DISCRETE MAGNETIC SCHROEDINGER OPERATORS AND TREE-WIDTH OF GRAPHS

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1. Introduction and results. —

We prove here some properties of a new integer-valued graph invariant which we call $\nu(G)$. This invariant is very close in spirit of our former $\mu(G)$ ([CV1]). But while $\mu(G)$ admits a linear upper bound in terms of the genus of G (recall in particular that $\mu(G) \leq 3$ if and only if G is a planar graph), $\nu(G)$ admits no such upper bound.

Instead of working with real symmetric matrices, we work with complex Hermitian matrices. In some sense which we not intend to make more explicit (see however [CV-NO]), it's like adding a magnetic field to our Schrödinger equations. It's well known that the magnetic Schrödinger operator on $\mathbf{R}^2$ with constant magnetic field has a ground state whose eigenspace is infinite dimensional. In other words, planarity is not related to multiplicities of ground states!! We will make this remark more quantitativ in this paper.

On the other hand, in their famous series of papers *Graphs Minors*, Robertson and Seymour have introduced the *tree-width* of a graph G which measures how close G to some tree is ([R-S 1], [R-S 2]). This invariant turns out to be rather universal for measuring the complexity of a planar graph: excluding a minor for a family $\mathcal{F}$ of planar graphs implies an upper bound of the tree-width on $\mathcal{F}$.

In this paper, we present some sharp relationships between $\nu(G)$ and the tree-width $TW(G)$.

Now let us come to more precise statements.

Let $G = (V, E)$ be a finite non-oriented connected graph without loops and multiple edges. If $(X, (.|.))$ is a N-dimensional complex Hermitian vector space, let $Herm(X)$ be the N^2-dimensional real vector space of Hermitian endomorphisms of X. Let $M_G \subset Herm(\mathbf{C}^V)$ (we endow $\mathbf{C}^V$ with the canonical Hermitian structure) be defined by:

DEFINITION 1. — *$A = (a_{i,j}) \in M_G$ if and only if*

i) $a_{i,j} \neq 0$ if $\{i, j\} \in E$,

ii) $a_{i,j} = 0$ if $\{i, j\} \notin E$ and $i \neq j$.

M_G is a submanifold of dimension $|V|+2|E|$ of $Herm(\mathbf{C}^V)$.

Let us denote by $W_l \subset Herm(\mathbf{C}^V)$ the submanifold of non-negativ matrices A (ie $(Ax|x) \geq 0 \forall x$) whith l-dimensional kernel.

DEFINITION 2. — *If G is a finite graph, let us define $\nu(G)$ as the sup of those l for which there exists $A \in M_G \cap W_l$ such that both manifolds intersect transversally at A.*

Our first result is an easy adaptation of [CV1]:

THEOREM 1. — *If G' is a minor of G ($G' < G$), then $\nu(G') \leq \nu(G)$.*

We study now a family of planar graphs P_N for which we can compute $\nu(G)$ explicitly. P_N is embedded in $\mathbf{C}$: $V_N = V(P_N) = \{n + m\omega | 1 \leq n, m \leq N, n+m \leq N+1\}$, where $\omega = e^{i\pi/3}$. $E_N = E(P_N)$ is the set of pairs $z, z' \in V_N$ such that $|z - z'| = 1$. In other words P_N is a triangulation of some equilateral triangle by equilateral triangles all of size 1.

Fig. 1: The graph P_5.

THEOREM 2. — $\nu(P_N) = N$

As a corollary, ν is not bounded for planar graphs. This is in sharp contrast with μ which is smaller than 3 if and only if the graph is planar.

Now we come to the *tree-width* for which we give a slightly different (but very close to) definition than Robertson-Seymour's.

First, if G_1, G_2 are two graphs, we define the *product* $G = G_1 \times G_2$ by $V = V_1 \times V_2$ and $\{(a_1, a_2), (b_1, b_2)\} \in E$ if and only if either ($a_1 = b_1$ and $\{a_2, b_2\} \in E_2$) or ($a_2 = b_2$ and $\{a_1, b_1\} \in E_1$).

DEFINITION 3. — *We define the* tree-width $TW(G)$ *of the graph* G *as the smallest* N *for which there exists a tree* T *such that* G *is a minor of the product* $T \times K_N$ *(here* K_N *is the complete graph with* N*-vertices).*

Now we come to our next results:

THEOREM 3. — $\nu(G) = 1$ *if and only if* G *is a tree.*

THEOREM 4. — *For every* G, $\nu(G) \leq TW(G)$.

