Vanishing cycles of Irregular D-modules

Yves LAURENT

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Université de Grenoble
Institut Fourier
UMR 5582 CNRS/UJF
BP 74
38402 St Martin d'Hères Cedex

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Introduction.

The aim of this paper is to define vanishing cycles of non regular $\mathcal{D}$-modules and use them to study the solutions of these modules.

The vanishing cycles of a $\mathcal{D}$-module were originally introduced in the case of regular holonomic $\mathcal{D}$-modules [12][23]. The definition was extended to non holonomic $\mathcal{D}_X$-module, to complex of $\mathcal{D}_X$-modules and to microdifferential equations in several papers [27][28] [21] [22].

If X is a complex manifold and Y a smooth hypersurface of X, the vanishing cycles of a $\mathcal{D}_X$-module $\mathcal{M}$ is a $\mathcal{D}_Y$-module $\Phi(\mathcal{M})$. Under a suitable condition ($\mathcal{M}$ "specializable"), the module $\Phi(\mathcal{M})$ is coherent. This condition is always satisfied for holonomic $\mathcal{D}_X$-modules and, then $\Phi(\mathcal{M})$ is holonomic.

In the regular holonomic case, the Riemann-Hilbert correspondence is compatible with vanishing cycles. This means that $\Phi(Sol(\mathcal{M})) = Sol(\Phi(\mathcal{M}))$. Here, $Sol(\mathcal{M})$ is the complex of holomorphic solutions of $\mathcal{M}$ that is $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and $\Phi(Sol(\mathcal{M}))$ is the sheaf of geometric vanishing cycles in the theory of Grothendieck-Deligne [7].

If $\mathcal{M}$ is holonomic but not regular, the solutions of $\Phi(\mathcal{M})$ correspond to formal solutions of $\mathcal{M}$, they have no connection with holomorphic solutions. This is, of course, related to the fact that formal solutions of non regular holonomic modules are not convergent. So, it appears that we have to define another notion of vanishing cycles which would be compatible via Riemann-Hilbert with the geometric vanishing cycles for any holonomic module.

On the other hand, the solutions of non regular differential equations in dimension 1 where studied in details by Ramis [25] [26]. He calculated the growth and the index of solutions from the Newton polygon of the equation that is from a finite sequence of rational slopes and of integral heights.

In this paper, we define a family $\Phi_{(r)}(\mathcal{M})$ of vanishing cycles indexed by a rational number r running from $r = 1$ (corresponding to holomorphic solutions) to $r = +\infty$ (formal solutions). They will not be coherent $\mathcal{D}_Y$-modules but perfect $\mathcal{D}_Y^\infty$-modules.

In the holonomic case, the family will be constant in r except for a finite number of values and each module will be a holonomic $\mathcal{D}_Y^\infty$-module (that is equal to $\mathcal{D}_Y^\infty \otimes_{\mathcal{D}_Y} \mathcal{N}$ for some holonomic $\mathcal{D}_Y$-module $\mathcal{N}$). These values of r generalize in higher dimension the slopes of the Newton polygon of Ramis while the integral heights are replaced here by the characteristic cycles of the $\mathcal{D}_Y^\infty$-module. We have $\Phi_{(\infty)}(\mathcal{M}) = \mathcal{D}_Y^\infty \otimes_{\mathcal{D}_Y} \Phi(\mathcal{M})$ and $\Phi(Sol(\mathcal{M})) = Sol(\Phi_{(1)}(\mathcal{M}))$. If $\mathcal{M}$ is regular holonomic $\Phi_{(r)}(\mathcal{M})$ is the same for all r.

More generally, we define a family of vanishing cycle for any $\mathcal{D}_X$-module $\mathcal{M}$ and any object of the derived category of $\mathcal{D}_X$-module. They are defined for each pair (r, s) of rational number such that $1 \leq s \leq r \leq +\infty$ as objects of the derived category of $\mathcal{D}_Y$-modules and denoted by $\Phi_{(r,s)}(\mathcal{M})$.

The properties of these sheaves are connected with the microcharacteristic varieties $Ch_{\Lambda(r,s)}(\mathcal{M})$ of [19]. We show that, under a geometric condition on $Ch_{\Lambda(r_0,s_0)}(\mathcal{M})$, $\Phi_{(r,s)}(\mathcal{M})$ is independent of (r, s) if $s_0 \leq s \leq r \leq r_0$. Moreover it is a $\mathcal{D}_Y^\infty$-module (not only an object of the derived category), and admits locally a finite resolution by free $\mathcal{D}_Y^\infty$-modules. We show that the functor $\Phi_{(r,s)}(.)$ is compatible with duality.

The formal power series used by Ramis are not adapted to the higher dimensional case. We use the sheaf $\mathcal{C}_{Y|X}^{\mathbb{R}}$ of microfunctions of Sato-Kawaï-Kashiwara [29] which lives on the conormal bundle T_Y^*X and some connected sheaves $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s)$ of [20] to control the growth. To work with them we use the sheaf of 2-microdifferential operators $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$.

This sheaf is in fact the sheaf of differential operators associated to $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ exactly as the sheaf $\mathcal{D}^{\infty}_X$ of differential operators of infinite order is associated to holomorphic functions.

We prove a kind of microlocal Cauchy theorem, that is an isomorphism between the solutions of $\mathcal{M}$ in the sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ and the solutions of $\Phi(r,s)(\mathcal{M})$ in the sheaf $\mathcal{O}_Y$ of holomorphic functions on Y.

This shows in particular that these solutions do not depend on (r,s) when $\Phi(r,s)(\mathcal{M})$ do not. We deduce from this a control on the growth of solutions of $\mathcal{M}$ in more familiar sheaves as holomorphic functions with essential singularities on Y or the formal completion of the sheaf of holomorphic functions along Y(see corollary 4.4.3).

With holonomic modules, the geometric condition is always satisfied when $r = s$. More precisely, we proved in [19] that, for each holonomic $\mathcal{D}_X$-module $\mathcal{M}$, there exists a finite number of indexes r, the critical indexes or slopes of $\mathcal{M}$, such that the condition is satisfied for (r,s) as soon as no slope is between r and s.

Let us denote by $r_0 = 1 < r_1 < \cdots < r_N = +\infty$ the sequence of slopes of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. We may therefore associate to each interval $[r_{i-1}, r_i]$ a sheaf of $\Phi_i(\mathcal{M})$ such that $\Phi(r,s)(\mathcal{M}) = \Phi_i(\mathcal{M})$ when $r_{i-1} \leq s \leq r \leq r_i$ and $s < r_i$. This family will be called the family of vanishing cycles of the irregular holonomic module $\mathcal{M}$.

In fact, we use more convenient notations. If $r = r_i$ is a slope then $\Phi\{r\}(\mathcal{M})$ denotes the sheaf $\Phi_i(\mathcal{M})$ and $\Phi(r)(\mathcal{M})$ denotes the sheaf $\Phi_{i+1}(\mathcal{M})$ and if $r_{i-1} \leq r \leq r_i$ then $\Phi\{r\}(\mathcal{M}) = \Phi(r)(\mathcal{M}) = \Phi_i(\mathcal{M})$.

