MINIMAL SIEGEL MODULAR THREEFOLDS

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ABSTRACT. In this paper we study the maximal extension Γ_t^* of the subgroup Γ_t of $\mathrm{Sp}_4(\mathbb{Q})$ which is conjugate to the paramodular group. The index of this extension is $2^{\nu(t)}$ where $\nu(t)$ is the number of prime divisors of t. The group Γ_t^* defines the minimal modular threefold $\mathcal{A}_t^*$ which is a finite quotient of the moduli space $\mathcal{A}_t$ of $(1,t)$-polarized abelian surfaces. We show that a certain degree 2 quotient of $\mathcal{A}_t$ is a moduli space of lattice polarized $K3$ surfaces. Using the action of Γ_t^* on the space of Jacobi forms we show that many spaces between $\mathcal{A}_t$ and $\mathcal{A}_t^*$ posess a non-trivial 3-form, i.e. the Kodaira dimension of these spaces is non-negative. Finally we determine the divisorial part of the ramification locus of the finite map $\mathcal{A}_t \to \mathcal{A}_t^*$ which is a union of Humbert surfaces. We interprete the corresponding Humbert surfaces as Hilbert modular surfaces.

INTRODUCTION

The moduli space $\mathcal{A}_t$ of abelian surfaces with a $(1,t)$-polarization is the quotient of the Siegel upper half plane $\mathbb{H}_2$ by a subgroup Γ_t of $\mathrm{Sp}_4(\mathbb{Q})$ which is conjugate to the paramodular group $\widetilde{\Gamma}_t$. In § 1 we define an isomorphism between the symplectic group and the special orthogonal group $\mathrm{SO}(3,2)$ over the integers.

This exhibits $\Gamma_t/\{\pm E_4\}$ as a subgroup of the orthogonal group $\mathrm{SO}(L_t)$ where L_t is the lattice of rank 5 equipped with the form $< 2t > \oplus 2U$ (here U denotes the hyperbolic plane). Let $\widehat{L}_t$ be the dual lattice of L_t. The image of Γ_t in $\mathrm{O}(L_t)$ acts trivially on $\widehat{L}_t/L_t$. The orthogonal group $\mathrm{O}(\widehat{L}_t/L_t)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\nu(t)}$ where $\nu(t)$ is the number of prime divisors of t. For every $d||t$ (i.e. $d|t$ and $(d,t/d) = 1$) we construct an element V_d in $\mathrm{Sp}_4(\mathbb{R})$. These elements V_d define a normal extension Γ_t^* of Γ_t of index $2^{\nu(t)}$ such that $\Gamma_t^*/\Gamma_t \cong \mathrm{O}(\widehat{L}_t/L_t)$. It turns out that Γ_t^* is the maximal normal extension of Γ_t as an arithmetic group. Hence we can consider the moduli space $\Gamma_t^*\backslash\mathbb{H}_2$ as "minimal" Siegel modular threefolds. In § 1 we also give a geometric interpretation of the action of V_d on the moduli space $\mathcal{A}_t$. In particular V_t identifies a polarized abelian surface with its dual. It also turns out that the space $(\Gamma_t \cup \Gamma_t V_t)\backslash\mathbb{H}_2$ is isomorphic to the moduli space of lattice polarized $K3$ surfaces with a polarization of type $< 2t > \oplus 2E_8(-1)$. Lattice polarized $K3$ surfaces have been studied by Dolgachev [D] and Nikulin [N2]. They play a role in mirror symmetry for $K3$ surfaces.

In § 2 we study the action of the elements V_d on the space of Jacobi forms. This gives rise to a decomposition of the space of Jacobi forms which was originally found by Eichler and Zagier [EZ]. Using lifting results due to the first author this

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enables us to prove that many moduli spaces lying between $\mathcal{A}_t$ and the minimal Siegel modular threefold $\mathcal{A}_t^*$ are not unirational, resp. have non-negative Kodaira dimension. This method, however, unfortunately does not give us information about the Kodaira dimension of $\mathcal{A}_t^*$ itself.

If one wants to determine the Kodaira dimension of $\mathcal{A}_t^*$ one needs precise information about the ramification locus of the finite map $\mathcal{A}_t \to \mathcal{A}_t^*$. This turns out to be a difficult problem. In § 3 we determine the divisorial part of this ramification locus for square free t (the general case can be treated by the same method). The divisorial part of this ramification locus is a finite union of Humbert surfaces. To determine these surfaces we reexamine the theory of Humbert surfaces from the point of view of the orthogonal group. This turns out to be a very useful way of studying Humbert surfaces. An example, originally due to Brasch, shows that the ramification locus can also contain curve components. We finally interprete the Humbert surfaces in the ramification locus as Hilbert modular surfaces.

§ 1. The symplectic and orthogonal groups

The local isomorphism between the symplectic group $\mathrm{Sp}_4(\mathbb{R})$ and the special orthogonal group $\mathrm{SO}(3,2)_{\mathbb{R}}$ of signature $(3,2)$ is well known. In this section we define this isomorphism over $\mathbb{Z}$.

Let us fix a lattice

$$L = e_1\mathbb{Z} \oplus e_2\mathbb{Z} \oplus e_3\mathbb{Z} \oplus e_4\mathbb{Z}.$$

We identify $l \in L$ with a column-vector in the basis $\{e_i\}$. $L^2 = L \wedge L$ is the lattice of integral bivectors, which is isomorphic to the lattice of integral skew-symmetric matrices. The bivector $e_i \wedge e_j$ corresponds to the elementary skew-symmetric matrix E_{ij}, which has only two non-zero elements $e_{ij} = 1$ and $e_{ji} = -1$. Any linear transformation $g : L \to L$ induces a linear map $\wedge^2 g : L \wedge L \to L \wedge L$ on the $\mathbb{Z}$-lattice of bivectors. If g is represented with respect to the basis $\{e_i\}$ by the matrix G, then

$$(\wedge^2 g)(X) = GX\,{}^tG \qquad \text{for any } \ X = \sum_{i<j} x_{ij} e_i \wedge e_j \in L \wedge L.$$

One can define a symmetric bilinear form (X, Y) on $L \wedge L$

$$X \wedge Y = (X, Y)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^4 L.$$

It is known, that $(X, X) = 2\,\mathrm{Pf}(X)$, where $\mathrm{Pf}(X)$ is the Pfaffian of the matrix X, and $\mathrm{Pf}(MX\,{}^tM) = \mathrm{Pf}(X)\det M$.

Definition. The group

$$\widetilde{\Gamma}_t = \{g : L \to L \mid \ \wedge^2 g\,(W_t) = W_t, \quad \text{where } \ W_t = e_1 \wedge e_3 + te_2 \wedge e_4\} \tag{1.1}$$

is called the integral paramodular group of level t.

The lattice $L_t = W_t^{\perp}$ consisting of all elements of $L \wedge L$ orthogonal to W_t has the following basis

$$L_t = (e_1 \wedge e_2,\ e_2 \wedge e_3,\ e_1 \wedge e_3 - te_2 \wedge e_4,\ e_4 \wedge e_1,\ e_4 \wedge e_3)\,\mathbb{Z}^5.$$

We fix this basis for the rest of the paper. The symmetric bilinear form $(\cdot,\cdot)$ defines a quadratic form S of signature $(3,2)$ on the lattice L_t, which has the following form in the given basis

$$S_t = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2t & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (1.2)$$

The group of the real points of the paramodular group is conjugate to $\mathrm{Sp}_4(\mathbb{R})$. Thus the determinant of any element of the paramodular group equals one and $\wedge^2 g$ keeps the bilinear form on $L \wedge L$.

It gives us a homomorphism from the symplectic group in the orthogonal group of the isometries of the lattice L_t

$$\wedge^2 : \widetilde{\Gamma}_t \to \mathrm{O}(L_t).$$

The paramodular group $\widetilde{\Gamma}_t$ is conjugate to a subgroup of the usual rational symplectic group:

$$\Gamma_t := I_t^{-1}\widetilde{\Gamma}_t I_t = \left\{ \begin{pmatrix} * & * & * & t* \\ t* & * & t* & t* \\ * & * & * & t* \\ * & t^{-1}* & * & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\},$$

where all entries $*$ denote integers and $I_t = \mathrm{diag}\,(1,1,1,t)$.

The quotient space

$$\mathcal{A}_t = \Gamma_t \backslash \mathbb{H}_2$$

is the coarse moduli space of abelian surfaces with a polarization of type $(1,t)$.

The composition of the conjugation with the homomorphism $\wedge^2$ defines a homomorphism

$$\Psi : \Gamma_t \to \mathrm{O}(L_t) \qquad \text{where} \quad \Psi(g) = \wedge^2(I_t\, g\, I_t^{-1}). \qquad (1.3)$$

One can extend Ψ to the real symplectic group $\Gamma_t(\mathbb{R}) \cong \mathrm{Sp}_4(\mathbb{R})$.

Let $\widehat{L}_t = \{u \in L_t \otimes \mathbb{Q} \mid \forall\ l \in L_t\ (l,u) \in \mathbb{Z}\}$ be the dual lattice of L_t. The discriminant group

$$A_t := \widehat{L}_t / L_t = (2t)^{-1}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/2t\mathbb{Z}$$

is a finite abelian group equipped with a quadratic form

$$q_t : A_t \times A_t \to (2t)^{-1}\mathbb{Z}/2\mathbb{Z} \qquad\qquad q_t(l,l) \equiv (l,l)_{\widehat{L}_t} \bmod 2\mathbb{Z}$$

(see [N1] for a general definition). Any $g \in \mathrm{O}(L_t)$ acts on the finite group A_t. By

$$\widehat{\mathrm{O}}(L_t) = \{g \in \mathrm{O}(L_t) \mid \forall \ell \in \widehat{L}_t \quad g\ell - \ell \in L_t\}$$

we denote the subgroup of the orthogonal group consisting of elements which act identically on the discriminant group.

One can easily prove the next lemma (see [G1]).

Lemma 1.1. *The following relations are valid*

1. $\Psi(\Gamma_t) \subset \widehat{\mathrm{SO}}(L_t) = \widehat{\mathrm{O}}(L_t) \cap \mathrm{SO}(L_t)$;
2. $\mathrm{Ker}\,\Psi = \{\pm E_4\}$.

The finite orthogonal group $\mathrm{O}(A_t)$ can be described as follows. For every $d||t$ (i.e. $d|t$ and $(d, \frac{t}{d}) = 1$) there exists a unique (mod $2t$) integer ξ_d satisfying

$$\xi_d = -1 \bmod 2d, \quad \xi_d = 1 \bmod 2t/d.$$

All such ξ_d form the group

$$\Xi(t) = \{\, \xi \bmod 2t \mid \xi^2 = 1 \bmod 4t \,\} \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}, \tag{1.4}$$

where $\nu(t)$ is the number of prime divisors of t. It is evident that $\mathrm{O}(A_t) \cong \Xi(t)$.

