

On exotic algebraic structures on affine spaces

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We review the recent development on the subject, emphasizing its analytic aspects and pointing out some open problems. The study of exotic \mathbf{C}^n -s is at the very beginning, and hopefully this survey would be useful to learn more about these unusual and beautiful objects.

All algebraic varieties considered below are usually assumed being smooth, reduced, irreducible, and defined over \mathbf{C} . *Isomorphism* means biregular isomorphism of algebraic varieties and is denoted by \simeq .

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1. Product structures

1.1. Definition. Let X be a smooth affine algebraic variety. It is called *an exotic*

\mathbf{C}^n if X is diffeomorphic to \mathbf{R}^{2n} but not isomorphic to \mathbf{C}^n .

The following characterization of \mathbf{C}^2 , due to C. P. Ramanujam [Ram] shows that there is no exotic \mathbf{C}^2 .

1.2. Ramanujam's Theorem. *A smooth affine algebraic surface X is isomorphic to \mathbf{C}^2 iff the groups $H_2(X; \mathbf{Z})$, $\pi_1(X)$, and $\pi_1^\infty(X)$ all are trivial. In particular, if X is homeomorphic to \mathbf{C}^2 , then X is isomorphic to \mathbf{C}^2 .*

Here $\pi_1^\infty(X)$ denotes the fundamental group at infinity of X . If X is the interior of a compact real manifold with the boundary ∂X , then $\pi_1^\infty(X) = \pi_1(\partial X)$.

To recognize an exotic \mathbf{C}^n one has to verify two properties from Definition 1.1. As for the first one, the next criterion could be useful.

1.3. Ramanujam–Dimca's Theorem ([Ram; Di], see also [Ka 1, Lemma 1; tD 1, Theorem 3.14]). *Let X be a smooth affine algebraic variety of complex dimension $n \geq 3$. Then X is diffeomorphic to \mathbf{R}^{2n} iff X is contractible, or, equivalently, iff $\pi_1(X) = \mathbf{1}$ and $H_i(X; \mathbf{Z}) = 0$ for $i = 1, \dots, n$.*

The proof is based on the h-cobordism theorem. The main point is to show that being contractible, X possesses a smooth simply connected boundary ∂X . The latter follows from the Lefschetz Hyperplane Section Theorem. In the case when X as below is a product of two contractible varieties, instead of Lefschetz-type arguments one can apply the Van Kampen Theorem.

As follows from [Ram] the statement of Theorem 1.3 does not hold for $n = 2$. Indeed, Ramanujam constructed an example of a smooth contractible affine surface S_0 which is not homeomorphic to \mathbf{R}^4 and, moreover, has an infinite group $\pi_1^\infty(S_0)$. See e.g. [GuMi, FlZa, PtD 3] for further information on such surfaces.

1.4. We recall Zariski's Cancellation Problem:

Given an isomorphism $X \times \mathbf{C}^k \simeq \mathbf{C}^{n+k}$, does it follows that $X \simeq \mathbf{C}^n$?

For $n \geq 3$ the problem is still open. For $n = 2$ the positive answer was obtained by the Miyanishi-Sugie and Fujita Theorem [MiSu, Fu 1]. This theorem provides us, in certain cases, with a tool to distinguish exotic \mathbf{C}^n -s. Despite of the fact that it was

proven later on, the first examples of exotic \mathbf{C}^n -s for any $n \geq 3$ had been alluded to already in [Ram]: take $X^n = S_0 \times \mathbf{C}^{n-2}$, where S_0 is the Ramanujam surface. See [Za 1,2,4] for some further examples of exotic structures of product type.

1.5. Exotic \mathbf{C}^n -s of log-general type. More generally, let $X_i, i = 1, \dots, m$, be smooth contractible affine varieties. Then by Ramanujam–Dimca’s Theorem 1.3 $X := (\prod_{i=1}^m X_i) \times \mathbf{C}^r$ is diffeomorphic to \mathbf{R}^{2n} as soon as $n := \dim_{\mathbf{C}} X$ is at least 3. To distinguish such a product structure X from the standard \mathbf{C}^n , the logarithmic Kodaira dimension \bar{k} could be available (see [Li 1]). But, if $r > 0$, then $\bar{k}(X) = \bar{k}(\mathbf{C}^n) = -\infty$. So, assume for the moment that $r = 0$. Then $\bar{k}(X) = \sum_{i=1}^m \bar{k}(X_i) \geq 0$ as far as $\bar{k}(X_i) \geq 0$ for each $i = 1, \dots, m$. Therefore, in this case X is an exotic \mathbf{C}^n .

For instance, put $X = (S_0)^m, m > 1$, where S_0 is the Ramanujam surface. Then X is an exotic \mathbf{C}^n of log-general type, i.e. $\bar{k}(X) = n$, where $n = 2m = \dim_{\mathbf{C}} X$ (indeed, by [Li 2] we have $\bar{k}(S_0) = 2$). Using other contractible surfaces of log-general type as factors, one may construct more such examples¹, and up to now these are the only known ones. Note that all of them are of even dimensions.

1.6. Remark. Let S be a contractible, or at least acyclic smooth surface of non-negative Kodaira dimension. Then $\bar{k}(S) = 1$ or 2 by the Fujita classification of open surfaces with $\bar{k} = 0$ [Fu 2]². Thus, for the product structure $X = \prod_{i=1}^m S_i$ obtained by means of surfaces S_i as above its log-Kodaira dimension takes the values in the interval $m \leq \bar{k}(X) \leq 2m = \dim_{\mathbf{C}} X$. Later on we will see that there exist exotic \mathbf{C}^n -s X with $1 \leq \bar{k}(X) \leq [n/2]$.

1.7. Examples. A source of examples provide contractible affine surfaces of log-Kodaira dimension one. The complete list of them was obtained by R. V. Gurjar and M. Miyanishi [GuMi]. T. tom Dieck and T. Petrie [PtD 1] realized some of them as hypersurfaces in \mathbf{C}^3 . Namely, put $p_{k,l}(x, y, z) = ((xz+1)^k - (yz+1)^l)/z \in \mathbf{C}[x, y, z]$, where $(k, l) = 1, k, l \geq 2$. Then all the fibres of the polynomial $p_{k,l}$ are contractible surfaces in \mathbf{C}^3 . All its non-zero fibres are smooth surfaces of log-Kodaira dimension one, isomorphic to $S_{k,l} = p_{k,l}^{-1}(1)$ (the zero fibre has non-isolated singularities).

¹this remark is due to T. tom Dieck [tD 1]

²by the Miyanishi-Sugie and Fujita Theorem [MiSu, Fu 1] (see also [Ru 2]) \mathbf{C}^2 is the only contractible, or even acyclic, surface of $\bar{k} = -\infty$.

Due to Theorem 1.8 below, the product $S_{k,l} \times \mathbf{C}^{n-2}$ is a hypersurface in \mathbf{C}^{n+1} which is an exotic \mathbf{C}^n . Moreover, all the fibres of $p_{k,l}$ regarded as polynomial on \mathbf{C}^{n+1} are contractible hypersurfaces, and all but the zero one are exotic \mathbf{C}^n -s.

Recently Sh. Kaliman and L. Makar–Limanov [KaML 3] have shown that all the log-Kodaira dimension one contractible surfaces from the Gurjar–Miyayoshi list admit embeddings into \mathbf{C}^3 , thus providing deformation families of contractible hypersurfaces (see (1.9) below).

Proposition (Sh. Kaliman, L. Makar–Limanov). *Any smooth contractible surface S of log-Kodaira dimension one can be given as a surface in \mathbf{C}^3 with the equation $p_{k,l,m,f}(x, y, z) = 0$, where*

$$p_{k,l,m,f}(x, y, z) = [(z^m x + f(z))^k - (z^m y + g(z))^l - z]/z^m ,$$

$(k, l) = 1$, $k, l \geq 2$, $m \geq 1$, $f, g \in \mathbf{C}[z]$, $\deg f, \deg g < m$, $f(0) = g(0) = 1$ and where g is uniquely defined by f (which can be taken arbitrary) in view of the assumption that $p_{k,l,m,f}$ is a polynomial.

1.8. Besides the log-Kodaira dimension, one may equally use other invariants to distinguish exotic product-structures, for instance the logarithmic plurigenera $\bar{P}_{m_1, \dots, m_l}$ [Li 1,2]. Indeed, if $X := (\prod_{i=1}^m X_i)$ and at least one of the factors X_i has a non-zero log-plurigenus, then by Künneth's Formula the corresponding log-plurigenus of X does not vanish. Moreover, we have the following theorem.

Iitaka–Fujita's Strong Cancellation Theorem [IiFu]. *Let X, Y, A_1, A_2 be algebraic varieties such that $\dim_{\mathbf{C}} X = \dim_{\mathbf{C}} Y$, $\bar{k}(X) \geq 0$, and the log-plurigenera $\bar{P}_{m_1, \dots, m_l}(A_i)$ all vanish, $i = 1, 2$. Then, given an isomorphism $\Phi : X \times A_1 \rightarrow Y \times A_2$, there exists an isomorphism $\varphi : X \rightarrow Y$ making the following diagram commutative:*

$$\begin{array}{ccc} X \times A_1 & \xrightarrow{\Phi} & Y \times A_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & Y \end{array} \quad (1)$$

(the vertical arrows are the canonical projections).