If $\mathcal{M}$ is holonomic, the sheaves $\Phi_{(r)}(\mathcal{M})$ are $\mathcal{D}^{\infty}_Y$-holonomic and we prove that the characteristic cycle of $\Phi_{(r)}(\mathcal{M})$ is equal to the corresponding microcharacteristic cycle of $\mathcal{M}$.

Using the microlocal Cauchy theorem, we deduce that the growth of solutions is given by the slopes of $\mathcal{M}$ and we get an index theorem analogous to Kashiwara's theorem [11]. More precisely, we prove that the complex of microfunction solutions of the holonomic module $\mathcal{M}$ are perverse sheaves whose Euler characteristic may be calculated from the microcharacteristic cycles of $\mathcal{M}$ and this is true for any value of r in $[1, +\infty]$. From this, we show analogous results for holomorphic functions with singularities on Y and Nilsson classes.

The family $\Phi(r)(\mathcal{M})$ makes thus the connection between the geometric invariants defined in [19] and the solutions.

In the one-dimensional case, a detailed study of growth of solutions in sectors of $\mathbb{C}$ through Fourier transform has been performed by Malgrange in [24].

In the case of holonomic modules, N. Honda [9][10], using the techniques of Kashiwara-Kawaï[14], proved the isomorphism between solutions in $\mathcal{C}^{\mathbb{R}}_{Y|X}(s,1)$ and $\mathcal{C}^{\mathbb{R}}_{Y|X}$ when s is lower than some s_0. This number s_0 is independant of the manifold Y. If we compare to our results, it seems to be the maximum of the first slopes r_1 over all manifolds Y of X.

The paper is divided in five sections. In the first one, we recall several definitions that will be used later, namely the Newton polygon of an operator, the microcharacteristic varieties, the holomorphic microfunctions and the 2-microlocal operators.

In section 2, we recall the definition of several sheaves of micofunctions and of their symbols. We show that in some special cases we may restrict microfunctions to a hypersurface.

The key result of the paper is in the third section, namely theorem 3.1.1. This theorem is of the same kind than a theorem of [11] showing that an operator with principal symbol t^m is equivalent to the operator t^m. The theorem of Kashiwara uses microlocal operators of $\mathcal{E}_X^{\mathbb{R}}$. Our theorem replaces the principal symbol by a microlocal symbol and the sheaf $\mathcal{E}_X^{\mathbb{R}}$ by the sheaf $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$.

In section 4, we apply this theorem to define the sheaves of vanishing cycles $\Phi(r,s)(\mathcal{M})$. We study their properties, prove the microlocal Cauchy theorem and deduce results about non holonomic $\mathcal{D}_X$-modules.

In section 5, are the results about holonomic modules, their solutions and the index theorems.

We have added an appendix in which we give another proof of the results of part which does not use the results of Kashiwara-Kawaï on regular holonomic modules.

1 Microcharacteristic varieties of a $\mathcal{D}$-module.

1.1 Some notations.

Let X be a complex analytic manifold and Y a submanifold of X. We will denote by $\Lambda = T_Y^*X$ the conormal bundle to Y with canonical projection $\pi : \Lambda \to Y$ and by $T^*\Lambda$ the cotangent bundle to Λ.

Local coordinates of X, when used, will be denoted by $(x_1, \dots, x_{n-d}, t_1, \dots, t_d)$ and chosen such that $Y = \{t = 0\}$. In this case, we have $\Lambda = T_Y^*X = \{(x, t, \xi, \tau) \in T^*X / t = 0, \xi = 0\}$. We will denote by (x, τ, x^*, τ^*) the corresponding coordinates on $T^*\Lambda$.

The sheaf of holomorphic functions on X will be denoted by $\mathcal{O}_X$ and the sheaf of differential operators with holomorphic coefficients on X by $\mathcal{D}_X$. The sheaf of microdifferential operators of [29] on T^*X will be denoted by $\mathcal{E}_X$.

We recall that the restriction to the zero section X of T^*X of the sheaf $\mathcal{E}_X$ is $\mathcal{D}_X$ and that $\pi^{-1}\mathcal{D}_X$ is a subsheaf of $\mathcal{E}_X$.

1.2 Newton Polygon of a differential operator.

The Newton Polygon of a differential or microdifferential operator near Y is a convex subset of $\mathbb{R}^2$ defined in [18] and [19]. We recall here its definition in local coordinates.

A differential operator P defined in a neighborhood of Y or a microdifferential operator P defined in a neighborhood of Λ has a symbol

$$P = \sum_{j \le m} P_j(x, t, \xi, \tau)$$

where each P_j is a holomorphic function near Λ homogeneous of degree j in (ξ, τ) (in the differential case, they are polynomial in (ξ, τ) and j is positive). They have a representation in Taylor's series :

$$P_j(x, t, \xi, \tau) = \sum_{\alpha, \beta} P_j^{\alpha\beta}(x, \tau) t^\alpha \xi^\beta$$

and we define the following functions on $T^*\Lambda$:

$$P_{ij}(x, \tau, x^*, \tau^*) = \sum_{|\alpha|+|\beta|=i} P_j^{\alpha\beta}(x, \tau)(-\tau^*)^\alpha (x^*)^\beta$$

The Newton Polygon $N_\Lambda(P)$ of P along Λ is the convex hull of the union of the sets

$$S_{ij} = \{(\lambda,\mu) \in \mathbb{R}^2 / \lambda + \mu \leq j, \mu \leq j - i\}$$

for all (i,j) such that $P_{ij} \neq 0$.

For each rational number r such that $1 \leq r \leq +\infty$, D_r is the supporting line of $N_\Lambda(P)$ with slope $-1/r$. It is the line with equation $\{i + r(j-i) = a\}$ which meets the boundary of $N_\Lambda(P)$. The principal symbols of the operator P are defined in the following way :

For $1 < r < +\infty$, $\sigma_\Lambda^{(r)}(P)$ (resp. $\sigma_\Lambda^{\{r\}}(P)$) is the function P_{ij} where $(i, j-i)$ is the point of $N_\Lambda(P) \cap D_r$ such that i is maximum (resp. minimum).

It is proved in [19] that these functions are well defined (i.e. independent of coordinates) on $T^*\Lambda$ and they are multiplicative, that is $\sigma_\Lambda^{(r)}(PQ) = \sigma_\Lambda^{(r)}(P)\sigma_\Lambda^{(r)}(Q)$.

For $1 \leq s < r \leq +\infty$, the symbol $\sigma_\Lambda^{(r,s)}(P)$ is defined as $P_{i,i+k}$ if the intersection of D_r and D_s has integral coordinates (i,k) and it is 0 otherwise.

Another way to define these symbols is to consider the 'critical indexes' of P. A number r such that $1 \leq r \leq +\infty$ is a critical index if $-1/r$ is a slope of $N_\Lambda(P)$ that is if $D_r \cap N_\Lambda(P)$ is *not* reduced to one point.

