alternative proof in the most general case. Since a by-product of the main theorem is a proof of Mostow's rigidity theorem for negatively curved locally symmetric spaces, a by-product of the proof presented here is a quick and unified proof of this rigidity result. The main idea is that we work with measures; we, in a way, represent our manifolds as submanifolds of a space of probability measures.

The spirit of this article is not to be exhaustive but descriptive. The original proof and the corollaries are described in full details in [B-C-G 1]. For the sake of simplicity we shall assume that all manifolds are orientable.

The first time that this question about the minimal value of the entropy was asked is the article by A. Katok [Kat]. In this text A. Katok proved the main theorem stated above when $Y = X$ and g is conformal to g_0; the approach developed was dynamical theoretic. The question was then included in the riemannian geometric context by M. Gromov in [Gro 1] and [Gro 2]; in particular the connection with the minimal volume question was made (see [Gro 1] and [B-C-G 1]). The differences in the approach rely on a slight modification in the invariant used; indeed, M. Gromov asked the question of finding the minimal value of the volume-entropy (the invariant studied in this article) whereas A. Katok worked with the topological entropy of the geodesic flow; it is a classical fact that they coincide on manifolds with non-positive curvature and that, in general, the volume-entropy is not greater than the topological entropy (see [Man]). The consequence is that our main theorem gives, as a corollary, the analogous result on the topological entropy and hence has consequences both for the riemannian geometry and for the study of the geodesic flow. In [Gro 1] M. Gromov introduced a topological invariant, the simplicial volume, which bounds from below the quantity $h^n(g)\,\text{vol}(X, g)$ and he showed that this invariant is non-zero when X carries a negatively curved metric; this in turn shows that the quantity $h^n(g)\,\text{vol}(X, g)$ is bounded away from zero, and hence the question of finding its minimal value among all riemannian metrics is meaningful. Unfortunately, one paid for the universality of the simplicial volume: the lower bound that one obtained for $h^n(g)\,\text{vol}(X, g)$ was not sharp, even in dimension two.

In [B-C-G 0] we introduced our original method (different from the one presented here). We manufactured an invariant of the differentiable structure of X, $S(X)$, that M. Gromov encouraged us to call the spherical volume. We compared $S(X)$ to the simplicial volume and showed that it bounds below (with some universal constant) the quantity $h^n(g)\,\text{vol}(X, g)$. Similarly we gave an alternative proof of Katok's result on the conformal class of a locally symmetric metric of negative curvature (see [B-C-G 0]). Using this technique we proved in [B-C-G 1] the main theorem as stated above and derived the various corollaries. At this stage, it is worth mentioning another proof of Katok's result in the conformal class of g_0 using an elegant idea: the volume-entropy is a convex function of the metric. This proof is due to G. Robert [Rob]. The present text is devoted to describing a different proof of the main theorem which yields a unique and easy proof of Mostow's strong rigidity theorem for the rank one case; easy enough to be taught eventually at the undergraduate level!

The authors are delighted to thank A. Katok for asking for this article and for his interest in our work.

2. The Patterson-Sullivan measure

Let (Y, g) be a connected n-dimensional riemannian manifold, where the metric g is assumed to have negative curvature. The (arbitrary) choice of an origin 0 in the universal cover $\widetilde{Y}$ of Y, allows to identify $\widetilde{Y}$ with the unit ball in $\mathbf{R}^n$, the geometric boundary $\partial\widetilde{Y}$ being identified with the unit sphere. In the case of dimension 2 and of constant curvature, S.J. Patterson ([Pat]) gave a construction of a family of measures on $\partial\widetilde{Y}$ indexed by the points $y \in \widetilde{Y}$. It has been extended to higher dimension by D. Sullivan ([Sul]). The variable curvature case is described by G. Knieper ([Kni]) and C. B. Yue ([CBY]). Let us notice that, when X is compact (and negatively curved), which is the case we would like to discuss in this article, this family of measures, suitably interpreted, yields the measure of maximal entropy for the geodesic flow, the so-called Bowen-Margulis measure, as described by G. A. Margulis (see [Mar] and also [Kni] and [CBY]).

In the sequel Y will be assumed to be *compact*. To each $y \in \widetilde{Y}$ we associate a measure on $\partial\widetilde{Y}$, denoted by ν_y. For y and y' in $\widetilde{Y}$, the measures ν_y and $\nu_{y'}$ are in the same class of density-measures and, for $\theta \in \partial\widetilde{X}$,

$$\frac{d\nu_y}{d\nu_{y'}}(\theta) = e^{-h(g)B_{y'}(y,\theta)}$$

where $B_{y'}(y, \theta)$ is the Busemann function computed at the point (y, θ) and normalized by $B_{y'}(y', \cdot) \equiv 0$. For the sake of simplicity we shall write $B(y, \theta)$ for $B_0(y, \theta)$, where 0 is the origin chosen in $\widetilde{Y}$. Let $\frac{1}{c(y)} = \int_{\partial\widetilde{Y}} e^{-h(g)B(y,\theta)}\, d\nu_0(\theta)$, then $\mu_y = c(y)\nu_y$ is a probability measure on $\partial\widetilde{Y}$. Furthermore, the map

$$\begin{aligned} \widetilde{Y} &\longrightarrow \mathcal{M}_1(\partial\widetilde{Y}) \\ y &\longmapsto \mu_y \end{aligned}$$

is equivariant, which means that, for any isometry γ of $\widetilde{Y}$ (for example $\gamma \in \Gamma \simeq \pi_1(Y)$), one has

$$\mu_{\gamma(y)} = \gamma_*(\mu_y)$$

(here $\mathcal{M}_1(\partial\widetilde{Y})$ denotes the compact space of probability measures on $\partial\widetilde{Y}$).

The construction of this family of measures follows the original idea of S. J. Patterson (see [Nic] for a detailed exposition. Briefly it goes as follows: let $g_s(y, z)$ =

$\sum_{\gamma\in\Gamma} e^{-s\,d(y,\gamma(z))}$ be the Poincaré series of Γ (acting on $\widetilde{Y}$ by isometries). It converges for $s > h(g)$ and diverges for $s \leq h(g)$. Now for $s > h(g)$ let us define

$$\nu_{y,z}(s) = \frac{\sum_{\gamma\in\Gamma} e^{-s\,d(y,\gamma(z))}\delta_{\gamma(z)}}{\sum_{\gamma\in\Gamma} e^{-s\,d(y,\gamma(y))}}$$

where d is the distance on $\widetilde{Y}$ associated to $\tilde{g}$, where $\tilde{g}$ is the metric pulled back from the metric g on Y and where $\delta_{\gamma(z)}$ is the Dirac measure at $\gamma(z)$. This defines a family of measures on $\widetilde{Y} \cup \partial\widetilde{Y}$ and we obtain ν_y by taking a weak limit for a subsequence when s goes to $h(g)$. The fact that the denominator diverges when $s = h(g)$, ensures that ν_y is concentrated on the set of accumulation points of the orbit $\Gamma(z)$, *i.e.* on the whole boundary $\partial\widetilde{Y}$ (let us recall that Γ is cocompact). The other properties can be checked following [Nic] and this is left to the reader. Let us point out the following remarks:

i) In our situation, namely Γ is cocompact, the Patterson-Sullivan measure is unique and thus the family $\nu_{y,z}(s)$ converges. Uniqueness is irrelevant in our construction.

ii) The other classical fact is that (again in our situation) it has no atom.

iii) To check the behaviour of the Poincaré series one can compare it to the behaviour of the integral $\int_{\widetilde{Y}} e^{-s\,d(y,z)}dv_{\tilde{g}}(z)$ (here $dv_{\tilde{g}}(z)$ is the volume element on $\widetilde{Y}$). This suggests that we could use another family of measures involving these integrals. In fact, in the cocompact case, this makes no difference. The fundamental reason is that, in the definition of the volume-entropy (see the paragraph 1), the volume of big balls, which is computed with respect to the riemannian measure of $\widetilde{Y}$, may be replaced by the measure of the same balls, computed with respect to any measure $\tilde{\mu}$ on $\widetilde{Y}$ which is the pulled-back of some measure μ on the basis Y. The point of view of Poincaré series corresponds to the choice μ = Dirac measure at some point in Y, while replacing Poincaré series by the above integral corresponds to replacing the Dirac measure by the riemannian measure dv_g. This last point of view (replacing Poincaré series by integrals) was used in [B-C-G 1] and will be developed in the paragraph 6.