Applying this theorem to the case when $A_1 = A_2 = \mathbf{C}^r$ we see that, as soon as $\bar{k}(X) \geq 0$, the factor \mathbf{C}^r cancels. Therefore, if X and Y are contractible varieties such that $n := \dim_{\mathbf{C}} X + r \geq 3$ and $0 \leq \bar{k}(X) \neq \bar{k}(Y)$, then $X \times \mathbf{C}^r$ is an exotic \mathbf{C}^n non-isomorphic to $Y \times \mathbf{C}^r$. This allows us to distinguish some exotic product structures of negative log-Kodaira dimension.

1.9. Deformable exotic product structures

Theorem (Flenner–Zaidenberg) [FlZa]. *For any $n \geq 3$ and $m \geq 1$ there exists a family of versal deformations $\tilde{f} : \tilde{X}^{n+m} \rightarrow B^m$ of exotic product structures on \mathbf{C}^n over a smooth quasiprojective base B^m of dimension m .*

Here \tilde{f} is a smooth morphism such that for any $b \in B^m$ the fibre $X_b = \tilde{f}^{-1}(b)$ is an exotic product structure on \mathbf{C}^n and $X_b \not\cong X_{b'}$ if $b \neq b'$ are generic points of B^m .

The construction uses families of versal deformations of contractible surfaces of log-Kodaira dimension one. If $f : \tilde{S}^{2+m} \rightarrow B^m$ is such a family, then by the Iitaka–Fujita Theorem 1.8 we can take $\tilde{X}^{n+m} = \tilde{S}^{2+m} \times \mathbf{C}^{n-2}$. Due to [KaML 3] (see 1.7) this leads to families of exotic hypersurfaces in \mathbf{C}^{n+1} .

In the next section we will give another construction of deformable exotic \mathbf{C}^n -s due to Sh. Kaliman. But still we do not know the answer to the following

Question. *Does there exist a deformable exotic \mathbf{C}^n of log-general type?*³

1.10. Analytically exotic \mathbf{C}^n -s. We say that an exotic \mathbf{C}^n is *analytically exotic* if it is not even biholomorphic to \mathbf{C}^n . The following result shows that among the product structures constructed above there are analytically exotic ones.

Strong Analytic Cancellation Theorem [Za 2]. *Let X, Y be smooth quasiprojective varieties of log-general type. If $\Phi : X \times \mathbf{C}^r \rightarrow Y \times \mathbf{C}^m$ is a biholomorphism, then $m = r$ and there exists an isomorphism $\varphi : X \rightarrow Y$ which makes diagram (1) commutative, where $A_i, i = 1, 2$ are replaced by \mathbf{C}^r .*

Thus, if S is a contractible surface of log-general type and $n \geq 3$, then $X = S \times \mathbf{C}^{n-2}$ is an analytically exotic \mathbf{C}^n . The same is true for the product structures $X = S \times M^{n-2}$,

³cf. the Rigidity Conjecture for acyclic surfaces of log-general type in [FlZa].

where S is as above and M^{n-2} is any contractible affine variety of dimension $n - 2$. Indeed, for such an X its Eisenman–Kobayashi intrinsic 2–measure form $E_X^{(2)}$ does not vanish identically (this useful remark is due to Sh. Kaliman [Ka 2]). More generally, one can consider the maximal value of k for which $E_X^{(k)}$ does not vanish identically. This yields a coarse analytic invariant which replaces the log–Kodaira dimension as it has been used in 1.8 above, and so it permits to distinguish certain analytically exotic \mathbf{C}^n -s up to a biholomorphism.

Another remarkable property of the above exotic product structures on \mathbf{C}^n is that they contain no copy of \mathbf{C}^{n-1} .

1.11. Theorem [Za 2; Ka 2]. *Let $X = S \times \mathbf{C}^{n-2}$ be an exotic product structure on \mathbf{C}^n , where S is a contractible surface of log-general type. Then there is no regular injection $\mathbf{C}^{n-1} \rightarrow X$; in particular, there is no algebraic hypersurface in X isomorphic to \mathbf{C}^{n-1} . Moreover, there is no proper holomorphic injection $\mathbf{C}^{n-1} \rightarrow X$, and therefore there is no closed analytic hypersurface in X biholomorphic to \mathbf{C}^{n-1} .*

Theorem 1.10 shows that the following is likely to be true.

1.12. Conjecture. *Any exotic \mathbf{C}^n is analytically exotic.*

1.13. Problem. *Does there exist a pair of biholomorphic but not isomorphic exotic \mathbf{C}^n -s? Does there exist a non-trivial deformation family of exotic \mathbf{C}^n -s with the same underlying analytic structure?*

We even do not know whether the deformation families of exotic product–structures constructed in the proof of Theorem 1.9 are versal in the analytic sense. To this point, the knowledge of the collection of entire curves (i.e. holomorphic images of \mathbf{C}) in the contractible surfaces of log–Kodaira dimension one could be useful. What is the set of tangent directions of such curves in the tangent bundle? Does the degeneration locus of the Kobayashi pseudo–distance provide a non-trivial analytic invariant of such surfaces?

2. Kaliman’s modification

2.1. Definition (cf. [Ka 2]). Consider a triple (X, H, C) consisting of an algebraic

variety X , an irreducible hypersurface H in X and a closed subvariety C of H with $\text{codim}_X C \geq 2$. Let $\sigma_C : \tilde{X} \rightarrow X$ be the blow-up of X at the ideal sheaf of C . Let $E \subset \tilde{X}$ be the exceptional divisor of σ_C and $\tilde{H} \subset \tilde{X}$ be the proper transform of H . The Kaliman modification⁴ consists in replacing the triple (X, H, C) by the pair (X', E') , where $X' = \tilde{X} \setminus \tilde{H}$ and $E' = E \setminus \tilde{H}$. We will also say that X' is the Kaliman modification of X along H with center C .

A triple (X, H, C) resp. a pair (X', E') as above will be called a *smooth contractible affine triple* resp. a *smooth contractible affine pair* if all its members are smooth contractible affine varieties.

2.2. Theorem [Ka 2, Theorem 3.5]. *The Kaliman modification of a smooth contractible affine triple is a smooth contractible affine pair.*

The statement of the theorem remains valid under the assumption that the hypersurface H (not necessarily smooth any more) is a topological cell and $C \subset \text{reg } H$, while all other conditions on X and C being preserved (see [Ka 2, Theorem 3.5]). However, we do not know whether the smoothness of C is essential⁵.

2.3. Examples. The Kaliman modification produces new examples of analytically exotic \mathbf{C}^n -s and of their versal deformations. Let (X, H, C) be a smooth contractible affine triple, where X is an exotic \mathbf{C}^n such that certain Eisenman–Kobayashi form $E_X^{(k)}$ is different from zero at the points of an open subset $U \subset X$. Performing the Kaliman modification we arrive again at an exotic \mathbf{C}^n , call it X' , which has a non-zero form $E_{X'}^{(k)}$. Indeed, by Theorems 1.3 and 2.2 X' is diffeomorphic to \mathbf{R}^{2n} . Furthermore, the restriction $\sigma_C|_{X'} : X' \rightarrow X$ is a dominant holomorphic mapping which is a contraction with respect to the Eisenman–Kobayashi forms, i.e. $\sigma_C^* E_X^{(k)} \leq E_{X'}^{(k)}$, and whence $E_{X'}^{(k)} \not\equiv 0$. Thus, X' is an analytically exotic \mathbf{C}^n .

For instance, fix a point s_0 in a smooth contractible affine surface S of log-general type and put $X = S \times \mathbf{C}^{n-2}$, $H = S \times \mathbf{C}^{n-3}$ and $C = \{s_0\} \times \mathbf{C}^k$, where $0 \leq k \leq n-3$. By Sakai's Theorem [Sak] S is measure hyperbolic, i.e. $E_S^{(2)}$ is positive on a subset

⁴in the surface case a similar transform was called a half-point attachment (detachment) in [Fu 2].

⁵See [Ka 2] for an example which shows that the theorem does not work without the condition $C \subset \text{reg } H$, even with a smooth C .

of S whose complement has measure zero. Therefore, $E_X^{(2)} = \text{pr}_S^* E_S^{(2)}$ is different from zero at the points of a massive subset of X (where $\text{pr}_S : X \rightarrow S$ is the natural projection). Performing the Kaliman modification, by Theorem 2.2, we obtain a smooth contractible pair (X', E') , where X' is an exotic \mathbf{C}^n which have a non-zero form $E_{X'}^{(2)}$. It is easily seen that $E' \simeq \mathbf{C}^{n-1}$. Hence, X' is not biholomorphic to any $\tilde{X} = \tilde{S} \times \mathbf{C}^{n-2}$ as above. Indeed, by Theorem 1.11 such an \tilde{X} does not contain any biholomorphic image of \mathbf{C}^{n-1} . And also it is not biholomorphic to any exotic $\tilde{X} = \tilde{S} \times \mathbf{C}^{n-2}$, where $\bar{k}(\tilde{S}) = 1$, because for the latter product the form $E_{\tilde{X}}^{(2)}$ vanishes at a Zariski open subset. ⁶ This proves the following

2.4. Proposition. *For $n = 3$ X' as above is an exotic \mathbf{C}^3 which is not biholomorphic to any exotic product-structure on \mathbf{C}^3 .*

2.5. Deformable analytically exotic \mathbf{C}^n

By iterating the construction used in the preceding example, Sh. Kaliman obtained the following result (cf. Theorem 1.9 above).

