

# OPEN PROBLEMS ON OPEN ALGEBRAIC VARIETIES

(MONTREAL 1994 PROBLEMS)

A lively session held at the end of a conference on Open Algebraic Varieties organized at the Centre de Recherches en Mathématiques in December 1994 produced a list of open problems that the participants would like to make available to the mathematical community. Thanks are due to the contributors, to D.-Q. Zhang, who undertook the initial collecting of the problems, and to M. Zaidenberg, who was the guiding spirit behind the efforts to prepare the collection for electronic distribution.

Peter Russell

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## R.V. Gurjar

**PROBLEM.** Let  $f : \mathbf{C}^n \rightarrow V$  be a proper (that is, a finite) morphism from the affine  $n$ -space onto a normal affine variety  $V$ . Show that  $V$  is contractible.

I proved long time ago that

(i)  $V$  is simply-connected, and

(ii) All homology groups of  $V$  are finite and the top homology group  $H_n(V; \mathbf{Z}) = (0)$ .

In particular, if  $n = 2$  then  $V$  is contractible. In fact, Miyanishi and independently Shastri-Gurjar proved that in this case  $V$  is isomorphic to a quotient of  $\mathbf{C}^2$  by a finite group. Later on Shravan Kumar made a very nice improvement on this. He proved that if  $n = 3$  then also  $V$  is contractible. More generally, he proved that the group  $H_{n-1}(V; \mathbf{Z})$  is trivial. His proof uses Smith theory in a nice way. The proof has appeared in [K].

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## Shulim Kaliman

### PROBLEM ABOUT CLASSIFICATION OF POLYNOMIALS IN TWO VARIABLES WITH A $\mathbf{C}^*$ -FIBER

Let  $p(x, y)$  and  $q(x, y)$  be polynomials in two complex variables. We shall say that these polynomials are equivalent if there exist a polynomial automorphism  $\alpha$  of  $\mathbf{C}^2$  and an affine automorphism  $\beta$  of  $\mathbf{C}$  for which  $p = \beta \circ q \circ \alpha$ .

**PROBLEM.** Find the list of non-equivalent polynomials such that every primitive polynomial  $p$  with a  $\mathbf{C}^*$ -fiber is equivalent to one of polynomials from this list.

The answer to this question is known if one of the following additional conditions holds [K2]

- (i)  $p$  is a rational polynomial, i.e. the genus of its generic fibers is 0; or
- (ii) there exists a contractible curve in  $\mathbf{C}^2$  which does not meet the  $\mathbf{C}^*$ -fiber of  $p$ .

It is worth mentioning that P. Russell constructed a polynomial with a  $\mathbf{C}^*$ -fiber which satisfies neither of these conditions.

More generally, consider the set  $S_R$  of polynomials which have a fiber isomorphic to a given affine algebraic curve  $R$ . It is natural to look for a list  $L_R$  of non-equivalent polynomials such that every polynomial from  $S_R$  is equivalent to one of the polynomials from the list  $L_R$ . If such a list exists we shall say that there is a classification of polynomials with this fiber  $R$ . This problem is equivalent to the problem of classification of all smooth polynomial embeddings of  $R$  into  $\mathbf{C}^2$  up to a polynomial automorphism. The Abhyankar-Moh-Suzuki theorem [AM], [Su] says that all smooth polynomial embeddings of the complex line into  $\mathbf{C}^2$  are equivalent to linear embeddings. Moreover, V. Lin and M. Zaidenberg [LZ] obtained the classification of polynomial injections of  $\mathbf{C}$  into  $\mathbf{C}^2$  (i.e. they found a description of all polynomials whose zero fiber is homeomorphic to  $\mathbf{C}$ ). Later W. Neumann and L. Rudolph [NR] reproved these theorems and W. Neumann obtained the classification for all polynomials whose zero fiber is diffeomorphic to a once-punctured Riemann surface of genus  $\leq 2$  [N].

The papers [AM], [NR], and [N] use essentially the following theorem [AS]:

*If the zero fiber of a polynomial is a once-punctured Riemann surface, then every other fiber of this polynomial is once punctured.*

The Lin-Zaidenberg theorem is based on the following elegant fact.

*If a polynomial has at most one degenerate fiber (and it is so in the case of a con-*

tractible fiber) then the polynomial is isotrivial, i.e. its generic fibers are pairwise isomorphic.

Isotrivial polynomials form a narrow class and its classification was obtained later in [K1], [Z1], [Z2].

If  $R$  has more than one puncture none of the above approaches works. The number of punctures on the generic fiber of the corresponding polynomial may be arbitrary and the polynomial may have a second degenerate fiber. This makes the problem difficult; that is why we suggest above to consider  $R$  isomorphic to  $\mathbf{C}^*$  which is the simplest case of a twice punctured Riemann surface.