3. The barycentre

In [D-E], the authors define a map that associates to any measure on S^1 (verifying some assumptions) a point in the unit disc, called the barycentre (or centre of mass). This map is equivariant with respect to the actions of the Möbius group on the unit disc and on S^1 (considered as the boundary of the unit disc). Such a map still exists in the n-dimensional hyperbolic case (see [D-E]), but also if we consider any metric on the n-ball whose curvature is negative and bounded away from zero (see [B-C-G 1], Appendix A for this obvious generalization that was afterwards also extended to CAT(-1) spaces in [B-M]). We give below a quick description of the construction, the details can be read in [B-C-G 1], Appendix A. Let us also notice that this idea is contained, in a less elaborated form, in the seminal works of H. Furstenberg ([Fur]).

As before (X, g_0) denotes a compact negatively curved manifold (in the sequel g_0 will be a locally symmetric metric of negative curvature), $\widetilde{X}$ is identified with the unit ball in $\mathbf{R}^n$ and $\partial\widetilde{X}$ with the unit sphere by choosing an origin 0. In the sequel, let $B_0(x, \theta)$, for $x \in \widetilde{X}$ and $\theta \in \partial\widetilde{X}$, be the Busemann function on $\widetilde{X}$, associated to g_0 and normalized at the origin, *i.e.* $B_0(0, \theta) = 0$ for all $\theta \in \partial\widetilde{X}$. One can think of $B_0(x, \theta)$ as the "distance" between x and the point at infinity θ. Now, if λ is a measure on $\partial\widetilde{X}$, let us define the function

$$\mathcal{B}(x) = \int_{\partial\widetilde{X}} B_0(x, \theta)\, d\lambda(\theta)$$

for $x \in \widetilde{X}$. It is the "average distance" between x and $\partial\widetilde{X}$. It turns out that there is a point which is the closest possible to infinity in the sense of the λ-average. More precisely,

3.1. Theorem. — *If λ has no atom, the function $\mathcal{B}$ is strictly convex on $\widetilde{X}$. Furthermore $\mathcal{B}(x)$ goes to infinity when x goes to $\theta \in \partial\widetilde{X}$ along a geodesic. Hence $\mathcal{B}$ has a unique critical point in $\widetilde{X}$ which is a minimum.*

3.2. Definition. — *The point where $\mathcal{B}$ achieves its minimum in $\widetilde{X}$ is called the barycentre of the measure λ and is denoted by* $\mathrm{bar}(\lambda)$.

Idea of the proof. — Since the metric $\tilde{g}$ on $\widetilde{X}$ is negatively curved, for each $\theta \in \partial\widetilde{X}$ the function $x \mapsto B_0(x, \theta)$ is convex and therefore $\mathcal{B}$, which is an average of such functions is also convex. It is in fact not difficult to show that $\mathcal{B}$ is strictly convex, indeed

$$Dd\mathcal{B}_x(\cdot,\cdot) = \int_{\partial\widetilde{X}} DdB_{(x,\theta)}(\cdot,\cdot)\, d\lambda(\theta)$$

is positive definite at each $x \in \widetilde{X}$ if λ is different from the most degenerate case which is $\alpha_1\delta_{\theta_1} + \alpha_2\delta_{\theta_2}$ ($\alpha_1 \geq 0$ and $\alpha_2 \geq 0$), where θ_1 et θ_2 are two different points on $\partial\widetilde{X}$.

One shows furthermore that $\mathcal{B}(x) \underset{x\to+\infty}{\longrightarrow} +\infty$ (*i.e.* when x goes to infinity along a geodesic). The reader is referred to [B-C-G 1], Appendix A for the details.

Remarks.

i) The barycentre of a measure λ is defined if the measure has no atom of weight greater or equal to $1/2$. We gave here a weak version which is sufficient for our purpose.

ii) Let us notice that the barycentre x_0 of a measure λ, as a critical point of $\mathcal{B}$, is defined by the implicit vector equation:

$$\int_{\partial \widetilde{X}} dB_{0_{(x_0,\,\theta)}}(\cdot) d\lambda(\theta) = 0.$$

4. The natural map

Let (Y, g) and (X, g_0) be two n-dimensional compact and negatively curved manifolds. We assume that they are homotopically equivalent, *i.e.* that there exists two continuous maps

$$f : Y \to X \ \text{ and } \ h : X \to Y$$

such that $f \circ h$ is homotopic to id_X and $h \circ f$ is homotopic to id_Y. Since both Y and X are $K(\pi, 1)$ (indeed, they carry negatively curved metrics) this hypothesis is equivalent to saying that their fundamental groups are isomorphic as abstract groups (see [B-P], p. 84).

In this paragraph we intend to construct, in this set up, a smooth map $F : Y \to X$, which we call the "natural map"; its definition is highly geometric and hence it becomes the most natural candidate for being an isometry between (Y, g) and (X, g_0) when this is expected. Its construction relies on some basic and classical facts that can be read in full details in [B-P] (for example).

1st step (see [B-P], p. 84)**. —** It is well known that if Y and X are homotopically equivalent, one can lift the map f (and h) to a map between the universal covers $\widetilde{Y}$ and $\widetilde{X}$ of Y and X respectively in such a way that

$$\tilde{f}(\gamma(y)) = \rho(\gamma)\tilde{f}(y)$$

for all $y \in \widetilde{Y}$ and $\gamma \in \pi_1(Y)$. Here $\pi_1(Y)$ and $\pi_1(X)$ acts on $\widetilde{Y}$ and $\widetilde{X}$ respectively by deck transformations and ρ is the isomorphism between $\pi_1(Y)$ and $\pi_1(X)$ induced by f. Furthermore, by regularization, $\tilde{f}$ (and $\tilde{h}$) can be taken to be C^1 maps. One can then show that $\tilde{f}$ is a quasi-isometry between $\widetilde{Y}$ and $\widetilde{X}$ (see [B-P], p. 86) here the compactness of Y and X is crucial. Finally a quasi-isometry gives rise to an homeomorphism between the boundaries at infinity

$$\bar{f} : \partial\widetilde{Y} \longrightarrow \partial\widetilde{X}$$

satisfying also $\bar{f} \circ \gamma = \rho(\gamma) \circ \bar{f}$, where the action of the fundamental group on $\widetilde{Y}$ (resp. $\widetilde{X}$) is extended trivially to an action on $\partial\widetilde{Y}$ (resp. $\partial\widetilde{X}$).

2nd step. — The Patterson-Sullivan measure described in the chapter 2 gives an equivariant map $y \mapsto \mu_y$ from $\widetilde{Y}$ to the space $\mathcal{M}_1(\partial\widetilde{Y})$ of probability measures on $\partial\widetilde{Y}$. As mentioned before, for each y, μ_y has no atom. We can now push forward each measure μ_y by the continuous map $\bar{f}$ and thereby construct a map

$$\begin{aligned} \widetilde{Y} &\longrightarrow \mathcal{M}_1(\partial\widetilde{X}) \\ y &\longmapsto \bar{f}_*(\mu_y) \ . \end{aligned}$$

The equivariance property of $\bar{f}$ with respect to the actions of $\pi_1(Y)$ on $\partial\widetilde{Y}$ and on $\partial\widetilde{X}$ via the isomorphism ρ shows that this map is equivariant with respect to the actions of $\pi_1(Y)$ on $\widetilde{Y}$ and on $\mathcal{M}_1(\partial\widetilde{X})$ via ρ. Finally, since $\bar{f}$ is a homeomorphism, the measures $\bar{f}_*(\mu_y)$ are well defined and have no atom.

3rd step. — We can now define the map $\widetilde{F}$ by

$$\widetilde{F}(y) = \mathrm{bar}(\bar{f}_*(\mu_y)).$$

It clearly satisfies the equivariance relation

$$\widetilde{F}(\gamma(y)) = \rho(\gamma)\widetilde{F}(y).$$

It gives rise to a map $F : Y \to X$. Its regularity will be studied in the next chapter. Let us notice that F induces also the isomorphism ρ between the two fundamental groups and hence is homotopic to f.

Let us emphasize the fact that we only require that $\bar{f}$ be continuous and that we do not need finer regularity properties of this map. In fact, the only thing we shall need on $\bar{f}$, in order to prove the regularity of F, is the fact that $\bar{f}_* : \mathcal{M}_1(\partial\widetilde{Y}) \to \mathcal{M}_1(\partial\widetilde{X})$ exists, is linear and sends a measure with no atom on a measure of the same type. We shall see in the next section the flexibility of this construction.

5. Volume and entropy: a particular case

We now proceed to the proof of the main theorem in a particular case. More precisely, let (Y, g) and (X, g_0) be two compact negatively curved riemannian n-manifolds. We assume furthermore that (X, g_0) is locally symmetric (of rank one since negatively curved) and that Y and X are homotopically equivalent (notice that we don't assume Y to be locally symmetric neither to admit a locally symmetric metric). We then have:

5.1. THEOREM. — *If $n = \dim X = \dim Y \geq 3$, we have*

i) $h^n(g)\,\mathrm{vol}(Y, g) \geq h^n(g_0)\,\mathrm{vol}(X, g_0)$.

ii) The equality case, namely $h(g) = h(g_0)$ and $\mathrm{vol}(X, g_0) = \mathrm{vol}(Y, g)$ occurs if and only if (Y, g) is isometric to (X, g_0).