Theorem [Ka 2, sect.4]. *For any $n \geq 3$ there exist versal deformation families of analytically exotic \mathbf{C}^n -s with any given number of moduli.*

The proof proceeds as follows. Start with an exotic \mathbf{C}^n , $X = S \times \mathbf{C}^{n-2}$, as above. Fix m distinct points $s_1, \dots, s_m \in S$ and m disjoint affine hyperplanes A_1, \dots, A_m in \mathbf{C}^{n-2} . Put $H_i = S \times A_i$, $C = \{s_i\} \times A_i$, and perform the Kaliman modifications along H_i with centers C_i for $i = 1, \dots, m$. Then we result with a family $X' = X'(s_1, \dots, s_m, A_1, \dots, A_m)$ of analytically exotic \mathbf{C}^n -s endowed each one with m disjoint hypersurfaces E'_1, \dots, E'_m isomorphic to \mathbf{C}^{n-1} . And they are the only biholomorphic images of \mathbf{C}^{n-1} in X' for fixed data [Ka 2, Lemma 4.1]. Now it is not difficult to check that the positions of the points s_1, \dots, s_m in S and of the hyperplanes A_1, \dots, A_m in \mathbf{C}^{n-2} provide the moduli of these exotic structures considered up to a biholomorphism.

⁶we are grateful to Sh. Kaliman for this remark.

EXOTIC AFFINE HYPERSURFACES

In Sections 3–5 below we review some explicit constructions of contractible hypersurfaces in \mathbf{C}^{n+1} . We discuss different approaches to the recognition problem for exotic hypersurfaces.

By the Abhyankar–Moh and Suzuki Theorem [AM, Su] the only smooth irreducible simply connected curves in \mathbf{C}^2 are those obtained from the affine line $x = 0$ by means of polynomial coordinate changes. By the Lin–Zaidenberg Theorem [LiZa] the only simply connected affine plane curves, up to the action of the group of biregular automorphisms $\text{Aut } \mathbf{C}^2$, are the quasihomogeneous ones. In particular, each irreducible singular such curve is equivalent to one and the only one from the sequence $\Gamma_{k,l} = \{x^k - y^l = 0\}$, $(k, l) = 1$, $k > l \geq 2$. Thus, in the case $n = 2$ this classifies completely the contractible hypersurfaces in \mathbf{C}^n .

In contrast, we have already seen in Section 1 above that there are even deformation families of smooth contractible surfaces of log–Kodaira dimension one in \mathbf{C}^3 , and so for any $n \geq 3$ there are deformation families of hypersurfaces in \mathbf{C}^{n+1} which are exotic \mathbf{C}^n -s. They are far from being classified in any sense. In particular, *no exotic hypersurface in \mathbf{C}^{n+1} of log-general type is known.*

3. Hyperbolic modification

In [tD 1] T. tom Dieck introduced a general construction which, under certain conditions, represents a given topological resp. complex manifold Z with possible singularities as an algebraic quotient of an action of a real resp. complex Lie group G on another such manifold X , canonically defined by Z and a given representation of G . This representation should have the unique fixed point, which should be of hyperbolic type, and so the correspondence $Z \mapsto X$ was called *the hyperbolic modification*.

When $Z \subset \mathbf{C}^n$ is an affine algebraic variety and $G = \mathbf{C}^*$, the hyperbolic modification X of Z is an affine algebraic variety in \mathbf{C}^{n+1} endowed with a \mathbf{C}^* -action, and $Z \simeq X//\mathbf{C}^*$. The n -th iterate X_n of the hyperbolic modification of Z is endowed with an action of the n -torus $T_n = (\mathbf{C}^*)^n$, and $Z \simeq X_n//T_n$.

The main advantage of this transform is that it leads, in the case of hypersurfaces, to new amazing examples of exotic families.

Proposition 3.1 [tD 1, KaML 3]. *For any $n \geq 3$ there exist effectively defined polynomials $p_{k,l}^{(n)}$ on \mathbf{C}^{n+1} , where $(k, l) = 1, k > l \geq 2$, such that all the fibres $(p_{k,l}^{(n)})^{-1}(c)$, $c \in \mathbf{C}$, are exotic \mathbf{C}^n -s.*

Since we are interested mainly in the hypersurface case, and it is simpler, we give the precise definition only in this case.

3.2. Definition [tD 1]. Let $p \in \mathbf{C}[x_1, \dots, x_n] \setminus \{0\}$ be a polynomial on \mathbf{C}^n such that $p(\bar{0}) = 0$. The polynomial $\tilde{p}(x, z) = p(xz)/z \in \mathbf{C}[x, z]$ on \mathbf{C}^{n+1} is called *the hyperbolic modification of p* , and its zero fibre $X_0 = \tilde{p}^{-1}(0) \subset \mathbf{C}^{n+1}$ is called *the hyperbolic modification of the zero fibre $Z_0 = p^{-1}(0) \subset \mathbf{C}^n$ of p* .

In the case of the simply connected curves $\Gamma_{k,l} = \{x^k - y^l = 0\}$ the hyperbolic modification was already used in [PtD 1]. The origin having been placed at the smooth point $(1, 1) \in \Gamma_{k,l}$, the hyperbolic modification gives rise to the Petrie–tom Dieck polynomials $p_{k,l} = ((xz + 1)^k - (yz + 1)^l)/z$ (see 1.6; see also [PtD 2] for some related constructions).

3.3. Some properties of the hyperbolic modification. Let $\tilde{p}(x, z)$ be the hyperbolic modification of a polynomial $p(x)$ on \mathbf{C}^n . Consider the \mathbf{C}^* -action $G(\lambda, x, z) = (\lambda x, \lambda^{-1}z)$ on \mathbf{C}^{n+1} . It is easily seen that \tilde{p} is a quasi-invariant of G of weight 1, i.e. $\tilde{p} \circ G_\lambda = \lambda \tilde{p}$, $\lambda \in \mathbf{C}^*$. Denote $X_1 = \tilde{p}^{-1}(1) \subset \mathbf{C}^{n+1}$. Then the restriction $G|(\mathbf{C}^* \times X_1)$ yields an isomorphism $\mathbf{C}^* \times X_1 \simeq \mathbf{C}^{n+1} \setminus X_0$. In particular, X_1 is a smooth hypersurface, and all the fibres $X_c = \tilde{p}^{-1}(c)$, $c \in \mathbf{C} \setminus \{0\}$, are isomorphic to X_1 .

Thus, being applied to a hypersurface Z_0 in \mathbf{C}^n , the hyperbolic modification produces actually a pair of distinct hypersurfaces X_0 and X_1 in \mathbf{C}^{n+1} . The zero fibre X_0 inherits the \mathbf{C}^* -action $G|X_0$. The ring of G -invariants coincides with the subring $\mathbf{C}[zx_1, \dots, zx_n] \subset \mathbf{C}[x, z]$, and $\pi : \mathbf{C}^{n+1} \ni (x, z) \mapsto xz \in \mathbf{C}^n$ is the canonical quotient morphism, as well as the restriction $\pi|X_0 : X_0 \rightarrow Z_0$. Thus, $Z_0 = X_0//\mathbf{C}^*$ is the algebraic quotient. Note that X_0 is smooth iff Z_0 is so.

For the morphism $\pi : X_0 \rightarrow Z_0$ the Kawamata Addition Theorem [Kaw] and the Iitaka inequality [Ii 1] imply that $\bar{k}(Z_0) \leq \bar{k}(X_0) \leq \dim_{\mathbf{C}} Z_0$. The same holds for any iterated hyperbolic modification $X_0^{(n)}$ of Z_0 .

The restriction $\pi|_{X_1} : X_1 \rightarrow \mathbf{C}^n$ is a birational morphism. From Proposition 3.6 in [tD 1] it follows that the generic fibre X_1 is the Kaliman modification of \mathbf{C}^n along Z_0 with center at the origin (see 2.1). Combining several statements from [tD 1, (1.1), (1.3), (2.1), (3.1); Ka 2, (3.5)] we obtain the following ⁷

3.4. Theorem. *Let $Z_0 = p^*(0)$, where $p \in \mathbf{C}[x_1, \dots, x_n]$. Assume that Z_0 is an irreducible reduced divisor on \mathbf{C}^n , which contains the origin and is smooth at the origin. Let X_0 resp. $X_1 \subset \mathbf{C}^{n+1}$ be the zero fibre resp. the generic fibre of the hyperbolic modification \tilde{p} of the polynomial p .*

a) Let Z_0 be smooth. Then both the hyperbolic modification X_0 and the Kaliman modification X_1 of Z_0 are acyclic resp. contractible as soon as Z_0 is acyclic resp. contractible.

b) Let Z_0 be a topological manifold and has at most isolated singularities. Then X_1 is acyclic resp. contractible if Z_0 is acyclic resp. a topological cell.

