A REPRESENTATION OF ISOMETRIES ON FUNCTION SPACES

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INTRODUCTION

This paper contains a proof of the main result previously announced in [Za1] and of its generalization to a class of ideal spaces. Namely, we prove the validity of a weighted shift representation of the surjective isometries between Banach function spaces which satisfy some minor restrictions. This class encludes at least all rearrangement-invariant (r.i.) spaces. Our proof follows Lumer's scheme [Lu] and uses some ideas due to A. Pelczynski (see [Ro]; cf. also [BrSe] and [SkZa1,2]). Recall that in [Za1] all the spaces are considered over the field **C** of complex numbers.

Due to a recent revival of interest in the isometric theory of Banach function spaces I have been asked several times during the last few years by my colleagues working on the subject about the proofs of the results announced in [Za1]. As a matter of fact, the proofs of Theorem 1 and Proposition 1 from [Za1] were published in [Za2], a Russian journal with a rather limited circulation. An English translation of that journal though prepared has never appeared due to some circumstances. Nevertheless, a translation of my article [Za2] was circulating among a small number of experts (see e.g. references in [KaRa2]). In order to satisfy numerous requests of my colleagues I am reproducing here this translation, certainly updaiting and modifying it.

Let me briefly mention several further developements. A generalization of Theorem 1 in [Za1] to the real case has been recently obtained in [KaRa1,2]. Papers [KaRa1,2] contain also a new proof of Theorem 4 in [Za1], which yields a characterization of the L_p-spaces as r.i. spaces with non-standard isometry groups (my original proof of this theorem covered both the real and complex cases but it was never published). Some corollaries of the main results of [Za1] are extended to the real case by different methods in [AbZa]. New proofs of the remaining statements from [Za1] can be found in the forthcoming paper [PKL]. Some additional information can be also found in [FlJe].

HERMITIAN OPERATORS

A Banach space E of measurable functions on a measure space (Ω, Σ, μ) is called *an ideal space* if $f \in E$, $|g| \leq |f|$ and $g \in L^0(\Omega, \Sigma, \mu)$ imply that $g \in E$ and

$||g|| \leq ||f||$.

If additionally, the equimeasurability of functions $|f|$ and $|g|$, where $f \in E$ and $g \in L^0$, implies that $g \in E$ and $||g|| = ||f||$, then the space E is called *symmetric* or rearrangement-invariant.

Let $\Sigma_0 = \{\sigma \in \Sigma \mid \mu(\sigma) < \infty\}$. We will always assume that the characteristic function $\chi_\sigma \in E$ of every set $\sigma \in \Sigma_0$ belongs to E. Let ρ_σ be the projector $\rho_\sigma(x) = \chi_\sigma x$, $x \in E$. The image $\rho_\sigma E$ is called *a band*, or *a component* of the space E.

DEFINITION 1. — An ideal space E is called *projection provided*, if for each finite collection $\overline{\omega} = \{\omega_i\}_{i=1}^n$ of disjoint sets $\omega_i \in \Sigma_0$, $i = 1, \dots, n$, there exists a projector $\rho^c_{\overline{\omega}}$ of norm 1 on the subspace of step-functions

$$E^n(\overline{\omega}) = \Big\{\sum_{i=1}^n c_i \chi_{\omega_i},\ c_i \in \mathbf{C},\ i = 1, \dots, n\Big\}$$

that commutes with the projectors ρ_{ω_i}, $i = 1, \dots, n$.

Any symmetric space is projection provided; we can take for $\rho^c_{\overline{\omega}}$ the Haar's projector of the conditional expectation ([SMB], p. 95).

DEFINITION 2. — We will say that the space E is *free from L_2-components* if for each component the norm of E does not coincide with the norm of $L_2(\nu)$, where ν is a positive measure on Σ.

An operator H on E is called *Hermitian* if $||e^{irH}|| = 1$, $\forall t \in R$. The set of Hermitian operators is denoted as $\mathrm{Herm}(E)$. By $LM^r_\infty(E)$ we denote the subset of the operators of multiplication by bounded real functions.

For simplicity we assume in the paper that Ω is $[0,1]$ or a line with the Lebesgue measure μ.

PROPOSITION 1. — *Let E be a projection provided Banach ideal space. If E is free from L_2-components, then* $\mathrm{Herm}(E) = LM^r_\infty(E)$.