Remark. — When (Y, g) is also locally symmetric, inequality *i)* also works in the converse sense, so we are in the equality case and (Y, g) is isometric to (X, g_0). So the paragraph 5 gives a very short proof of Mostow's strong rigidity theorem.

The proof relies on the study of the natural map and its behaviour with respect to volume elements. More precisely we shall show that this map contracts the volumes up to the factor $\left(\frac{h(g)}{h(g_0)}\right)^n$. If $\mathrm{Jac}\,F(y)$ denotes the jacobian of F computed with respect to the volume elements on T_yY and $T_{F(y)}X$, then

5.2. PROPOSITION. — *The natural map F is of class C^1 (at least). Furthermore, one has*

i) $|\,\mathrm{Jac}\,F(y)| \leq \left(\frac{h(g)}{h(g_0)}\right)^n$ *for all $y \in Y$.*

ii) If for some $y \in Y$, $|\,\mathrm{Jac}\,F(y)| = \left(\frac{h(g)}{h(g_0)}\right)^n$ then the differential D_yF of F at y is a homothety (of ratio $\frac{h(g)}{h(g_0)}$).

Proof of the theorem 5.1. — Let us assume the proposition 5.2 and recall that we proved in the paragraph 4 that F is a homotopy equivalence and hence is a map of degree one. Let ω_0 be the volume form of the (oriented) manifold (X, g_0) and ω the volume form of (Y, g), then

$$\int_Y F^*(\omega_0) = \int_X \omega_0 = \mathrm{vol}(X, g_0)$$

and the inequality *i)* of the proposition 5.2 gives

$$\mathrm{vol}(X, g_0) \leq \int_Y |F^*(\omega_0)| = \int_Y |(\mathrm{Jac}\,F)\omega| \leq \left(\frac{h(g)}{h(g_0)}\right)^n \int_Y \omega \leq \left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{vol}(Y, g)$$

which proves the theorem 5.1 *i)*. In the equality case then $|\,\mathrm{Jac}\,F(y)| = \left(\frac{h(g)}{h(g_0)}\right)^n = 1$ for all $y \in Y$ and hence D_yF is a homothety of ratio 1, *i.e.* an isometry for all $y \in Y$.

5.3. *Remark.* — Instead of the degree theory, we could have used the co-area formula (see [Fed], p. 241) and the fact that F is surjective. This shows that the orientability hypothesis can be dropped.

Let us now proceed to the proof of the main proposition.

Proof of the Proposition 5.2. — For the sake of simplicity we shall use the same notation for the natural map F and its pull back to the universal covers. The estimations are done on the universal covers $\widetilde{Y}$ and $\widetilde{X}$ of Y and X respectively, but since they are pointwise they can be thought of as being on Y and X.

Let us recall that $(\mu_y)_{y\in\widetilde{Y}}$ is the family of Patterson measures on $\partial\widetilde{Y}$. From the paragraphs 3 and 4 we see that the natural map is defined by the implicit equation

$$\int_{\partial\widetilde{X}} dB_{0_{(F(y),\,\theta)}}(\cdot)d(\bar{f}_*(\mu_y))(\theta) = 0,$$

which is a vector-valued equation. Equivalently one has

$$(*) \qquad \int_{\partial\widetilde{Y}} dB_{0_{(F(y),\,\bar{f}(\alpha))}} e^{-h(g)B(y,\alpha)} d\mu_0(\alpha) = 0.$$

Let us insist on the fact that, in the above equation, B_0 (resp. B) is the Busemann function of $(\widetilde{X}, \tilde{g}_0)$ [resp. of $(\widetilde{Y}, \tilde{g})$], which is the riemannian universal covering of (X, g_0) [resp. of (Y, g)].

Here we simplify by the function $c(y)$ which is positive and thus do not play any role in the equation defining $F(y)$; it is in fact obvious that the barycentre of a measure is the same for all non-zero multiple of this measure.

We choose now a frame $(e_i(z))_{i=1,\dots,n}$ of $T_z\widetilde{X}$ depending smoothly on $z \in \widetilde{X}$. Let us define the functions:

$$G_i(z,y) = \int_{\partial\widetilde{Y}} dB_{0_{(z,\,\bar{f}(\alpha))}}(e_i(z)) e^{-hB(y,\alpha)} d\mu_0(\alpha)$$

$$G : \widetilde{X} \times \widetilde{Y} \longrightarrow \mathbf{R}^n$$

$$(z,y) \longmapsto (G_1(z,y), \dots, G_n(z,y))$$

then $(*)$ reads

$$G(F(y), y) = 0.$$

Since the Busemann functions B_0 and B are smooth with respect to their first variable and $\partial\widetilde{Y}$ is compact, it is not difficult to see that G is a smooth map. Then the proof of the fact that F is C^1 is a simple application of the implicit function theorem; the details are left to the reader but let us point out that the invertibility condition on the partial differential of G with respect to z is implied by the definite-positiveness of $Dd\mathcal{B}$ (see the chapter 3).

As usual the implicit function theorem gives the existence of the differential of the implicitly defined function F and a formula for this differential. So, if one differentiates $(*)$, one gets:

$$(**)\qquad \int_{\partial\widetilde{Y}} dB_{0_{(F(y),\, \bar{f}(\alpha))}}(D_yF(\cdot),\cdot)d\mu_y(\alpha) = h(g)\int_{\partial\widetilde{Y}} dB_{0_{(F(y),\, \bar{f}(\alpha))}}(\cdot)dB_{(y,\alpha)}(\cdot)d\mu_y(\alpha)$$

this equality is to be understood as an equality between bilinear forms. Let us introduce the following quadratic forms K and H on $T_{F(y)}\widetilde{X}$, that we express with respect to the metric g_0 as symmetric endomorphisms:

$$g_0(K_{F(y)}(u),u) = \int_{\partial\widetilde{X}} DdB_{0_{(F(y),\,\theta)}}(u,u)d(\bar{f}_*\mu_y)(\theta),$$
$$g_0(H_{F(y)}(u),u) = \int_{\partial\widetilde{X}} dB^2_{0_{(F(y),\,\theta)}}(u)d(\bar{f}_*\mu_y)(\theta).$$

For $u\in T_{F(y)}\widetilde{X}$ and $v\in T_y\widetilde{Y}$, the Cauchy-Schwarz inequality gives

$$(***)\qquad \left|g_0\left(K_{F(y)}\circ D_yF(v),u\right)\right| \le h(g)\left(g_0(H_{F(y)}(u),u)\right)^{1/2}\left(\int_{\partial\widetilde{Y}} dB^2_{(y,\alpha)}(v)d\mu_y(\alpha)\right)^{1/2}.$$

For the sake of simplicity we shall omit the subscripts in $K_{F(y)}$ and $H_{F(y)}$ and use the notation $\langle\cdot,\cdot\rangle_0$ instead of $g_0(\cdot,\cdot)$.

We remark that the symmetric endomorphism K is invertible since the bilinear form $\langle K\cdot,\cdot\rangle_0$ is the hessian of the strictly convex function $\mathcal{B}$ introduced in the paragraph 3. This allows to compute the Jacobian of F; let us recall that it is the determinant of D_yF computed with respect to orthonormal basis of $(T_y\widetilde{Y},g)$ and $(T_{F(y)}\widetilde{X},g_0)$.