This leads to the following result [tD 1, Theorem 3.12].

3.5. Theorem (T. tom Dieck). *If $Z_0 = p^{-1}(0)$ is a smooth contractible hypersurface in \mathbf{C}^n , $n \geq 3$, then all the fibres of the hyperbolic modification \tilde{p} of p are smooth hypersurfaces in \mathbf{C}^{n+1} diffeomorphic to \mathbf{R}^{2n} . If, furthermore, $\bar{k}(Z_0) \geq 0$, then the zero fibre X_0 of \tilde{p} is an exotic \mathbf{C}^n of non-negative log-Kodaira dimension.*

Starting with the Petrie–tom Dieck surface $S_{k,l}$ in \mathbf{C}^3 of log-Kodaira dimension one (see 1.7) and iterating the hyperbolic modification, for any given $n \geq 3$ one can effectively find a polynomial $p_{k,l}^{(n)}$ on \mathbf{C}^{n+1} such that all its fibres are smooth and diffeomorphic to \mathbf{R}^{2n} , and the zero fibre is an exotic \mathbf{C}^n . In fact, in this particular case all of them are exotic \mathbf{C}^n -s, as has been recently shown in [KaML 3]. More precisely, it was shown that none of these fibres is dominated by \mathbf{C}^n . This proves Proposition 3.1 above. The Brieskorn–Pham polynomials provide another examples of this type (in this case one has to apply (b) of Theorem 3.4; see [tD 1, Section 4]).

To manage the general case, it would be useful to prove the following conjecture,

⁷Another approach to the proof of (a certain part of) this theorem, based on Kempf–Ness and Neeman results on algebraic group actions (see e.g. [Kr 1, §4] and references therein) was proposed in the lectures of T. Petrie at the Workshop on Open Algebraic Varieties, CRM, Montréal, December 5–9, 1994; see also [Pe].

which seems to be interesting by itself. It is easily checked for $n = 2$.

3.6. Conjecture. *Let X_0 be a special fibre and X_1 be a generic fibre of a primitive polynomial $p \in \mathbf{C}[x_1, \dots, x_n]$. Let X'_0 be the desingularization of an irreducible component of X_0 . Then $\bar{k}(X'_0) \leq \bar{k}(X_1)$. In other words, the log-Kodaira dimension is lower semi-continuous on the fibres of a polynomial in \mathbf{C}^n .*

Note that for $n \geq 5$ the exotic \mathbf{C}^n which is the zero fibre $X_0 = (p_{k,1}^{(n)})^{-1}(0)$ is different from the exotic product structures on \mathbf{C}^n considered in 1.5 above, since here $1 \leq \bar{k}(X_0) \leq 2$.

4. Dimca's and Kaliman's examples of exotic hypersurfaces

As we have seen in the preceding section, the hyperbolic modification \tilde{p} of a polynomial p on \mathbf{C}^n is a quasi-invariant of weight 1 of a linear \mathbf{C}^* -action G on \mathbf{C}^{n+1} of mixed type (the latter means that the \mathbf{C}^* -action G has one fixed point only and the linear part of G in the fixed point has weights of different signs). In the examples considered below the defining polynomials of exotic hypersurfaces in \mathbf{C}^{n+1} will be quasi-invariants of weights > 1 of regular \mathbf{C}^* -actions of mixed type. The generic fibre X_1 of such a polynomial does not need to be contractible any more. Its zero fibre X_0 will be presented as a cyclic branched covering of \mathbf{C}^n ramified along a hypersurface Z_0 with certain properties, which ensure that X_0 is contractible.

4.1. Dimca's list [Di, ChoDi]. A. Libgober [Lib] discovered that the (singular) projective hypersurface

$$\bar{H}_{n,d} = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + \dots + x_{n-1}^{d-1}x_n + x_{n+1}^d = 0\} \subset \mathbf{P}^{n+1}$$

has the same homology groups as \mathbf{P}^n , n odd. In fact, $\bar{H}_{n,d}$ is a completion of \mathbf{C}^n ; namely, $H_{n,d} \simeq \mathbf{C}^n$, where $H_{n,d} = \bar{H}_{n,d} \setminus \{x_0 = 0\}$.

Generalizing this example, G. Barthel and A. Dimca [BaDi] found some others homology \mathbf{P}^n -s with isolated singularities⁸. These are the projective closures in \mathbf{P}^{n+1} of the affine hypersurfaces $H_{n,d,a} \subset \mathbf{C}^{n+1}$ with the equations

$$p_{n,d,a}(x) = x_1 + x_1^{d-1}x_2 + \dots + x_{n-2}^{d-1}x_{n-1} + x_{n-1}^{d-a}x_n^a + x_{n+1}^d = 0$$

⁸and in particular, all homology \mathbf{P}^2 -s which are normal surfaces in \mathbf{P}^3 endowed with a \mathbf{C}^* -action.

where n is odd and $1 \leq a < d$, $(a, d) = (a, d - 1) = 1$.

4.2. Proposition [Di, Propositions 5, 7; ChoDi, Theorem 5, Proposition 6].

- a) $H_{n,d,a} \subset \mathbf{C}^{n+1}$ is diffeomorphic to \mathbf{R}^{2n} .
- b) For $a = 1$ the map $p_{n,d,a} : \mathbf{C}^{n+1} \rightarrow \mathbf{C}$ is a smooth fibre bundle with the fibre diffeomorphic to \mathbf{R}^{2n} .
- c) For $a > 1$ the generic fibre $X_1 = p_{n,d,a}^{-1}(1)$ is not contractible (in fact, its Euler characteristic is different from 1).
- d) $H_{3,d,1} \simeq \mathbf{C}^3$ and the fibration $p_{3,d,1} : \mathbf{C}^4 \rightarrow \mathbf{C}$ is algebraically trivial.⁹

A. Dimca posed the question:

Is it true that for $a > 1$ all the hypersurfaces $H_{3,d,a} = \{x + x^{d-1}y + y^{d-a}z^a + t^d = 0\} \subset \mathbf{C}^4$ as above are exotic \mathbf{C}^3 -s?

The positive answer was done by [KaML 2], see Theorem 5.10 below. It is still unknown whether the same is true in higher dimensions.

The following criterion of contractibility of cyclic coverings, proposed in [Ka 1, Theorem A] (see also [tD 2]), allows one to establish that the hypersurfaces like those in the previous examples and more general ones are contractible.

4.3. Theorem (Kaliman). *Let a polynomial $q \in \mathbf{C}[x_1, \dots, x_n]$ be a quasi-invariant of a positive weight l of a regular \mathbf{C}^* -action G on \mathbf{C}^n . Suppose that the zero fibre $Z_0 = q^*(0)$ is a smooth, reduced, and irreducible divisor in \mathbf{C}^n such that*

- i) $\pi_1(\mathbf{C}^n \setminus Z_0) \approx \mathbf{Z}^{10}$;
- ii) for some prime k coprime with l , $H_i(Z_0; \mathbf{Z}/k\mathbf{Z}) = 0$, $i = 1, \dots, n$, i.e. the fibre Z_0 is $\mathbf{Z}/k\mathbf{Z}$ -acyclic.

Then the zero fibre $X_0 = p^{-1}(0) \subset \mathbf{C}^{n+1}$ of the polynomial $p(x, z) = q(x) + z^k$ is diffeomorphic to \mathbf{R}^{2n} .

Note that the polynomial p is a quasi-invariant of weight kl of the \mathbf{C}^* -action $\tilde{G}(\lambda, x, z) = (G(\lambda^k, x), \lambda^l z)$ on \mathbf{C}^{n+1} . The morphism $\pi : X_0 \ni (x, z) \mapsto x \in \mathbf{C}^n$ represents X_0 as a k -fold branched cyclic covering over \mathbf{C}^n ramified along Z_0 . This covering is equivariant with respect to the actions G on \mathbf{C}^n and $\tilde{G}|_{X_0}$ on X_0 .

⁹Sh. Kaliman [Ka 1] noted that for $d = 3$ it is trivialized by the Nagata automorphism.

¹⁰see footnote 11 below.

The generic fibre $X_1 = p^{-1}(1)$ topologically is the joint $Z_1 * \mathbf{Z}/k\mathbf{Z}$, where $Z_1 = q^{-1}(0) \subset \mathbf{C}^n$ is the generic fibre of q [Ne]. Therefore, X_1 is not contractible as soon as Z_1 is not contractible [Ka 1, Lemma 9].

The assumption (i) is always fulfilled for a generic fibre of a primitive polynomial [Ka 1, Lemma 8]. Though the zero fibre of a \mathbf{C}^* -quasi-invariant is usually non-generic, there exist non-trivial examples where this and all the other conditions of Theorem 4.3 are satisfied. ¹¹

4.4. Proposition [Ka 1, Theorem 10]. *Put*

$$q(x, y, z) = x + x^a y^b + y^c z^d,$$

$l = bd$ and $G(\lambda, x, y, z) = (\lambda^l x, \lambda^{-r} y, \lambda^s z)$, where $r = d(a - 1)$, $s = c(a - 1) + b$. If $(s, d) = 1$, then the polynomial q and the \mathbf{C}^* -action G verify all the assumptions of Theorem 4.3. Furthermore, the Euler characteristic of the generic fibre $Z_1 = q^{-1}(1)$ is different from 1 as soon as $b, d \geq 2$.