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## N. Mohan Kumar

Let  $A$  be an Artin local ring over an algebraically closed field  $k$  of characteristic 0, with  $A$  finite dimensional over  $k$  (as a vector space). For any finitely generated module  $M$  over  $A$ , let  $D(M)$  denote the module of  $k$ -derivations from  $A$  to  $M$ . Also let  $l$  stand for length of a module.

**QUESTION 1.** Is it true that  $l(\Omega_A^1) \geq l(A) - 1$  and equality if and only if  $A$  is isomorphic to  $k[x]/x^n$  ?

**QUESTION 2.** More generally, is it true that  $l(D(M)) \geq l(M) - l(s(M))$  where  $s(M)$  denotes the socle of  $M$  and equality if and only if  $A$  is isomorphic to  $k[x]/x^n$  and  $M$  free ?

## Masayoshi Miyanishi

### VECTOR FIELDS ON FACTORIAL SCHEMES

Let  $K$  be a field of characteristic zero, let  $R$  be a noetherian  $K$ -algebra domain and let  $L$  be the quotient field of  $R$ . Let  $\delta$  be a  $K$ -derivation on  $R$ . We call an ideal  $I$  of  $R$  a  **$\delta$ -integral ideal** if  $\delta(I) \subseteq I$ . If a principal ideal  $I = fR$  is a  $\delta$ -integral ideal,  $f$  is called a  *$\delta$ -integral element*. When we restrict ourselves to the case  $R = K[x, y]$ , we call  $f$  a  *$\delta$ -integral curve*. If  $f$  is a  $\delta$ -integral element, we can write  $\delta(f) = f\chi(f)$  with  $\chi(f) \in R$ .

An element  $t$  of  $R$  is called a  $\delta$ -integral factor if there exists a  $\delta$ -integral element  $f$  such that  $t = \chi(f)$ . The set of all  $\delta$ -integral factors in  $R$  is denoted by  $X_\delta(R)$  or simply by  $X_\delta$ . We note that an invertible element of  $R$  is a  $\delta$ -integral element.

LEMMA 1. (1) *Let  $t$  be a  $\delta$ -integral factor, and let  $A_t$  be the set of  $\delta$ -integral elements in  $R$  with the same  $\delta$ -integral factor  $t$ . Then  $A_t$  is a  $K$ -vector space.*

(2)  *$X_\delta$  is an abelian monoid under the addition of  $R$ , i.e.,  $X_\delta$  is closed under the addition and contains the zero element.*

(3) *Let  $A$  be the subalgebra of  $R$  generated by all  $\delta$ -integral elements over  $K$ . Then  $A = \sum_{t \in X_\delta} A_t$  and  $A_s \cdot A_t \subseteq A_{s+t}$ , while  $\sum_{t \in X_\delta} A_t$  is not necessarily a direct sum. We call  $A$  the  $\delta$ -integral ring of  $R$ .*

Let  $L = Q(R)$  be the quotient field of  $R$ . Then the  $K$ -derivation  $\delta$  on  $R$  is naturally extended to a  $K$ -derivation on  $L$ . An element  $\xi$  of  $L$  is, by definition, a  $\delta$ -integral element in  $L$  if  $\chi(\xi) = \delta(\xi)/\xi \in R$ , and  $\chi(\xi)$  is then a  $\delta$ -integral factor. The set of all  $\delta$ -integral factors in  $L$  is an abelian group under the addition, which we denote by  $\widetilde{X}_\delta$ . Clearly,  $X_\delta \subseteq \widetilde{X}_\delta$  as monoids. The next lemma shows that  $\widetilde{X}_\delta = X_\delta - X_\delta$ , i.e.,  $\widetilde{X}_\delta$  is an abelian group generated by  $X_\delta$  provided  $R$  is a factorial domain (= a unique factorization domain).

LEMMA 2. *Assume that  $R$  is a factorial domain. Then we have:*

(1) *Let  $\xi$  be a  $\delta$ -integral element in  $L$  and write  $\xi = fg^{-1}$  with mutually prime elements  $f, g$  of  $R$ . Then  $f$  and  $g$  are  $\delta$ -integral elements in  $R$ , and  $\chi(\xi) = \chi(f) - \chi(g)$ .*

(2) *Let  $f$  be a  $\delta$ -integral element in  $R$ . Then any prime factor as well as any divisor of  $f$  is a  $\delta$ -integral element in  $R$ . Furthermore, the  $\delta$ -integral ring  $A$  is generated over  $K$  by invertible elements of  $R$  and  $\delta$ -integral elements which are prime elements of  $R$ .*

We then pose the following:

**QUESTION 1.** *With the notations and assumptions as above, is  $\widetilde{X}_\delta$  a finitely*

generated abelian group provided  $R$  is finitely generated over  $K$  ?

Later, we shall present one result which asserts that  $\widetilde{X}_\delta$  is finitely generated.

DEFINITION. Let  $X$  be an abelian monoid.