In the last section, we formulate without proof the analogues for infinite graphs of the previous results.

I would like to thank very much Lex Schrijver for inviting me to visit CWI where he told me about tree-width and where most of the results in this paper were discovered.

2. Discrete holomorphic functions and $\nu(P_N) = N$. —

We will prove theorem 2.

First, for any $A \in M_{P_N}$, $dim(Ker(A)) \leq N$: otherwise there would exists a nonzero function in $\operatorname{Ker} A$ vanishing on the vertices $1+\omega, 2+\omega, \cdots, N+\omega$. Such a function vanishes identically since we can compute (using A) it's values line after line moving up.

For the converse we need an explicit A. The simplest one has real coefficients and is:

$$Af(z) = \sum_{z' \sim z} f(z') + \frac{d(z)}{2} f(z) \ ,$$

where $d(z)$ is the degree of z ($d(z) = 2, 4$ or 6 depending on the position of z).

Now, if $D : \mathbf{C}^{V_N} \to \mathbf{C}^{V_N}$ is defined by

$$Df(z) = f(z) + f(z+1) + f(z+\omega)$$

(or $Df(z) = 0$ if $z+1$ and $z+\omega$ do not belong to V_N), then it's easy to check that $A = D^\star D$. This implies that A is non negativ and $Ker(A) = Ker(D)$. Moreover an easy exercise shows that $dim(Ker(D)) = N$. More precisely $\operatorname{Ker} D$ admits a basis

$$\varphi_l \ (l = 1, \cdots, N)$$

where $\varphi_l(i+\omega) = \delta_{i,l}$.

Now we can check transversality.

LEMMA 1. — *The support of φ_l consists of $z's$ in V_N which satisfy:*

$$z = l + (n+1)\omega + m\omega^2, m, n \geq 0 \ .$$

Easy.

By the same kind of arguments as in [CV3], transversality means that the Hermitian forms

$$q_i(\varphi, \psi) = \varphi(i)\bar{\psi}(i) \ (i \in V) \ ,$$

$$r_{i,j}(\varphi, \psi) = \operatorname{Re}(\varphi(i)\bar{\psi}(j)) \ , \ s_{i,j}(\varphi, \psi) = \operatorname{Im}(\varphi(i)\bar{\psi}(j)) \ (\{i,j\} \in E),$$

restricted to $KerA$ generate $Herm(KerA)$.

Now we can prove the transversality by induction on N. In fact the restrictions of $\varphi_1, \cdots, \varphi_{N-1}$ to V_{N-1} are exactly the $\varphi_l' s$ for P_{N-1}.

We consider only $q_z, \ z = 1+\omega, \cdots, z = N+\omega$, and $r_{z,z-\omega}, \ s_{z,z-\omega}, \ \{z, z-\omega\} \in E_N$ which form a set of N^2 Hermitian forms.

So we need only to prove that, if we restrict them to $Ker(D_N)$, they are independant. Assume a linear relation between them. By induction we can stay with a linear relation between $q_{N+\omega}, r_{N+\omega+j\omega^2}, s_{N+\omega+j\omega^2}, j = 1, \cdots, N-1$. Now, if we test on $(\varphi_N, \varphi_l), l = 1, \cdots, N-1$, we get a triangular invertible matrix because of the support of $\varphi_l' s$.

□

We started with a (slighly) more complicated example with is gauge equivalent to this one. Let's say that a triangle of P_N is black if it's of the form $(z, z+1, z+\omega)$. Then let's define an holomorphic function on P_N by asking the image of any black triangle to be equilateral. In other terms if $D_1 f(z) = f(z+\omega) - f(z) - \omega(f(z+1) - f(z))$, we ask $Df = 0$. Now we put $B = D_1^\star D_1$. Then B is unitarily equivalent to A (by gauge transform).

Fig.2: holomorphic map on P_4.

Question: given some $A \in M_G$, we may define the flux of the magnetic field through each triangular face as a number in $\mathbf{R}/2\pi\mathbf{Z}$ which is the argument of the product $\Pi_i a_{i,i+1}$ on the oriented boundary. In our case the flux is π for each face.

Uur question is now: does there exist any upper bound of $dim(Ker A)$ for $A \in M_G$ in terms of information on the flux? For example, if the flux is 0 for every cycle, then the multiplicity of the lowest eigenvalue is 1 by Perron-Frobenius.