One can take an element in $\mathrm{SO}(L_t)$ realising the multiplication by ξ_d on A_t. It gives us an element in $\mathrm{Sp}_4(\mathbb{R})$ with integral Ψ-image. For example, for every $d||t$ we can define $x, y \in \mathbb{Z}$ (which are not uniquely determined) such that

$$xd - yt_d = 1 \quad \text{where } t_d = \frac{t}{d}.$$

The matrix

$$\widetilde{V}_d = \begin{pmatrix} dx & -1 & 0 & 0 \\ -yt & d & 0 & 0 \\ 0 & 0 & d & yt \\ 0 & 0 & 1 & dx \end{pmatrix}$$

is an integral symplectic similitude of degree d. We put

$$V_d = \frac{1}{\sqrt{d}} \widetilde{V}_d \in \mathrm{Sp}_4(\mathbb{R}).$$

V_d has the following Ψ-image

$$\Psi(V_d) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & d & -2yt & y^2 t_d & 0 \\ 0 & -1 & dx + t_d y & -xy & 0 \\ 0 & t_d & -2tx & x^2 d & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.5}$$

We note here that

$$xd + t_d y = -1 \bmod 2d \qquad \text{and} \qquad xd + t_d y = 1 \bmod 2t_d,$$

thus V_d induces the multiplication by ξ_d on A_t.

It is easy to see, that for all V_d $(d||t)$

$$V_d^2 \in \Gamma_t, \qquad V_d \Gamma_t V_d = \Gamma_t.$$

I.e. V_d are involutions modulo Γ_t. Therefore one can define the following normal extension of the paramodular group Γ_t.

Definition. Γ_t^* is the group generated by the elements of Γ_t and V_d for all $d||t$.

In accordance with Lemma 1.1 any element in $\Psi(V_d\Gamma_t)$ defines the same automorphism of A_t, thus

$$\Gamma_t^*/\Gamma_t \cong \mathrm{O}(\widehat{L}_t/L_t) \cong \Xi(t) \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}. \tag{1.6}$$

The real orthogonal group $\mathrm{O}_{\mathbb{R}}(L_t) = \mathrm{O}(L_t \otimes \mathbb{R})$ acts on a domain lieing on a projective quadric, more exactly on

$$\mathbb{PH}_t^3 = \mathbb{PH}_{L_t}^3 = \{Z \in \mathbb{P}(L_t \otimes \mathbb{C}) \mid (Z,Z) = 0,\ (Z,\overline{Z}) < 0\} = \mathbb{PH}_t^+ \cup \overline{\mathbb{PH}_t^+},$$

where

$$\mathbb{PH}_t^+ = \{Z = {}^t\big((tz_2^2 - z_1z_3),\, z_3,\, z_2,\, z_1,\, 1\big) \cdot z_0 \in \mathbb{H}_t^3 \mid \mathrm{Im}\,(z_1) > 0\}. \tag{1.7}$$

This is a classical homogeneuos domain of type IV. The condition $(Z,\overline{Z}) < 0$ is equivalent to

$$y_1y_3 - ty_2^2 > 0 \qquad \text{where} \quad y_i = \mathrm{Im}\,(z_i).$$

Taking $z_0 = 1$ one gets the corresponding cylindric domain in the affine coordinates $(z_i)_{1\le i\le 3}$

$$\mathbb{H}_t^+ = \{Z = {}^t(z_3,\, z_2,\, z_1) \in \mathbb{C}^3 \mid\ y_1y_3 - ty_2^2 > 0, \quad y_1 = \mathrm{Im}\,(z_1) > 0\}.$$

The domain $\mathbb{H}_t^+$ for $t = 1$ coincides with the Siegel upper half-plane $\mathbb{H}_2$. For a general t one can define the following isomorphism of the complex domains

$$\psi_t : \mathbb{H}_2 \to \mathbb{H}_t^+ \qquad \psi_t\left(\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}\right) = {}^t\left(\frac{\tau_3}{t},\, \frac{\tau_2}{t},\, \tau_1\right). \tag{1.8}$$

The linear action of the real orthogonal group $\mathrm{O}_{\mathbb{R}}(L_t)$ on $\mathbb{PH}_t^+$ defines "fractional-linear" transformations on $\mathbb{H}_t^+$. By $\mathrm{O}_{\mathbb{R}}^+(L_t)$ we denote the subgroup of index 2 of the orthogonal group consisting of elements which leave $\mathbb{PH}_t^+$ invariant. (This is the subgroup of the elements with real spin norm equal one.)

Proposition 1.2. *Let t be square free. Then Ψ defines the following isomorphisms*

$$\Psi :\ \Gamma_t^*/\{\pm E_4\} \to \mathrm{SO}^+(L_t),$$

where $\mathrm{SO}^+(L_t) = \mathrm{SO}(L_t) \cap \mathrm{O}_{\mathbb{R}}^+(L_t)$*, and*

$$\Psi :\ \Gamma_t/\{\pm E_4\} \to \widehat{\mathrm{SO}}^+(L_t),$$

where $\widehat{\mathrm{SO}}^+(L_t) = \mathrm{SO}^+(L_t) \cap \widehat{\mathrm{SO}}(L_t)$*. Moreover the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{H}_2 & \xrightarrow{\ g\ } & \mathbb{H}_2 \\ {\scriptstyle \psi_t}\downarrow & & {\scriptstyle \psi_t}\downarrow \\ \mathbb{H}_t^+ & \xrightarrow{\Psi(g)} & \mathbb{H}_t^+. \end{array}$$

Proof. The diagram is commutative for any $g \in \mathrm{Sp}_4(\mathbb{R})$. To prove this one has to calculate the images of standard generators of $\mathrm{Sp}_4(\mathbb{R})$ under Ψ.

It is known that for a square free t, the group $P\Gamma_t^* \cong \Gamma_t^*/\{\pm E_4\}$ is a maximal discrete subgroup of the group of the analytical automorphisms of $\mathbb{H}_2$ and $[\Gamma_t^* : \Gamma_t] = 2^{\nu(t)}$ (see for example [Al], [Gu]). From the description of the finite orthogonal group $O(A_t)$ given in (1.4) we obtain that $[\mathrm{SO}^+(L_t) : \widehat{\mathrm{SO}}^+(L_t)] = 2^{\nu(t)}$. The statement of the proposition about the isomorphism of the groups follows from the maximality of Γ_t^* and Lemma 1.1.
□

The coset $V_t\Gamma_t$ (in the case $d = t$ we may take $x = 0, y = -1$) can also be written in the form

$$V_t\Gamma_t = \begin{pmatrix} 0 & \sqrt{t}^{-1} & 0 & 0 \\ \sqrt{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t} \\ 0 & 0 & \sqrt{t}^{-1} & 0 \end{pmatrix} \Gamma_t.$$

According to (1.3) and (1.5) $\Psi(V_t)$ defines the multiplication by -1 on $\widehat{L_t}/L_t$, i.e. V_t corresponds to the element $\xi_t = -1$ of $\Xi(t)$ (see (1.4)). Therefore

$$-\Psi(V_t) \in \widehat{\mathrm{O}}(L_t).$$

Elements M and $-M \in \mathrm{O}(L_t)$ define the same transformation of the domain $\mathbb{H}_t^+$. Thus we have

Corollary 1.3. *Let t be square free. The groups*

$$\Psi(\Gamma_t \cup \Gamma_t V_t) \qquad \text{and} \qquad \mathrm{O}^*(L_t) = \widehat{\mathrm{O}}(L_t) \cap \mathrm{O}^+_{\mathbb{R}}(L_t)$$

coincide, if we consider them as groups of analytic transformations of $\mathbb{P}\mathbb{H}_t^+$.

Proposition 1.4. *The quotient*

$$(\Gamma_t \cup \Gamma_t V_t) \setminus \mathbb{H}_2$$

is isomorphic to the moduli space of polarized $K3$ surfaces with a polarization of type $< 2t > \oplus 2E_8(-1)$.

Proof. It is known that a moduli space of polarized $K3$ surfaces is a quotient of a 19-dimensional homogeneous domain of type IV by an arithmetic group. In the proposition we consider polarized $K3$ surfaces with a condition on the Picard group or equivalently on the lattice of its trancendental cycles. To formulate these conditions we need some definitions (see [D], [N2]).

Let X be a $K3$ surface. Let us take a sublattice $D_t =< 2t > \oplus E_8(-1) \oplus E_8(-1)$ of the lattice

$$L_{K3} = U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \cong H^2(X, \mathbb{Z}),$$

where $< 2t >$ $(t \in \mathbb{Z})$ denotes the one dimensional lattice generated by a vector l such that $l^2 = 2t$, U is the hyperbolic plane with quadratic form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $E_8(-1)$

is the even unimodular lattice of dimension 8 with the negative definite quadratic form. We note that sign $(D_t) = (1, 16)$ and $D_t^{\perp} \cong U \oplus U \oplus < -2t >= L'_t = L_t(-1)$. (Notation $L_t(-1)$ means that we multiply the quadratic form S_t on the lattice L_t by -1.) We recall (see [N1]) that the orthogonal group

$$\mathrm{O}(D_t,\, L_{K3}) = \{\, g : L_{K3} \to L_{K3} \mid g\,|_{D_t} \equiv \mathrm{id}\}$$

is isomorphic to the group

$$\widehat{\mathrm{O}}(L'_t) = \{\, g :\, L'_t \to L'_t \mid \forall\, l \in \widehat{L}'_t \;\; gl - l \in L'_t \,\}.$$

A marked D_t-polarized $K3$-surface is defined by the following datum (see [D] and [N2] for more details): a surface X, a fundamental domain $C(M)^+$ of a group generated by some 2-reflections of the lattice D_t acting on a connected component $V(D_t)^+$ of the cone $V(D_t) = \{v \in D_t \otimes \mathbb{R} \mid (v,v) > 0\}$ and an isomorphism of the lattices $\phi : H^2(X,\mathbb{Z}) \to L_{K3}$, such that $\phi^{-1}(D_t) \subset \mathrm{Pic}\,(X)$, $\phi^{-1}(V(D_t)^+) \subset V(X)^+$ and $\phi^{-1}(C(M)^+)$ contains at least one numerically effective divisor class. By $V(X)^+$ one denotes the connected component of the cone

$$V(X) = \{v \in H^{1,1}_{\mathbb{R}}(X) \mid (v,\, v) > 0\}$$

containing the cohomology class of a Kähler form on X.