Then if r is not a critical index we have $\sigma_\Lambda^{(r)}(P) = \sigma_\Lambda^{\{r\}}(P) = P_{ij}$ where $(i, j-i)$ is the unique point where D_r meets $N_\Lambda(P)$. If r is a critical index, then $\sigma_\Lambda^{(r)}(P)$ (resp. $\sigma_\Lambda^{\{r\}}(P)$) is equal to $\sigma_\Lambda^{(r-\varepsilon)}(P)$ (resp. $\sigma_\Lambda^{(r+\varepsilon)}(P)$) for $0 < \varepsilon << 1$. The function $\sigma_\Lambda^{(r,s)}(P)$ is equal to $\sigma_\Lambda^{(r)}(P)$ if there is no critical index between r and s and 0 otherwise.

These definitions are independent of local coordinates and they may be extended to any conic lagrangian submanifold Λ of T^*X.

1.3 Microcharacteristic varieties.

The microcharacteristic varieties were defined in [19] where we refer for details about the definitions recalled here.

Let Λ be a lagrangian conic submanifold Λ of T^*X and $T^*\Lambda$ its cotangent bundle. When we consider $\mathcal{D}_X$-modules, we assume that Λ is the conormal bundle $T^*_Y X$ to a submanifold Y of X.

The manifold Λ is provided with a canonical action of $\mathbb{C}^*$ which defines an action of $\mathbb{C}^*$ on $T^*\Lambda$ denoted by H_∞. On the other hand, $T^*\Lambda$ is a vector bundle on Λ and is therefore provided with another action of $\mathbb{C}^*$ denoted by H_0.

If r is a nonnegative rational number, written as $r = p/q$ with relatively prime integers p and q, we set $H_r = H_\infty^p H_0^q$.

In local coordinates H_r is given by :

$$\begin{aligned} H_0(\lambda) &: (x,\tau,x^*,\tau^*) \longrightarrow (x,\tau,\lambda x^*,\lambda\tau^*) \\ H_\infty(\lambda) &: (x,\tau,x^*,\tau^*) \longrightarrow (x,\lambda\tau,x^*,\lambda^{-1}\tau^*) \\ H_r(\lambda) &: (x,\tau,x^*,\tau^*) \longrightarrow (x,\lambda^p\tau,\lambda^q x^*,\lambda^{q-p}\tau^*) \end{aligned}$$

We denote by $\mathcal{O}_{[T^*\Lambda]}(i,j)$ the sheaf of holomorphic functions on $T^*\Lambda$ which are homogeneous of degree j for H_1 and i for H_0 and are polynomial in the fibers of $\pi : T^*\Lambda \to \Lambda$ and by $\mathcal{O}_{[T^*\Lambda]}$ the sum $\oplus\mathcal{O}_{[T^*\Lambda]}(i,j)$.

The restriction to Y of the sheaf $\mathcal{D}_X$ is provided with two canonical filtrations. The first one is the usual filtration by the order of operators witch is denoted by $(\mathcal{D}_{X,m})_{m\geq 0}$ and the second one is defined by [12] :

$$V_k\mathcal{D}_X = \{P \in \mathcal{D}_X|_Y / \forall j \in \mathbb{Z},\;\; P\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\}$$

where $\mathcal{I}_Y$ is the ideal of definition of Y and $\mathcal{I}_Y^j = \mathcal{O}_X$ if $j \leq 0$.

These definitions extend to the sheaf $\mathcal{E}_X$ and any lagrangian conic manifold Λ in the following way [15],[19] : the usual filtration denoted by $(\mathcal{E}_{X,m})$, we define $V_0\mathcal{E}_X$ as the sub-algebra of $\mathcal{E}_X|_\Lambda$ generated by the operators of order 1 whose principal symbol vanishes on Λ and $V_k\mathcal{E}_X = \mathcal{E}_{X,k}V_0\mathcal{E}_X$.

Now if $r = p/q$ is a rational number such that $1 < r < \infty$, we define a F_r-filtration on $\mathcal{E}_X|_\Lambda$ by :

$$F_r^k\mathcal{E}_X = \sum_{(p-q)m+qn=k} \mathcal{E}_{X,n} \cap V_m\mathcal{E}_X$$

The associated graded ring $gr_{F_r}\mathcal{E}_X$ is isomorphic to $\pi_*\mathcal{O}_{[T^*\Lambda]}$.

Now we may define the microcharacteristic varieties of a coherent $\mathcal{E}_X$-module or of coherent $\mathcal{D}_X$-module like the classical definition of the characteristic variety of a coherent $\mathcal{D}_X$-module. A filtration of a coherent $\mathcal{E}_X$-module $\mathcal{M}$ is a good F_r-filtration if it is locally of finite type that is of the form $\mathcal{M}_k = F_r^k\mathcal{E}_X u_1 + \cdots + F_r^k\mathcal{E}_X u_p$ for some local sections $u_1, \ldots, u_p$ of $\mathcal{M}$.

If $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module, it has (locally) a good F_r-filtration and the associated graded ring $gr_{F_r}\mathcal{M}$ is a coherent $\pi_*\mathcal{O}_{[T^*\Lambda]}$-module which defines an analytic subvariety $\Sigma_\Lambda^{(r)}(\mathcal{M})$ of $T^*\Lambda$ and a positive analytic cycle $\widetilde{\Sigma}_\Lambda^{(r)}(\mathcal{M})$ with support this subvariety.

They are independent of the good filtration and we call them the microcharacteristic variety and the microcharacteristic cycle of type r.

It is proved in [19] that $\Sigma_\Lambda^{(r)}(\mathcal{M})$ is involutive and homogeneous for H_r and that there exists a finite sequence $1 = r_0 < r_1 < \cdots < r_N = +\infty$ of rational numbers such that $\Sigma_\Lambda^{(r)}(\mathcal{M})$ is independent of $r \in]r_i, r_{i+1}[$ for $i = 1 \ldots N-1$. The numbers r_i are called the critical indexes or slopes of $\mathcal{M}$.

If r is not one of the critical indexes the variety $\Sigma_\Lambda^{(r)}(\mathcal{M})$ is homogeneous for several r hence for any r, we will say that it is bihomogeneous.

So, the family $\Sigma_\Lambda^{(r)}(\mathcal{M})$ for $1 < r < +\infty$ is the union of two finite families, the first one indexed by the critical indexes $r_1 \ldots r_N$ and the second one by the open intervals $]r_i, r_{i+1}[$. The members of the second family are bihomogeneous.

Because of their relations to the solutions of the module, which we will study in this paper, it is convenient to rename these varieties in the following way :

$$Ch_{\Lambda\{r\}}(\mathcal{M}) = \Sigma_\Lambda^{(r+\varepsilon)}(\mathcal{M}) \quad \text{and} \quad Ch_{\Lambda(r)}(\mathcal{M}) = \Sigma_\Lambda^{(r-\varepsilon)}(\mathcal{M})$$

where $0 < \varepsilon << 1$.