For that problem, it's interesting to compare [L-L].

3. Minors. —

I'll give a sketch of a proof of theorem 1 wich is along the lines already described in [BA-CV] and [CV2].

First, we need to consider a compactification of $Herm(X)$ which is a slight modification of the compactification already considered in [CV2].

Let $(X, (.|.))$ be a complex Hermitian space of dimension N and ω the sesquilinear form on $X \oplus X$ defined by:

$$\omega(x + x', y + y') = (x|y') - (x'|y) \ .$$

The graph of a linear map A from X to X is ω-isotropic if and only if A is Hermitian.

Now we define the manifold Λ_X as the set of all $N-$dimensional complex subspaces of $X \oplus X$ which are ω-isotropic. It's well known that Λ_X is a smooth compact manifold of dimension N^2 whose tangent space at the point L is canonically $Herm(L)$.

W_l admits a smooth compactification in Λ_X which we denote by $\bar{W}_l$ and which is the set of all $L \subset \{(x,\xi)|(x|\xi) \geq 0\}$ such that $\dim(L \cap (X \oplus 0)) = l$.

Moreover, there is a bijection of Λ_X with the set of pairs (Y, B) where Y is a subspace of X and $B \in Herm(Y)$. To any pair (Y, B) we associate $L = \{(y,\xi)|y \in Y,\ \forall y' \in Y,\ (\xi|y') = (Ay|y')\}$.

Minors: let us recall that G' is a minor of G ($G' < G$) if we can relate G to G' by elementary moves consisting of *contracting an edge* $\{a, b\}$: $G' = C_{a,b}G$ or *removing an edge* $\{a, b\}$: $G' = R_{a,b}G$. So it's enough to prove that

(i) $\nu(C_{a,b}G) \leq \nu(G)$ and

(ii) $\nu(R_{a,b}G) \leq \nu(G)$.

The proof of ii) is rather easy because we can use smooth perturbation theory; denote by $\Delta_{a,b} \in Herm(\mathbf{C}^V)$ the map defined by

$$\Delta_{a,b}(x_a, x_b, x') = (x_a - x_b, x_b - x_a, x')$$

then, if

$$G' = R_{a,b}G$$

and $A \in M_{G'}$,

$$\forall \varepsilon \neq 0,\ A + \varepsilon M_{a,b} \in M_G \ .$$

So, for ε small enough, if $M_{G'}$ intersects W_l transversally at some point A_o, then $M_{G'} + \varepsilon\Delta_{a,b}$ intersects W_l transversally at some point $A_\varepsilon \in M_G$, and a fortiori M_G intersect W_l transversally at A_ε.

Assertion (i) is more difficult because we cannot embed $M_{G'}$ into $Herm(\mathbf{C}^V)$ but we can easily embed it into $\Lambda_{\mathbf{C}^V} = \Lambda_V$ and mimic the proof of (i).

Let's give some details.

Assume $G' = C_{1,2}G$. If $B \in M_{G'}$, we associate to it a pair $j(B) = (Y, C) \in \Lambda_V$ in the following way: $Y = \{x_1 = x_2\}$ and $C(x_0, x_0, x') = B(x_0, x')$. We denote by Z_0 the submanifold of Λ_V which consists of all those C: $Z_0 = j(M_{G'})$.

We claim that, if W_l' intersect $M_{G'}$ transversally at B_o, then $\bar{W}_l$ intersect Z_0 transversally at $j(B_o)$.

Now we will mimic the proof of (ii) by looking at

$$A_\varepsilon = A + \frac{1}{\varepsilon}\Delta_{1,2} \ .$$

We prove now that L_{A_ε} converges smoothly in Λ_X as ε tends to 0 and compute the limit.

Assume A is given by:

$$\xi_1 = A_1(x_1, x_2, x'),\ \xi_2 = A_b(x_1, x_2, x'),\ \xi' = A'(x_1, x_2, x') \ ,$$

and so we can rewrite the equations of L_{A_ε} as:

$$\xi_1 + \xi_2 = (A_1 + A_2)(x_1, x_2, x'),\ x_1 - x_2 = \varepsilon(\xi_1 - A_1(x_1, x_2, x')),\ \xi' = A'(x_1, x_2, x') \ .$$