Let us denote by ω_X a holomorphic 2-form which generates $H^{2,0}(X)$. Its image under the isometry ϕ belongs to the following domain in the projective space $\mathbb{P}^4$

$$\phi(\omega_X) \in \mathcal{D}_m = \{v \in \mathbb{P}(L'_t \otimes \mathbb{C}) :\; (v,v) = 0, \quad (v,\bar{v}) > 0\}.$$

This domain is an example of the domains of type IV. Its connected components are isomorphic to the domain $\mathcal{H}_t^+$ introduced in (1.7).

The quotient

$$\mathcal{M}(< 2t > \oplus\, 2E_8(-1)) = \mathrm{O}^*(L'_t) \setminus \mathcal{H}_t^+$$

is the moduli space of isomorphic classes of $< 2t > \oplus\, 2E_8(-1)$-polarized $K3$ surfaces. In accordance with Corollary 1.3

$$\mathcal{M}(< 2t > \oplus\, 2E_8(-1)) \cong (\Gamma_t \cup \Gamma_t V_t) \setminus \mathbb{H}_2.$$

□

Our next aim is to interprete the involutions V_d geometrically. Because of Proposition 1.2 V_d induces a map from $\mathcal{A}_t$ to itself.

Let (A, H) be a $(1,t)$-polarized abelian surface. The polarization H defines an isogeny

$$\begin{aligned} \lambda_H :\quad & A \to \widehat{A} = \mathrm{Pic}^0 A \\ & x \mapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \end{aligned}$$

where $\mathcal{L}$ is a line bundle representing H and T_x denotes translation by x. The map λ_H only depends on H, not on the line bundle $\mathcal{L}$. There is a (non-canonical) isomorphism $\ker\lambda_H \cong \mathbb{Z}_t \times \mathbb{Z}_t$. For every divisor d of t there is a unique subgroup $G(d) \subset \ker\lambda_H$ which is isomorphic to $\mathbb{Z}_d \times \mathbb{Z}_d$. This subgroup defines a quotient

$$\lambda_d : \quad A \to A/G(d) = A'.$$

If A is given by the period matrix

$$\Omega = \begin{pmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & t & \tau_2 & \tau_3 \end{pmatrix}, \qquad \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2$$

then A' is given by

$$\Omega' = \begin{pmatrix} d & 0 & d\tau_1 & \tau_2 \\ 0 & t_d & \tau_2 & \tau_3/d \end{pmatrix}.$$

The abelian surface A' carries a uniquely determined polarization H' with

$$dH = \lambda_d^*(H').$$

The polarization H' is of type $e \cdot (1, t/e^2)$ where $e = (d, t_d)$. Altogether this shows that we have a morphism of moduli spaces

$$\begin{aligned} \Phi = \Phi(d) : \quad & \mathcal{A}_t \to \mathcal{A}_{t/e^2} \\ & (A, H) \mapsto (A', H'). \end{aligned}$$

If $d = t$ we obtain as a special case the map

$$\begin{aligned} \Phi(t) : \quad & \mathcal{A}_t \to \mathcal{A}_t \\ & (A, H) \mapsto (\widehat{A}, \widehat{H}) \end{aligned}$$

which maps an abelian surface to its dual polarized abelian surface.

Proposition 1.5. *Let d be a divisor of t with $(d, t_d) = 1$. Then the map*

$$\Phi(d) : \quad \mathcal{A}_t \to \mathcal{A}_t$$

is the map induced by V_d.

Proof. For $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2$ we have the following formula for the action

$$V_d < \tau >= \begin{pmatrix} x & -1 \\ -yt_d & d \end{pmatrix} \begin{pmatrix} d\tau_1 & \tau_2 \\ \tau_2 & \tau_3/d \end{pmatrix} \begin{pmatrix} x & -yt_d \\ -1 & d \end{pmatrix}.$$

Now consider the matrix

$$\begin{pmatrix} 1 & t_d & 0 & 0 \\ y & xd & 0 & 0 \\ 0 & 0 & x & -yt_d \\ 0 & 0 & -1 & d \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Z}).$$

This matrix transforms the symplectic form $de_1 \wedge e_3 + t_d e_2 \wedge e_4$ into W_t (see (1.1)). The claim now follows from the equality

$$\begin{aligned} & \begin{pmatrix} x & -1 \\ -yt_d & d \end{pmatrix} \begin{pmatrix} d & 0 & d\tau_1 & \tau_2 \\ 0 & t_d & \tau_2 & \tau_3/d \end{pmatrix} \begin{pmatrix} 1 & t_d & 0 & 0 \\ y & xd & 0 & 0 \\ 0 & 0 & x & -yt_d \\ 0 & 0 & -1 & d \end{pmatrix} \\ & = \begin{pmatrix} 1 & 0 & \\ 0 & t & V_d < \tau > \end{pmatrix}. \end{aligned}$$

□

§ 2. Nonunirationality of some quotient spaces

In accordance with Proposition 1.4 a moduli space of special $K3$ surfaces is a quotient of a moduli space of polarized abelian surfaces. It gives us a double covering

$$\mathcal{A}_t \to (\Gamma_t \cup \Gamma_t V_t) \setminus \mathbb{H}_2 .$$

The degree of the covering $\mathcal{A}_t \to \mathcal{A}_t^* = \Gamma_t^* \setminus \mathbb{H}_2$ has order $2^{\nu(t)}$, where $\nu(t)$ is the number of prime divisors of t. Since for square free t the extension $\mathrm{P}\Gamma_t^*$ is the maximal discrete subgroup of $\mathrm{PSp}_4(\mathbb{Z})$ containing $\mathrm{P}\Gamma_t$ the quotient $\mathcal{A}_t^*$ is the minimal Siegel threefold associated to the polarization $(1,t)$.

There are $2^{\nu(t)}-2$ other threefolds between $\mathcal{A}_t$ and $\mathcal{A}_t^*$. Let us take for example two primes $p \neq q$ and let $t = pq$. The involutions V_p and V_q give rise to the following moduli spaces

$$\mathcal{A}_{pq}^{(p)} = \mathcal{A}_{pq}/ < V_p >,\ \mathcal{A}_{pq}^{(q)} = \mathcal{A}_{pq}/ < V_q >,\ \mathcal{A}_{pq}^{*} = \mathcal{A}_{pq}/ < V_p, V_q > = \Gamma_{pq}^*\backslash\mathbb{H}_2$$

resp. a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{A}_{pq} & & \\ & \swarrow & & \searrow & \\ \mathcal{A}_{pq}^{(p)} & & & & \mathcal{A}_{pq}^{(q)} \\ & \searrow & & \swarrow & \\ & & \mathcal{A}_{pq}^{*} & & \end{array}$$

where all maps are 2:1. Using the modular forms constructed in [G1] we can obtain information about the geometrical type of some of these moduli spaces.

By $J_{k,t}^{cusp}$ we denote the space of Jacobi cusp forms of weight k and index t. In [G1] a lifting was constructed which associates to a Jacobi form $\Phi \in J_{k,t}^{cusp}$ a cusp form $F_\Phi \in \mathcal{M}_k(\widehat{\Gamma}_t)$ of weight k with respect to the group

$$\widehat{\Gamma}_t = \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & *t^{-1} \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\},$$

where all entries $*$ denote integers.

The groups Γ_t and $\widehat{\Gamma}_t$ are conjugate. Indeed if $C_t = \mathrm{diag}\,(1,\, t^{-1},\, 1,\, t)$ then $\widehat{\Gamma}_t = C_t \Gamma_t C_t^{-1}$. For

$$\widehat{V}_t = {}^tV_t = \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 \\ \sqrt{t}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t}^{-1} \\ 0 & 0 & \sqrt{t} & 0 \end{pmatrix}$$

it was proved in [G1, formula (2.8)] that

$$F_\Phi(Z) = (-1)^k F_\Phi|_k\, \widehat{V}_t(Z).$$

or equivalently

$$F_\Phi(\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}) = F_\Phi(\begin{pmatrix} t\tau_3 & \tau_2 \\ \tau_2 & t^{-1}\tau_1 \end{pmatrix}).$$

In this section we describe the behavior of F_Φ with respect to the group

$$\widehat{\Gamma}_t^* = <\widehat{\Gamma}_t,\, \widehat{V}_d \mid d||t>,$$

where $\widehat{V}_d = C_t V_d C_t^{-1}$.

Eichler and Zagier [EZ, § 5], have constructed a decomposition of the space of Jacobi forms

$$J_{k,t}^{cusp} = \bigoplus_{\epsilon} J_{k,t}^{\epsilon}$$

where ϵ runs over all characters of the group

$$\Xi(t) = \{\, \xi \bmod 2t \mid \xi^2 \equiv 1 \bmod 4t \,\} \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}$$

satisfying $\epsilon(-1) = (-1)^k$. For any d with $d||t$ one can define an operator W_d acting on $J_{k,t}$ in the following way [EZ, §5]. For

$$\Phi(\tau_1,\tau_2) = \sum_{\substack{n,\, l\in\mathbb{Z} \\ 4nt>l^2}} f(n,l)\exp(2\pi i(n\tau_1 + l\tau_2)) \in J_{k,t}^{cusp}$$

we put

$$(\Phi \,|\, W_d)(\tau_1,\tau_2) = \sum_{\substack{n,\, l\in\mathbb{Z} \\ 4nt>l^2}} f(n',l')\exp(2\pi i(n\tau_1 + l\tau_2)) \in J_{k,t}^{cusp}$$

where l', n' are determined by

$$l' \equiv -l \bmod 2d, \quad l' \equiv l \bmod 2t/d, \quad 4n't - l'^2 = 4nt - l^2.$$

All W_d are involutions. They form a group isomorphic to $\Xi(t)$. The subspaces $J_{k,t}^{\epsilon}$ are eigenspaces of the operation W_d, namely

$$J_{k,t}^{\epsilon} = \{\Phi \in J_{k,t}^{cusp} \mid \Phi \,|\, W_d = \epsilon(W_d)\,\Phi\}.$$