By definition, these microcharacteristic varieties are all bihomogeneous and if r is not a critical index we have :

$$Ch_{\Lambda\{r\}}(\mathcal{M}) = Ch_{\Lambda(r)}(\mathcal{M}) = \Sigma_\Lambda^{(r)}(\mathcal{M})$$

The microcharacteristic cycles $\widetilde{Ch}_{\Lambda\{r\}}(\mathcal{M})$ and $\widetilde{Ch}_{\Lambda(r)}(\mathcal{M})$ are defined in the same way from $\widetilde{\Sigma}_{\Lambda}^{(r)}(\mathcal{M})$.

If $\mathcal{I}$ is a coherent ideal of $\mathcal{E}_X$, the microcharacteristic varieties of $\mathcal{M} = \mathcal{E}_X/\mathcal{I}$ are :

$$Ch_{\Lambda\{r\}}(\mathcal{M}) = \{\theta \in T^*\Lambda / \ \forall P \in \mathcal{I}, \ \ \sigma_{\Lambda}^{\{r\}}(P) = 0\} \tag{1.3.1}$$

$$Ch_{\Lambda(r)}(\mathcal{M}) = \{\theta \in T^*\Lambda / \ \forall P \in \mathcal{I}, \ \ \sigma_{\Lambda}^{(r)}(P) = 0\} \tag{1.3.2}$$

The same is true for the varieties $\Sigma_{\Lambda}^{(r)}$ with the functions $\sum_{(i,j)\in D_r} P_{ij}$.

As we will see later, the microcharacteristic variety $\Sigma_{\Lambda}^{(r)}$ characterizes the solutions of a given growth (r). If we want to prove a relation between solutions of growth (r) and (s), we need double indexed varieties $Ch_{\Lambda(r,s)}(\mathcal{M})$. Their definition, given in [19], uses the bifiltration associated to the two filtrations F_r and F_s :

$$F_{rs}^{kl}\mathcal{E}_X = F_r^k\mathcal{E}_X \cap F_s^l\mathcal{E}_X$$

Then the definitions of a good filtration and of the associate bigraded module are similar to the previous one, this bigraded module is a coherent $\pi_*\mathcal{O}_{[T^*\Lambda]}$-module which defines a microcharacteristic variety $Ch_{\Lambda(r,s)}(\mathcal{M})$.

A point $\theta \in T^*\Lambda$ is not in $Ch_{\Lambda(r,s)}(\mathcal{M})$ if and only if for each germ of $\mathcal{M}$ at $\pi(\theta)$ there is an operator P such that $Pu = 0$ and $\sigma_{\Lambda}^{(r,s)}(P)(\theta) \neq 0$.

The functor $Ch_{\Lambda(r,s)}(.)$ is additive, that is $Ch_{\Lambda(r,s)}(\mathcal{M}) = Ch_{\Lambda(r,s)}(\mathcal{M}'') \cup Ch_{\Lambda(r,s)}(\mathcal{M}')$ for any exact sequence :

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

There is a relation of inclusion between these varieties :

Lemma 1.3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module and r, r', s, s' be four numbers such that $r \geq r' > s' \geq s$. Then*

$$Ch_{\Lambda(r)}(\mathcal{M}) \subset Ch_{\Lambda(r',s')}(\mathcal{M}) \subset Ch_{\Lambda(r,s)}(\mathcal{M})$$

and the first inclusion is an equality if $0 < r - s << 1$. We have also :

$$\Sigma_{\Lambda}^{(r')}(\mathcal{M}) \subset Ch_{\Lambda(r,s)}(\mathcal{M})$$

This result was proved in [19, prop. 3.3.1.]. The proof is easy in the case of one operator and the extension to the case of a module follows from 1.3.1 and the corresponding formulas for $\Sigma_{\Lambda}^{(r')}(\mathcal{M})$ and $Ch_{\Lambda(r,s)}(\mathcal{M})$.

1.4 Critical indexes and holonomy.

In the case of one operator P, the critical indexes and the functions $\sigma_{\Lambda}^{(r,s)}(P)$ are closely related. In the case of $\mathcal{D}_X$- or $\mathcal{E}_X$-module, such relations are still true under additional assumptions, for example holonomy.

There is a canonical hypersurface in $T^*\Lambda$ which is denoted by S_Λ. It is the characteristic variety of the Euler vector field associated to the action of $\mathbb{C}^*$ on Λ. In local coordinates we have :

$$S_\Lambda = \{(x,\tau,x^*,\tau^*)/ \ <\tau,\tau^*>=0\}$$

This hypersurface is important here because our main hypothesis on coherent $\mathcal{D}_X$- or $\mathcal{E}_X$-modules will be $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$.

This condition is equivalent to "for each section u of $\mathcal{M}$ there is an operator P such that $Pu=0$ and $\sigma_\Lambda^{(r,s)}(P)$ is a local equation for S_Λ".

This condition is satisfied by large classes of $\mathcal{D}_X$- and $\mathcal{E}_X$-modules as shown by the following results.

Proposition 1.4.1. *If $\mathcal{M}$ is a holonomic $\mathcal{E}_X$-module, the microcharacteristic varieties $Ch_{\Lambda(r)}(\mathcal{M})$ and $Ch_{\Lambda\{r\}}(\mathcal{M})$ are subvarieties of S_Λ for any $r \in [1,+\infty]$.*

This proposition was proved in [19]. More precisely, we proved that the microcharacteristic varieties of a holonomic module are lagrangian bihomogeneous subvarieties of $T^*\Lambda$ [19, theorem 4.1.1.] and that all lagrangian bihomogeneous subvarieties of $T^*\Lambda$ are contained in S_Λ [19, proposition 4.3.1.].

Proposition 1.4.2. *If r_k and r_{k+1} are two consecutive critical indexes of a coherent $\mathcal{E}_X$-module $\mathcal{M}$ then for any (r,s) such that $r_k \geq r > s \geq r_{k+1}$ we have :*

$$Ch_{\Lambda(r,s)}(\mathcal{M}) = Ch_{\Lambda(r)}(\mathcal{M}) = Ch_{\Lambda(r_k)}(\mathcal{M})$$

This proposition will be proved at the end of the section. It implies in particular that, if r_k and r_{k+1} are two consecutive critical indexes and if $Ch_{\Lambda(r)}(\mathcal{M}) \subset S_\Lambda$ for some $r \in]r_k, r_{k+1}[$, then $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$ for any (r,s) with $r_k \geq r > s \geq r_{k+1}$. Conversely, we have :

Proposition 1.4.3. *If $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module such that $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$, $\mathcal{M}$ has no critical index for $\mathcal{M}$ in $]s,r[$.*

Proof. We know from lemma 1.3.1 that $\Sigma_\Lambda^{(r')}(\mathcal{M}) \subset Ch_{\Lambda(r,s)}(\mathcal{M})$ if $r > r' > s$ hence by the hypothesis we have $\Sigma_\Lambda^{(r')}(\mathcal{M}) \subset S_\Lambda$. We conclude with [21, Lemma 4.5.1.] which asserts that if $\Sigma_\Lambda^{(r')}(\mathcal{M}) \subset S_\Lambda$ then $\Sigma_\Lambda^{(r')}(\mathcal{M})$ is bihomogeneous and thus r' is not a critical index. □