- (1)  $X$  is *positive* if  $X$  contains no abelian subgroups other than  $(0)$ .
- (2)  $X$  is *finitely generated* if  $\widetilde{X} := X - X$  is a finitely generated abelian group.
- (3)  $X$  is *good* if  $X$  is positive and if there exist elements  $t_1, \dots, t_r$  of  $X$  such that  $X = \mathbf{Z}_+t_1 + \dots + \mathbf{Z}_+t_r$  and  $\widetilde{X}$  is a free abelian group with free basis  $t_1, \dots, t_r$ , where  $\mathbf{Z}_+$  is the set of non-negative integers. We then write  $X = \mathbf{Z}_+t_1 \oplus \dots \oplus \mathbf{Z}_+t_r$  and call  $r$  the rank of  $X$ .

We shall consider the subring  $A_0 = \{x \in R; \delta(x) = 0\}$  of  $R$  and the subfield  $L_0 = \{\xi \in L; \delta(\xi) = 0\}$  of  $L$ . We say that  $A_0$  is an *inert* subring of  $R$  if  $a \in A_0$  and  $a = bc$  with  $b, c \in R$  implies  $b, c \in A_0$ .

LEMMA 3. *Let  $R$  be a factorial domain. Then we have:*

- (1)  $X_\delta$  is positive if and only if  $A_0$  is an inert subring of  $R$ .
- (2)  $L_0$  is algebraically closed in  $L$ . If  $A_0$  is an inert subring of  $R$  then  $Q(A_0)$  is algebraically closed in  $L$ .
- (3) Every element  $\xi$  of  $L_0$  is written as  $\xi = b/a$ , where  $a$  and  $b$  are  $\delta$ -integral elements in  $R$  with the same  $\delta$ -integral factor. Conversely, if  $a, b \in A_t$  with  $t \in X_\delta$  and  $a \neq 0$  then  $b/a \in L_0$ .

We have the following result.

THEOREM 4. *Let  $R$  be a noetherian  $K$ -algebra domain and let  $\delta$  be a  $K$ -derivation on  $R$ . Assume that  $R$  is a factorial domain and the monoid  $X_\delta$  of  $\delta$ -integral factors is good. Write  $X_\delta = \mathbf{Z}_+t_1 \oplus \dots \oplus \mathbf{Z}_+t_r$  with  $t_1, \dots, t_r \in R$ . Let  $f_i$  ( $1 \leq i \leq r$ ) be a  $\delta$ -integral element such that  $\chi(f_i) = t_i$ . Then the following assertions hold.*

(1) We may assume that  $f_1, \dots, f_r$  are prime elements of  $R$ .

(2)  $R^* \subseteq A_0$  and  $A_0$  is an inert subring of  $R$ .

(3) For the  $\delta$ -integral ring  $A$  of  $R$ , we have  $A \otimes_{A_0} L_0 = L_0[f_1, \dots, f_r]$ . If  $f_1, \dots, f_r$  are algebraically independent over  $L_0$ ,  $A = \sum_{t \in X_\delta} A_t$  is a graded ring. Namely, the decomposition

$\sum_{t \in X_\delta} A_t$  is a direct sum.

(4)  $L_0 = Q(A_0)$  if and only if  $A = A_0[f_1, \dots, f_r]$ . Furthermore, if  $L_0 = Q(A_0)$  and  $A = \sum_{t \in X_\delta} A_t$  is a graded ring, then  $f_1, \dots, f_r$  are algebraically independent over  $L_0$ .

LEMMA 5. Assume that  $L_0 = Q(A_0)$ ,  $X_\delta$  is positive and  $\widetilde{X}_\delta$  is finitely generated. Then  $X_\delta$  is good.

DEFINITION. We say that  $\delta$  is *locally nilpotent* if, for each  $x \in R$ ,  $\delta^n(x) = 0$  ( $n \gg 0$ ). Let  $T$  be an indeterminate. Define a mapping  $\varphi : R \rightarrow R[T]$  by  $\varphi(x) = \sum_{i \geq 0} \frac{1}{i!} \delta^i(x) T^i$ . Then  $\varphi$  is a homomorphism of  $K$ -algebras.

LEMMA 6. Suppose  $\delta$  is locally nilpotent. Then  $X_\delta = (0)$ ,  $A = A_0$  and  $L_0 = Q(A_0)$ .

Now, we assume that  $R$  is a factorial domain of dimension 2 which is finitely generated over  $K$ . For a  $K$ -derivation  $\delta$  of  $R$ , the set  $\delta(R) = \{\delta(x); x \in R\}$  generates an ideal  $R\delta(R)$ . The *divisorial part* ( $\delta$ ) of  $\delta$  is the greatest principal ideal of  $R$  which divides  $R\delta(R)$ . Set  $(\delta) = dR$  with  $d \in R$ . Then  $\delta' = d^{-1}\delta$  is a  $K$ -derivation of  $R$  such that the ideal  $R\delta'(R)$  has height 2; we then say that  $\delta'$  has *no divisorial part*. Let  $V = \text{Spec}(R)$ . The subset  $V(R\delta(R))_{\text{red}}$  is called the *zero set* of  $\delta$ , which we denote by  $Z(\delta)$ . If  $\delta$  has no divisorial part,  $Z(\delta)$  is a finite set.