Let $p(x, y, z, t) = q(x, y, z) + t^k$, where q is as above and k is a prime such that $(bd, k) = 1$. Then, by Theorem 4.3, $X_0 = X_{a,b,c,d,k} = p^{-1}(0)$ is a smooth contractible hypersurface in \mathbf{C}^4 . Later on we will see that most of these threefolds are exotic \mathbf{C}^3 -s.

The polynomial p being \tilde{G} -quasi-invariant, the threefold X_0 carries a \mathbf{C}^* -action $\tilde{G}|X_0$. In general, a polynomial f on \mathbf{C}^n is a quasi-invariant of a linear diagonalized \mathbf{C}^* -action iff its Newton diagram is linearly degenerate, i.e. if it lies in an affine hyperplane. This is always the case when f consists of n monomials only, like in the preceding examples. Since all the hypersurfaces discussed in this section carry \mathbf{C}^* -actions, none of them is of log-general type. However, as we will see, for some of them the log-Kodaira dimension is non-negative.

The next result provides an estimate from below of the log-Kodaira dimensions of ramified coverings, and so makes it possible, in certain cases, to recognize exotic \mathbf{C}^n -s among hypersurfaces as in Theorem 4.3.

4.5. Proposition [Ka 1, Corollaries 12, 13]. *Let $f : W \rightarrow V$ be a morphism of smooth quasiprojective varieties, which is a branched covering ramified over a divisor*

¹¹Sh. Kaliman has informed me that in fact the condition (i) of Theorem 4.3 is superfluous; see Appendix.

$R \subset V$ of simple normal crossing type. Assume that the Sakai analytic dimension k_c of the complement $V \setminus R$ is non-negative (see [Sa])¹². If the ramification order of f on each of the irreducible components of $f^{-1}(R) \subset W$ is high enough, then $\bar{k}(W) \geq \bar{k}(V \setminus R)$. Consequently, in this case W is of log-general type if $V \setminus R$ is so.

More careful analysis with the same type of arguments [Ka 1, Theorem B] leads to the conclusion that $\bar{k}(X_{a,b,c,d,k}) = 2$ if $k > a \geq 4$, $d = a - 1$, $(b, d) = (bd, k) = 1$ and $(b, c) > d^2k$. For instance, $\bar{k}(X) = 2$ for $X = X_{4,46,92,3,5} = \{x + x^4y^{46} + y^{92}z^3 + t^5 = 0\} \subset \mathbf{C}^4$. See [Ru 1, Ka 3] for some other cases when the log-Kodaira dimension of these threefolds is at least non-negative or more. Note that an exotic threefold of non-negative log-Kodaira dimension cannot be an exotic product structure on \mathbf{C}^3 , which is always of log-Kodaira dimension $-\infty$.

5. Russell's \mathbf{C}^* -threefolds and the Makar-Limanov invariant

By a \mathbf{C}^* -variety we mean a smooth irreducible algebraic variety endowed with a regular effective \mathbf{C}^* -action. Most of the exotic \mathbf{C}^n -s which are known, except some Kaliman modifications, are \mathbf{C}^* -varieties. Thus, we come to the following

5.1. Problem. *Classify contractible \mathbf{C}^* -varieties up to equivariant isomorphism.*

This includes the famous linearization problem for \mathbf{C}^* -actions on the affine space \mathbf{C}^3 , which is still open (see e.g. [KoRu, Kr 2]).

5.2. Koras-Russell bicyclic covering construction. Analysing Dimca's examples 4.2 from the point of view of the previous work with M. Koras [KoRu], P. Russell [Ru 1] came to a remarkable general method of constructing contractible \mathbf{C}^* -threefolds. In particular, it yields all of them of a certain restricted type (namely, *tame of mixed type*; see (5.4) and Theorem 5.5 below), including those of 4.2 and 4.4 above. We discuss here some principal points of this construction.

Denote by ω_r the cyclic group of the complex r -roots of unity. Let B be a smooth contractible algebraic variety and $Z_1, Z_2 \subset B$ be two smooth divisors which meet normally. For a pair (α_1, α_2) of coprime positive integers consider the bicyclic covering over B branched to order α_i over Z_i , $i = 1, 2$. We get a commutative diagram

¹²being non-negative k_c must coincide with the log-Kodaira dimension \bar{k} [Sa]

$$\begin{array}{ccc}
& X & \\
\omega_{\alpha_2} \swarrow & & \searrow \omega_{\alpha_1} \\
X_1 & & X_2 \\
\omega_{\alpha_1} \searrow & & \swarrow \omega_{\alpha_2} \\
& B &
\end{array}
\tag{2}$$

Question. *Under which assumptions on Z_1, Z_2 the resulting variety X is contractible?*

Here X is acyclic if both $\omega_{\alpha_1}, \omega_{\alpha_2}$ act trivially on the homologies $H_*(X; \mathbf{Z})$. Indeed, in this case the homologies would be α_i -torsions, $i = 1, 2$, and hence trivial [Ru 1]. This is so if B is a \mathbf{C}^* -variety and the divisors Z_1, Z_2 are invariant under the \mathbf{C}^* -action and, furthermore, they are given as $Z_i = q_i^{-1}(0)$, $i = 1, 2$, where $q_i \in \mathbf{C}[B]$ is a regular \mathbf{C}^* -quasi-invariant of weight which is relatively prime with α_i , $i = 1, 2$. Indeed, under these assumptions (2) is a diagram of equivariant morphisms of \mathbf{C}^* -varieties, and $\omega_{\alpha_i}, i = 1, 2$, acts on X via the \mathbf{C}^* -action. Therefore, it induces the trivial action in the homologies. We also need, of course, X to be simply connected (cf. Appendix).

From now on we restrict the consideration by \mathbf{C}^* -threefolds, and namely by those of mixed (or hyperbolic) type. We say that a \mathbf{C}^* -threefold X with a \mathbf{C}^* -action G is of *mixed type* if G has the unique fixed point $X^G = \{x_0\}$ and the weights of the diagonalized linear action $dG|_{T_{x_0}X}$ are of different signs. One may assume that

$$dG(\lambda, x, y, z) = (\lambda^{-a}x, \lambda^b y, \lambda^c z), \quad \lambda \in \mathbf{C}^*,$$

where $a, b, c > 0$. Since G is effective, a, b, c are relatively prime: $(a, b, c) = 1$. The triple (a, b, c) is called *reduced* if, moreover, $(a, b) = (b, c) = (a, c) = 1$.

We will say that a mixed \mathbf{C}^* -threefold X is *tame* if the algebraic quotient $X//G$ is isomorphic to $\mathbf{C}^3//dG$, or, what is the same, to one of the surfaces $\mathbf{C}^2//\omega_r$, where ω_r acts diagonally in \mathbf{C}^2 . X is said to be *linearizable* if it is equivariantly isomorphic to \mathbf{C}^3 with a linear \mathbf{C}^* -action. The following statement is unexplicitly contained in [KoRu, Ko].

5.3. Proposition. *If X is a tame contractible affine \mathbf{C}^* -threefold of mixed type with*

a reduced triple of weights at the fixed point, then X is linearizable.

The next example due to P. Russell shows the importance of the assumption on X being tame.

Example. Let S be a contractible surface of non-negative log-Kodaira dimension. Put $Y = S \times \mathbf{C}$, and let X be the Kaliman modification of Y along $S \times \{0\}$ with center at a point $\{s_0\} \times \{0\}$. The tautological \mathbf{C}^* -action on \mathbf{C} lifts to a \mathbf{C}^* -action on Y , which in turn induces a \mathbf{C}^* -action of mixed type on X with the reduced triple of weights $(-1, -1, 1)$. Here $X//\mathbf{C}^* \simeq S \not\simeq \mathbf{C}^2//\omega_r$, and being an exotic \mathbf{C}^3 , X is not linearizable.

Now the idea of [KoRu, Ru 1] can be described as follows. Starting with a tame contractible \mathbf{C}^* -threefold X with a non-reduced triple of weights $(-a, b, c)$, we may factorise it by a cyclic subgroup of \mathbf{C}^* to make the triple of weights being reduced. By Proposition 5.3, we arrive in this way to \mathbf{C}^3 with a linear \mathbf{C}^* -action. Namely, put

$$\alpha = (b, c), \beta = (a, c), \gamma = (a, b) \quad \text{and} \quad a' = a/\beta\gamma, b' = b/\alpha\gamma, c' = c/\alpha\beta.$$

Then both (α, β, γ) and (a', b', c') are reduced triples and $(a', \alpha) = (b', \beta) = (c', \gamma) = 1$. Furthermore, $B = X/\omega_{\alpha\beta\gamma}$ is a tame \mathbf{C}^* -threefold of mixed type with the reduced triple of weights $(-a', b', c')$, which is, due to Proposition 5.3, isomorphic to \mathbf{C}^3 with a linear \mathbf{C}^* -action of the same type.