5.4. Lemma. — *With the above notations*

$$|\operatorname{Jac}F(y)| \le \frac{h^n(g)}{n^{n/2}}\frac{(\det H)^{1/2}}{\det K}.$$

Proof of the lemma. — If D_yF has not maximal rank, then $\operatorname{Jac}F(y)=0$ and the inequality is obvious. We can therefore assume, without loss of generality that D_yF is invertible. Let us take (u_i) an orthonormal basis of $T_{F(y)}\widetilde{X}$ which diagonalizes the endomorphism H. Let now

$$v_i' = (K\circ D_yF)^{-1}(u_i)$$

(the maps K and D_yF are invertible), the orthonormalization process of Schmidt, applied to v_i' yields an orthonormal basis (v_i) of $T_y\widetilde{Y}$. The matrix of $K\circ D_yF$ written in the basis (v_i) for $T_y\widetilde{Y}$ and (u_i) for $T_{F(y)}\widetilde{X}$ is then triangular so that

$$\det(K \circ D_y F) = (\det K)(\mathrm{Jac}\, F(y)) = \prod_{i=0}^{n} \langle K(D_y F(v_i)), u_i\rangle_0 .$$

Here we identify endomorphisms with matrices using the basis involved. The previous inequality $(***)$ then gives

$$(\det K)|\,\mathrm{Jac}\, F(y)| \leq h^n(g) \prod_{i=1}^{n} \langle H u_i, u_i\rangle_0^{1/2} \prod_{i=1}^{n} \left(\int_{\partial\widetilde{Y}} dB^2_{(y,\alpha)}(v_i) d\mu_y(\alpha) \right)^{1/2} .$$

By the choice of the basis $\{u_i\}$ and $\{v_i\}$,

$$\prod_{i=1}^{n} \langle H u_i, u_i\rangle^{1/2} = (\det(H))^{1/2}$$

$$\prod_{i=1}^{n} \left(\int_{\partial\widetilde{Y}} dB^2_{(y,\alpha)}(v_i) d\mu_y(\alpha) \right)^{1/2} \leq \left(\frac{\sum \int_{\partial\widetilde{Y}} dB^2_{(y,\alpha)}(v_i) d\mu_y(\alpha)}{n} \right)^{n/2} \leq \frac{1}{n^{n/2}}$$

since $\sum dB^2_{(y,\alpha)}(v_i) = ||dB^2_{(y,\alpha)}||^2_g = 1$ and μ_y is a probability measure. Finally, one gets

$$|\,\mathrm{Jac}\, F(y)| \leq \frac{h^n(g)}{n^{n/2}} \frac{(\det H)^{1/2}}{(\det K)}$$

which is the desired inequality. ■

End of the proof of the proposition 5.2. — Let us remark now that H and K are related to the locally symmetric metric of negative curvature g_0, the bilinear form $\langle K_{F(y)}\cdot,\cdot\rangle_0$ is an average of the second fundamental form of the horospheres passing through $F(y)$ and similarly $\langle H_{F(y)}\cdot,\cdot\rangle_0$ is an average of $dB^2_{0_{(F(y),\, \tilde{f}(\theta))}}(\cdot)$. The metric g_0 is locally symmetric so it is real, complex or quaternionic hyperbolic or a Cayley hyperbolic space of real dimension 16. Let d be the real dimension of the field or the ring under consideration (*i.e.* $d = 1, 2, 4$ or 8 respectively when we consider the real, complex, quaternionic or Cayley hyperbolic spaces, and let $J_1(x), \dots, J_{d-1}(x)$ be the orthogonal endomorphisms at each point defining the complex, quaternionic or Cayley structure. They are parallel and satisfy

$$J_i^2(x) = -\,\mathrm{id}\,.$$

It is a classical fact (see [B-C-G 1]) that if g_0 is normalized so that its curvature lies between -4 and -1, then one has the following equality

$$DdB_0(\cdot,\cdot) = g_0(\cdot,\cdot) - dB_0(\cdot)dB_0(\cdot) + \sum_{k=1}^{d-1} dB_0(J_k(\cdot))dB_0(J_k(\cdot))$$

which by integration gives

$$K = I - H - \sum_{k=1}^{d-1} J_k H J_k .$$

By choosing a (orthonormal) basis at $F(y)$ we can think of K, H and J_ℓ as being matrices instead of endomorphisms. The last remark that we need to make is that the matrix H satisfies

$$\operatorname{trace} H = 1,$$

indeed, if $\{u_i\}$ is an orthonormal basis of $(T_{F(y)}\widetilde{X}, g_0)$,

$$\begin{aligned}\operatorname{trace}(H) &= \sum_{i=1}^{n} \langle H_{F(y)}(u_i), u_i\rangle_0 \\ &= \int_{\partial \widetilde{X}} \left(\sum_{i=1}^{n} dB^2_{0_{(F(y),\,\theta)}}(u_i) \right) d(\bar{f}_* \mu_y)(\theta) \\ &= 1\end{aligned}$$

since $||dB_0||_0 = 1$ and $\bar{f}_*(\mu_y)$ is a probability measure.

5.5. LEMMA. — *If H is a symmetric positive definite $n \times n$ matrix whose trace is equal to 1 then, if $n > 3$,*

i) $\dfrac{(\det H)^{1/2}}{\det\left(I - H - \sum_{k=1}^{d-1} J_k H J_k\right)} \leq \left(\frac{\sqrt{n}}{n+d-2}\right)^n$ *i.e. the maximal value of this functional is achieved for $H = \frac{1}{n} I$.*

ii) The equality implies that $H = \frac{1}{n} I$.

The proof of this lemma is given in [B-C-G 1], Appendix B. Let us notice that, in [B-C-G 1], the lemma 5.5 is proved under the assumption $n \geq d+2$ which, in our geometric context, is equivalent to the assumption $n \geq 3$. In fact, every complex, quaternionic or Cayley hyperbolic space which is not isometric to a real hyperbolic space has dimension at least $2d$. Let us also remark that $n+d-2$ is the entropy $h(g_0)$ of the metric g_0 normalized as before. One then obtains the inequality of the proposition 5.2 by combining the two previous lemmas 5.4 and 5.5.

In the equality case, *i.e.* if $|\operatorname{Jac} F(y)| = \left(\frac{h(g)}{h(g_0)}\right)^n$, the inequality of the lemma 5.5 is an equality, namely

$$H_{F(y)} = \frac{1}{n} I \text{ and } K_{F(y)} = \frac{(n+d-2)}{n} I = \frac{h(g_0)}{n} I$$

for all $y \in \widetilde{Y}$; hence, the inequality $(***)$ becomes

$$|\langle D_y F(v), u\rangle_0| \leq \sqrt{n} \frac{h(g)}{h(g_0)} ||u||_0 \left(\int_{\partial \widetilde{Y}} dB^2_{(y,\alpha)}(v) d\mu_y(\alpha) \right)^{1/2}$$

for all $v \in T_y\widetilde{Y}$ and $u \in T_{F(y)}\widetilde{X}$. By taking the supremum in $u \in T_{F(y)}\widetilde{X}$ such that $||u||_0 = 1$, one gets

$$||D_yF(v)||_0 \leq \sqrt{n}\frac{h(g)}{h(g_0)}\left(\int_{\partial\widetilde{Y}} dB^2_{(y,\alpha)}(v)d\mu_y(\alpha)\right)^{1/2}$$

for all $v \in T_y\widetilde{Y}$. Let L be the endomorphism of $T_y\widetilde{Y}$ defined by $L = (D_yF)^* \circ (D_yF)$ and (v_i) a g-orthonormal basis of $T_y\widetilde{Y}$, we have

$$\begin{aligned}\text{trace}\, L = \sum_{i=1}^n \langle Lv_i, v_i\rangle_g &= \sum_{i=1}^n \langle D_yF(v_i), D_yF(v_i)\rangle_0 \\ &\leq n\left(\frac{h(g)}{h(g_0)}\right)^2,\end{aligned}$$

where we still use the fact that $||dB||_g = 1$.

Now

$$\left(\frac{h(g)}{h(g_0)}\right)^{2n} = |\,\text{Jac}\, F(y)|^2 = \det L \leq \left(\frac{\text{trace}\, L}{n}\right)^n \leq \left(\frac{h(g)}{h(g_0)}\right)^{2n}$$

which implies that $\det L = \left(\frac{\text{trace}\, L}{n}\right)^n$ and that

$$L = \left(\frac{h(g)}{h(g_0)}\right)^2 \cdot I.$$

This exactly means that D_yF is an isometry (between $T_y\widetilde{Y}$ and $T_{F(y)}\widetilde{X}$) composed with a homothety of ratio $\frac{h(g)}{h(g_0)}$. ■

This gives a quick proof of the main theorem in this particular case which emphasizes the role played by the Patterson-Sullivan measure. After describing the proof in the general case we shall make some comments (paragraph 7) on the whole construction.

6. Volume and entropy: the general case

In the general case, namely when (Y, g) is not assumed any more to have negative curvature and when the map $f : Y \to X$ is just supposed to have non-zero degree, the previous construction fails. In fact, if (Y, g) has not negative curvature, the universal cover $\widetilde{Y}$ cannot any more be identified with a ball in $\mathbf{R}^n$ and we cannot use the boundary-to-boundary map. We thus have to modify the construction; the idea is clearly to work on $\widetilde{Y}$ and $\widetilde{X}$ directly, without using the boundaries; we pay for that by the fact that the inequality is proved by taking a limit and hence the equality case is more difficult (technically) to treat because the natural map F (see the paragraph 4) is then defined as a limit of a sequence of maps F_s defined as follows.