Note that if

$$\Phi(\tau_1,\tau_2) = \sum_{\mu \bmod 2t} \varphi_\mu(\tau_1)\theta_{t,\mu}(\tau_1,\tau_2) \in J_{k,t}^{\epsilon}$$

is the standard decomposition of the Jacobi form Φ with respect to the theta-functions $\theta_{t,\mu}(\tau_1,\tau_2)$ then for $\xi_d \in \Xi(t)$

$$\varphi_{\xi_d\mu}(\tau_1) = \epsilon(\xi_d)\varphi_\mu(\tau_1).$$

Theorem 2.1. *Let $\Phi \in J^{\epsilon}_{k,t}$ be a Jacobi form and $F_\Phi \in \mathcal{M}_k(\widehat{\Gamma}_t)$ be its lifting. For any divisor d of t with $(d, t_d) = 1$ the following equality holds*

$$F_\Phi|_k \widehat{V}_d = \epsilon(\xi_d) F_\Phi .$$

Proof. Let us recall the definition of the lifting F_Φ in terms of the Fourier expansion [G2]. If

$$\Phi(\tau_1, \tau_2) = \sum_{\substack{n,\, l \in \mathbb{Z} \\ 4nt > l^2}} f(n, l) \exp(2\pi i(n\tau_1 + l\tau_2)) \in J^{cusp}_{k,t}$$

then

$$F_\Phi(Z) = \sum_{N \in \mathfrak{A}_t} b(N) \exp(2\pi i \ tr(NZ))$$

where summation is taken over all positive definite symmetric matrices of the following form

$$N \in \mathfrak{A}_t = \left\{ \begin{pmatrix} n & l/2 \\ l/2 & mt \end{pmatrix} > 0 \mid n, l, m \in \mathbb{Z} \right\}$$

and

$$b(\begin{pmatrix} n & l/2 \\ l/2 & mt \end{pmatrix}) = \sum_{a \,|\, (n,l,m)} a^{k-1} \ f\left(\frac{nm}{a^2}, \frac{l}{a}\right).$$

The action of $\widehat{V}_d$ on F_Φ is given by

$$(F_\Phi|_k \widehat{V}_d)(Z) = F_\Phi(d^{-1} A_d \ Z \ {}^t A_d) \quad \text{ where } A_d = \begin{pmatrix} dx & -t \\ -y & d \end{pmatrix}$$

or

$$\begin{aligned}(F_\Phi|_k \widehat{V}_d)(Z) &= \sum_{N \in \mathfrak{A}_t} b(N) \exp(2\pi i \ tr(d^{-1} \ {}^t A_d N A_d Z)) \\ &= \sum_{N \in \mathfrak{A}_t} b(d^{-1} \ {}^t \tilde{A}_d N \tilde{A}_d) \exp(2\pi i \ tr(NZ))\end{aligned}$$

where $\tilde{A}_d = dA_d^{-1} = \begin{pmatrix} d & t \\ y & dx \end{pmatrix}$. Let $\widetilde{N} = d^{-1} \ {}^t\tilde{A}_d N \tilde{A}_d = \begin{pmatrix} \tilde{n} & \tilde{l}/2 \\ \tilde{l}/2 & \widetilde{m}t \end{pmatrix}$. Clearly $\det N = \det \widetilde{N}$. It is easy to see that the elements n, l, m and $\tilde{n}$, $\tilde{l}$, $\widetilde{m}$ have the same set of common divisors. Moreover

$$\tilde{l} = l(yt_d + dx) + 2(nt + xymt) \quad \text{and} \quad \tilde{l} \equiv \begin{cases} -l & \mod 2d \\ \ \ l & \mod 2t_d \end{cases}$$

Hence, by the definition of $J^{\epsilon}_{k,t}$ we have

$$b(d^{-1} \ {}^t\tilde{A}_d N \tilde{A}_d) = \sum_{a \,|\, (n,l,m)} a^{k-1} f\left(\frac{\tilde{n}\widetilde{m}}{a^2}, \frac{\tilde{l}}{a}\right) = \epsilon(\xi_d) b(N)$$

which proves the theorem.
□

This result can be used to gain some information on the Kodaira dimension of moduli spaces. Whenever we speak of the *Kodaira dimension* of some moduli space $\mathcal{A}$ we mean the Kodaira dimension of a desingularization of a projective compactification of $\mathcal{A}$.

Corollary 2.2. *Let $p \neq q$ be primes ≥ 5. Then the Kodaira dimension of at least one of the spaces $\mathcal{A}_{pq}^{(p)}$ or $\mathcal{A}_{pq}^{(q)}$ is ≥ 0.*

Proof. The Eichler-Zagier decomposition gives a decomposition

$$J_{3,pq} = J_{3,pq}^{(+,-)} \oplus J_{3,pq}^{(-,+)}.$$

If $p, q \geq 5$ then there exists a cusp form in $J_{3,pq}$ and hence in $J_{3,pq}^{(+,-)}$ or $J_{3,pq}^{(-,+)}$. By [G1], [G2] this can be lifted to a weight 3 cusp form with respect to $< \Gamma_t, V_p >$, resp. $< \Gamma_t, V_q >$. By Freitag's extension theorem this defines a differential form on any desingularization of a projective compactification of $\mathcal{A}_{pq}^{(p)}$, resp. $\mathcal{A}_{pq}^{(q)}$. This gives the result.
□

For any integer t let us take a character of the group $\Xi(t)$ isomorphic to the orthogonal group of the discriminant group of L_t (see (1.4))

$$\epsilon : \ \Xi(t) \to \{\pm 1\}.$$

We define a set $U(\epsilon) = \{\, V_d \in \Gamma_t^* |\ \epsilon(\xi_d) = 1 \,\}$ and a subgroup

$$\Gamma_t^\epsilon = < \Gamma_t,\ \xi_d \mid \xi_d \in U(\epsilon) > \subset \Gamma_t^*$$

of Γ_t^*. Theorem 2.1 and the method of the proof of Corollary 2.2 gives us the next result

Corollary 2.3. *If $\dim(J_{3,t}^\epsilon) > 0$, then the Kodaira dimension of the quotient space*

$$\mathcal{A}_t^\epsilon = \Gamma_t^\epsilon \backslash \mathbb{H}_2$$

is nonnegative.

Remark. Corollary 2.3 gives information about $\mathcal{A}_t^\epsilon$ only if $V_t \notin U(\epsilon)$. If $\epsilon(V_t) = \epsilon(-1) = 1$, then $\dim(J_{3,t}^\epsilon) = 0$.

Corollary 2.4. *Let $t \geq 21$ ($t \neq 30, 36$) and let its number of prime divisors $\nu(t) \geq 2$. Then there exists a finite quotient of $\mathcal{A}_t$ of degree $2^{\nu(t)-1}$ which is not unirational.*

Proof. For any integer t from the corollary the dimension of $J_{3,t}^{cusp}$ is positive. Thus there is a character ϵ of $\Xi(t)$ such that $\nu(\xi_t) = -1$ and $\dim J_{3,t}^\epsilon > 0$. $\Xi(t) \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}$ therefore $[\Gamma_t^\epsilon : \Gamma_t] = 2^{\nu(t)-1}$.
□

Using dimension formulae for the spaces $J_{k,t}^\epsilon$ one can obtain more precise results. It is easy to get an exact dimension formula using the trace formula of the operator W_d on the space $J_{k,t}^{cusp}$ given in [SZ]. By definition of W_d we have

$$\mathrm{tr}\,(W_d,\ J_{3,t}^{cusp}) = \sum_{\epsilon} \epsilon(\xi_d) \dim(J_{3,t}^\epsilon),$$

where the sum is taken over all characters of $\Xi(t)$. Therefore

$$\dim(J^{\epsilon}_{3,t}) = 2^{-\nu(t)} \sum_{d||t} \epsilon(\xi_d)\,\mathrm{tr}\,(W_d,\ J^{cusp}_{3,t}).$$

For weight 3 the trace formula of W_d on $J^{cusp}_{3,t}$ (we recall that $d|t$ and $(d,t_d)=1$, where $t_d = \dfrac{t}{d}$) proved in [SZ, Theorem 1] can be reduced to the following expression

$$\begin{aligned}\mathrm{tr}\,(W_d,\ J^{cusp}_{3,t}) = \frac{1}{4}\sum_{e|d} H_{t_d}(-4e) - \frac{1}{4}\sum_{e'|t_d} H_d(-4e') + \frac{3}{2}(H_d(0) - H_{t_d}(0))\\ + \frac{1}{2}\big(\delta_2(t_d)H_d(-4) - \delta_2(d)H_{t_d}(-4)\big) + \big(\delta_3(t_d)H_d(-3) - \delta_3(d)H_{t_d}(-3)\big)\\ + \frac{1}{4}\Big((Q(t_d),2)Q(d) - (Q(d),2)Q(t_d)\Big).\end{aligned}$$

We denote by $Q(n)$ the greatest integer whose square divides n; $\delta_a(b)=1$ or 0 if $a|b$ or $a \not| b$ and $H_n(\Delta)$ is a generalization of the Hurwitz-Kronecker class number, i.e $H_1(0) = -\frac{1}{12}$ and $H_1(\Delta)$ for $\Delta<0$ is the number of equivalence classes with respect to $\mathrm{SL}_2(\mathbb{Z})$ of integral, positive definite, binary quadratic forms of discriminant Δ, counting forms equivalent to a multiple of x^2+y^2 (resp. x^2+xy+y^2) with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$). For $n \ge 2$ with $(n,\Delta) = a^2 b$ and square free b

$$H_n(\Delta) = \begin{cases} a^2 b\left(\dfrac{\Delta/a^2b^2}{n/a^2b}\right) H_1(\Delta/a^2b^2) & \text{if } a^2b^2|\Delta \\ 0 & \text{otherwise,}\end{cases}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the generalized Kronecker symbol.

We note that the trace formula has the simplest form for square free t coprime to 6:

$$\mathrm{tr}\,(W_d,\ J^{cusp}_{3,t}) = \frac{1}{4}\left(\sum_{e|d}\left(\frac{-4e}{t_d}\right)H_1(-4e) - \sum_{e'|t_d}\left(\frac{-4e'}{d}\right)H_1(-4e')\right) + \frac{t_d - d}{8}.$$

Example 2.5. The calculation gives us only thirteen different threefolds of type $\mathcal{A}^{\epsilon}_t$ with t having only two prime divisors ($t = p^a q^b$), whose geometric genus could be equal to zero. They correspond to the following *trivial* subspaces of $J^{cusp}_{3,t}$ of type J^{-+}_{3,p^aq^b} (this notation means that $\epsilon(\xi_{p^a}) = -1$ and $\epsilon(\xi_{q^b}) = 1$):

$$J^{-+}_{3,2\cdot 11},\ J^{-+}_{3,2\cdot 13},\ J^{-+}_{3,2\cdot 17},\ J^{-+}_{3,2\cdot 19},\ J^{-+}_{3,2\cdot 25},\ J^{-+}_{3,2\cdot 27},$$
$$J^{-+}_{3,3\cdot 7},\ J^{-+}_{3,3\cdot 13},\ J^{-+}_{3,3\cdot 16},\ J^{-+}_{3,5\cdot 7},\ J^{-+}_{3,5\cdot 8},\ J^{-+}_{3,7\cdot 4},\ J^{-+}_{3,7\cdot 8}.$$

We may add to this list twenty threefolds $\mathcal{A}_t$ with $t = 1, \dots, 12, 14, 15, 16, 18, 20, 24, 30, 36$ (see [G2]) whose geometric genus could be zero. According to classical results and new results of M. Gross and S. Popescu (see [GP]) it really is for $t = 1, \dots, 12, 14, 16, 18, 20$.