Proof. — It is enough to prove that any Hermitian operator H on E holds the following property:

$$\chi_{\Omega\setminus\omega} \cdot H\chi_\omega = 0, \quad \forall \omega \in \Sigma_0. \tag{1}$$

Indeed, if (1) is true, then

$$\chi_{\omega_1} \cdot H\chi_{\omega_2} = \chi_{\omega_2} \cdot H\chi_{\omega_1}\ (= H\chi_{\omega_1\cap\omega_2}), \quad \forall \omega_1, \omega_2 \in \Sigma_0. \tag{2}$$

So, the equality

$$\chi_\omega h = H\chi_\omega, \quad \omega \in \Sigma_0 \tag{3}$$

determines a measurable function h on Ω such that

$$e^{itH}(\chi_\omega) = \chi_\omega e^{ith}. \tag{4}$$

By Theorem 2 of [Za3], (4) implies that

$$e^{itH}(f) = e^{ith} \cdot f, \quad \forall f \in E, \tag{5}$$

and therefore

$$H(f) = h \cdot f, \quad \forall f \in E, \tag{6}$$

and $\mathrm{Im}\, h = 0$, $||H|| = ||h||_{L_\infty}$ [Lu, Lemma 7].

Suppose that there exists an operator $H_0 \in \mathrm{Herm}(E)$ which does not hold (1), that means that $\chi_{\Omega\setminus\omega_0} H_0 \chi_{\omega_0} \neq 0$ for some $\omega_0 \in \Sigma_0$. Choose disjoint sets $\omega_1, \ldots, \omega_n \in \Sigma_0$ ($\omega_i \cap \omega_j = \emptyset$, $i \neq j$, $i, j = 0, \ldots, n$) such that

$$|\chi_{\omega_i} H_0 \chi_{\omega_0} - \lambda_i \chi_{\omega_i}| < |\lambda_i|, \tag{7}$$

where $\lambda_i \neq 0$, $i = 1, \ldots, n$. Let $\overline{\omega} = \{\omega_i\}_{i=0}^n$. Due to Lumer's Lemma [Lu, Lemma 8], the operator $H_0^{\overline{\omega}}$ determined by the equality

$$H_0^{\overline{\omega}} = \rho_{\overline{\omega}}^c \cdot H_0 \rho_{\overline{\omega}}^c$$

is a Hermitian operator on the subspace $E^{n+1}(\overline{\omega})$. Since E is an ideal space, $||\rho_{\overline{\omega}}^c|| = 1$, and $\rho_{\overline{\omega}}^c(\chi_{\omega_i} x) = \chi_{\omega_i} \rho_{\overline{\omega}}^c(x)$, $i = 0, 1, \ldots, n$, from (7) it follows that

$$\begin{aligned} ||\chi_{\omega_i} H_0^{\overline{\omega}} \chi_{\omega_0} - \lambda_i \chi_{\omega_i}|| &= ||\rho_{\overline{\omega}}^c(\chi_{\omega_i} H_0 \chi_{\omega_0} - \lambda_i \chi_{\omega_i})|| \\ &\leq ||\chi_{\omega_i} H_0 \chi_{\omega_0} - \lambda_i \chi_{\omega_i}|| \\ &< |\lambda_i| \cdot ||\chi_{\omega_i}||, \quad i = 1, \ldots, n, \end{aligned}$$

and whence

$$\chi_{\omega_i} H_0^{\overline{\omega}} \chi_{\omega_0} \neq 0, \quad i = 1, \ldots, n. \tag{8}$$

Next we show that the subspace $E_{\overline{\omega}}^{n+1}$ is Euclidean, and

$$\Big\| \sum_{i=0}^n c_i \chi_{\omega_i} \Big\|^2 = \sum_{i=0}^n |c_i|^2 ||\chi_{\omega_i}||^2. \tag{9}$$

To this point we use the following lemma (*).

Lemma 1. — *Let E^{n+1} be an ideal Minkowski space (**) over $\mathbf{C}$, $\{e_i\}_0^n$ be the standard basis in E^{n+1}, and H be a Hermitian operator on E^{n+1} such that*

$$(He_0, e_k) \neq 0, \quad k = 1, \ldots, n. \tag{8'}$$

(*) Similar statements can be found, for instance, in [KaWo] (complex case), [SkZa1,2] (real case); see also the bibliography therein.