For $s > h(g)$ we consider the following family of measures on $\widetilde{Y}$ (see the paragraph 2):

$$\mu_{s,y} = \frac{e^{-s\,d(y,z)}dv_{\tilde{g}}}{\int_{\widetilde{Y}} e^{-s\,d(y,z)}dv_{\tilde{g}}(z)}$$

where, as before, Γ is the fundamental group of Y acting on $\widetilde{Y}$ by deck transformations, d is the distance on $\widetilde{Y}$ induced by the metric $\tilde{g}$ and 0 is a fixed origin on $\widetilde{Y}$. This is a family of probability measures which, in the case where (Y, g) has negative curvature, converges to the Patterson-Sullivan measure μ_y on $\partial\widetilde{Y}$. In the general case, for obvious compactness reasons, every sequence $\mu_{s_i,y}$ admits a subsequence which converges, but this subsequence may depend on y and the limits may be not unique. Let $\tilde{f}$ be the continuous map between $\widetilde{Y}$ and $\widetilde{X}$ induced from f, the idea is to push-forward the family $\mu_{s,y}$, using $\tilde{f}$, to a family of measures on $\widetilde{X}$ and then make the convolution with the harmonic measure on $(\widetilde{X}, \tilde{g}_0)$ in order to regularize it. We thus get, by this method, the family of probability measures $\sigma_{s,y}$ on $\partial\widetilde{X}$ defined by:

$$\sigma_{s,y} = \left(\frac{\int_{\widetilde{Y}} e^{-s\,d(y,z)}e^{-h(g_0)B_0(\tilde{f}(z),\theta)}dv_{\tilde{g}(z)}}{\int_{\widetilde{Y}} e^{-s\,d(y,z)}dv_{\tilde{g}}(z)}\right)d\theta.$$

It is a family of measures on $\partial\widetilde{X}$, with density with respect to $d\theta$; with our normalization, namely $d\theta$ is a probability measure, then $\sigma_{s,y}$ is also a probability measure (see [B-C-G 1]). We then define for each $s > h(g)$ an equivariant map:

$$\widetilde{F}_s : \widetilde{Y} \longrightarrow \widetilde{X}$$

$$y \longmapsto \widetilde{F}_s(y) = \mathrm{bar}(\sigma_{s,y})$$

which gives rise to a map, denoted by F_s, between Y and X. The main theorem follows then from the two propositions.

6.1. Proposition. — *For each $s > h(g)$ the map F_s is a C^1 map and satisfies*

$$|\,\mathrm{Jac}\,F_s(y)| \le \left(\frac{s}{h(g)}\right)^n$$

for all $y \in Y$.

Proof. — As in the paragraph 5 (using the same notations), the map F_s is defined by the implicit equation $G(F_s(y), y) = 0$ where $G : \widetilde{X} \times \widetilde{Y} \to \mathbf{R}^n$ is the application whose coordinates are

$$G_i(x, y) = \int_{\partial\widetilde{X}}\int_{\widetilde{Y}} dB_{0_{(x,\,\theta)}}[e_i(x)]e^{-s\,d(y,z)}e^{-h_0\cdot B_0(\tilde{f}(z),\theta)}dv_{\tilde{g}}(z)\cdot d\theta.$$

The only difference with the paragraph 5 lies in the fact that, when the curvature is not supposed to be negative, the function $y \mapsto d(y, z)$ is not C^1 any more, but only

Lipschitz (and, moreover, C^1 on the complementary of the cut-locus of z). However, let us notice that, for any curve $t \mapsto y(t)$ parametrized by arc-length, both functions $-\frac{1}{t}\left(e^{-sd(y(t),z)} - e^{-sd(y(0),z)}\right)$ and $s\langle \nabla d_{(y(t),z)}, y'(t)\rangle e^{-sd(y(t),z)}$ are dominated (for every z) by $\frac{1}{t}(e^{ts} - 1)e^{-sd(y(0),z)}$ and by $se^{ts}e^{-sd(y(0),z)}$ respectively; moreover, the limit of these two functions when t goes to zero is, for every z which does not lie in the cut-locus of $y(0)$, equal to $s\langle \nabla d_{(y(0),z)}, y'(0)\rangle e^{-sd(y(0),z)}$. This implies that the assumptions of Lebesgue dominated convergence theorem are fulfilled and, using it twice, we prove that G is C^1.

Applying the implicit function theorem, we get an inequality which is the analogous of the inequality $(***)$ of the paragraph 5 and which writes:

$$(***)' \qquad \left| g_0\left(K_{F_s(y)} \circ D_y F_s(v), u\right)\right| \leq s\left(g_0(H_{F_s(y)}(u), u)\right)^{1/2}\left(\int_{\widetilde{Y}} g(\nabla d_{(y,z)}, v)^2 d\mu_{s,y}(z)\right)^{1/2}$$

where K and H are defined by the same integral formulas as in paragraph 5, where the measure $\bar{f}_* \mu_y$ is replaced now by $\sigma_{s,y}$.

The proof of the proposition is then identical to the proof of the inequality *i)* of the proposition 5.2. The distance d replaces the Busemann function B; it shares the same fundamental property (for our purpose), namely that its gradient has norm equal to one with respect to the metric g. The number s plays the role of the entropy $h(g)$. The inequality in the main theorem is obtained by integrating, as before, $F_s^*(\omega_0)$ on Y, using the above proposition and letting s go to $h(g)$. ■

As we mentioned previously, since the main inequality is proved by a limiting process, the equality case is naturally more difficult. In fact we cannot use the equality case of the pointwise estimate given in the proposition 6.1.

6.2. PROPOSITION. — *If* $h(g) = h(g_0)$ *and* $\mathrm{vol}(Y, g) = |\deg f|\,\mathrm{vol}(X, g_0)$ *then a subsequence* F_{s_k} *of the family of maps* F_s *converges uniformly (when* s_k *goes to* $h(g)$*) to a riemannian covering* $F : Y \to X$*, homotopic to* f*.*

Ideas of the proof. — The proof described in [B-C-G 1] is in a slightly different context. Let us briefly summarize the different steps:

i) From the equality we deduce first that there is a sequence s_k going to $h(g) = h(g_0)$ when k goes to infinity such that $|\,\mathrm{Jac}\, F_{s_k}|$ converges to 1 almost everywhere.

ii) If H_s is the symmetric endomorphism defined by

$$\langle H_{s,y} u, u\rangle_0 = \int_{\partial \widetilde{X}} dB_0^2{}_{(\widetilde{F_s}(y),\,\theta)}(u)\, d\sigma_{s,y}(\theta)$$

for $u \in T_{\widetilde{F_s}(y)}\widetilde{X}$, then using step *i)* we show that $H_{s_k,y}$ goes to $\frac{1}{n}I$ for almost every $y \in \widetilde{Y}$ and that this convergence is uniform on a ε-dense subset of $\widetilde{Y}$.

iii) By studying the variation of $H_{s_k,y}$ with respect to y, we then prove that the greatest eigenvalue of $H_{s_k,y}$, denoted by $\lambda_n^k(y)$, is uniformly bounded away from 1; indeed we show that for all $y \in \widetilde{Y}$,

$$0 \leq \lambda_n^k(y) \leq 1 - \frac{1}{n} < 1$$

(the value $1 - \frac{1}{n}$ is irrelevant, it could be any number less that 1 and greater than $\frac{1}{n}$). This in turns allows to bound uniformly the differential of F_{s_k}, so that this family of C^1 maps is shown to be equicontinuous. By the standard compactness theorem we then deduce that a subsequence, again denoted by F_{s_k}, converges uniformly to a continuous (even Lipschitz) map F between Y and X.

iv) We then bound the (covariant) derivatives of H_{s_k} along the geodesics of (X, g_0) and deduce that H_{s_k} (and thus K_{s_k}) converge to $\frac{1}{n}I$ (resp. to $\frac{h(g_0)}{n}I$), uniformly on Y. Plugging this result in the inequality $(***)'$, we get that the trace (with respect to g) of $F_{s_k}^*(g_0)$ goes to n uniformly on Y ; recalling that the jacobian of F_{s_k} goes to 1 almost everywhere, we get that $F_{s_k}^* g_0$ is everywhere bounded and almost everywhere convergent to the initial metric g on Y. We deduce that the Lipschitz constant of F is at most 1.

v) It is easy to show that, as before, each map F_s is homotopic to f (by construction) and hence F too, since it is the uniform limit of F_{s_k}. Let us assume that deg f is positive, which can always be obtained by changing the orientation, then F is a distance contracting map of degree (deg f) between the riemannian manifolds (Y, g) and (X, g_0) which satisfy

$$\mathrm{vol}(Y, g) = (\deg f)\, \mathrm{vol}(X, g_0);$$

if F were of class C^1 it would be easy to show that it is a riemannian covering. But in our case it is just a Lipschitz map, hence it is technically a little bit more difficult to conclude that it is a local isometry (hence a covering), (see [B-C-G 1], Appendix C). This finishes the proof. ■

7. Miscellaneous comments

In this last paragraph we shall make a series of comments on our result and the method used to prove it and announce some of our works in progress.