Example 2.6. We have the following quotients of order 4 and 8 which are not unirational. One has

$$\dim J^{cusp}_{3,42} = \dim J^{++-}_{3,2\cdot 3\cdot 7} = 1.$$

Thus for $\mathcal{A}^{(2,3)}_{42} =< V_2, V_3 > \setminus \mathcal{A}_{42}$ we obtain $h^{3,0}(\mathcal{A}^{(2,3)}_{42}) \geq 1$.

The geometric genus of all four threefolds of type $< V_a, V_b, V_c > \setminus \mathcal{A}_{210}$, where $a,\ b,\ c \in \{2,\ 3,\ 5,\ 7\}$ is positive.

§ 3. The Humbert surfaces and the ramification locus

If one wants to determine the Kodaira dimension of the variety $\mathcal{A}^*_t = \Gamma^*_t \backslash \mathbb{H}_2$ it is important to know the ramification locus of the covering map $\mathcal{A}_t \to \mathcal{A}^*_t$, i.e. the locus where the stabilizer of the finite group Γ^*_t/Γ_t is not trivial. Unfortunately this turns out to be a difficult question. Here we shall give a partial answer, i.e. we shall determine the divisorial part of the ramification locus which is a union of a finite numbers of Humbert surfaces. We shall restrict ourselves to t square free.

First we collect some known facts about divisors on the homogeneous domain $\mathbb{P}\mathbb{H}^+_t$. For any $v \in L_t \otimes \mathbb{R}$ we set

$$\mathcal{H}_v = \{Z \in \mathbb{P}\mathbb{H}^+_t \mid v \cdot Z = 0\},$$

where $v \cdot u = (v, u)_t$ is the bilinear product corresponding to the quadratic form S_t (see (1.2)).

Lemma 3.1.

1. *$\mathcal{H}_{gv} = g^{-1}\mathcal{H}_v$ for any $g \in \mathrm{O}^+_{\mathbb{R}}(L_t)$.*
2. *Let $v \neq 0$, then $\mathcal{H}_v \neq \varnothing$ if and only if $v^2 > 0$.*
3. *$\mathcal{H}_v \cap \mathcal{H}_u \neq \varnothing$ if and only if the matrix $\begin{pmatrix} v^2 & v \cdot u \\ v \cdot u & u^2 \end{pmatrix}$ is positive define.*

Proof. 1. The first property is trivial.

2. The orthogonal group $\mathrm{O}^+_{\mathbb{R}}(L_t)$ acts transitively on $\mathbb{P}\mathbb{H}^+_t$. Thus any $Z \in \mathbb{P}\mathbb{H}^+_t$ can be reduced to $Z_{\mathrm{i}} = {}^t(1, i, 0, i, 1)$ in the coordinates (z_i) from (1.7). If $v = (a, b, c, d, e)$ and $v \cdot Z_{\mathrm{i}} = 0$, then $a = -e$ and $b = -d$. Thus $v^2 = 2a^2 + 2b^2 + 2tc^2 > 0$.

Let $L_t \otimes \mathbb{R} = \mathbb{R}v \oplus V^\perp$. One has $\mathrm{sign}\,(V^\perp) = (2, 2)$. The group $\mathrm{SO}^+(V^\perp) = \mathrm{SO}^+_{\mathbb{R}}(2, 2)$ is locally isomorphic to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Thus

$$\mathcal{H}_v \cong \mathbb{P}\mathbb{H}^+_{V^\perp} \cong \mathbb{H}_1 \oplus \mathbb{H}_1\,,$$

where $\mathbb{H}_1$ is the usual upper half-plane. This proves the second statement.

3. Let us suppose that $\mathcal{H}_v \cap \mathcal{H}_u \neq \varnothing$. It follows from $(xu - v)^2 > 0$ that the matrix in 3 is positive definite.

If the symmetric bilinear form on the plane $P = \mathbb{R}v \oplus \mathbb{R}u$ is positive definite, then $\mathrm{sign}\,(P^\perp) = (1, 2)$. The group $\mathrm{SO}^+(P^\perp) = \mathrm{SO}^+(1, 2)_{\mathbb{R}}$ is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and

$$\mathcal{H}_v \cap \mathcal{H}_u \cong \mathbb{P}\mathbb{H}^+_{P^\perp} \cong \mathbb{H}_1\,.$$

□

Remark. For $l \in L_t$ ($l^2 > 0$) the group $\widehat{\mathrm{SO}}^+(l^\perp)$ is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ or to a subgroup of a Hilbert modular group.

Definition. Let $\ell \in \widehat{L}_t$ be a vector in the dual lattice. The Humbert surface H_ℓ is defined by

$$H_\ell = \pi\,(\bigcup_{g\in \widehat{SO}^+(L_t)} \mathcal{H}_{g\ell}\,),$$

where $\pi:\ \mathbb{P}\mathbb{H}_t^+ \to \widehat{\mathrm{SO}}^+(L_t)\setminus \mathbb{P}\mathbb{H}_t^+$ is the natural projection.

$\mathcal{H}_\ell$ depends only on the one dimensional lattice $\mathbb{Z}\ell$, thus we can restrict ourselves to primitive vectors $\ell \in \widehat{L}_t$. The primitivity means that $\ell/d \notin \widehat{L}_t$ for any interger $d>1$. The first statement of Lemma 3.1 says that there is a one to one correspondence between the $\widehat{\mathrm{SO}}^+(L_t)$-orbits of primitive vectors $\ell \in \widehat{L}_t$ with positive norm and the Humbert surfaces.

It is well known that for any even integral lattice L with two hyperbolic planes (in particular for L_t) the $\widehat{\mathrm{SO}}(L)$-orbit of any $l\in L$ depends only on the norm of l and its canonical image $l^* := l/\mathrm{div}\,(l)$ in the discriminant group $\widehat{L}/L$, where the *divisor* $\mathrm{div}\,(l)\in\mathbb{N}$ of l is the positive generator of the ideal $\{(x,l)_L \mid x\in L\}$. As a corollary we have

Lemma 3.2. *Let $\ell_1,\ \ell_2\in\widehat{L}_t$ be two primitive vectors with the same image in the discriminant group (i.e. $\ell_1-\ell_2\in L_t$). If $\ell_1^2=\ell_2^2$, then $H_{\ell_1}=H_{\ell_2}$.*

Proof. If $\ell_1-\ell_2\in L_t$, then $\mathrm{div}\,(2t\ell_1)=\mathrm{div}\,(2t\ell_2)$ and $\widehat{\mathrm{SO}}^+(L_t)\,\ell_1=\widehat{\mathrm{SO}}^+(L_t)\,\ell_2$. □

Definition. Let ℓ be a primitive vector of $\widehat{L}_t$. The integer $\Delta(\ell)=2t\ell^2$ is called the discriminant of H_ℓ.

From the isomorphism $\widehat{L}_t/L_t\cong\mathbb{Z}/2t\mathbb{Z}$ one gets

Corollary 3.3. *The number of surfaces H_ℓ with fixed discriminant $\Delta=2t\ell^2$, which are not Γ_t-equivalent, is equal to the number of solutions*

$$\#\,\{b \bmod 2t \mid\ b^2\equiv\Delta \bmod 4t\}.$$

The standard definition of the Humbert surfaces (see [vdG], [F]) is given in terms of the moduli space of abelian surfaces with polarization $(1,t)$. Let us compare both definitions.

According to (1.8) and Proposition 1.2 we may rewrite the definition of $\mathcal{H}_\ell$ with $\ell=(e,a,-\frac{b}{2t},c,f)\in\widehat{L}_t$ $((e,a,b,c,f)=1)$ in coordinates (τ_i) of $\mathbb{H}_2$:

$$\mathcal{H}'_x=\psi_t^{-1}(\mathcal{H}_\ell)=\{\begin{pmatrix}\tau_1&\tau_2\\ \tau_2&\tau_3\end{pmatrix}\in\mathbb{H}_2\mid(\tau_1\tau_3-\tau_2^2)f+c\tau_3+b\tau_2+ta\tau_1+te=0\,\},$$

where $x=(te,ta,b,c,f)$. The number

$$2t\ell^2=b^2-4f(te)-4c(ta)=\Delta(\mathcal{H}'_x).$$

is by definition the discriminant of $\mathcal{H}'_x$. Let us introduce a lattice

$$N_t=\{(e,a,b,c,f)\in\mathbb{Z}^5\mid\ e,\ a\equiv 0 \bmod t\}.$$

In accordance with Proposition 1.2 and Lemma 3.1 we have the following decomposition of the usual (in sense of [vdG], [F]) Humbert surface $H_\Delta \subset \mathcal{A}_t = \Gamma_t \setminus \mathbb{H}_2$:

$$H_\Delta = \pi_t \Big(\bigcup_{\substack{x\in N_t,\ primitive \\ \Delta(x)=\Delta}} \mathcal{H}'_x \Big) \cong \pi \Big(\bigcup_{2t\ell^2=\Delta} \ \bigcup_{g\in \widehat{SO}^+(L_t)} \mathcal{H}_{g\ell} \Big),$$

where one takes the summation over representatives ℓ from the distinct orbits and π_t is the natural projection $\pi_t : \mathbb{H}_2 \to \mathcal{A}_t$. Thus the surface $\mathcal{H}_\ell$ defined above corresponds to an irreducible component of surface H_Δ. Corollary 3.3 tells us that the number of the irreducible components of the H_Δ is equal to

$$\#\{b \bmod 2t \mid b^2 \equiv \Delta \bmod 4t\}.$$

This gives a new proof of Theorem 2.4 in [vdG] (see p. 212).