LEMMA 7. Let  $K \subset K_1 \subset L := Q(R)$  be a subfield such that  $\text{tr.deg}_K K_1 = 1$ . Then there exists a non-trivial  $K$ -derivation  $\delta$  of  $R$  determined uniquely up to an invertible element of  $R$  such that  $\delta$  has no divisorial part and  $L_0 := \{\xi \in L; \delta(\xi) = 0\} \supset K_1$ .



Let  $\varphi : V = \text{Spec} : R : \cdots \rightarrow C$  be a rational mapping onto a smooth algebraic curve  $C$ . Then  $\varphi$  may not be defined in a finite set  $\Sigma$  of  $V$ . Namely,  $\varphi^0 := \varphi|_{V-\Sigma} : V - \Sigma \rightarrow C$  is a morphism. For  $P \in C$ , the schematic closure of the fiber  $(\varphi^0)^{-1}(P)$  is called the fiber of  $\varphi$  over  $P$  and denoted by  $\varphi^{-1}(P)$ . We say that a rational mapping  $\varphi : V : \cdots \rightarrow C$  is an *irreducible pencil* parametrized by  $C$  if  $\varphi^0$  is surjective and general fibers of  $\varphi$  are irreducible and reduced. A rational mapping  $\varphi : V : \cdots \rightarrow C$  is equivalently defined by giving a subfield  $K(C)$  of  $L := Q(R)$ . Then  $\varphi$  is an irreducible pencil if and only if  $K(C)$  is algebraically closed in  $L$ . Given a rational mapping  $\varphi : V : \cdots \rightarrow C$ , the Stein factorization  $\varphi : V : \overset{\sim}{\varphi} \rightarrow \tilde{C} \xrightarrow{\sigma} C$  gives an irreducible pencil  $\tilde{\varphi} : V : \cdots \rightarrow \tilde{C}$ . An irreducible pencil  $\varphi : V : \cdots \rightarrow C$  is called a *fibration* if  $\varphi$  is a morphism.

Let  $\delta$  be a  $K$ -derivation of  $R$ . We say that  $\delta$  is *composed* of a pencil  $\varphi : V : \cdots \rightarrow C$  if  $\delta(I_F) \subseteq I_F$  for every general fiber  $F$  of  $\varphi$ , where  $I_F$  signifies the defining ideal of  $F$ . We also say that  $\delta$  is *of fibered type* if  $\delta$  is composed of some pencil.

LEMMA 8 *Let the notations be the same as above. Assume that  $R^* = K^*$ , where  $R^*$  is the group of invertible elements of  $R$  and that  $\delta$  is composed of an irreducible pencil  $\varphi : V : \cdots \rightarrow C$ . Then the following assertions hold:*

- (1)  *$C$  is either  $\mathbf{A}^1$  or  $\mathbf{P}^1$ .*
- (2) *In case  $C = \mathbf{A}^1$ , there exists an invertible element  $f$  of  $R$  with  $\delta(f) = 0$  such that  $C = \text{Spec} : K[f]$  and  $\varphi$  is a morphism associated with the inclusion  $K[f] \hookrightarrow R$ . We then have  $A_0 \supset K$ .*
- (3) *In case  $C = \mathbf{P}^1$ , there exists two irreducible elements  $f, g$  of  $R$  such that  $\chi(f) = \chi(g), \gcd(f, g) = 1$  and  $\varphi : V : \cdots \rightarrow \mathbf{P}^1$  is the natural extension of a fibration  $D(g) \rightarrow \text{Spec} : K[f/g] = \mathbf{A}^1$ , where  $\mathbf{P}^1 = \text{Proj} : K[f, g]$ . We then have  $L_0 \supset A_0 = K$ .*

LEMMA 9. *Assume that  $R^* = K^*$ . Let  $f$  be an irreducible element of  $R$  and let  $\varphi : V \rightarrow \mathbf{A}^1$  be a fibration defined by the inclusion  $K[f] \hookrightarrow R$ . Let  $\delta$  be a  $K$ -derivation of  $R$  such that  $L_0 \supseteq K(f)$  and  $\delta$  has no divisorial parts (cf. Lemma 7). Then the following*

assertions hold:

- (1)  $\delta$  is composed of the fibration  $\varphi$ .
- (2) The following conditions are equivalent to each other:
  - (i)  $\text{tr.deg: } {}_K Q(A) = 1$ .
  - (ii)  $A = A_0 = K[f]$ .
  - (iii)  $X_\delta = (0)$ .
  - (iv)  $X_\delta$  is positive.
  - (v) Every fiber of  $\varphi$  is irreducible.