The proposition 5.2 may be seen as a real version of the Schwarz's lemma. In fact, let $(\widetilde{X}, \widetilde{g}_0)$ and $(\widetilde{Y}, \widetilde{g})$ be two simply connected riemannian manifolds, without conjugate points. In this case, their ideal boundaries $\partial\widetilde{X}$ and $\partial\widetilde{Y}$ are well defined. Let Γ be any (eventually empty) subset of $\mathrm{Isom}(\widetilde{Y},\widetilde{g})$ and ρ a (eventually trivial) homomorphism: $\Gamma \to \mathrm{Isom}(\widetilde{X},\widetilde{g}_0)$. We shall denote by $\bar{\gamma}$ and $\overline{\rho(\gamma)}$ the actions of γ and $\rho(\gamma)$ induced on the ideal boundaries $\partial\widetilde{Y}$ and $\partial\widetilde{X}$. Let us suppose that

i) The manifold $(\widetilde{X},\widetilde{g}_0)$ is a rank one symmetric space.

ii) It is possible to choose some point 0 in $\widetilde{Y}$, a positive real number h and a measure μ_0 on $\partial\widetilde{Y}$ such that the family of measures $\mu_y = e^{-hB(y,\cdot)}\mu_0$ (where B is the Busemann function on $(\widetilde{Y},\widetilde{g})$ with origin at the point 0) satisfies:

(a) $\bar{\gamma}_*\mu_0 = \mu_{\gamma(O)}$ for every $\gamma \in \Gamma$.

(b) $\lim_{y\to\theta}\mu_y$ exists and is proportional to the Dirac measure δ_θ (here y goes to θ non-tangentially).

With the same proof as in the proposition 5.2, we obtain the:

7.1. Proposition. — *If $\widetilde{X},\widetilde{Y},\Gamma$ satisfy i) and ii), then, for any homeomorphism $f:\partial\widetilde{Y}\to\partial\widetilde{X}$ such that $f\circ\bar{\gamma} = \overline{\rho(\gamma)}\circ f$ for any $\gamma\in\Gamma$, the map $F:\widetilde{Y}\to\widetilde{X}$, given by the formula $F(y) = \mathrm{bar}(f_*\mu_y)$, has the following properties:*

(1) $F\circ\gamma = \rho(\gamma)\circ F$ *for any* $\gamma\in\Gamma$,

(2) $F(y)$ *goes to* $f(\theta)$ *when* y *goes to* θ *non-tangentially,*

(3) $|\mathrm{Jac}\,F(y)| \le \left(\frac{h}{h_0}\right)^n$ *for any* $y\in\widetilde{Y}$.

Moreover, equality holds, for some $y\in\widetilde{Y}$, in the inequality (3) iff D_yF is an homothety from $(T_y\widetilde{Y},g)$ onto $(T_{F(y)}\widetilde{X},g_0)$ whose ratio is $\frac{h}{h_0}$.

Remark. — When $(\widetilde{Y},\widetilde{g})$ admits a cocompact isometry group (containing Γ), the Patterson-Sullivan measure satisfies *(a)*, which immediately implies *(b)* for $\gamma^n_*(\mu_y)$ goes to δ_θ when $\gamma^n(y)$ goes to θ. This works in particular when $(\widetilde{Y},\widetilde{g})$ and $(\widetilde{X},\widetilde{g}_0)$ are both symmetric spaces and the proposition 7.1 applies. The interesting fact here is that we do not really need that Γ is cocompact, but only that $\mathrm{Isom}(\widetilde{Y},\widetilde{g})$ is cocompact, it is then possible to study the case where $\widetilde{Y}/\Gamma$ is not compact; the problem is then to find a substitute to the degree theory in order to establish an analogous of the theorem 5.1. In a forthcoming paper ([B-C-G 3]), we shall apply this idea in the finite volume case.

7.2. — The natural map F which is defined in the paragraph 6 (as a limit of maps F_s) occurs to be the same as the one defined in [B-C-G 1]. However the proofs of the inequality between entropies (proposition 6.1) and of the equality case (proposition 6.2) work differently. In fact, in [B-C-G 1], the maps F_s were splitted in a product of two applications $M_s : \widetilde{Y} \longrightarrow L^2(\partial\widetilde{X}, d\theta)$ and $\pi_0 : L^2(\partial\widetilde{X}, d\theta) \longrightarrow \widetilde{X}$. In [B-C-G 1], these two maps were studied separately: $M_s(y)$ was the square root of the density (with respect to $d\theta$) of the measure $\sigma_{s,y}$ defined in the paragraph 6, it only depends on the geometry of $(\widetilde{Y}, \tilde{g})$ via the fact that $s > h(g)$. On the contrary, π_0 only depends on the geometry of $(\widetilde{X}, \tilde{g}_0)$, namely $\pi_0(f) = \text{bar}(f^2(\theta)d\theta)$.

It was very easy to show that the energy of M_s is bounded by $\frac{s^2}{4}$, so that the volume of the image by M_s of a fundamental domain U in $\widetilde{Y}$ (measured with respect to the L^2-metric) is smaller than the initial volume of $(U, \tilde{g})$ multiplied by $\left(\frac{s^2}{4n}\right)^{n/2}$.

The implicit function theorem proved that π_0 is a C^1 submersion and an easy computation gave that the determinant of its differential (restricted to the horizontal subspace) is equal to $\frac{2^n(\det H_{\pi_0(f)})^{1/2}}{\det K_{\pi_0(f)}}$, where H and K are defined by the same integral formulas as in paragraphs 5 and 6, where the measure $\bar{f}_*\mu_y$ is replaced by $f^2(\theta)d\theta$. So the most remarkable property of π_0 is a corollary of the lemma 5.5: the differential of π_0 (restricted to the horizontal subspace) has determinant bounded by $\left(\frac{4n}{h_0^2}\right)^{n/2}$; moreover, this upper bound is achieved iff this differential is a homothety from the horizontal subspace onto $T_{\pi_0(f)}\widetilde{X}$ (see [B-C-G 1], Chapter 5). Turning to the fact that $F_s = \pi_0 \circ M_s$, this immediately proves the proposition 6.1 and the inequality between entropies.

Let us point out that this method, developed in [B-C-G 1], although rather simple in its principle, is technically delicate to apply. This comes from the fact that we are working with maps from (or to) infinite dimensional Hilbert spaces (namely $L^2(\partial\widetilde{X})$), so the regularity of M_s and π_0 and the convergence of the family F_s to some F (in the equality case) are not as easy to establish compared to the present paragraph 6. In fact, in this paragraph, working directly with the F_s's, we reduce the problem to establish regularity and convergence for maps between finite dimensional spaces.

7.3. — The comparison between [B-C-G 1] and the present paragraph 5 points out the fact that [B-C-G 1] obeyed to a different philosophy. In order to describe it in a simple case, let us again assume that (Y, g) has negative curvature and f is a homotopy equivalence. We use the notations of the paragraph 2; let us denote by S^∞ the unit sphere of $L^2(\partial\widetilde{X}, d(\bar{f}_*\mu_0))$ (we recall that μ_y is the family of Patterson-Sullivan measures and $0 \in \widetilde{Y}$ is an origin). Then the map

$$M : \widetilde{Y} \longrightarrow S^\infty$$
$$y \longmapsto (\theta \longmapsto \exp\left(-\frac{h(g)}{2}B(y, \bar{f}^{-1}(\theta))\right)$$

is an equivariant C^1 map. The action of $\pi_1(Y)$ on $L^2(\partial\widetilde{X}, d(\bar{f}_*\mu_0))$ comes from the representation ρ and the extension to $L^2(\partial\widetilde{X}, d(\bar{f}_*\mu_0))$ of the natural action of $\pi_1(X)$ on $\partial\widetilde{X}$, more precisely, if $\ell \in L^2(\partial \widetilde{X}, d(\bar{f}_* \mu_0))$ and $\gamma \in \pi_1(Y)$,

$$\begin{aligned}(\gamma\ell)(\theta) &= \ell\left(\overline{\rho(\gamma^{-1})}(\theta)\right)\left(\frac{d\left[\overline{\rho(\gamma)}_* \bar{f}_*(\mu_0)\right]}{d(\bar{f}_*(\mu_0))}\right)^{1/2} \\ &= \ell\left(\overline{\rho(\gamma^{-1})}(\theta)\right)\left(\frac{d(\bar{f}_*(\mu_{\gamma(0)}))}{d(\bar{f}_*(\mu_0))}\right)^{1/2} \\ &= \ell\left(\overline{\rho(\gamma^{-1})}(\theta)\right)\exp\left[-\frac{h(g)}{2}B\left(\gamma(0), \bar{f}^{-1}(\theta)\right)\right].\end{aligned}$$

We remark that the map M is "contracting", namely the energy of M (measured by the riemannian metric $\tilde{g}$ on the source $\widetilde{Y}$ and by the L^2-scalar product on the target space S^∞) is bounded by $\frac{h(g)^2}{4}$.