In the holonomic case, these results take a very simple form :

Corollary 1.4.4. *If $\mathcal{M}$ is a holonomic $\mathcal{E}_X$-module, the condition $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$ is true if and only if there is no critical index between r and s.*

To prove proposition 1.4.2, we will use the following lemma :

Lemma 1.4.5. *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module and r, r', s, s' be four numbers such that $r > r' > s' > s$. Then*

$$Ch_{\Lambda(r,s)}(\mathcal{M}) = Ch_{\Lambda(r,s')}(\mathcal{M}) \cup Ch_{\Lambda(r',s)}(\mathcal{M})$$

Proof. The proof of the lemma make full use of the results of [18], in particular $Ch_{\Lambda(r,s)}(\mathcal{M})$ is the support of the sheaf $\mathcal{E}^2_{\Lambda(r,s)} \otimes_{q^{-1}\mathcal{E}_X} q^{-1}\mathcal{M}$ with $q : T^*\Lambda \longrightarrow T^*X$.

The sheaf $\mathcal{E}^2_{\Lambda(r,s)}$ is a sheaf of rings on $T^*\Lambda$ whose definition will be recalled in 2.3, but here we use only two propeties of this sheaf :

1)$\mathcal{E}^2_{\Lambda(r,s)}$ is flat on $q^{-1}\mathcal{E}_X$ ([18, theorem 2.6.10.])

2)If $r > r' > s$, then $\mathcal{E}^2_{\Lambda(r',s)}/\mathcal{E}^2_{\Lambda(r,s)}$ is independant of s ([18, theorem 2.3.1.]), we denote it here by $\mathcal{E}[r',r]$.

So we get an exact sequence :

$$0 \longrightarrow \mathcal{E}^2_{\Lambda(r,s)} \longrightarrow \mathcal{E}^2_{\Lambda(r',s)} \longrightarrow \mathcal{E}[r',r] \longrightarrow 0$$

As $\mathcal{E}^2_{\Lambda(r',s)}$ is flat on $q^{-1}\mathcal{E}_X$ we get an exact sequence :

$$0 \longrightarrow \mathcal{T}\mathrm{or}_1(\mathcal{E}[r',r], \mathcal{M}) \longrightarrow \mathcal{E}^2_{\Lambda(r,s)} \otimes \mathcal{M} \longrightarrow \mathcal{E}^2_{\Lambda(r',s)} \otimes \mathcal{M}$$

and the same with s replaced by s'.

This shows that $Ch_{\Lambda(r,s)}(\mathcal{M})$, which is the support of $\mathcal{E}^2_{\Lambda(r,s)} \otimes \mathcal{M}$ is contained in the union of $Ch_{\Lambda(r',s)}(\mathcal{M})$ and of the support of $\mathcal{T}\mathrm{or}_1(\mathcal{E}[r',r], \mathcal{M})$ which itself is contained in $Ch_{\Lambda(r,s')}(\mathcal{M})$, hence :

$$Ch_{\Lambda(r,s)}(\mathcal{M}) \subset Ch_{\Lambda(r,s')}(\mathcal{M}) \cup Ch_{\Lambda(r',s)}(\mathcal{M})$$

Lemma 1.3.1 shows that this inclusion is an equality. □

Let us now come to the proof of proposition 1.4.2. We have to prove that $r_0 = r_k$, where r_0 is the highest r in $[r_{k+1}, r_k]$ such that :

$$Ch_{\Lambda(r,r_{k+1})}(\mathcal{M}) = Ch_{\Lambda(r)}(\mathcal{M})$$

Lemma 1.3.1 says that $Ch_{\Lambda(r)}(\mathcal{M}) \subset Ch_{\Lambda(r,s)}(\mathcal{M})$ and that the inclusion is an equality if $0 < r - s << 1$, so $r_0 > r_{k+1}$.

On the other hand, if s is not a critical index for $\mathcal{M}$ then $\Sigma^{(s)}_\Lambda(\mathcal{M})$ is biconic and the same proof than that of [19, prop. 3.3.1.] shows that there exists some $\varepsilon > 0$ such that $Ch_{\Lambda(s+\varepsilon,s-\varepsilon)}(\mathcal{M}) = Ch_{\Lambda(s)}(\mathcal{M})$. This property and lemma 1.4.5 show that $r_0 = r_k$.

2 Microlocalization and second microlocalization

We will use several families of sheaves which were defined in previous papers. Let us try first to classify them and then we will recall as briefly as possible their definitions. So sections 2.1 to 2.4 will simply remind known definitions while the next sections will introduce new results for multivalued sections of some of these sheaves.

There are three levels to consider: local, microlocal and 2-microlocal.

At the local level, that is on the complex variety X, are the sheaves of holomorphic functions, holomorphic hyperfunctions and differential operators. The principal results of this paper, at least the most explicit ones, are given at this level in section 5 but they are corollaries of microlocal results.

At the microlocal level, that is on the cotangent bundle T^*X to X, are the sheaves of holomorphic microfunctions, real holomorphic microfunctions and microdifferential operators. There is also a class of 2-differential operators. This level is the natural one for our results, for example to define the vanishing cycles.

The 2-microlocal level is the cotangent bundle $T^*\Lambda$ to a lagrangian submanifold Λ of T^*X as in section 1. It will appear in section 3.2 and will be fully used in the appendix.

At each level, the sheaves are indexed by a couple (r,s) of rational numbers giving the growth of the symbols.

2.1 Holomorphic hyperfunctions.

We keep the notations of §1.1 and denote by d the codimension of Y in X.

The sheaves $\mathcal{B}^\infty_{Y|X}$ and $\mathcal{B}_{Y|X}$ of holomorphic hyperfunctions on Y have been defined primitively in [29] (see also [31]). The sheaf $\mathcal{B}^\infty_{Y|X}$ is the d-th local cohomology group of $\mathcal{O}_X$ with support on Y while $\mathcal{B}_{Y|X}$ is the corresponding algebraic cohomology group :

$$\mathcal{B}^\infty_{Y|X} = \mathcal{H}^d_Y(\mathcal{O}_X) \qquad\qquad \mathcal{B}_{Y|X} = \mathcal{H}^d_{[Y]}(\mathcal{O}_X)$$

If Y has codimension 1, $\mathcal{B}^\infty_{Y|X}$ is thus the sheaf of holomorphic functions on $X-Y$ modulo holomorphic functions on X while $\mathcal{B}_{Y|X}$ is the subsheaf generated by meromorphic functions on Y.