In § 1 we fixed a basis of the lattice L_t such that

$$L_t = U(-1) \oplus U(-1) \oplus < 2t >,$$

where $U(-1)$ is the integral hyperbolic plane with the quadratic form $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ and $< 2t >$ is the one dimensional $\mathbb{Z}$-lattice with even quadratic form $2t$. By $L_t^{(3)}$ we denote the orthogonal component in L_t of the first hyperbolic plane

$$L_t^{(3)} = \big(e_2 \wedge e_3,\ e_1 \wedge e_3 - te_2 \wedge e_4,\ e_4 \wedge e_1\big)\mathbb{Z}^3 \subset L_t.$$

It is easy to see that in any orbit $\widehat{\mathrm{SO}}(L_t)\ell$ there is a vector from $L_t^{(3)}$ and $\widehat{\mathrm{SO}}^+(L_t)\ell = \widehat{\mathrm{SO}}(L_t)\ell$. Thus any Humbert surfaces can be given in the form

$$\mathcal{H}_\ell = \{az_1 + bz_2 + cz_3 = 0\} \subset \mathbb{P}\mathbb{H}_t^+, \qquad \Delta(\mathcal{H}_\ell) = 2t\ell^2$$

or

$$\mathcal{H}'_x = \{ta\tau_1 + b\tau_2 + c\tau_3 = 0\} \subset \mathbb{H}_2, \qquad \Delta(\mathcal{H}'_x) = b^2 - 4tac = 2t\ell^2$$

where $\ell = {}^t(0, a, -\frac{b}{2t}, c, 0) \in \widehat{L}_t$ and $x = (ta, b, c) \in N_t$.

For any $d||t$ we define the following subgroup of Γ_t^* and the corresponding quotient space of the moduli space $\mathcal{A}_t$

$$\Gamma_t^{(d)} = \Gamma_t \cup \Gamma_t V_d, \qquad \mathcal{A}_t^{(d)} = \Gamma_t^{(d)} \setminus \mathbb{H}_2.$$

The ramification locus of the map $\mathcal{A}_t \to \mathcal{A}_t^{(d)}$ can consist of components of different dimension. In the next theorem we describe its *divisorial part* $D_t^{(d)}$.

Theorem 3.4. *Let t be square free, $d>1$ and $t_d = \dfrac{t}{d}$. Then*

$$D_t^{(d)} = \begin{cases} H_{4d}\cup H_d & \text{if } \left(\dfrac{d}{4t_d}\right)=1 \\ H_{4d} & \text{if } \left(\dfrac{d}{4t_d}\right)\neq 1 \text{ and } \left(\dfrac{d}{t_d}\right)=1, \end{cases}$$

where $\left(\dfrac{a}{b}\right)$ is the generalized Kronecker symbol of the quadratic residue.

Remarks. 1. For $d=t$

$$D_t^{(t)} = \begin{cases} H_{4t}\cup H_t & \text{if } t\equiv 1 \bmod 4 \\ H_{4t} & \text{otherwise.} \end{cases}$$

In particular $D_t^{(t)}$ is irreducible if $t\equiv 2$ or $3 \bmod 4$ (see Corollary 3.3).

2. For $d=1$ Theorem 3.4 is still true if we denote by $D_t^{(1)}$ the divisorial part of the branch locus of the covering $\mathbb{H}_2\to\mathcal{A}_t$. We note that $D_t^{(1)}$ was found in [Br] by another method.

Corollary 3.5. *Let t be square free. The divisorial part D_t^* of the ramification locus of the map $\mathcal{A}_t\to\mathcal{A}_t^*$, where $\mathcal{A}_t^*=\Gamma_t^*\setminus\mathbb{H}_2$ is the "minimal" Siegel modular threefold corresponding to polarization of type $(1,t)$, is the union of the following Humbert surfaces*

$$D_t^* = \bigcup_{d|t} \left(\varepsilon_1(d)H_{4d}\cup\varepsilon_2(d)H_d\right),$$

where $\varepsilon_1(d)=1$ if $\left(\dfrac{d}{t_d}\right)=1$, $\varepsilon_2(d)=1$ if d is odd and $\left(\dfrac{d}{4t_d}\right)=1$ and they equal 0 in all other cases. Moreover none of the above Humbert surfaces are Γ_t-equivalent.

Proof of Corollary. We have to prove only the last statement. If ℓ_1 and $\ell_2\in\widehat{L}_t$ are two primitive vectors with norms $\ell_1^2=2/t_{d_1}$, $\ell_2^2=1/(2t_{d_2})$, then $\ell_1^2\neq\ell_2^2$, since t is square free.
□

We break up the proof of Theorem 3.4 into several lemmas.

Let us consider a reflection with respect to a vector $v\in L_t\otimes\mathbb{R}$:

$$\sigma_v(x) = x - \frac{2(x,v)}{(v,v)}\,v.$$

It is known that $\sigma_v\in \mathrm{O}_{\mathbb{R}}^+(L_t)$ if and only if $v^2>0$. (This follows from the definition of the real spin norm.) If $\sigma_v\in\mathrm{O}_{\mathbb{R}}^+(L_t)$, then the set $\mathrm{Fix}_{\mathbb{P}\mathbb{H}_t^+}(\sigma_v)$ of fix points of σ_v on $\mathbb{P}\mathbb{H}_t^+$ is a complex surface $\mathcal{H}_v$. The opposite statement is also true.

Lemma 3.6. *Let us suppose that the set of fix points of $\sigma \in \mathrm{SO}^+_{\mathbb{R}}(L_t)$ on $\mathbb{P}\mathbb{H}^+_t$ is a complex surface. Then $-\sigma$ is a reflection with respect to a vector $v \in L_t \otimes \mathbb{R}$.*

Proof. Over $\mathbb{R}$ one can reduce the quadratic form S_t to $S = \mathrm{diag}\,(E_3, -E_2)$. The maximal compact subgroup $K_{\mathbb{R}}$ of the orthogonal group $\mathrm{SO}^+_{\mathbb{R}}(S)$ is isomorphic to $\mathrm{SO}(3) \times \mathrm{SO}(2)$ consisting of all elements which fix the point $Z_{\mathrm{i}} = {}^t(0,0,0,i,1) \in \mathbb{P}\mathbb{H}^+_t$. Since the group $\mathrm{SO}^+_{\mathbb{R}}(S)$ acts transitively on the homogeneous domain we can suppose that $\sigma = \mathrm{diag}\,(A, B) \in K$ where $A \in \mathrm{SO}(3)$ and $B \in \mathrm{SO}(2)$. If $B \neq \pm E_2$, then B has only one fix point $\mathbf{i} = {}^t(i,1)$ on the projective line. If $\sigma = \mathrm{diag}\,(A,B)$ has at least three fixed points, then B has an eigenvalue λ of order two. A and B are orthogonal, thus all eigenvalues of σ are equal to ± 1.

There are two possibilities for the set of eigenvalues of σ

$$\{\,\lambda(\sigma)\,\} = \{\,1,-1,-1,-1,-1\} \quad \text{or} \quad \{\,1,1,1,-1,-1\}.$$

In the first case $-\sigma$ is a reflection. In the second case σ can be written as a product of two reflections $\sigma_v\sigma_u$ with orthogonal u and v. Thus $\mathrm{Fix}_{\mathbb{P}\mathbb{H}^+_t}(\sigma) = \mathcal{H}_u \cap \mathcal{H}_v$ and we have proved the lemma for non-trivial B.

If $B = \pm E_2$, then the same arguments show that σ is conjugate to

$$D = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm E_2 \end{pmatrix} \qquad B_1 \in \mathrm{SO}(2) \tag{3.1}$$

if σ has at least two fixed points. If $B_1 \neq \pm E_2$, then $\mathrm{Fix}_{\mathbb{P}\mathbb{H}^+_t}(D)$ is a subset of $\mathcal{H}_x \cap \mathcal{H}_y$, where x and y form an orthogonal basis of the plane of rotation of $B_1 \in \mathrm{SO}(2)$.
□

Lemma 3.7. *There is a one to one correspondence between the irreducible components H of the divisorial part $D_t^{(d)}$ and the surfaces H_ℓ defined by reflections $\sigma_\ell \in \Gamma_t V_d$.*

Proof. By $\mathcal{H}$ we denote an irreducible surface in $\mathbb{H}_2$ whose image is H. Let us suppose that $\mathcal{H} = \mathrm{Fix}_{\mathbb{H}_2}(G)$ with $G \in \Gamma_t V_d$. In accordance with Proposition 1.2 and Lemma 3.6 $\Psi(G) = \sigma_\ell$ is a reflection. Moreover $\psi_t(\mathcal{H}) = \mathcal{H}_\ell \subset \mathbb{P}\mathbb{H}^+_t$ and σ_ℓ induces multiplication by ξ_d on the discrimnant group A_t.

The reflection

$$\sigma_\ell(x) = x - \frac{2(x,\ell)}{(\ell,\ell)}\,\ell$$

depends only on the line $< \ell >$ defined by $\ell \in L_t$. It follows from the definition that σ_ℓ keeps the lattice L_t invariant if and only if $\ell^2 \,|\, 2D$ where $D = \mathrm{div}\,(\ell)$ (see the definition before Lemma 3.1). The surface H depends only on the class $\{\,\gamma G \gamma^{-1} \,|\, \gamma \in \Gamma_t\}$ and

$$\Psi(\gamma G \gamma^{-1}) = \beta \sigma_\ell \beta^{-1} = \sigma_{\beta\ell} \qquad \big(\gamma \in \Gamma_t,\ \beta = \Psi(\gamma) \in \widehat{\mathrm{SO}}^+(L_t)\big).$$

□

Therefore in order to find all Humbert surfaces in the divisorial part we have to classify the $\widehat{\mathrm{SO}}(L_t)$-orbits of vectors $\ell \in L_t$ with the additional condition $\ell^2 \,|\, 2\mathrm{div}\,(\ell)$.