For the same reason, let us denote S_0^∞ the unit sphere of $L^2(\partial \widetilde{X}, d\theta)$ and M_0 the map from $\widetilde{Y}$ in S_0^∞ given by

$$M_0(y) = \left(\theta \longmapsto \exp\left(-\frac{h_0}{2}B_0(f(y), \theta)\right)\right).$$

As f can be assumed to be a C^1 map, M_0 is a C^1 equivariant map. Moreover, M_0 is an homothetic embedding, whose energy is constant equal to $\frac{h_0^2}{4}$.

Now the idea is to use the calibration theory to prove that

$$\text{vol}\, M(U) \geq \text{vol}\, M_0(U),$$

where U is any fixed fundamental domain for the action of $\pi_1(Y)$ on $\widetilde{Y}$.

The first problem is that the two L^2-spaces involved are different. One way to circumvent this difficulty is to see M as a "limit" of a maps M_s from $\widetilde{Y}$ into S_0^∞ given by

$$M_s(y) = \left(\theta \longmapsto \left(\frac{\int_{\widetilde{Y}} e^{-s\,d(y,z)} e^{-h(g_0)B_0(\bar{f}(z),\theta)} dv_{\tilde{g}}(z)}{\int_{\widetilde{Y}} e^{-s\,d(y,z)} dv_{\tilde{g}}(z)}\right)^{1/2}\right)$$

which is the square root of the density (with respect to $d\theta$) of the measure $\sigma_{s,y}$ defined in the chapter 6. Let us remark that, when s goes to $h(g)$, $\sigma_{s,y}$ converges to μ_y (see also the section 7.2) what we really prove is that

$$\text{vol}\, M_s(U) \geq \text{vol}\, M_0(U)$$

and this suffices to prove the main inequality (we don't need to show that M_s converges to M, although it is probably true in some reasonable sense). The reader is referred to [B-C-G 1], Chapter 5, for precise proofs. In order to prove the last inequality we use calibration theory; let us consider the map

$$\begin{aligned}\pi_0 : S_0^\infty &\longrightarrow \widetilde{X} \\ \varphi &\longmapsto \text{bar}(\varphi^2(\theta)\, d\theta)\end{aligned}$$

then the calibrating n-form is $\pi_0^*(\omega_0)$ (see [B-C-G 1]).

As in the section 7.2, π_0 is a submersion whose horizontal space at φ is the vector space $\mathcal{H}_\varphi$ spanned by the functions

$$\theta \longmapsto dB_{0_{(\pi_0(\varphi),\,\theta)}}(u_i)\varphi(\theta), \qquad i = 1, 2, \ldots, n$$

where (u_i) is a g_0-orthonormal basis of $T_{\pi_0(\varphi)}\widetilde{X}$.

Furthermore the differential of π_0 (restricted to this horizontal subspace) has determinant bounded by $\left(\frac{4n}{h_0^2}\right)^{n/2}$. Moreover this upper bound is achieved when φ lies in the image of M_0 and, in that case, the tangent space to the image of M_0 coincides with the horizontal subspace $\mathcal{H}_\varphi$. These are exactly the assumptions of the calibration theory for the calibrating form $\pi_0^*\omega_0$, where ω_0 is the riemannian volume-form of $(\widetilde{X}, \tilde{g}_0)$; in fact the above properties of the determinant of $d\pi_0$ write:

$$\left|(\pi_0^*\omega_0)_\varphi(u_1, \ldots, u_n)\right| \leq \left(\frac{4n}{h_0^2}\right)^{n/2} \|u_1 \wedge \cdots \wedge u_n\|_{L^2(\partial\widetilde{X})},$$

for any $u_1, \ldots, u_n \in L^2(\partial\widetilde{X})$ tangent to S_0^∞ at φ, with equality when $\varphi \in$ Image(M_0) and when $u_1, \ldots, u_n$ are tangent to the image of M_0.

The equality case amounts to saying that the image of M_s tends to be horizontal (*i.e.* tangent to the horizontal distribution at each point) and that, for each y, $M_s(y) \in S_0^\infty$ is a point where the differential of π_0 tends to be an homothety from the horizontal subspace at $M_s(y)$ onto $T_{\pi(M_s(y))}\widetilde{X}$, in such a way that $(\pi \circ M_s) : \widetilde{Y} \longrightarrow \widetilde{X}$ tends to be an equivariant homothety giving rise to an isometry (after renormalization) between (Y, g) and (X, g_0).

This point of view seems to have disappeared in the proof given in this article, but if one considers the equality $(**)$ of the paragraph 5, then the right hand side is

$$h(g) \int_{\partial\widetilde{Y}} dB_{0_{(F(y),\, \bar{f}(\alpha))}}(u) dB_{(y,\alpha)}(v)\, d\mu_y(\alpha)$$

for $u \in T_{F(y)}\widetilde{X}$ and $v \in T_y\widetilde{Y}$; this is nothing but the $L^2(\partial\widetilde{X}, d(\bar{f}_*\mu_0))$ scalar product, at the point $\varphi = M(y) \in S^\infty$ (defined by $\varphi(\theta) = \exp\left[-\frac{h(g)}{2}B(y, \bar{f}^{-1}(\theta))\right]$) of the vector $-2D_yM(v) \in T_\varphi(S^\infty)$, which is defined by $-2D_yM(v)(\theta) = dB_{(y,\bar{f}^{-1}(\theta))}(v) \cdot \varphi(\theta)$, with the vector $\theta \longmapsto dB_{0_{(\pi(\varphi),\,\theta)}}(u) \cdot \varphi(\theta)$, which lies in the π-horizontal subspace $\mathcal{H}_\varphi$ (notice that $\pi(\varphi) = F(y)$ for $F = \pi \circ M$). In this construction, π is the map:

$$\begin{aligned} \pi : S^\infty &\longrightarrow \widetilde{X} \\ \varphi &\longmapsto \text{bar}[\varphi^2(\theta)\, d(\bar{f}_*\mu_0)] \end{aligned}.$$

7.4. — When we specify our proof of the paragraph 5 to the case where (Y, g) is also locally symmetric of negative curvature we obtain a very quick and simple proof of Mostow's strong rigidity theorem in that context. It is simpler than the original proof (see [Mos]) and unified for all locally symmetric spaces of negative curvature. We do not need to use and hence to show any regularity property of the boundary-to-boundary map $\bar{f}$ since we only need to have it acting on a family of measures. In fact, in that case the map M gives a realisation of $\widetilde{Y}$ in the space $\mathcal{M}_1(\partial\widetilde{X})$ of probability measures on $\partial\widetilde{X}$. We also do not use the ergodicity of the action of $\pi_1(Y)$ on $\partial\widetilde{Y}$ (see [Mos]). This proof is thus very flexible. The proof of Mostow's strong rigidity theorem derives from that given in the paragraph 5, but it may be done even much more simply. The reason is that, when (Y, g) is locally symmetric, the Patterson-Sullivan measure coincides with the Lebesgue measure, so that the inequality $(***)$ of the paragraph 5 writes in this case:

$$|g_0(K_{F(y)} \circ D_yF(v), u)| \leq \frac{h(g)}{\sqrt{n}} \big(g_0(H_{F(y)}(u), u)\big)^{1/2} g(v, v)^{1/2},$$

which immediately proves the lemma 5.4.

The equality case is also much simpler. In fact the above inequality and the equality case in the lemma 5.5 immediately proves that $\left(\frac{h(g_0)}{h(g)} D_yF\right)$ is a linear contraction from (T_yY, g) onto $(T_{F(y)}X, g_0)$; as, by assumption, it has determinant equal to 1, it is an isometry. It is then obvious that the proof of Mostow's rigidity theorem reduces to the proof of the lemma 5.5. The reader is referred to [B-C-G 1], chapter 9 for more rigidity results.

7.5. — We used the natural map when the target manifold is locally symmetric but it exists in a more general context, for example when X is endowed with a negatively curved metric since the barycentre exists in this context (see [B-C-G 1], Appendix A). It gives for any homotopy class of maps between two negatively curved manifolds (for example) a representative which has good properties with respect to the volume elements. In that respect it should be compared to the harmonic map in the given homotopy class (see [E-L] for a survey of this subject); harmonic maps are defined analytically, our construction is more geometric and moreover explicit and, as we have shown in this article, it is easy to differentiate. The harmonic maps minimize an energy, a L^2-norm of the differential; our map minimizes a volume, a L^n-norm of the differential. As such, our proof of Mostow's rigidity theorem contains some of the features of the original proof of G. Mostow given in [Mos] where is used a notion of conformal capacity (see [Mos] for the details).