If $d>1$, there is a similar representation if we write Y as a complete intersection $Y = Y_1 \cap \dots Y_d$ of smooth hypersurfaces: let V be a domain of holomorphy in Y and U any holomorphic neighborhood of V in X. Let $\widehat{U}$ be the complementary in U of $\cup_{i=1,\dots,d} Y_i$ and for $i=1,\dots,d$, $\widehat{U}_i$ be the complementary in U of $\cup_{k\neq i} Y_k$. Then

$$\Gamma(V,\mathcal{B}^\infty_{Y|X}) = \mathcal{O}_X(\widehat{U}) \Big/ \oplus_{i=1\dots d}\mathcal{O}_X(\widehat{U}_i)$$

In [18], we generalized these definitions to sheaves $\mathcal{B}_{Y|X(r)}$ and $\mathcal{B}_{Y|X\{r\}}$ for all rational $r\geq 1$. The sheaf $\mathcal{B}_{Y|X(r)}$ (resp. $\mathcal{B}_{Y|X\{r\}}$) is the subsheaf of $\mathcal{B}^\infty_{Y|X}$ corresponding to functions whose growth is less than $\exp(C/|t|^{\frac{1}{r-1}})$ for some $C>0$ (resp. any $C>0$) if $t=(t_1,\dots,t_d)$ is a local system of equations of Y.

To unify the notations, we set $\mathcal{B}_{Y|X\{1\}} = \mathcal{B}^\infty_{Y|X}$ and $\mathcal{B}_{Y|X(\infty)} = \mathcal{B}_{Y|X}$.

Let us fix local coordinates $(x_1,\dots,x_{n-d},t_1,\dots,t_d)$ such that $Y=\{t=0\}$. A function f of $\mathcal{O}_X(\widehat{U})$ may be represented in Laurent series as :

$$f(x,t) = \sum_{\alpha\in\mathbb{Z}^n} f_\alpha(x) t_1^{\alpha_1}\dots t_d^{\alpha_d}$$

Let us denote as in [29] :

$$\Phi_n(t) = \frac{1}{2i\pi}\frac{n!}{(-t)^{n+1}}$$

As positive powers of any t_k gives 0 in $\mathcal{B}^\infty_{Y|X}$, a section of $\mathcal{B}^\infty_{Y|X}$ on V, hence of $\mathcal{B}_{Y|X(r)}$ or $\mathcal{B}_{Y|X\{r\}}$ for any r, has a unique representation as :

$$g(x,t) = \sum_{\alpha\in\mathbb{N}^n} g_\alpha(x)\Phi_{\alpha_1}(t_1)\dots\Phi_{\alpha_d}(t_d)$$

By definition, the symbol of the hyperfunction represented by $g(x,t)$ is the function :

$$u(x,\tau)=\sum_{\alpha\in\mathbb{N}^n} g_\alpha(x)\tau^\alpha$$

The growth conditions on the functions give the following characterization of the sheaves of holomorphic hyperfunctions through their symbols [29],[18] :

The sections of $\mathcal{B}_{Y|X}(r)$ on an open set V of Y are the holomorphic functions $u(x,\tau)$ on $V\times\mathbb{C}^d$ such that :

$$\forall K\subset\subset V,\ \exists C_0, C>0,\ \forall (x,\tau)\in K\times\mathbb{C}^d,\ |u(x,\tau)|<C_0\exp(C|\tau|^{1/r})$$

while the sections of $\mathcal{B}_{Y|X}\{r\}$ are the holomorphic functions $u(x,\tau)$ on $V\times\mathbb{C}^d$ such that :

$$\forall K\subset\subset V,\ \forall\varepsilon>0,\ \exists C_\varepsilon>0,\ \forall (x,\tau)\in K\times\mathbb{C}^d,\ |u(x,\tau)|<C_\varepsilon\exp(\varepsilon|\tau|^{1/r})$$

If $r=+\infty$, the function u is polynomial in τ.

The sheaf $\mathcal{D}_X^\infty$ of differential operators of infinite order on X may be defined as the cohomology group :

$$\mathcal{D}_X^\infty=\mathcal{B}^\infty_{X|X\times X}\otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X=\mathcal{H}^n_X(\mathcal{O}_{X\times X})\otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X \qquad (2.1.1)$$

where X is identified to the diagonal of $X\times X$, p_2 is the second projection $X\times X\longrightarrow X$ and Ω_X is the sheaf of holomorphic differential forms of maximum degree $n=\dim X$.

This definition means that a differential operator is represented by its kernel which is a holomorphic form on $X\times X$ with singularities on the diagonal X. The sheaf of differential operators of finite order on X is :

$$\mathcal{D}_X=\mathcal{B}_{X|X\times X}\otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$

The representation by symbols given for holomorphic hyperfunctions gives the ordinary symbols for differential operators. This means that the form

$$\Phi_k(t-t')dt'=\frac{(-1)^{k+1}}{2i\pi}\frac{k!}{(t-t')^{k+1}}dt'$$

is the kernel of the operator $\left(\frac{\partial}{\partial t}\right)^k$ whose symbol is τ^k. This is nothing else than the Cauchy formula.

2.2 Real holomorphic microfunctions.

The sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ of real holomorphic microfunctions has been defined in [29] as the microlocalization of the sheaf $\mathcal{O}_X$ of holomorphic functions, that is the Fourier transform of the localization of $\mathcal{O}_X$ along Y (see [16] for a detailed study of microlocalization). By definition, it is a sheaf on T^*_YX which is conic for the canonical action of $\mathbb{R}^*_+$.

A symbolic calculus for these microfunction was established by Boutet de Monvel [5] and Aoki [1]. In [20], we defined a family of sheaves $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ for $+\infty\geq r\geq s\geq 1$ so that $\mathcal{C}^{\mathbb{R}}_{Y|X}(1,1) = \mathcal{C}^{\mathbb{R}}_{Y|X}$. The definition did not use microlocalization but was given by the symbols :

We consider an open subset U_0 of $T^*_Y X$ with coordinates (x,τ). An open subset U of U_0 is said to be $\mathbb{R}$-conic if it is invariant under the action of $\mathbb{R}^*_+$ in τ and convex if the fiber over each x is convex. For such sets, $U' \subset\subset U$ means that $U' \cap \{(x,\tau)/|\tau| = 1\}$ is relatively compact in U. Then we set [20] :

(i) $\mathcal{S}_+(c,r,U)$ is the set of holomorphic functions on U such that :

$$\forall U' \subset\subset U,\ \exists C > 0,\ \forall (x,\tau) \in U', |f(x,\tau)| < Ce^{c|\tau|^{1/r}}$$

(ii) $\mathcal{S}_-(s,U)$ is the set of holomorphic functions on U such that :

$$\forall U' \subset\subset U,\ \exists \delta > 0,\ \exists C > 0,\ \forall (x,\tau) \in U', |f(x,\tau)| < Ce^{-\delta|\tau|^{1/s}}$$

If U is a $\mathbb{R}$-conic convex open subset of U_0, if $+\infty > r > s \geq 1$ the set of sections of $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ on U is equal to :

$$\Gamma(U, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)) = \varprojlim_{U' \subset\subset U} \varinjlim_{c>0} \mathcal{S}_+(c,r,U)\,/\mathcal{S}_-(s,U) \tag{2.2.1}$$

If $r = s$ we take the same formula, the injective limit on $c > 0$ being replaced by the projective limit :

$$\Gamma(U, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,r)) = \varprojlim_{U' \subset\subset U} \varprojlim_{c>0} \mathcal{S}_+(c,r,U)\,/\mathcal{S}_-(r,U)$$

If $r = +\infty$ and s is finite, $\mathcal{S}_+(c,r,U)$ is the set of holomorphic functions with polynomial growth. If $r = s = +\infty$, the definition of [20] is a little more complicated and will not be used in this paper.

The sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ is now given on $\mathbb{R}$-conic convex open subset of U_0 and this generates a unique sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ on U_0. To define sheaf on $T^*_Y X$, we have need the formulas connecting the symbols in two different coordinate systems and for this we refer to [20, Prop 2.2.5.].

Real holomorphic microfunctions have also a representation by an alternate system of symbols which, after Boutet de Monvel, we call formal symbols :

(i) $\widehat{\mathcal{S}}_+(c,r,s,U)$ is the set of series $\sum_{k\geq 0} f_k$ of holomorphic functions on U such that :

$$\forall U' \subset\subset U,\ \exists C > 0,\ \exists A > 0,\ \forall (x,\tau) \in U',\ \forall k \geq 0,\ |f_k(x,\tau)| < CA^k|\tau|^{-k}(k!)^s e^{c|\tau|^{1/r}}$$

(ii) $\widehat{\mathcal{S}}_-(c,r,s,U)$ is the subset of $\widehat{\mathcal{S}}_+(c,r,s,U)$ of series $\sum f_k$ such that $\sum g_k$ with :

$$g_k(x,\tau) = \sum_{i=0}^{k-1} f_i(x,gt)$$

is still an element of $\widehat{\mathcal{S}}_+(c,r,s,U)$.

If U is a $\mathbb{R}$-conic open subset of U_0, if $+\infty > r > s \geq 1$ the set of sections of $\mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}$ on U is equal to :

$$\Gamma(U, \mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}) = \varprojlim_{U' \subset\subset U} \varinjlim_{c>0} \widehat{\mathcal{S}}_+(c,r,s,U) \Big/ \widehat{\mathcal{S}}_-(c,r,s,U)$$

Note that here U does not need to be convex as before. If U is convex, the isomorphism between the two kinds of symbol is induced by the natural map $\mathcal{S}_+(c,r,U) \to \widehat{\mathcal{S}}_+(c,r,s,U)$ which associates to a function f the series given by $f_0 = f$ and $f_k = 0, i > 0$ [20].

The sheaves $\mathcal{C}_{Y|X(r,s)}$ of holomorphic microfunctions on $T^*_Y X$ are simply the global section of real holomorphic microfunctions, that is :

$$\mathcal{C}_{Y|X(r,s)}|_Y = \mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}|_Y \qquad \mathcal{C}_{Y|X(r,s)}|_{\dot{T}^*_Y X} = \gamma^{-1}\gamma_* \left(\mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}|_{\dot{T}^*_Y X} \right) \tag{2.2.2}$$

where $\dot{T}^*_Y X = T^*_Y X - Y$ and γ is the canonical projection of $\dot{T}^*_Y X$ to the associated projective bundle $\mathbb{P}^*_Y X$.

They were defined in [18], generalizing the definitions of [29] with the correspondence :

$$\mathcal{C}_{Y|X(1,1)} = \mathcal{C}^{\infty}_{Y|X} \qquad \mathcal{B}_{Y|X(1)} = \mathcal{B}^{\infty}_{Y|X}$$
$$\mathcal{C}_{Y|X(\infty,1)} = \mathcal{C}_{Y|X} \qquad \mathcal{B}_{Y|X(\infty)} = \mathcal{B}_{Y|X}$$

If $T^*_Y X = \{(x,t,\xi,\tau) \in T^*X / t = 0, \xi = 0\}$, a section of $\mathcal{C}_{Y|X(r,s)}$ on an open set U of $T^*_Y X$ is a formal series $\sum_{j\in\mathbb{Z}} f_j(x,\tau)$ of holomorphic functions on U such that :

(i) f_j is homogeneous of degree j in τ.

(ii) $\forall K \subset\subset U,\ \exists C > 0,\ \forall j < 0,\ \forall (x,\tau) \in K,\ |f_j(x,\tau)| < C^{-j}(-j)!^s$

(iii) $\forall K \subset\subset U,\ \exists C > 0,\ \forall j \geq 0,\ \forall (x,\tau) \in K,\ |f_j(x,\tau)| < C^{j+1}\frac{1}{(j!)^r}$

If $r = s$ then (iii) is replaced by

(iii)' $\forall K \subset\subset U,\ \forall \varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall j \geq 0,\ \forall (x,\tau) \in K,\ |f_j(x,\tau)| < C_\varepsilon \varepsilon^j \frac{1}{(j!)^r}$

and if $r = +\infty$ then (iii) is replaced by

(iii)” $\exists m \in \mathbb{Z},\ \forall j > m,\ f_j \equiv 0$

(if $s = +\infty$ condition (ii) is void).

From this definition, we can see that $\mathcal{B}_{Y|X(r)}$ is the restriction to the zero section Y of $T^*_Y X$ of the sheaf $\mathcal{C}_{Y|X(r,s)}$ when $r > s$ and that $\mathcal{B}_{Y|X\{r\}}$ is the restriction of $\mathcal{C}_{Y|X(r,s)}$ when $r = s$.

The canonical morphism from $\mathcal{C}^{\infty}_{Y|X(r,s)}$ to $\mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}$ is injective. To a symbol $\sum_{j\in\mathbb{Z}} f_j$ of $\mathcal{C}^{\infty}_{Y|X(r,s)}$ we associate a formal symbol $\sum_{k\geq 0} g_k$ of $\mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}$ given by $g_0 = \sum_{j\geq 0} f_j$ and $g_k = f_k$ if $k < 0$.

Microlocal operators and microdifferential operators of finite or infinite order are defined from holomorphic microfunctions as in (2.1.1) :

$$\mathcal{E}^{\mathbb{R}}_X = \mathcal{C}^{\mathbb{R}}_{X|X\times X} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$
$$\mathcal{E}^{\infty}_X = \mathcal{C}^{\infty}_{X|X\times X} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$
$$\mathcal{E}_X = \mathcal{C}_{X|X\times X} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$

2.3 Second microlocalization.

The sheaf $\mathcal{D}_X^\infty$ of differential operators of infinite order on X may be defined as a cohomology group by (2.1.1). It is shown in [20] that $\mathcal{O}_{X\times X}$ may be replaced in this definition by real holomorphic microfunctions to define "2-microlocal operators" :

$$\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s) = \mathcal{H}^n_{T_Y^*X}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X}(r,s)\right)\otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$

and by holomorphic microfunctions to define "2-microdifferential operators" :

$$\mathcal{D}_\Lambda^{2(\infty,\infty)}(r,s) = \mathcal{H}^n_{T_Y^*X}\left(\mathcal{C}^{\infty}_{Y\times Y|X\times X}(r,s)\right)\otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$

This is possible because the cohomological properties of $\mathcal{C}^\infty_{Y|X}(r,s)$ and $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ are very similar to those of holomorphic functions ([20]). We remark that the definitions together with formula (2.2.2) give immediately

$$\mathcal{D}_\Lambda^{2(\infty,\infty)}(r,s) = \gamma^{-1}\gamma_*\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$$

We get in this way sheaves of rings on $\Lambda = T_Y^*X$. There are canonical inclusion of sheaves of rings $\mathcal{E}_X^\infty|_\Lambda \subset \mathcal{D}_\Lambda^{2(\infty,\infty)}(r,s) \subset \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ and $\mathcal{E}_X^{\mathbb{R}}|_\Lambda \subset \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ where $\mathcal{E}_X$ and $\mathcal{E}_X^{\mathbb{R}}$ are the sheaves of microdifferential and microlocal operators of [29].