(**) i.e. a finite dimensional Banach space

Then E^{n+1} *is a Euclidean space and*

$$\left\|\sum_{i=0}^{n} c_i e_i\right\|^2 = \sum_{i=0}^{n} |c_i|^2 \|e_i\|^2. \tag{9'}$$

Proof. — Let G_0 be the connected component of unity in the isometry group Iso (E^{n+1}), and let $\langle\cdot,\cdot\rangle$ be a G_0-invariant scalar product such that $\langle e_0, e_0\rangle = 1$. The orbit $G_0 e_0$ is a connected G_0-invariant submanifold in E^{n+1}, which is contained in the intersection of the Minkowski sphere $S(E^{n+1})$ and the Euclidean sphere S^{2n+1} (indeed, $G_0 \subset U(n)$ by our choice of scalar product). The tangent space T to the orbit $G_0 e_0$ at point e_0 is invariant with respect to the stationary subgroup $G_0^{e_0}$ of e_0. Note that the operator $t_k(\varphi)$ of the rotation of k–th coordinate on angle φ (i.e. $t_k(\varphi)e_k = e^{i\varphi}e_k,\ t_k(\varphi)e_i = e_i,\ i \neq k$) belongs to the subgroup $G_0^{e_0}$ and, therefore, $t_k(\varphi)T = T$. The latter is possible either if the k–th coordinate of any vector of T is equal to zero, or if $e_k \in T$ and $ie_k \in T$. Since the tangent vector iHe_0 to the curve $e^{itH}(e_0) \subset G_0 e_0$ belongs to the subspace T, in view of $(8')$, the k–th coordinate of this vector is non–zero. Therefore, $e_k \in T$ and $ie_k \in T$ $(k = 1, \ldots, n)$. Besides that, the subspace T contains vector ie_0 tangent to the curve $e^{itH}(e_0) \subset G_0 e_0$. Thus, the real dimension of the subspace T and, hence, also of the orbit $G_0 e_0$ is equal to $2n + 1$. Since $G_0 e_0$ is a compact connected manifold, we have

$$G_0 e_0 = S(E^{n+1}) = S^{2n+1}\ , \tag{10}$$

that proves that E^{n+1} is Euclidean.

Because eigen-vectors e_k and e_ℓ $(\ell \neq k)$ of the operator $t_k(\pi) \in G_0$ are orthogonal, $\{e_k\}_0^n$ is an orthogonal basis in E^{n+1}. This implies (9'). The lemma is proved.

Let us come back to the proof of Proposition 1. Let $S = \bigcup_{i=1}^{n} \omega_i$, $E^b(S)$ be the closure in $\rho_S E$ of the set of finite-valued (step) functions. Passing to the limit over subpartitions, we can prove that $E^b(S)$ is a Hilbert space. Due to reflexivity of the space $E^b(S)$, the norm on $E^b(S)$ is absolutely continuous [Zab, Theorem 30], *i.e.* $\|\chi_\sigma\| \to 0$ as $\mu(\sigma) \to 0$, and $E^b(S) = (E^b(S))''$. Therefore, $E^b(S) = \rho_S E$. Absolute continuity of the norm permits to get the equality

$$\|\chi_\sigma\|^2 = \sum_{i=1}^{\infty} \|\chi_{\sigma_i}\|^2, \tag{11}$$

where $\sigma_i,\ \sigma \in \Sigma(S) = \Sigma \cap S,\ \sigma_i \cap \sigma_j = \emptyset,\ i \neq j,\ \sigma = \bigcup_{i=1}^{\infty} \sigma_i$.

Let $\nu(\sigma) = \|\chi_\sigma\|^2$. Due to (11), ν is a positive (absolutely continuous with respect to μ) σ–additive measure on the algebra $\Sigma(S)$. Previous arguments show that $\rho_S E = L_2(\nu)$, that contradicts to our assumption. The proposition is proved.

PROPOSITION 2. — *Let E be a symmetric space such that the norm on E is not proportional to the norm of the space $L_2(\Omega, \Sigma, \mu)$. Then* $\mathrm{Herm}(E) = LM_\infty^r(E)$.