7.6. — We should point out that we used the family of measures called the Patterson-Sullivan measures, because they are the one that naturally occurs in the entropy problem; we could of course also construct a natural map using any reasonable family of measures on $\partial\widetilde{Y}$, the only requirements are that

i) the assignments

$$\widetilde{Y} \longrightarrow \mathcal{M}_1(\partial\widetilde{Y})$$
$$y \longmapsto \mu_y$$

is equivariant with respect to the actions of $\pi_1(Y)$ both on $\widetilde{Y}$ and on $\partial\widetilde{Y}$.

ii) the measures μ_y are in the same measure class on $\partial\widetilde{Y}$, *i.e.*

$$\frac{d\mu_y}{d\mu_0} = \exp(a(y, \cdot))$$

for a function a on $\widetilde{Y} \times \partial\widetilde{Y}$.

If a is regular enough we can construct a natural map which is C^1 and if we differentiate the implicit equation that defines it, the right hand side of $(**)$ becomes

$$\int_{\partial\widetilde{Y}} dB_{0_{(F(y),\, \bar{f}(\alpha))}}(\cdot)\; da_{(y,\alpha)}(\cdot)\; d\mu_y(\alpha).$$

The question is now to estimate Jac F. One important feature of the Patterson-Sullivan measures is that the function $a = -h(g)B$ satisfies

$$||dB_{(y,\alpha)}||_g = 1$$

for all $(y, \alpha) \in \widetilde{Y} \times \partial\widetilde{Y}$.

7.7. — We also can work with manifolds Y and X which do not have the same dimension. Precisely, we can show that the natural map also contracts the p-dimensional volumes when $3 \leq p \leq n$. This idea will be used in [B-C-G 4] to study quasi-fuchsian deformations of lattices of rank one symmetric spaces.

7.8. — The natural question that is raised by the previous work is to extend the main theorem to the case when (X, g_0) is a higher rank locally symmetric space. We shall give a first answer to this question in [B-C-G 5], namely prove the main theorem when $(\widetilde{X}, \tilde{g}_0)$ is a product of symmetric spaces of negative curvature, each of which has dimension greater than 2. Let us emphasize that whether the lattice acts irreducibly or not is irrelevant.

7.9. — Let us make some comments concerning a previous work [B-C-G 2] in which we proved a local version of the main theorem. The technique used was completely different; we manufactured a family of functionals each of which bounds from below the entropy and we reduced the problem to showing a similar inequality on these functionals. In order to do so we had to study the space of metrics near a hyperbolic metric g_0. We showed a slice theorem in a neighbourhood of g_0, namely we proved that we could find a submanifold (of infinite dimension) in the space of metrics, containing g_0, which is transversal to the conformal changes of metrics and to the action of the diffeomorphism group (see the article [B-C-G 2] for the details). Such a slice amounts to finding a good (adapted to the question) reparametrization φ of the manifold an to replacing the initial metric g by the element $\varphi^* g$ of the slice. Now if our natural map were a diffeomorphism it could be interpreted as a reparametrization of Y. If we then replace the reparametrization that we used in [B-C-G 2] by the reparametrization by the natural map, the proof of [B-C-G 2] still works. In fact, the reparametrization by the natural map F allows a comparison of the volume elements of (Y, g) and (X, g_0). Indeed, the proposition 5.2 is

$$|F^*\omega_g| \leq \left(\frac{h(g)}{h(g_0)}\right)^n |\omega_0|$$

where ω_0 (resp. ω_g) is the volume form of (Y, g) (resp. (X, g_0)).

7.10. — Finally, we can interpret our result as follows: let $\Omega(X)$ be the infimum, among all metrics g on X, of $h(g)^n \operatorname{vol}(X, g)$, if X carries a locally symmetric metric of negative curvature g_0, then

i) $\Omega(X) = h(g_0)^n \operatorname{vol}(X, g_0)$.

ii) If $f : Y \to X$ is a map of non-zero degree, then $\Omega(Y) \geq |\deg f|\ \Omega(X)$.

Now, by combining this with some results of Y. Babenko ([Bab]), it can be shown that $\Omega(Y) = \Omega(X)$ when f is a degree 1 map which induces an isomorphism between the fundamental groups of Y and X (see [Sam] for this result and further developments).

Bibliography

[Bab] Babenko Y. — *Closed geodesics, asymptotic volume, and characteristics of group growth*, Math. USSR-Izv. **33** (1989), 1–37.

[B-C-G 0] Besson G., Courtois G., Gallot S. — *Le volume et l'entropie minimal des espaces localement symétriques*, Invent. Math. **103** (1991), 417–445.

[B-C-G 1] Besson G., Courtois G., Gallot S. — *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, to appear in G.A.F.A..

[B-C-G 2] Besson G., Courtois G., Gallot S. — *Les variétés hyperboliques sont des minima locaux de l'entropie topologique*, Invent. Math. **117** (1994), 403–445.

[B-C-G 3] Besson G., Courtois G., Gallot S. — *Entropie et rigidité des espaces localement symétriques de volume fini*, In preparation.

[B-C-G 4] Besson G., Courtois G., Gallot S. — *Entropie et rigidité des représentations quasi-fuchsiennes*, In preparation.

[B-C-G 5] Besson G., Courtois G., Gallot S. — *Entropie et rigidité des quotients de produits d'espaces symétriques*, In preparation.

[B-M] Burger M., Mozes S. — *CAT(-1) spaces, divergence groups and their commensurators*, Preprint, 1995.

[B-P] Benedetti R., Petronio C. — *Lectures on hyperbolic geometry*, Springer-Verlag, Universitext, 1992.

[CBY] Cheng Bo Yue. — *The ergodic theory of discrete isometry groups on manifolds of variable negative curvature*, Preprint Penn State University.

[D-E] Douady A., Earle C. — *Conformally natural extension of homeomorphisms of the circle*, Acta. Math. Soc. **27** (1992), 139–142.

[E-L] Eells J., Lemaire L. — *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), 1–68.

[Fed] Federer H. — *Geometric measure theory*, Springer-Verlag, Grundlehren Band 153, 1969.

[Fur] Furstenberg H. — *A Poisson formula for semi-simple Lie group*, Annals of Math. **77** (1963), 335–386.

[Gro 1] Gromov M. — *Volume and bounded cohomology*, Publ. Math. Inst. Hautes Étud. Sci. **56** (1981), 213–307.

[Gro 2] Gromov M. — *Filling Riemannian manifolds*, J. Diff. Geom. **18** (1983), 1–147.

[Kat] Katok A. — *Entropy and closed geodesics*, Ergodic Theory Dynamical Systems **2** (1982), 339–367.

[Kni] Knieper G. — *Spherical means on compact Riemannian manifolds of negative curvature*, Diff. Geometry and its applications **4** (1994), 361–390.

[Man] Manning A. — *Topological Entropy for geodesic flows*, Ann. Math. **110** (1979), 567–573.

[Mar] Margulis G. A. — *Applications of ergodic theory to the investigation of manifolds of negative curvature*, Funct. Anal. Appl. **3** (1969), 335–336.

[Mos] Mostow G. D. — *Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms*, Publ. Math. Inst. Hautes Étud. Sci. **34** (1968), 53–104.

[Nic] Nichols P. J. — *The ergodic theory of discrete groups*, London Math. Soc. Lecture note ser. **143**, 1989.

[Pat] Patterson S. J. — *The limit set of a Fuchsian group*, Acta Math. **136** (1976), 241–273.

[Rob] Robert G.. — *Invariants topologiques et géométriques reliés aux longueurs des géodésiques et aux sections harmoniques de fibrés*, Thèse de Doctorat de l'Université de Grenoble I, soutenue le 25 octobre 1994.

[Sam] Sambusetti A. — In preparation,.

[Sul] Sullivan D. — *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. Inst. Hautes Étud. Sci. **50** (1979), 171–202.

G. BESSON
Institut Fourier -C.N.R.S.
U.R.A. 188
B.P. 74
38402 Saint Martin d'Hères Cedex - France

G. COURTOIS
École Polytechnique - C.N.R.S.
U.R.A. 169
Centre de Mathématiques
91128 Palaiseau Cedex - France

S. GALLOT
École Normale Supérieure de Lyon
U.M.R. 128
46, Allée d'Italie
69364 LYON Cedex 07 - France

(29 aout 1995)