From the second one we get:

$$x_2 = L(\varepsilon, x_1, \xi_1, x') \ ,$$

where is smooth with respect to ε and using the others 2 we get:

$$\xi_2 = M(\varepsilon, x_1, \xi_1, x'),\ x_2 = L(\varepsilon, x_1, \xi_1, x'),\ \xi' = N(\varepsilon, x_1, \xi_1, x') \ ,$$

where L, M, N are smooth with respect to ε. Now we can compute the limit:

$$x_2 = x_1,\ \xi_2 + \xi_1 = (A_1 + A_2)(x_1, x_1, x'),\ \xi' = A'(x_1, x_1, x') \ .$$

Now, it is not always true that the limit is in Z_0 if $A = (a_{i,j})$ is in M_G: in the case where 1 and 2 have a commun neighbourhood i it may happen that:

$$a_{1,i} + a_{2,i} = 0 \ .$$

Fig. 3: Z_0, Z_ε and $\bar{W}_l$.

But we can easily build a submanifold Z_ε of M_G which converges smoothly to Z_0: assume that we split vertices close to 1 or 2 into 3 parts:

U consisting of vertices $i \neq 2$ such that $\{i, 1\} \in E$ and $\{i, 2\} \notin E$, V the same by permuting 1 and 2 and W the set of $j \in V$ for which $\{j, 1\}$ and $\{j, 2\}$ are in E. Now given $B \in M_{G'}$, call 0 the vertex obtained by identification of 1 and 2 and keep the same labelling for all others vertices. To B we associate $w(B) = A \in Herm(\mathbf{C}^V)$ by taking:

(i) if $i \in U$, $a_{1,i} = b_{0,i}$, $a_{2,i} = 0$,

(ii) if $i \in V$, $a_{2,i} = b_{0,i}$, $a_{1,i} = 0$,

(iii) if $i \in W$, $a_{1,i} = a_{2,i} = \frac{1}{2} b_{0,i}$,

(iv) $a_{1,1} = b_{0,0}$, $a_{2,2} = 0$,

and keep all others coefficients with no indices in $\{0, 1, 2\}$.

Then

$$w(M_{G'}) + \frac{1}{\varepsilon}\Delta_{1,2} \subset M_G$$

and $w(B) + \frac{1}{\varepsilon}\Delta_{1,2}$ converges smoothly to B.

Now the proof is the same as that of (ii).

Fig 4: contracting $\{1,2\}$ to 0; U, V, W

4. Vector bundles on trees. —

Let $G=(V,E)$ be a finite graph as before. A Hermitian vector bundle $L=(L_i)_{i\in V}$ over G is a collection of isomorphic Hermitian vector spaces L_i. To these data we associate the Hermitian space of sections which is $H=\oplus_{i\in V}L_i$.

A matrix $\Omega=(\Omega_{i,j})$ which is a Hermitian endomorphism of H is said to be *(G,L)-admissible* or simply *admissible* if we have:

i) $\Omega_{i,j}=\Omega^\star_{j,i}$ is invertible for each edge $\{i,j\}$;

ii) $\Omega_{i,j}=0$ if $\{i,j\}\notin E$ and $i\neq j$;

iii) $\Omega_{i,i}$ is self-adjoint.

Basic example:

G' is another graph and $L_i=\mathbf{C}^{V'}$. Then $H=\mathbf{C}^{V\times V'}$ and any $A\in M_{G\times G'}$ is admissible: if $\{i,j\}$ is an edge of G, $\Omega_{i,j}$ is diagonal with non-zero entries.

Now, we have the following basic lemma:

LEMMA. — *If Ω is admissible, nonnegativ and G is a tree, then $dim(Ker(\Omega))\leq N=\dim(L_i)$.*

Proof of the lemma:

By contradiction. If the conclusion isn't true, there exists a $\varphi\in Ker(\Omega)$ and an edge $\{i_0,j_0\}$ such that $\varphi(i_0)=0,\varphi(j_0)\neq 0$. Let $T_1,\cdots,T_l$ be the collection of disjoint trees such that $T\setminus i_0=T_1\cup\cdots\cup T_l$ and assume $j_0\in T_1$.

Fig. 5: $T=\{i_0\}\cup_{j=1}^{l}T_j$.

Let $\omega_1,\cdots,\omega_l$ be complex numbers of modulus 1 and φ_ω be defined by $\varphi_\omega(i)=\omega_j\varphi(i)$ if $i\in T_j$ and $\varphi_\omega(i_0)=0$.