(3) Assume that  $R$  is a polynomial ring  $K[x, y]$ . Then  $\delta = d^{-1} \left( \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right)$ , where  $d = \text{gcd} \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ . Furthermore, if  $d = 1$ , the following conditions are equivalent to each other:

- (i)  $\delta$  has no zeroes, i.e.,  $Z(\delta) = \emptyset$ .
- (ii) Every fiber of  $\varphi$  is smooth.

If  $X_\delta = (0)$  then  $d = 1$ .

LEMMA 10. Assume that  $R^* = K^*$ . Let  $f, g$  be two (distinct) irreducible elements of  $R$  such that  $1 \notin Kf + Kg$  and  $fR + gR = R$ , and let  $\varphi : V : \cdots \rightarrow \mathbf{P}^1$  be a rational mapping defined by  $P \mapsto (f(P) : g(P))$ . Let  $\delta$  be a non-trivial  $K$ -derivation of  $R$  such that  $\delta$  has no divisorial parts and  $L_0 \supset K(f/g)$  (cf. Lemma 7). Then the following assertions hold:

- (1)  $\delta$  is composed of the pencil  $\varphi$ .
- (2)  $L_0 = K(f/g)$  and  $A_0 = K$ .
- (3) In the case where  $R$  is a polynomial ring  $K[x, y]$ ,  $\delta$  is determined, up to a constant multiple, as

$$\delta = d^{-1} \left( g \left( \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y} \right) - f \left( \frac{\partial g}{\partial y} \frac{\partial}{\partial x} - \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right) \right),$$

where

$$d = \text{gcd} \left( g \frac{\partial f}{\partial y} - f \frac{\partial g}{\partial y}, g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x} \right).$$

Furthermore, if  $d = 1$  and  $J(f, g) \in K^*$  then  $Z(\delta) = \emptyset$ , where  $J(f, g)$  is the Jacobian of  $f, g$  with respect to  $x, y$ .

We say that a  $K$ -derivation  $\delta$  of  $R$  is locally nilpotent along an irreducible pencil fibration  $\varphi : V : \cdots \rightarrow C$  if  $\delta$  is composed of  $\varphi$  and if the restriction  $\delta_F$  of  $\delta$  onto  $F$  is a locally nilpotent  $K$ -derivation of  $R/I_F$ , where  $F$  is a general fiber of  $\varphi$  and  $I_F$  is the defining ideal of  $F$ .

**THEOREM 11.** *Let  $R$  be a factorial domain of dimension two which is finitely generated over  $K$  and let  $\delta$  be a non-trivial  $K$ -derivation. Assume that  $R^* = K^*$ . Then the following conditions are equivalent to each other:*

- (1)  $R = K[x, y]$ , a polynomial ring, and  $\delta(y) \in K[x]$ .
- (2)  $\delta$  is locally nilpotent.
- (3) There exists a fibration  $\varphi : V \rightarrow C$  such that  $\delta$  is locally nilpotent along  $\varphi$ .

**THEOREM 12.** *Let  $R = K[x, y]$  be a polynomial ring, let  $f$  be an irreducible polynomial in  $R$  and let  $\delta = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$ . Assume that  $\delta$  has no divisorial parts. Then  $\delta$  is locally nilpotent if and only if  $X_\delta = (0)$  and there exists a curve  $C$  which is defined by  $g = 0$  with a  $\delta$ -integral element  $g$  of  $R$  and isomorphic to  $\mathbf{A}^1$ .*

The following result gives a partial answer to the question on finite generation of  $\tilde{X}_\delta$ .

**THEOREM 13.** *Let  $R$  be a factorial domain of dimension two which is finitely generated over  $K$  and let  $\delta$  be a non-trivial  $K$ -derivation of fibered type on  $R$ . Assume that  $R^* = K^*$ . Then  $\tilde{X}_\delta (= X_\delta - X_\delta)$  is finitely generated.*

If  $A_0 \supset K$  or  $L_0 \supset K$ , a derivation  $\delta$  of fibered type. We say that a  $K$ -derivation  $\delta$  is of general type if  $A_0 = L_0 = K$ . Then our second question is the following:

**QUESTION 2.** *Classify all  $K$ -derivations of general type on  $R = K[x, y]$  and*

describe them in terms of the  $\delta$ -integral ring  $A$ .

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**Peter Russell**

**QUESTION 1.** Let  $A$  be a purely inseparable form of  $k^{[n]}$ , the polynomial ring in  $n$  variables over the field  $k$ , i.e.  $A$  is isomorphic to  $K^{[n]}$  over the perfect closure  $K$  of  $k$ . Is  $A$  trivial if  $A$  is  $k$ -rational, i.e. if the quotient field  $L$  of  $A$  is isomorphic to  $k^{(n)}$ , or, more strongly, if  $A$  is birationally contained in  $k^{[n]}$  ?