Lemma 3.8. *Let t be an arbitrary positive integer and $d||t$. There is a one to one correspondence between the $\widehat{\mathrm{SO}}(L_t)$-conjugacy classes of reflections σ_ℓ in the coset $(-\Gamma_t V_d)$ and the orbits of the primitive vectors in L_t, which satisfy the following conditions:*

$$\ell^2 = 2d \quad \textit{and} \quad \begin{cases} \operatorname{div}(\ell) = 2t_d & \textit{if } \left(\dfrac{d}{4t_d}\right) = 1 \\ \operatorname{div}(\ell) = t_d & \textit{if } \left(\dfrac{d}{t_d}\right) = 1. \end{cases}$$

Proof. One can suppose that $\ell = {}^t(0, a, b, c, 0) \in L_t$ and $(a, b, c) = 1$. For such ℓ the matrix of σ_ℓ has the following form

$$\sigma_\ell = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 + \frac{2ca}{\ell^2} & -\frac{4tba}{\ell^2} & \frac{2a^2}{\ell^2} & 0 \\ 0 & \frac{2cb}{\ell^2} & 1 - \frac{4tb^2}{\ell^2} & \frac{2ab}{\ell^2} & 0 \\ 0 & \frac{2c^2}{\ell^2} & -\frac{4tbc}{\ell^2} & 1 + \frac{2ac}{\ell^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\ell^2 = 2tb^2 - 2ac$. $-\sigma_\ell \in \mathrm{SO}^+_{\mathbb{R}}$ if and only if $\ell^2 > 0$. On the discriminant group $-\sigma_\ell$ defines multiplication by

$$\xi(\ell) = \frac{4t}{\ell^2} b^2 - 1.$$

By definition $D = \operatorname{div}(\ell) = (a,\, 2tb,\, c)$, therefore $D\,|\,2t$ and $D\,|\,\ell^2\,|\,2D$. We put

$$\ell = {}^t(0, Da_1,\, b,\, Dc_1, 0) \qquad \text{with } \ (a_1,\, \frac{2tb}{D},\, c_1) = 1.$$

We have to consider four cases:

$$\ell^2 = D \quad \text{or} \quad \ell^2 = 2D \qquad \text{and} \qquad D\,|\,t \quad \text{or} \quad D \not|\, t.$$

1). Let us suppose that $D\,|\,t$ ($t_D = \dfrac{t}{D}$). Then we have

$$\ell^2 = 2D \iff 1 = t_D b^2 - Da_1c_1 \iff (D, t_D) = 1\,\&\left(\frac{t_D}{D}\right) = 1,$$

$$\xi(-\sigma_\ell) = 2t_D b^2 - 1 = \begin{cases} -1 & \bmod\ 2t_D \\ 1 & \bmod\ 2D \end{cases} \implies -\sigma_\ell \in V_{t_D}\Gamma_t$$

(see (1.5)). The case $\ell^2 = D$ leads trivially to a contradiction.

2). Let us suppose that $D\,|\,2t$, but $\dfrac{t}{D} \notin \mathbb{Z}$. In this case $D = 2D_1$, $D_1|\,t$ and t_{D_1} is odd. For such D we have

$$\ell^2 = D \iff 1 = t_{D_1} b^2 - 4D_1 a_1 c_1 \iff (D_1, t_{D_1}) = 1\,\&\,(t_{D_1} \text{ odd})\,\&\left(\frac{t_{D_1}}{4D_1}\right) = 1,$$

$$\xi(-\sigma_\ell) = 2t_{D_1} b^2 - 1 = \begin{cases} -1 & \bmod\ 2t_{D_1} \\ 1 & \bmod\ 2D_1 \end{cases} \implies -\sigma_\ell \in V_{t_{D_1}}\Gamma_t.$$

The case $\ell^2 = 2D$ leads to a contradiction to the primitivity of ℓ.
$\square$

For square free t the system of the surfaces $\{H_{\ell^*}\}$, where $\ell^* = \dfrac{\ell}{\operatorname{div}(\ell)}$ and ℓ satisfies the condition of Lemma 3.8, contains all irreducible components of the Humbert surfaces from Theorem 3.4. This finishes the proof of Theorem 3.4.
$\square$

The next corollary follows immediately from the proof of Lemma 3.8.

Corollary 3.9. *Let t be square free and $d||t$. If $\left(\dfrac{d}{t_d}\right) = 1$ and $\left(\dfrac{d}{4t_d}\right) \neq 1$ then there is, up to conjugation with respect to Γ_t, exactly one, and if $\left(\dfrac{d}{4t_d}\right) = 1$ then there are exactly two involutions in $\Gamma_t V_d$. They are $\Psi^{-1}(-\sigma_{\ell_1})$ (in the both cases), and $\Psi^{-1}(-\sigma_{\ell_2})$ (in the second case), where*

$$\ell_1 = {}^t(0, a_1, \frac{b_1}{t_d}, c_1, 0) \in \widehat{L}_t, \qquad (a_1, b_1, c_1) = 1, \quad db_1^2 - t_d a_1 c_1 = 1$$
$$\ell_2 = {}^t(0, a_2, \frac{b_2}{2t_d}, c_2, 0) \in \hat{L}_t, \qquad (a_2, b_2, c_2) = 1, \quad db_2^2 - 4t_d a_2 c_2 = 1.$$

Remarks. 1. It is possible to apply Lemma 3.8, which has been proved for any integer t, to classify the divisorial part of the ramification locus of the covering $\mathcal{A}_t \to \mathcal{A}_t^*$ for any integer t.

2. Using the same method one can construct divisors on a homogeneous domain of type IV of any dimension n.

The ramification locus can also have components of smaller dimension. The proof of Lemma 3.6 shows us that the orthogonal group can contain a rotation in the positive definite subplane of the lattice L_t.

Example. (Brasch) The following example is due to Brasch. It shows that in general the ramification locus of the map $\mathcal{A}_t \to \mathcal{A}_t^*$ contains other components apart from the divisorial part described above. Let $t \equiv 1 \bmod 4$. For an integer $f > 0$ put

$$c = -f^2 t - 1, \quad g = f^2.$$

Then the matrix

$$N = \begin{pmatrix} -f\sqrt{t} & 1/\sqrt{t} & 0 & f\sqrt{t} \\ c\sqrt{t} & 0 & f\sqrt{t} & f^2 t\sqrt{t} \\ c\sqrt{t} & 0 & f\sqrt{t} & -c\sqrt{t} \\ 0 & 1/\sqrt{t} & -1/\sqrt{t} & 0 \end{pmatrix}$$

is an element of Γ_t^*. One immediately checks that $N^2 = -E_4$. The fixed point set Fix N is a curve (cf. [B, Hilfssatz 2.5.3]). Moreover the curve Fix N is not contained in the fixed point set Fix I of an involution I in Γ_t^*. This would namely imply that $IN = -NI$. Now using the explicit form for N which follows from [B, Hilfssatz 2.8], a lengthy but straightforward calculation shows $c = -1$, a contradiction.

As a next step we want to interprete the surfaces H_t, resp. H_{4t} as moduli spaces of abelian surfaces with real multiplication. It is well known that there is a close connection between Hilbert modular surfaces and Humbert surfaces [F], [vdG, chapter IX]. Here we want to determine precisely which Hilbert modular surfaces correspond to H_t, resp. H_{4t}. Consider the ring $\mathfrak{o}$ of integers in the number field $\mathbb{Q}(\sqrt{t})$ and recall that

$$\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\omega, \quad \omega = \frac{1}{2}(1+\sqrt{t}) \quad \text{if } t \equiv 1 \bmod 4$$

resp.

$$\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\omega, \quad \omega = \sqrt{t} \quad \text{if } t \not\equiv 1 \bmod 4.$$

The *Hilbert modular group* $\mathrm{SL}_2(\mathfrak{o})$ acts on $\mathbb{H}_1 \times \mathbb{H}_1$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (z_1, z_2) = \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right)$$

where $'$ denotes the Galois automorphism $\sqrt{t} \mapsto -\sqrt{t}$. The quotient space $Y = \mathrm{SL}_2(\mathfrak{o}) \backslash \mathbb{H}_1 \times \mathbb{H}_1$ is the standard *Hilbert modular surface* associated to $\mathbb{Q}(\sqrt{t})$. Let σ be the involution which interchanges the two factors of $\mathbb{H}_1 \times \mathbb{H}_1$, i.e.

$$\sigma(z_1, z_2) = (z_2, z_1).$$

Then the *symmetric* Hilbert modular group is

$$\mathrm{SL}_2^\sigma(\mathfrak{o}) = \mathrm{SL}_2(\mathfrak{o}) \cup \sigma\, \mathrm{SL}_2(\mathfrak{o})$$

and $Y^\sigma = \mathrm{SL}_2^\sigma(\mathfrak{o}) \backslash \mathbb{H}_1 \times \mathbb{H}_1$ is the corresponding symmetric Hilbert modular surface. We shall first consider the Humbert surface H_t. In particular we assume that $t \equiv 1 \bmod 4$. To every point $(z_1, z_2) \in \mathbb{H}_1 \times \mathbb{H}_1$ one can associate the lattice

$$\Lambda_{(z_1,z_2)} = \mathfrak{o} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathfrak{o} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbb{Z} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \eta z_1 \\ \eta' z_2 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} -\eta' \backslash \sqrt{t} \\ \eta \backslash \sqrt{t} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \sqrt{t} \\ -\sqrt{t} \end{pmatrix}$$

where $\eta = \sqrt{t}\omega$. The form

$$E((x_1, x_2), (y_1, y_2)) = \mathrm{Im} \left(\frac{x_1 \bar{y}_1}{\mathrm{Im}\, z_1} + \frac{x_2 \bar{y}_2}{\mathrm{Im}\, z_2} \right)$$

defines a Riemann form for Λ_t. With respect to the basis given above this is just the alternating form W_t. The torus

$$A_{(z_1,z_2)} = \mathbb{C}^2 / \Lambda_{(z_1,z_2)}$$

is hence a $(1,t)$-polarized abelian surface with real multiplication in $\mathfrak{o}$. The Hilbert modular surface Y is the moduli space of these objects. The 2:1 cover $Y \to Y^\sigma$ identifies abelian surfaces whose real multiplication differs by the Galois conjugation. We have a “forgetful” map

$$\Phi : Y^\sigma \to \mathcal{A}_t.$$

Theorem 3.10. *Assume $t \equiv 1 \bmod 4$. The Humbert surface H_t is the image of the symmetric Hilbert modular surface Y^σ under the natural morphism $\Phi : Y^\sigma \to \mathcal{A}_t$ which is of degree 1 onto its image.*

Before giving the proof we turn to the Humbert surfaces H_{4t}. Consider the ring

$$\mathfrak{o}_2 = \mathbb{Z} + \mathbb{Z}\sqrt{t}.$$

Note that this is an order in $\mathfrak{o}$ if $t \equiv 1 \bmod 4$, whereas $\mathfrak{o} = \mathfrak{o}_2$ if $t \not\equiv 1 \bmod 4$. Let

$$\tilde{\mathfrak{o}}_2 = \frac{1}{2}\mathbb{Z} + \frac{1}{2}\sqrt{t}\mathbb{Z}, \quad \tilde{\mathfrak{o}}_2^{-1} = 2\mathbb{Z} + 2\sqrt{t}\mathbb{Z}.$$

The group

$$\mathrm{SL}_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \mathfrak{o}_2,\ b \in \tilde{\mathfrak{o}}_2,\ c \in \tilde{\mathfrak{o}}_2^{-1},\ ad - bc = 1\}$$

acts on $\mathbb{H}_1 \times \mathbb{H}_1$ as well as its symmetric counterpart

$$\mathrm{SL}_2^\sigma(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) = \mathrm{SL}_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) \cup \sigma\, \mathrm{SL}_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2).$$