Then it is easy to check that, if $Q(\varphi_\omega) = (\Omega\varphi_\omega|\varphi_\omega)$, $Q(\varphi_\omega) = 0$ and hence $\varphi_\omega \in Ker\Omega$.

Now we can choose $v_0 \in L_{i_0}$ which satisfy $(A_{i_0,j_0}\varphi(j_0)|v_0) \neq 0$ ((i) imply that $A_{i_0,j_0}\varphi(j_0) \neq 0$).

So we can choose ω_j such that

$$\mathrm{Re}(A_{i_0,j_0}\varphi_\omega(j_0)|v_0) > 0$$

and

$$\mathrm{Re}(A_{i_0,j}\varphi_\omega(j)|v_0) \geq 0$$

for all $j \sim i_0$.

We see now that

$$Q(\varphi_\omega + \varepsilon v_0 \delta_{i_0}) = a\varepsilon + O(\varepsilon^2)$$

where $a > 0$. So, by taking a small $\varepsilon < 0$, we get a contradiction with the fact that $Q \geq 0$. □

As a corollary and using the basic example for $T \times K_N$ we obtain theorem 4.

In particular $\nu(T) = 1$ if T is a tree and the converse is true (theorem 3): if G is not a tree, then $P_2 < G$ and $\nu(G) \geq 2$.

5. Infinite graphs. —

We will now extend the theory to graphs which are not necessarily finite.

If $G = (V, E)$, we assume only connectivity and local finiteness of G.

Of course in that case we need to define operators carefully. We take as Hilbert space $H_V = l^2(V, \mathbf{C})$. Now, if A is in M_G, $A = (a_{i,j})$ is an essentially selfadjoint operator (not necessarily bounded) in H_V, with the same algebraic conditions as for finite graphs on $a_{i,j}'^s$.

In that case, the spectrum is in general not discrete; so we need to add another hypothesis which we can formulate in the following way: $A = A_o + K$ where there exists $m > 0$ such that $A_o - mId \geq 0$ and K is compact.

For example

$$(Ax|x) = Q(x) = \sum_{i \in V} V_i |x_i|^2 + \sum_{\{i,j\} \in E} c_{i,j} |x_i - e^{\sqrt{-1}\theta_{i,j}} x_j|^2$$

where $c_{i,j} > 0$ and $\liminf_{i \to \infty} V_i \geq m$.

Under those hypothesis, the spectrum below m is discrete and each eigenvalue is of finite multiplicity. So we define M_G by both conditions.

We can now define $\nu(G)$ for infinite graphs. Along the same lines, it's possible to define $\mu(G)$.

We need to take another definition of minors; a minor of G is given in the following way: take any partition of V into connected components. You get a new graph whose vertices are the subsets of the partition and edges connecting two subsets having elements already connected by some edge of G. A minor is obtained by removing from this new graph an arbitrary number of edges.

We have:

THEOREM 1'. — *If G' is a minor of G, then $\nu(G') \leq \nu(G)$.*

THEOREM 2'. — $\nu(P_\infty) = +\infty$.

THEOREM 3'. — *$\nu(G) = 1$ if and only if G is a tree.*

THEOREM 4'. — *For every G, $\nu(G) \leq TW(G)$.*

For μ, we need to define the notion of planarity which consists of being embeddable in the plane, allowing accumulation points.

Then we have:

THEOREM 5. — *$\mu(G) \leq 3$ if and only if G is planar.*

All these results are proved using the same patterns as for finite graphs. We may also prove that there exists always a finite minor G' of G such that $\nu(G') = \nu(G)$.

6. Some open problems on μ and ν. —

Here is a selection of open questions which were presented at CWI.

1. Computability questions. —

The question is to find algorithms producing $\mu(G)$ and $\nu(G)$ from G. Theoritically, there are algorithms because everything is about intersecting algebraic manifolds. Of course what would be nice is to have a computer programm computing these numbers.

2. Maximizing the gap. —

Now we come back to real case. For many purposes it's interesting to have matrices A in O_Γ with a large gap (gap(A)=$\lambda_2 - \lambda_1$). The problem is to find an appropriate normalisation condition which insures that the problem is well posed. Moreover, it seems reasonnable that if A maximize the gap then the multiplicity of $\lambda_2(A)$ is the largest possible. Compare [NA] for the continuous case.

3. $\nu(G)$ and $TW(G)$. —

From general results of Robertson-Seymour, there exists an upper bound

$$TW(G) \leq F(\nu(G))$$

which holds for planar graphs. The question is to find an explicit $F : \mathbf{N} \to \mathbf{N}$.