Not much is known about purely inseparable forms of  $k^{[n]}$  except under strong homogeneity assumptions, and even for  $n = 1$  our information is quite incomplete [KMT] [KM] [R1]. Our question, easily answered for  $n = 1$ , arose in efforts to extend the results of [BR] on the behaviour of birational subrings of  $k^{[2]}$  under basefield extension to the purely inseparable case.

**QUESTION 2.** Let  $(A, M)$  be a regular local ring and  $B = A[a/b]$  with  $a$  in  $M \setminus M^2$  and  $b$  in  $M^2$  and  $GCD(a, b) = 1$ . (These conditions are there to make  $B$  regular.) Under what conditions is  $MB$  a complete intersection in  $B$  ? (An obvious sufficient condition is that  $a$  can be extended to a regular system of parameters  $a, u, \dots, v$  with  $b$  in  $(u, \dots, v)A$ . If  $\dim(A) = 2$ , this condition is also necessary.)

Let  $A = \mathbf{C}[x, y, z, t]$  with  $x + x^2y + z^2 + t^3 = 0$ . The problem came out of efforts to decide whether the ideal  $(x, z, t)$  is a complete intersection in  $A$ . This is of interest since it is known that  $\text{Spec}(A)$  is a smooth contractible threefold [KR] not isomorphic to  $\mathbf{C}^{[3]}$  [ML].

**QUESTION 3.** If the polynomial  $p$  in two variables over  $\mathbf{C}$  has smooth, irreducible rational zero set, is  $p$  a factor of a field generator (or generically rational polynomial), i.e. does there exist a polynomial  $q$  such that the generic fibre of  $pq$  is rational ?

To "effectively" describe all polynomials  $p$  in two variables with smooth rational zero set is certainly a formidable task. There is a bit more hope for those  $p$  that give a fibration of the plane with rational generic fibre. (Such  $p$  have been widely studied [MS] [S] [R2] [R3]. See also the problem proposed by Sh. Kaliman and the references given there.) It is known that every irreducible factor  $q$  of  $p - c$ ,  $c \in \mathbf{C}$ , has smooth rational zero set, but it is not hard to find examples where then the zeros of  $q - c$  are non rational for general  $c \in \mathbf{C}$ . We propose to investigate whether some converse might be true.

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**Fumio Sakai**

**PROBLEM 1.** Classify all rational and elliptic cuspidal plane curves.

**PROBLEM 2.** Find the maximal number of cusps among all the rational cuspidal plane curves.

**David Wright**

**QUESTION.** Can one prove there does not exist a counterexample  $(f, g)$  to the Jacobian Conjecture such that

$$f, g \in \mathbf{C}[Y, XY, X^2Y, X^3Y - X]$$

Note: A positive answer to this question would be a step toward proving the 2-dimensional Conjecture in the case of smooth integral closure.

**Mikhail Zaidenberg**

## §1. ACYCLIC SURFACES

**1.1.** Let  $X$  be a smooth contractible complex affine algebraic surface. If  $\bar{\kappa}(X) = 1$ , where  $\bar{\kappa}$  denotes the logarithmic Kodaira dimension, then there is only one simply connected curve  $l_X$  in  $X$ , and this curve is isomorphic to  $\mathbf{C}$  ( see [GuMi, GuPa, MiTs, Za

1(Addendum)). Let  $l_X = \{p = 0\}$ , where  $p$  is an irreducible regular function on  $X$ . Then all the other fibres  $F_c = p^{-1}(c)$ ,  $c \neq 0$ , of  $p$  are pairwise isomorphic smooth once-punctured curves. The image of any morphism from a once-punctured curve  $\Gamma$  into  $X$  is known to be contained in a fibre of  $p$  (Sh. Kaliman, L. Makar–Limanov [KML 1]). Thus,  $X$  contains, up to an isomorphism, only two such curves, namely  $l_X = F_0$  and  $F_X := F_1$ .

Consider now a smooth contractible surface  $X$  of log-general type. It is known that there is no simply-connected curve in  $X$  (M. Zaidenberg [Za 1]; see also [GuMi, MiTs]).

**QUESTION.** Does  $X$  contain any once-punctured curve? Does it contain only a finite number of such curves?

Of course, the same question has sense for  $\mathbf{Q}$ -acyclic surfaces of log-general type, as well as for embeddings of  $\mathbf{C}^*$  into  $X$  (the latter question has been proposed by M. Miyanishi; oral communication).

**1.2.** A smooth contractible algebraic surface  $X$  is known to be affine (T. Fujita [Fu]). Consider a proper embedding  $X \hookrightarrow \mathbf{C}^N$  and the plurisubharmonic function  $\varphi(x) = \|x\|^2$  on  $X$ . Put  $X_R = \{\varphi < R^2\}$ . Then for all sufficiently large  $R$ ,  $X_R$  is a strictly pseudoconvex domain in  $X$  with a real-algebraic boundary. Furthermore,  $X_R$  is diffeomorphic to  $X$  and hence contractible. Fix such an  $R$ . Note that  $X_R$  is homeomorphic to a bounded domain in  $\mathbf{R}^4$ .