5.4. Russell's threefolds. P. Russell [Ru 1] realized the converse procedure. Starting now with \mathbf{C}^3 with a linear diagonalized reduced mixed \mathbf{C}^* -action and passing to the corresponding tricyclic coverings, he reconstructed all possible tame contractible affine \mathbf{C}^* -threefolds of mixed type. In what follows we call them *Russell's threefolds*.

The principal point of Russell's construction is the choice of branching divisors $Z_0, Z_1, Z_2 \subset \mathbf{C}^3$ of the tricyclic covering. The first of them Z_0 appears naturally. Indeed, let X be a contractible \mathbf{C}^* -threefold of the mixed type $(-a, b, c)$ with the fixed point $x_0 \in X$. Put

$$X^+ = \{x \in X \mid \lim_{\lambda \rightarrow 0} G_\lambda x = x_0\},$$

$$X^- = \{x \in X \mid \lim_{\lambda \rightarrow 0} G_{\lambda^{-1}} x = x_0\}.$$

Then X^+ , X^- are isomorphic to \mathbf{C}^2 and \mathbf{C} respectively and meet transversally. We call them *the positive plane* resp. *the negative axis*. If $\sigma : X \rightarrow B$, $B = X/\omega_{\alpha\beta\gamma} \simeq \mathbf{C}^3$, is the quotient morphism, then $\sigma(X^\pm) = B^\pm$ are, respectively, the coordinate plane $x = 0$ and the coordinate axis $y = z = 0$, and σ is branched to order α along $Z_0 = B^+$.

The two other divisors Z_1, Z_2 are chosen as follows. In order to get $X_i \simeq \mathbf{C}^3$ in diagram (2) we take $Z_i \simeq \mathbf{C}^2$, $i = 1, 2$. Moreover, after passing to the first covering we would like to have in $X_i \simeq \mathbf{C}^3$ ($i = 1, 2$) the situation described in Theorem 4.3. To this point these three embedded planes $Z_0, Z_1, Z_2 \subset \mathbf{C}^3$ should satisfy the following conditions:

- i) Z_i is equivalent to a coordinate plane under a tame automorphism of \mathbf{C}^3 , $i = 1, 2$;
- ii) $Z_0 \cup Z_1 \cup Z_2$ is a normal crossing divisor;
- iii) Z_i are invariant under the \mathbf{C}^* -action $(\lambda, x, y, z) \mapsto (\lambda^{-a'}x, \lambda^{b'}y, \lambda^{c'}z)$ on \mathbf{C}^3 , $i = 0, 1, 2$;
- iv) locally at the origin the triple Z_0, Z_1, Z_2 determines a quasi-homogeneous coordinate system¹³
- v) the intersection $Z_1 \cap Z_2$ consists of the negative axis B^- and of $r - 1$ closed \mathbf{C}^* -orbits, where $r \geq 1$.

If Z_1, Z_2 can be linearized simultaneously (this corresponds to $r = 1$), then clearly $X \simeq \mathbf{C}^3$. Therefore, in interesting cases $r > 1$.

Put $C_i = Z_i \cap P$, $i = 1, 2$, where $P = \{x = 1\} \subset \mathbf{C}^3$. Note that the surface $Z_i \subset \mathbf{C}^3$ is the closure of the orbit of the curve C_i under the \mathbf{C}^* -action. The affine plane P is invariant with respect to the induced $\omega_{a'}$ -action, and by (iii) the curves $C_1, C_2 \subset P$ should be $\omega_{a'}$ -invariant, too. Furthermore, C_1, C_2 are isomorphic to \mathbf{C} and meet normally at the origin $P \cap B^-$ and in $r - 1$ other points. If C_i are given in the plane P by the equations $p_i(y, z) = 0$, $p_i \in \mathbf{C}[y, z]$, $i = 1, 2$, then the equations of Z_i are $\tilde{p}_i(x, y, z) = 0$, where $\tilde{p}_i \in \mathbf{C}[x, y, z]$, $i = 1, 2$, are defined as follows:

$$\tilde{p}_1(x^{a'}, y, z) = x^{-b'} p_1(x^{b'} y, x^{c'} z) \quad \text{and} \quad \tilde{p}_2(x^{a'}, y, z) = x^{-c'} p_2(x^{b'} y, x^{c'} z) \quad (3)$$

As soon as a pair of plane curves C_1, C_2 as above is chosen in such a way that \tilde{p}_i are polynomials, $i = 1, 2$, the corresponding triple Z_0, Z_1, Z_2 satisfies all the conditions (i)-(v).

¹³this follows, of course, from ii) and iii).

5.5. Theorem. a) [KoRu, Ru 1] Fix two reduced triples (a', b', c') and (α, β, γ) of positive integers such that $(a', \alpha) = (b', \beta) = (c', \gamma) = 1$. Let $C_i = p_i^{-1}(0)$, $Z_i = \tilde{p}_i^{-1}(0)$, $i = 1, 2$, and $Z_0 = \{x = 0\}$ be as above. Let $X \rightarrow \mathbf{C}^3$ be the tricyclic covering ramified to order α over Z_0 , to order β over Z_1 and to order γ over Z_2 ¹⁴. Then X is a Russell threefold. Conversely, any Russell threefold is obtained by the above construction.

b) [ML, KaML 1,2] X as above is an exotic \mathbf{C}^3 except in the cases when

$$(r - 1)(\beta - 1)(\gamma - 1) = 0.$$

5.6. Remark. Putting $p_2(y, z) = z$, which is possible, we may present a Russell threefold X as the hypersurface in \mathbf{C}^4 with the equation

$$\tilde{p}_1(x^\alpha, y, z^\gamma) + t^\beta = 0.$$

5.7. Examples. Put $(-a', b', c') = (-r + 1, 1, 1)$, where $r \geq 2$, and $p_1(y, z) = y + y^r + z$. Then $\tilde{p}_1(x, y, z) = y + xy^r + z$ and

$$X = \{y + x^\alpha y^r + z^\gamma + t^\beta = 0\} \subset \mathbf{C}^4.$$

In the simplest non-trivial case $r = 2, \alpha = 1, \beta = 3, \gamma = 2$ we get the affine cubic $X_0 \subset \mathbf{C}^4$ with the equation

$$y + xy^2 + z^2 + t^3 = 0,$$

which is an exotic \mathbf{C}^3 [ML].

The following result makes precise the statement of Theorem 5.5, b). It was obtained by a rather elementary method of analyzing the defining equations of Russell's threefolds [KaML 1,3],

5.8. Theorem [KaML 1]. Let X be a Russell threefold constructed by the data $(r, \alpha, \beta, \gamma)$, where $r > 1, \alpha \geq 2, \beta, \gamma \geq 4$. Then there is no dominant morphism $\mathbf{C}^3 \rightarrow X$.

¹⁴that is, $X = \text{spec } A$, where A is the extension of $\mathbf{C}[x, y, z]$ by the corresponding roots of x, \tilde{p}_1 and \tilde{p}_2 .

It was previously known [Ru 1] that $\bar{k}(X) = 2$ if $\alpha \geq a'\beta\gamma$ and $\beta, \gamma \gg 1$; furthermore, $\bar{k}(X) \geq 0$ if $a' = 1, \alpha \geq 2, \beta, \gamma \geq 4$. But $\bar{k}(X) = -\infty$ for the Russell's threefolds X as in 5.7 with $\alpha = 1$, since the complement $X \setminus \{y = 0\}$ is isomorphic to $\mathbf{C}^* \times \mathbf{C}^2$. Furthermore, the cubic X_0 from 5.7 is dominated by \mathbf{C}^3 .

5.9. The Makar–Limanov invariant

Definition. Recall that a derivation ∂ of a ring A is called *locally nilpotent* if each element $a \in A$ is vanished by an appropriate positive power $n = n(a)$ of ∂ , i.e. $\partial^n(a) = 0$. Denote $A^\partial = \text{Ker } \partial$; A^∂ is called *the ring of constants of ∂* . Let $LN(A)$ denote the set of all locally nilpotent derivations of A . Put $A_0 = \bigcap_{\partial \in LN(A)} A^\partial$. We call A_0 *the ring of absolute constants of A* . Note that $A_0 = \{\mathbf{C}\}$ if $A = \mathbf{C}[x_1, \dots, x_n]$ is a polynomial ring.

The subring $A_0 \subset A$ of the absolute constants is invariant under ring isomorphisms. It was introduced in [ML], where it was shown that it is non-trivial in the case of the algebra $A = \mathbf{C}[X_0]$ of the regular functions on the Russell cubic X_0 (see 5.7). We call A_0 *the Makar–Limanov invariant of A* .

5.10. Theorem [KaML 2, Theorem 8.3]. *Let $A = \mathbf{C}[X]$, where X is a Russell threefold. Then $A_0 = A$ (i.e. there is no locally nilpotent derivations on A) except in the following two cases:*

- i) if $X = \{x + x^r y + z^\beta + t^\gamma = 0\}$, then $A_0 = \mathbf{C}[x]$;*
- ii) if $X = \{x + (x^r + z^\beta)^l y + t^\gamma = 0\}$, then $A_0 = \mathbf{C}[x, z]$.*

This proves (b) of Theorem 5.5.