The proof is quite similar to that of Proposition 1. A partition $\overline{\omega}$ is chosen in a way that provides (8) just for $i = 1$, but we should require, besides that, that $\mu(\omega_i) = \mu(\omega_1)$, $i = 2, \ldots, n$. Using Lemma IX.8.4 of [Ro] instead of Lemma 1, we obtain the equality

$$\left\|\sum_{i=1}^{n} c_i \chi_{\omega_i}\right\|^2 = \|\chi_{\omega_1}\|^2 \sum_{i=1}^{n} |c_i|^2. \tag{9''}$$

Passing to the limit over partitions we can see that $\|\cdot\|_E = k\|\cdot\|_{L_2(\mu)}$ on any component $\rho_s E$, $s \in \Sigma_0$, where $k = \varphi_E(1)$ (here φ_E denotes the fundamental function of the symmetric space E, i.e. $\varphi_E(t) = \|\chi_\sigma\|$, where $\mu(\sigma) = t$). This yields a contradiction and completes the proof.

MAIN THEOREM

THEOREM 1.

(a) Let $E_i = E_i(\Omega_i)$ $(i = 1, 2)$ be projection provided ideal Banach spaces, free from L_2-components. Then for any isometric isomorphism $Q : E_1 \to E_2$ there exists a measurable function q and an invertible measurable transformation $\varphi : \Omega_2 \to \Omega_1$ such that

$$(Qf)(t) = q(t)f(\varphi(t)), \quad \forall f \in E_1. \tag{12}$$

(b) The same conclusion is true providing that E_i $(i = 1, 2)$ are symmetric spaces and the norm on E_1 is not proportional to the norm on $L_2(\mu)$.

Proof. — As it was proved in [Se], the norm of a Hilbert symmetric space is proportional to L_2-norm. Therefore, under the conditions in *(b)*, $\|\cdot\|_{E_2} \neq k\|\cdot\|_{L_2(\mu)}$. By Propositions 1 and 2, $\mathrm{Herm}(E_i) = LM_\infty^r(E_i)$, $i = 1, 2$. Consider the mapping $Q_* : \mathrm{Herm}(E_1) \to \mathrm{Herm}(E_2)$, $Q_*(H) = QHQ^{-1}$. It is not difficult to verify that Q_* is an algebraic isomorphism. It generates an isomorphism of Boolean algebras $\theta_* : \Sigma_1 \to \Sigma_2$, connected with Q by the relation $Q_*(\chi_\sigma) = \chi_{\theta_*\sigma}$, $\sigma \in \Sigma_1$. Define a measurable transformation $\varphi : \Omega_2 \to \Omega_1$ by the equalities

$$\chi_{\theta_*\sigma}\varphi = Q_*(t\chi_\sigma), \quad \sigma \in (\Sigma_1)_0. \tag{13}$$

Obviously,

$$Q_*f = f(\varphi)\ \forall f \in L_\infty^r(\Omega_1), \quad \chi_{\theta_*\sigma} = \chi_\sigma(\varphi),\ \sigma \in \Sigma_1\,. \tag{14}$$

(14) and the definition of Q_* imply:

$$Q(\chi_{\omega'}\chi_{\omega''}) = \chi_{\omega'}(\varphi)Q(\chi_{\omega''}), \quad \omega' \in \Sigma_1,\ \omega'' \in (\Sigma_1)_0. \tag{15}$$

Equalities (15) permit to define the unique measurable function q on Ω_2 such that

$$Q(\chi_\sigma) = q\chi_\sigma(\varphi), \quad \sigma \in (\Sigma_1)_0 \tag{16}$$

By Theorem 2 of [Za3], (16) implies (12). Since the operator Q is inversible, the transformation φ is also inversible. The theorem is proved.

Comments.

1. The theorem is true for more general measure spaces. In particular, the proof did not use continuity of the measure.

2. To prove Theorem 1 in the real case, probably it is impossible simply to pass to the complexifications. Indeed, in general there is no universal definition of a norm in the complexification of an ideal function space that would be at the same time ideal and hold the property of "extension of isometries", *i.e.* such that all of them extend to isometries of the complexification. The simplest example of such situation is the Minkowski plane E^2, where the unit sphere is a regular octagon. This symmetric space has an extra isometry, namely the rotation on angle $\pi/4$ (***). It is proved in [BrSe] that extra isometries of real symmetric sequence spaces could exist in dimensions 2 and 4 only. Due to Theorem 3 from [Ta] any complex symmetric sequence space permits just standard isometries, i.e. permutations and rotations of coordinates.

3. It would be interesting to extend Theorem 1 to more general classes of Banach lattices; for instance, to describe the class of Banach lattices in which any invertible isometry is disjoint, *i.e.* maps disjoint elements to disjoint elements.

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(***) This example was also noted by Yu. Sokolovski.

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