4. Higher dimensional complexes. —

The question is to extend that kind of invariant to higher dimensionnal complexes, in particular 2d simplicial complexes and find relationship with Hodge-de Rham Laplace operators on forms.

5. Chromatic number. —

This problem is the most exciting: prove or disprove

$$C(G) \leq \mu(G) + 1 \ ,$$

where $C(G)$ is the chromatic number of G. It would imply 4-color theorem and is weaker than Hadwiger's conjecture.

6. Prescribing spectras. —

Describe all possible spectra for $A \in O_G$ or $A \in M_G$. Already for special graphs like trees this problem is not yet solved. It's solved for paths and for cyclic graphs.

For the cyclic graph $C_N = \mathbf{Z}/N\mathbf{Z}$, we have the following set of inequalities for any $A \in O_{C_N}$:

$$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \cdots \ .$$

It's known that for any graph G with N vertices and any set $\sigma = \{\lambda_1 < \lambda_2 < \cdots < \lambda_N\}$ there exists $A \in O_G$ such that

$$\sigma(A) = \sigma \ .$$

There is a general question: is it always true that the restrictions on possible spectra is given by restrictions on multiplicities of eigenvalues?

7. Lex Schriver's question. —

Is it always true that:

$$\mu(G) = \inf_{G<G'} m(G') \ ,$$

where $m(G')$ is the maximal multiplicity of the second eigenvalue for $A \in O_G$?

It's true for example for planar graphs, because $m(G) = 3$ if G is a triangulation of the 2-sphere ([HO]).

Same question for $\nu(G)$.

7. References. —

[BA-CV] R. BACHER AND Y. CDV . — *Multiplicités de valeurs propres et transformations étoile-triangle des graphes*, Bull Soc Math France, **123** (1995), 101-117.

[CV1] Y. CDV. — *Un nouvel invariant des graphes finis et application à un critère de planarité*, Journal of Combinatorial theory B , **50** (1990), 11–21.

(and English translation of the previous one:)

[CV1'] Y. CDV. — *A new graph invariant and a criterion for planarity*, in Graph Structure theory (Robertson-Seymour editors), AMS, **147** (1991), 137-148.

[CV2] Y. CDV. — *Réseaux électriques planaires I*, Commentarii Math. Helv., **69** (1994), 351-374.

[CV3] Y. CDV. — *Multiplicités de valeurs propres: laplaciens discrets et continus*, Rendiconti di Matematica, **VII, 13** (1993), 433-460.

[CV4] Y. CDV. — *Spectres des graphes*, Cours en cours de rédaction (3 chapitres écrits), polycopié IF, 1995.

[CV-TO] Y. CDV ET N. TORKI. — *Opérateur de Schrödinger avec champs magnétiques*, Séminaire de théorie spectrale et géométrie , **11** (1992"1993), 9-18.

[HO] H. VAN DER HOLST. — *A short proof of the planarity characterization of Colin de Verdière*, Preprint CWI, (1994), .

[L-L] E. LIEB, M. LOSS. — *Fluxes, Laplacians and Kasteleyen's theorem*, Duke Math. Journal, **71** (1993), 337-363.

[NA] N. NADIRASHVILI. — *Berger's Isoperimetric problem and Minimal Immersions of Surfaces*, Preprint ESI, (1995), .

[R-S 1] N. ROBERTSON, P. SEYMOUR. — *Graphs minors III: planar tree-width* , J. Comb. Theory B, **36** (1984), 49-64.

[R-S 2] N. ROBERTSON, P. SEYMOUR. — *Graphs minors IV: Tree-width and well quasi-ordering*, J. Comb. Theory B, **48** (1990), 227-254.

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Résumé. —

Dans cet article, nous introduisons un nouvel invariant numérique $\nu(G)$ des graphes très voisin de notre invariant $\mu(G)$: au lieu de considérer la multiplicité de la seconde valeur propre d'un opérateur de type Laplacien, nous considérons la multiplicité de l'état fondamental d'un opérateur de type Schrödinger avec champ magnétique.

L'invariant $\nu(G)$ (contrairement à $\mu(G)$) n'admet pas de borne supérieure sur les graphes planaires, mais est contrôlé par la *largeur d'arbre* de G.

Mots-Clés. —

Graph, tree-width, Schrödinger operator.

Classification mathématique. —

05C10, 05C50, 91Q10.