6. APPENDIX: Simply connectedness of \mathbf{C}^* -equivariant cyclic coverings

The results of this section are due to Sh. Kaliman¹⁵. The original presentation has been modified by P. Russell by picking out the group theoretic component (see Proposition 6.2). In particular, he used the following

¹⁵Letter to the author from 12.03.1995. We get them placed here with the kind permission of Sh. Kaliman and P. Russell.

6.1. Definition. Let G be a group. We say that a subgroup $H \subset G$ is *normally generated by elements* $a_1, \dots, a_n \in H$ if it is generated by the set of all elements conjugate to a_1, \dots, a_n . Thus, H is the minimal normal subgroup of G that contains a_1, \dots, a_n . We denote it by $\langle\langle a_1, \dots, a_n \rangle\rangle$.

We say that G is *normally one-generated* if $G = \langle\langle a \rangle\rangle$ for some element $a \in G$. If $A, B \subset G$, then $[A, B]$ denotes the subgroup generated by all the commutators $[a, b] = aba^{-1}b^{-1}$, where $a \in A, b \in B$.

6.2. Proposition. Put $K = [G, G]$. Assume that $G = \langle\langle a \rangle\rangle$ is normally one-generated and that $G_{\text{ab}} \simeq \mathbf{Z}$ where $G_{\text{ab}} = G/K$ is the abelianization of G . Then the following statements hold.

a) $K = \langle\langle [a, K] \rangle\rangle$.

b) Suppose that $[a^l, K] = \mathbf{1}$ for some $l \neq 0$. Fix any $k \in \mathbf{Z}$ with $(k, l) = 1$. Let $G_k = \rho^{-1}(H_k)$, where $\rho : G \rightarrow G_{\text{ab}} = G/K$ is the canonical epimorphism and $H_k \subset G_{\text{ab}}$ is the subgroup generated by $a^k(\text{mod } K)$. Then $G_k = \langle\langle a^k \rangle\rangle$.

Proof. a) Denote by A the cyclic subgroup generated by a : $A = \langle a \rangle \subset G$. Set $M_a = \langle\langle [A, G] \rangle\rangle = \langle\langle [a^k, G] \mid k \in \mathbf{Z} \rangle\rangle$.

Claim 1. $K = M_a$.

Proof. The abelianization G_{ab} of G is a free cyclic group generated by the class $a(\text{mod } K) \neq 0$. From the short exact sequence

$$\mathbf{1} \rightarrow K \rightarrow G \rightarrow G_{\text{ab}} \simeq \mathbf{Z} \rightarrow \mathbf{0}$$

it follows that the commutator subgroup $K = [G, G]$ consists of the elements

$$g = \prod_{i=1}^r c_i a^{m_i} c_i^{-1} \in G$$

such that $\sum_{i=1}^r m_i = 0$.

Set $k_0 = 0$ and $k_i = \sum_{j=1}^i m_j$, so that $m_i = -k_{i-1} + k_i$. Denote by $\sim b$ any element conjugate to b . Since $ca^{k+l}c^{-1} = (ca^k c^{-1})(ca^l c^{-1})$, with the above notation every

element $g \in G$ can be written as ¹⁶

$$g = \prod_{i=1}^r (\sim a^{m_i}) = \prod_{i=1}^r (\sim a^{-k_{i-1}+k_i}) = \left(\prod_{i=1}^{r-1} (\sim a^{k_i})(\sim a^{-k_i}) \right) a^{k_r},$$

where $k_r = 0$ iff $g \in K$. Furthermore, note that

$$(\sim a^k)(\sim a^{-k}) = ca^k c^{-1} da^{-k} d^{-1} = c(a^k c^{-1} da^{-k} d^{-1} c)c^{-1} = c[a^k, b]c^{-1} = \sim [a^k, b],$$

where $b = c^{-1}d$. Thus, we have

$$g = \left(\prod_{i=1}^{r-1} (\sim [a^{k_i}, b_i]) \right) a^{k_r}.$$

If $g \in K$, then $k_r = 0$ and, therefore, $g \in M_a$. This proves the inclusion $K \subset M_a$. Since, evidently, $M_a \subset K$, we have $K = M_a$. \square

Put

$$N_a = \langle\langle [a^\epsilon, G] \mid \epsilon = \pm 1 \rangle\rangle \subset M_a.$$

Claim 2. $M_a = N_a$.

Proof. From the identity

$$[a^k, b] = (a^{k-1}[a, b]a^{-k+1})[a^{k-1}, b], \quad k \in \mathbf{Z}_{>0},$$

we obtain by induction

$$[a^k, b] = (a^{k-1}[a, b]a^{-k+1})(a^{k-2}[a, b]a^{-k+2}) \dots (a[a, b]a^{-1})[a, b] \in N_a.$$

To show that $[a^{-k}, b] \in N_a$, $k \in \mathbf{Z}_{>0}$, it is enough to replace a by a^{-1} in the above identities. Thus, $M_a \subset N_a$. \square

Claim 3. $N_a = \langle\langle [a, K] \rangle\rangle$.

Proof. Since $G_{\text{ab}} = \langle a \pmod{K} \rangle$ is a cyclic group, any element $b \in G$ can be written as $b = a^m d$, where $d \in K$. Therefore,

$$[a^\epsilon, b] = a^\epsilon (a^m d) a^{-\epsilon} (a^m d)^{-1} = a^m (a^\epsilon d a^{-\epsilon} d^{-1}) a^{-m} = \sim [a^\epsilon, d] \in \langle\langle [a^\epsilon, K] \rangle\rangle.$$

¹⁶Although the equality $\sim a^{k+l} = (\sim a^k)(\sim a^l)$ is not symmetric any more, this does not cause problems, as well as the use of the non-commutative product symbol \prod .

Finally,

$$[a^{-1}, b]^{-1} = a^{-1}[a, b]a \in \ll [a, K] \gg,$$

and hence $[a^{-1}, b] \in \ll [a, K] \gg$. □

Now (a) follows from Claims 1–3. □

To prove (b), we start with the following

Claim 4. *Under assumptions of (b), $K = \ll [a^k, K] \gg$.*

Proof. Represent $1 = \mu l + \nu k$, where $\mu, \nu \in \mathbf{Z}$. Then for $d \in K$ we have

$$[a^\epsilon, d] = a^\epsilon d a^{-\epsilon} d^{-1} = (a^k)^{\epsilon\nu} d (a^k)^{-\epsilon\nu} d^{-1} \in \ll [a^k, K] \gg .$$

Since $\ll [a^k, K] \gg \subset K$ and by (a) $K = \ll [a, K] \gg$, the Claim follows. □

Furthermore, note that for any $c \in G_k$ we have $c = a^{mk}d$, where $d \in K$. Therefore, to prove (b) it suffices to show that $K \subset \ll a^k \gg$.

Take $[g, h] \in K$ arbitrary. Due to Claim 4 we have the presentation

$$\begin{aligned} [g, h] &= \prod_{i=1}^N d_i [a^{\nu_i k}, c_i] d_i^{-1} = \prod_{i=1}^N (d_i a^{\nu_i k} d_i^{-1}) ((d_i c_i) a^{-\nu_i k} (d_i c_i)^{-1}) \\ &= \prod_{i=1}^N (\sim a^{\nu_i k}) (\sim a^{-\nu_i k}) \in \ll a^k \gg . \end{aligned}$$

This proves (b). □

6.3. Notation. Let X be a smooth irreducible algebraic variety, $q \in \mathbf{C}[X]$, $F_0 = q^*(0)$ and $F_1 = q^{-1}(1)$. Assume that F_0 is a reduced and irreducible divisor. Fix a smooth complex disc $\Delta \subset X$ which meets F_0 normally at a smooth point of F_0 , and a small positive simple loop $\delta \subset \Delta$ around F_0 . It defines uniquely up to conjugacy an element $\alpha \in \pi_1(X \setminus F_0)$ ¹⁷. Following Fujita [Fu 2, (4.17)] we call such an α *the vanishing loop of the divisor F_0* , and the group $\ll \alpha \gg \subset \pi_1(X \setminus F_0)$ *the vanishing subgroup of F_0* , keeping in mind that $\ll \alpha \gg$ is contained in the kernel of the natural surjection $i_* : \pi_1(X \setminus F_0) \rightarrow \pi_1(X)$.

¹⁷indeed, this easily follows from the connectedness of the smooth part $\text{reg } F_0$ of F_0 .

The following statement should be well known. However, in view of the lack of references we sketch the proof.