**QUESTION.** Assume that  $X$  as above is not isomorphic to  $\mathbf{C}^2$ . Is it true that then  $X_R$  is not biholomorphic to a bounded (strictly) pseudoconvex domain in  $\mathbf{C}^2$ ?

The boundary  $S_R = \partial X_R$  is a homology sphere with a non-trivial perfect fundamental group (C. P. Ramanujam [Ra]). So, more generally we may ask

**QUESTION.** Let a homology sphere  $S$  be a boundary of a strictly pseudoconvex domain in  $\mathbf{C}^2$ . Is it true that then  $S$  is diffeomorphic to  $S^3$ ?

**1.3. RIGIDITY CONJECTURE** (H. Flenner-M. Zaidenberg [FlZa]). Any smooth  $\mathbf{Q}$ -acyclic surface  $X$  of log-general type is rigid.

This conjecture has been verified in a number of examples [FlZa].

Consider a minimal SNC-completion  $(V, D)$  of  $X$ , i.e.  $X = V - D$ , where  $V$  is a smooth projective surface and  $D = \sum_i D_i$  is a simple normal crossings divisor in  $V$ , which is minimal in this class. The Euler characteristic of the logarithmic tangent bundle of  $(V, D)$  can be expressed as follows:

$$\chi(\Theta_V\langle D \rangle) = K_V(K_V + D) = 10 - 3r - \sum_i D_i^2,$$

where  $K_V$  is the canonical divisor of  $V$  [FlZa]. The Rigidity Conjecture induces the following one [FlZa]:

**1.4. CONJECTURE.** For  $X, V, D$  as above  $K_V(K_V + D) = 0$ .

Since  $D$  is a rational tree, this is equivalent to the equality for the logarithmic Chern number  $\bar{c}_1^2(X) := (K_V + D)^2 = -2$ . Together with the evident equality for the topological Euler characteristic  $\bar{c}_2(X) = e(X) = 1$  this would imply that  $3\bar{c}_2(X) - \bar{c}_1^2(X) = 5$ , which was conjectured by T. tom Dieck [tD].

**1.5.** Let things be as above. Denote by  $\Gamma_D$  the (unweighted) dual graph of  $D$  (which is a tree), and by  $\check{\Gamma}_D$  the (unweighted) Eisenbud–Neumann diagram of  $D$ , i.e. a tree which is obtained from  $\Gamma_D$  by replacing every linear branch between two neighbouring branching points of  $\Gamma_D$  by a single line, and every extremal linear branch of  $\Gamma_D$  by an arrowhead.

**1.6. QUESTION.** Is the number of all Eisenbud–Neumann diagrams  $\check{\Gamma}_D$  of the boundaries  $D$  of the minimal smooth SNC-completions of the contractible (resp. acyclic, resp.  $\mathbf{Q}$ -acyclic) surfaces of log-general type finite ?

**1.7.** It is easily seen that for any non-constant polynomial  $p \in \mathbf{C}[x, y]$  there exists



another one  $q \in \mathbf{C}[x, y]$  such that the regular mapping  $F = (p, q) : \mathbf{C}^2 \rightarrow \mathbf{C}^2$  is proper. On the other hand, if  $X$  is a contractible surface of log–Kodaira dimension 1 and if  $l_X = p^*(0)$ ,  $p \in \mathbf{C}[X]$ , is the unique curve in  $X$  isomorphic to  $\mathbf{C}$ , then there is no  $q \in \mathbf{C}[X]$  such that  $F = (p, q) : X \rightarrow \mathbf{C}^2$  would be a proper mapping [Za 1, Addendum].

**QUESTION.** Let  $X$  be a smooth acyclic surface such that for any non–constant regular function  $p \in \mathbf{C}[X]$  there exists another one  $q \in \mathbf{C}[X]$  with the property that  $F = (p, q) : X \rightarrow \mathbf{C}^2$  is a proper morphism. Is it true that  $X$  is isomorphic to  $\mathbf{C}^2$  ?

## §2. EXOTIC $\mathbf{C}^n$ 's

**2.1. QUESTION.** Let  $n > 3$ . Does there exist any smooth contractible hypersurface of  $\mathbf{C}^n$  of log-general type ?

**2.2. QUESTION.** Let  $p \in \mathbf{C}[x_1, \dots, x_n]$  be such that the hypersurface  $X = \{p = 0\}$  in  $\mathbf{C}^n$  is contractible. Is then  $p$  an isotrivial polynomial, i.e. such that its generic fibres are pairwise isomorphic hypersurfaces in  $\mathbf{C}^n$  ?