Let

$$\widetilde{Y} = \mathrm{SL}_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) \backslash \mathbb{H}_1 \times \mathbb{H}_1, \quad \widetilde{Y}^\sigma = \mathrm{SL}_2^\sigma(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) \backslash \mathbb{H}_1 \times \mathbb{H}_1$$

be the corresponding Hilbert modular surfaces. Again the Riemann form E induces a $(1,t)$-polarization on the tori

$$A_{(z_1, z_2)} = \mathbb{C}^2 / \Lambda_{(z_1, z_2)}$$

where for $t \equiv 1 \bmod 4$

$$\Lambda_{(z_1,z_2)} = \mathbb{Z}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 2\eta z_1 \\ 2\eta' z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} -\eta'/\sqrt{t} \\ \eta/\sqrt{t} \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \sqrt{t}/2 \\ -\sqrt{t}/2 \end{pmatrix}$$

resp. $t \not\equiv 1 \bmod 4$

$$\Lambda_{(z_1,z_2)} = \mathbb{Z}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \omega z_1 \\ \omega' z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \omega/2 \\ \omega'/2 \end{pmatrix}$$

Hence $\widetilde{Y}$, resp. $\widetilde{Y}^\sigma$ are moduli spaces of $(1,t)$-polarized abelian surfaces with real multiplication in $\mathfrak{o}_2$ and as before we have a canonical map

$$\widetilde{\Phi} : \widetilde{Y}^\sigma \to \mathcal{A}_t.$$

Theorem 3.11. *The Humbert surface H_{4t} is the image of the symmetric Hilbert modular surface $\widetilde{Y}^{\sigma}$ under the natural map $\widetilde{\Phi} : \widetilde{Y}^{\sigma} \to \mathcal{A}_t$ which is of degree 1 onto its image.*

Proof of Theorems 3.10 and 3.11. We shall treat the case of H_{4t} and $t \equiv 1 \bmod 4$ in detail and then comment on the other cases. The proof is similar to the proofs in [HL §0], cf. also [F, Abschnitt 3]. Let

$$R = \begin{pmatrix} 1 & 2\eta \\ 1 & 2\eta' \end{pmatrix}$$

and consider the map

$$\begin{aligned} \widehat{\Phi} : \mathbb{H}_1 \times \mathbb{H}_1 &\to \mathbb{H}_2 \\ (z_1, z_2) &\mapsto {}^tR \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} R. \end{aligned}$$

Then

$$\operatorname{Im} \widehat{\Phi} = \{-(t^2 - t)\tau_1 + 2t\tau_2 - \tau_3 = 0\} = \mathcal{H}'_{4t}$$

and modulo

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_t$$

this is equivalent to $\mathcal{H}_{4t} = \{t\tau_1 - \tau_3 = 0\}$. Let $A_{\widehat{\Phi}(z_1,z_2)}$ be the abelian surface associated to the period matrix

$$\left({}^tR \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \; R \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \right).$$

Then $A_{(z_1,z_2)}$ and $A_{\widehat{\Phi}(z_1,z_2)}$ are isomorphic as polarized abelian surfaces since

$${}^tR \begin{pmatrix} z_1 & 2\eta z_1 & -\eta'/\sqrt{t} & \sqrt{t}/2 \\ z_2 & 2\eta' z_2 & \eta/\sqrt{t} & -\sqrt{t}/2 \end{pmatrix} = \left({}^tR \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \; R \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \right).$$

Hence $\widehat{\Phi}$ is a lift of the map $\widetilde{\Phi}$. Next we consider the homomorphism

$$\Psi : \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathrm{Sp}_4(\mathbb{R})$$

$$\Psi(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) = \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} d(a_1,a_2) & d(b_1,b_2) \\ d(c_1,c_2) & d(d_1,d_2) \end{pmatrix} \begin{pmatrix} {}^tR & 0 \\ 0 & R \end{pmatrix}$$

where $d(a_1, a_2) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, etc. Via the embedding

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Q}(\sqrt{t})) &\to \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto (\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}) \end{aligned}$$

this also defines a homomorphism

$$\widehat{\Psi} : \mathrm{SL}_2(\mathbb{Q}(\sqrt{t})) \to \mathrm{Sp}_4(\mathbb{R}).$$

For $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we find that

$$S = \begin{pmatrix} {}^tRJ\,{}^tR^{-1} & 0 \\ 0 & R^{-1}JR \end{pmatrix} \in \Gamma_t.$$

Clearly S is an involution. Setting $\widehat{\Psi}(\sigma) = S$ we can extend $\widehat{\Psi}$ to a homomorphism

$$\widehat{\Psi}_\sigma : \mathrm{SL}_2^\sigma(\mathbb{Q}(\sqrt{t})) \to \mathrm{Sp}_4(\mathbb{R})$$

and one checks easily that $\widehat{\Phi}$ is $\widehat{\Psi}_\sigma$-equivariant. Let G, resp. $G_{\mathbb{R}}$ be the stabilizer of $\mathcal{H}'_{4t}$ in Γ_t, resp. $\mathrm{Sp}_4(\mathbb{R})$. As in [HL, Lemma 0.8], cf. also [F, Korollar 3.2.8] one shows that $G_{\mathbb{R}}$ is the group generated by the image of Ψ and S. The result follows if we can prove that $G = \widehat{\Psi}_\sigma(\mathrm{SL}_2^\sigma(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2))$. For this it is now enough to prove the following.

Lemma 3.12. *Let* $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$; $i = 1, 2$ *and assume that*

$$M = \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} d(a_1,a_2) & d(b_1,b_2) \\ d(c_1,c_2) & d(d_1,d_2) \end{pmatrix} \begin{pmatrix} {}^tR^{-1} & 0 \\ 0 & R \end{pmatrix} \in \Gamma_t.$$

Then $a_1, d_1 \in \mathfrak{o}_2,\ b_1 \in \tilde{\mathfrak{o}}_2,\ c_1 \in \tilde{\mathfrak{o}}_2^{-1}$ *and* $a_2 = a'_1,\ b_2 = b'_1,\ c_2 = c'_1,\ d_2 = d'_2$.

Proof of the lemma. We write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Straightforward calculation gives

$$A = \frac{1}{2(\eta' - \eta)} \begin{pmatrix} 2\eta' a_1 - 2\eta a_2 & -a_1 + a_2 \\ 4\eta\eta'(a_1 - a_2) & -2\eta a_1 + 2\eta' a_2 \end{pmatrix}.$$

From $A_{12} \in \mathbb{Z}$ we find $(a_1 - a_2)/(2\sqrt{t}) \in \mathbb{Z}$, i.e.

$$a_2 = a_1 + 2n\sqrt{t}, \quad n \in \mathbb{Z}.$$

$A_{11} \in \mathbb{Z}$ gives $a_1\omega' + a_2\omega \in \mathbb{Z}$. Hence

$$a_1 + 2n\sqrt{t}\omega' = a_1(\omega + \omega') + 2n\sqrt{t}\omega' = a_1\omega + a_2\omega' \in \mathbb{Z}. \tag{3.2}$$

I.e. $a_1 \in \mathbb{Z} + \mathbb{Z}\sqrt{t} = \mathfrak{o}_2$. We can write $a_1 = \alpha + \beta\sqrt{t}$ with $\alpha, \beta \in \mathbb{Z}$. By (3.2) we have $a_1 + n(\sqrt{t} + t) \in \mathbb{Z}$ and hence $\beta = -n$. But then $a_2 = a_1 - 2\beta\sqrt{t} = \alpha - \beta\sqrt{t} = a'_1$. Note also that for $a_1 \in \mathfrak{o}_2$ and $a_2 = a'_1$ one has $A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Similarly we obtain

$$B = \begin{pmatrix} b_1 + b_2 & 2\eta b_1 + 2\eta' b_2 \\ 2\eta b_1 + 2\eta' b_2 & 4\eta^2 b_1 + 4\eta'^2 b_2 \end{pmatrix}.$$

Since $B_{11} \in \mathbb{Z}$ we find that $b_2 = n - b_1$ for some $n \in \mathbb{Z}$. Using this and $B_{12} \in t\mathbb{Z}$ one concludes that

$$(b_1\omega - b_2\omega') \in \frac{\sqrt{t}}{2}\mathbb{Z}.$$

Hence

$$b_1 - n\omega' = b_1(\omega + \omega') - n\omega' = b_1\omega - b_2\omega' \in \frac{\sqrt{t}}{2}\mathbb{Z}. \tag{3.3}$$

This shows $b_1, b_2 \in \tilde{\mathfrak{o}}_2$. Writing $b_1 = (\alpha + \beta\sqrt{t})/2$ with $\alpha, \beta \in \mathbb{Z}$ one has from (3.3)

$$\frac{\alpha}{2} + \frac{\beta}{2}\sqrt{t} - n\omega' \in \frac{\sqrt{t}}{2}\mathbb{Z}$$

i.e. $\alpha = n$. Hence $b_2 = n - b_1 = (\alpha - \beta\sqrt{t})/2 = b_1'$. Conversely if $b_1 \in \tilde{\mathfrak{o}}_2$ and $b_2 = b_1'$, then $B \in \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} \end{pmatrix}$. For C one computes

$$C = \frac{1}{4t}\begin{pmatrix} 4\eta'^2 c_1 + 4\eta^2 c_2 & -2\eta' c_1 - 2\eta c_2 \\ -2\eta' c_1 - 2\eta c_2 & c_1 + c_2 \end{pmatrix}.$$

Comparing this with the situation for B one finds that $c_1 \in \tilde{\mathfrak{o}}_2^{-1}, c_2 = c_1'$. Finally

$$D = \frac{1}{2(\eta' - \eta)}\begin{pmatrix} 2\eta' d_1 - 2\eta d_2 & 4\eta\eta'(d_1 - d_2) \\ -d_1 + d_2 & -2\eta d_1 + 2\eta' d_2 \end{pmatrix}.$$

This case is analogous to A and one obtains $d_1 \in \mathfrak{o}_2,\ d_2 = d_1'$. This proves Lemma 3.12.

□

The case of the Humbert surface H_t, $t \equiv 1 \bmod 4$ can be treated in the same manner if we choose $R = \begin{pmatrix} 1 & \eta \\ 1 & \eta' \end{pmatrix}$. This is analogous to [HL, §0] where the case $t = 5$ was done. Finally we have to deal with H_{4t} in case $t \not\equiv 1 \bmod 4$. Here we can choose $R = \begin{pmatrix} 1 & \omega \\ 1 & \omega' \end{pmatrix}, \omega = \sqrt{t}$. In this case

$$\operatorname{Im}\widehat{\Phi} = \{t\tau_1 - \tau_3 = 0\} = \mathcal{H}_{4t}$$

and the above arguments go through essentially unchanged.

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