6.4. Lemma. $\text{Ker } i_* = \langle\langle \alpha \rangle\rangle$.

Proof. Let a loop $\gamma : S^1 \rightarrow X \setminus F_0$ represents the class $[\gamma] \in \text{Ker } i_*$, i.e. γ is contractible in X . Fix a stratification of F_0 which satisfies the Whitney condition A and contains the regular part $\text{reg } F_0$ of F_0 as an open stratum. By the Thom Transversality Theorem the homotopy $S^1 \times [0, 1] \rightarrow X$ of $\gamma = \gamma_0$ to the constant loop $\gamma_1 \equiv \text{const}$ can be chosen being transversal to the stratification, and therefore such that its image meets the divisor F_0 in a finite number of its regular points only. We may also assume that these intersection points $p_1, \dots, p_n \in \text{reg } F_0$, $p_i \in \gamma_{t_i} \cap F_0$, correspond to different values $0 < t_1 < \dots < t_n < 1$ of the parameter of homotopy $t \in [0, 1]$. If $s_i \in [0, 1]$, $s_i < t_i < s_{i+1}$, and $\bar{\gamma}_i = \gamma_{s_i} : S^1 \rightarrow X \setminus F_0$, $i = 1, \dots, n+1$, then clearly $\bar{\gamma}_{i+1}^{-1} \cdot \bar{\gamma}_i \approx \delta_i^{\epsilon_i}$, i.e. $\bar{\gamma}_i \approx \bar{\gamma}_{i+1} \cdot \delta_i^{\epsilon_i}$, where δ_i is a vanishing loop of F_0 at the point p_i and $\epsilon_i = \pm 1$, and $\bar{\gamma}_{n+1} \approx \text{const}$. Thus, $[\gamma] = [\bar{\gamma}_1] = [\delta_n]^{\epsilon_n} \cdot \dots \cdot [\delta_1]^{\epsilon_1} \in \langle\langle \alpha \rangle\rangle$, and we are done. \square

6.5. Corollary. *If $\pi_1(X) = \mathbf{1}$, then the group $G = \pi_1(X \setminus F_0)$ is normally one-generated by the vanishing loop α of F_0 .*

6.6. Assume further that the restriction $q|(X \setminus F_0) : X \setminus F_0 \rightarrow \mathbf{C}^*$ is a smooth fibration. Then we have the exact sequence

$$\mathbf{1} \rightarrow \pi_1(F_1) \xrightarrow{i_*} \pi_1(X \setminus F_0) \xrightarrow{q_*} \mathbf{Z} \rightarrow \mathbf{0} \quad (4)$$

such that $q_*(\alpha) = 1 \in \mathbf{Z}$.

6.7. Lemma. *In the assumptions as above suppose additionally that X is simply connected. Then $i_*\pi_1(F_1) = K = [G, G]$, and $G_{\text{ab}} = H^1(X \setminus F_0) \simeq \mathbf{Z}$.*

Proof. By Corollary 6.5 we have $G = \langle\langle \alpha \rangle\rangle$. From the exact sequence (4) it follows that

$$i_*\pi_1(F_1) = \text{Ker } q_* = \left\{ g = \prod_{i=1}^r (\sim \alpha^{k_i}) \mid \sum_{i=1}^r k_i = 0 \right\} = K$$

(see the proof of Claim 1 in Proposition 6.2). This proves the first assertion. The second one follows by applying (4) once again. \square

6.8. Lemma. *Let, in the notation as above, q be a quasi-invariant of a positive weight l of a regular \mathbf{C}^* -action t on $X \setminus F_0$. Then $[\alpha^l, K] = \mathbf{1}$.*

Proof. Let $\varphi_l : Y_l \rightarrow X \setminus F_0$ be the cyclic covering of order l :

$$Y_l = \{(x, z) \in (X \setminus F_0) \times \mathbf{C} \mid z^l = q(x)\}.$$

Put $q_l = q \circ \varphi_l : Y_l \rightarrow \mathbf{C}^*$. Define a morphism $\theta : F_1 \times \mathbf{C}^* \rightarrow Y_l$ as follows:

$$\theta(x, \lambda) = (t(\lambda, x), \lambda), \quad x \in F_1, \lambda \in \mathbf{C}^*.$$

It is easily seen that $\theta : F_1 \times \mathbf{C}^* \rightarrow Y_l$ is an isomorphism. We have

$$\varphi_{*}\pi_1(Y_l) = \varphi_{*}(\pi_1(F_1) \times \mathbf{Z}) = \langle i_*\pi_1(F_1), \alpha^l \rangle.$$

This implies that α^l commutes with $K = i_*\pi_1(F_1)$. □

The next theorem is the main result of this section.

6.9. Theorem (Sh. Kaliman). *Let X be a simply connected smooth irreducible algebraic variety, $q \in \mathbf{C}[X]$ be a regular function on X such that*

- i) $F_0 = q^*(0)$ is a smooth reduced irreducible divisor, and*
- ii) $q|_{(X \setminus F_0)}$ is a quasi-invariant of weight $l > 0$ of a regular \mathbf{C}^* -action t on $X \setminus F_0$.*

Let $\sigma_k : X_k \rightarrow X$ be the cyclic covering branched to order k over F_0 :

$$X_k = \{(x, z) \in X \times \mathbf{C} \mid q(x) = z^k\}, \quad \sigma_k(x, z) = x.$$

If $(k, l) = 1$, then X_k is simply connected.

Proof. Put $q_k = q \circ \sigma_k \in \mathbf{C}[X_k]$ and $F_{k,0} = q_k^{-1}(0) \subset X_k$. Since $X_k \setminus F_{k,0} \rightarrow X \setminus F_0$ is a non-ramified k -sheeted cyclic covering, the induced homomorphism

$$(\sigma_k)_* : \pi_1(X_k \setminus F_{k,0}) \rightarrow \pi_1(X \setminus F_0) = G$$

is an injection onto a normal subgroup G_k of G of index k , and $G/G_k \simeq \mathbf{Z}/k\mathbf{Z}$. Clearly, $\alpha^k \in G_k$ is covered by a vanishing loop $\beta \in \pi_1(X_k \setminus F_{k,0})$ of the smooth divisor $F_{k,0} \subset X_k$, i.e. $(\sigma_k)_*(\beta) = \alpha^k$. Therefore, $\langle\langle \alpha^k \rangle\rangle \subset G_k$.

In fact, G_k has the same meaning that in Proposition 6.2(b), i.e. $G_k = q_*^{-1}(H_k)$, where $H_k = k\mathbf{Z} \subset \mathbf{Z} \simeq G_{\text{ab}}$. Indeed, by the universal property of the commutator subgroup, under the homomorphism $\tau : G \rightarrow G/G_k \simeq \mathbf{Z}/k\mathbf{Z}$ we have $K \subset \text{Ker } \tau = G_k$, and hence $G_k = q_*^{-1}(q_*(G_k))$. Furthermore, $q_*(G_k) \supset k\mathbf{Z} = H_k$, because $\alpha^k \in G_k$ and $q_*(\alpha^k) = 1 \in \mathbf{Z}$. Actually, $q_*(G_k) = H_k$, since $[G : G_k] = k$. It follows that $G_k = q_*^{-1}(H_k)$.

By Lemma 6.8, we have $[\alpha^l, K] = \mathbf{1}$, so that Proposition 6.2(b) can be applied. Due to this Proposition, $G_k = \langle\langle \alpha^k \rangle\rangle$. Or, what is the same, $\pi_1(X_k \setminus F_{k,0}) = \langle\langle \beta \rangle\rangle$. The inclusion $i : X_k \setminus F_{k,0} \hookrightarrow X_k$ induces an epimorphism $i_* : \pi_1(X_k \setminus F_{k,0}) \rightarrow \pi_1(X_k)$ with the kernel $\langle\langle \beta \rangle\rangle$ (see Lemma 6.4). Thus, $\pi_1(X_k) = \mathbf{1}$, as desired. \square

7. Concluding remarks

Of course, in such a short survey it is impossible to touch all the interesting related topics. Let us make just a few remarks.

7.1. Due to a lemma of T. Fujita [Fu 2, (2.4)] any smooth acyclic algebraic surface is affine. In general, this does not hold in higher dimensions. Indeed, J. Winkelmann [Wi] constructed a free regular \mathbf{C}_+ -action on \mathbf{C}^5 with the quotient $\mathbf{C}^5//\mathbf{C}_+ = Q \setminus Z$, where Q is a smooth affine quadric of complex dimension four and $Z \subset Q$ is a smooth codimension two subvariety. This quotient is diffeomorphic to \mathbf{R}^8 , but it is not Stein.

7.2. By the Gurjar–Shastri Theorem [GuSha] any smooth acyclic surface is rational. All the exotic \mathbf{C}^n -s in the present paper are rational as well. For instance, Russell’s threefolds are rational being \mathbf{C}^* -varieties with a rational quotient. The general problem (posed by Van de Ven [VdV]) *whether a smooth contractible quasiprojective variety is rational*, is still open (for this and the related Hirzebruch problem on compactifications of \mathbf{C}^n see e.g. [MS, Fur]).

7.3. At last, we remind the well known Abhyankar–Sathaye problem on equivalence of embeddings $\mathbf{C}^k \hookrightarrow \mathbf{C}^n$, where $k < n \leq 2k + 1$. Note that for $n \geq 2k + 2$ all such embeddings are equivalent (Jalonek–Kaliman–Nori–Srinivas; see e.g. [Ka 4, Sr] and references therein), while this is unknown already for the embeddings $\mathbf{C} \hookrightarrow \mathbf{C}^3$ and $\mathbf{C}^2 \hookrightarrow \mathbf{C}^3$.

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