See the problems of Sh. Kaliman above and the references therein on the description of the isotrivial polynomials on  $\mathbf{C}^2$ . By the way, is it possible to describe those polynomials on  $\mathbf{C}^2$  which are *projectively isotrivial* in the sense that their generic fibres all have isomorphic smooth projective models? All generically rational polynomials (i.e. *field generators*, cf. Question 3 of P. Russell) are in this class.

**2.3. QUASIHOMOGENEITY CONJECTURE.** If  $X$  as above has an isolated singular point, then the polynomial  $p$  is equivalent, up to an automorphism of  $\mathbf{C}^3$ , to a quasihomogeneous one.

Note that both the question and the conjecture above are confirmed for  $n = 2$ . Indeed, by the Lin–Zaidenberg Theorem [LiZa] any contractible curve in  $\mathbf{C}^2$  is equivalent to a

quasihomogeneous one. But we do not know whether or not an analogous statement holds in positive characteristic. So, we propose the following

**PROBLEM.** *Given a field  $k$  of positive characteristic, classify all possible regular injections of the affine line  $\mathbf{A}_k^1$  into the affine plane  $\mathbf{A}_k^2$ .*

Even for the embeddings this problem is wide open in  $\text{char } k > 0$ ; see [D] and references therein (this remark is due to P. Russell). Of course, in characteristic 0 this is the Epimorphism Theorem of Abhyankar and Moh and Suzuki.

**2.4.** By an *exotic  $\mathbf{C}^n$*  we mean a smooth affine algebraic variety diffeomorphic but non-isomorphic to  $\mathbf{C}^n$  (see [Za 2, 3]).

**QUESTION.** Does there exist a pair of exotic  $\mathbf{C}^n$ 's which are biholomorphic but not isomorphic? Does there exist a non-trivial deformation family of exotic  $\mathbf{C}^n$ 's with the same underlying analytic structure?

**2.5. CONJECTURE.** Let  $X$  be a smooth algebraic variety diffeomorphic to  $\mathbf{C}^n$ . If  $X$  is biholomorphic to  $\mathbf{C}^n$ , then it is isomorphic to  $\mathbf{C}^n$ .

Note that by Ramanujam's Theorem [Ra] this is true for  $n = 2$ .

**2.6. PROBLEM.** Verify that the hypersurface  $X = \{x + x^2y + z^3 + t^2 = 0\} \subset \mathbf{C}^4$ , which is known to be an exotic  $\mathbf{C}^3$  (L. Makar-Limanov [ML; KML 2]), is not biholomorphic to  $\mathbf{C}^3$ . The same for other known hypersurfaces in  $\mathbf{C}^n$ , which are exotic  $\mathbf{C}^{n-1}$ 's (see [Za 3]).

The next general problem is going back to F. Hirzebruch and A. Van de Ven [VdV]; the only known positive result is the Gurjar-Shastri Rationality Theorem for acyclic surfaces (or, what is the same, for complex homology planes) [GuSh].

**2.7. PROBLEM.** Does there exist a non-rational exotic  $\mathbf{C}^n$  ?

**2.8. QUESTION.** Let  $X$  be an exotic  $\mathbf{C}^n$ . Is it true that the action of the automorphism group  $\text{Aut } X$  on  $X$  is not transitive ? The same question for the group of analytic automorphisms.

### §3. ELLIPTICALLY CONNECTED KÄHLER MANIFOLDS

**3.1.** Note that an irreducible smooth quasiprojective curve  $C$  is non-hyperbolic iff the group  $\pi_1(C)$  is abelian. Let  $X$  be a complex manifold. We say that  $X$  is *elliptically connected* if any two points  $x, y \in X$  can be connected by a finite chain of holomorphic images of non-hyperbolic curves and of complex tori.

**CONJECTURE.** If  $X$  is an elliptically connected complete Kähler manifold, then the group  $\pi_1(X)$  is almost abelian (or, in a weaker form, almost nilpotent).

Here ‘*almost abelian*’ resp. ‘*almost nilpotent*’ means that  $\pi_1(X)$  has an abelian resp. nilpotent subgroup of a finite index.

Note that a *rationally connected* (i.e. connected by means of chains of rational curves) compact Kähler manifold is simply connected (F. Campana [Ca]). Note also that for quasiprojective surfaces the above conjecture (in its stronger form) has been verified by using the classification (L. Haddak–K. Oguiso–M. Zaidenberg). In higher dimensions, it is unknown even whether or not  $\pi_1(X)$  is almost solvable.

On the other hand, it seems reasonable to enlarge the notion of an elliptically connected manifold by admitting as members of chains the meromorphic images of tori, or even of quasi-compact complex manifolds with abelian fundamental groups.

And the last remark: the condition of kählerness is essential. As examples one can consider quotients of  $SL(2; \mathbf{C})$  by discrete cocompact subgroups (J. Winkelmann; oral

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