It is proved in [20] that $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ is a $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$-module and that :

$$\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s) = \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)\otimes_{\mathcal{E}_X}\mathcal{C}_{Y|X}$$

.

When $r \geq r' \geq s' \geq s$, the canonical morphism $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s) \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y|X}(r',s')$ induces a morphism of sheaves of rings $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s) \longrightarrow \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r',s')$ which makes $\mathcal{C}^{\mathbb{R}}_{Y|X}(r',s')$ a $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$-module.

We defined in [20] symbols for sections of $\mathcal{D}_\Lambda^{2(\infty,\infty)}(r,s)$ and $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ generalizing those of $\mathcal{E}_X^\infty$ and $\mathcal{E}_X^{\mathbb{R}}$. It is the same proof than the existence of symbols for $\mathcal{B}_{Y|X}$ in section 2.1. The coordinates are (x,τ) on Λ and (x,τ,x^*,τ^*) on $T^*\Lambda$.

Definition 2.3.1. Let (r,s) be two real numbers with $1 \leq s < r < +\infty$ and let V be an open subset of Λ, $\mathbb{R}$-conic in τ.

For each $a > 0$ $\mathcal{S}^2_+(r)(a,V)$ is the set of holomorphic functions $u(x,\tau,x^*,\tau^*)$ on $(V\cap\{|\tau|>a\})\times\mathbb{C}^n$ such that :

$$\exists c, C>0,\ \forall(x,\tau,x^*,\tau^*)\quad |u(x,\tau,x^*,\tau^*)| < C\exp(c|\tau|^{1/r} + a(|x^*|+|\tau||\tau^*|))$$

$\mathcal{S}^2_-(r)(a,V)$ has the same definition but with c replaced by $-c$.

$$\mathcal{S}^2(r,s)(V) = \varinjlim_{a>0}\varinjlim_{V'}\mathcal{S}^2_+(r)(a,V')\,/\,\mathcal{S}^2_-(s)(a,V')$$

where the limit is taken over all open subsets V' of V which are $\mathbb{R}^+$-conic in τ and such that $V'\cap\{|\tau|=1\}$ is relatively compact in V.

We proved in [20] that, if V is contained in some half space $\{\mathrm{Re} < \lambda, \tau > > 0\}$, there is an isomorphism between $\mathcal{S}^2{}_{(r,s)}(V)$ and $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}(V)$. The product of two operators of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$ is defined in [20] by a formula which is not easy to explicit in $\mathcal{S}^2{}_{(r,s)}(V)$. These symbols are issued from the symbols of $\mathcal{C}^{\mathbb{R}}_{Y|X}{}_{(r,s)}$ of section 2.2 while the good formula is defined in other symbols coming from formal symbols of $\mathcal{C}^{\mathbb{R}}_{Y|X}{}_{(r,s)}$.

However, if the symbols are independent of τ^* the formula is meaningful in ordinary symbols, the product of $P(x, x^*, \tau)$ by $Q(x, x^*, \tau)$ is $R(x, x^*, \tau)$ with :

$$R(x, x^*, \tau) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \left(\frac{\partial}{\partial x^*}\right)^\alpha P(x, x^*, \tau) \left(\frac{\partial}{\partial x}\right)^\alpha Q(x, x^*, \tau) \tag{2.3.1}$$

This product appears as the product of differential operators in x with holomorphic parameter τ.

In the same way, symbols of $\mathcal{C}^{\infty}_{Y|X}{}_{(r,s)}$ are used to define symbols of $\mathcal{D}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}$ [18][20], they have the following form :

If $r > s$, the set $\Gamma(U, \mathcal{D}_\Lambda^{2(\infty,\infty)}{}_{(r,s)})$ of sections of $\mathcal{D}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}$ on an open set U of Λ is in bijection with the formal series

$$P = \sum_{(i,k) \in \mathbb{N} \times \mathbb{Z}} P_{ik}(x, \tau, x^*, \tau^*)$$

such that $P_{ik}(x, \tau, x^*, \tau^*)$ is polynomial homogeneous of degree i in (x^*, τ^*) with coefficients holomorphic on U and homogeneous of degree $i + k$ in (τ, x^*) and satisfy the following inequalities :

$$\forall K \subset\subset U,\ \exists C > 0,\ \forall \varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall (x, \tau) \in K, \tag{2.3.2}$$

$$(i)\ |P_{ik}(x, \tau, x^*, \tau^*)| < (C_\varepsilon)^{-k} \varepsilon^i \frac{(-k)!^s}{i!} (|x^*| + |\tau^*|)^i \quad \text{if } i \geq 0, k < 0$$

$$(ii) |P_{ik}(x, \tau, x^*, \tau^*)| < C_\varepsilon \varepsilon^i C^k \frac{1}{k!^r\, i!} (|x^*| + |\tau^*|)^i \quad \text{if } i \geq 0, k \geq 0$$

Substituting $\mathcal{C}^{\mathbb{R}}_{Y|X}{}_{(r,s)}$ and $\mathcal{C}^{\infty}_{Y|X}{}_{(r,s)}$ to $\mathcal{O}_X$ in the definition of the sheaf $\mathcal{D}_X^\infty$ of differential operators gave us the sheaves $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$ and $\mathcal{D}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}$. We may use the same substitution in the definition of $\mathcal{E}_X^{\mathbb{R}}$ and $\mathcal{E}_X^\infty$ to get the four sheaves $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}{}_{(r,s)}$, $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$, $\mathcal{E}_\Lambda^{2(\infty,\mathbb{R})}{}_{(r,s)}$ and $\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}$. Each of them is a sheaf of rings on $T^*\Lambda$.

So, the definition is :

$$\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}{}_{(r,s)} = \mu_{T^*_Y X} \left(\mathcal{C}^{\mathbb{R}}_{Y \times Y | X \times X}{}_{(r,s)}\right) [n] \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$

where $\mu_{T^*_Y X}$ is the microlocalization along $T^*_Y X$ diagonal of $T^*_Y X \times T^*_Y X$.

We get $\mathcal{E}_\Lambda^{2(\infty,\mathbb{R})}{}_{(r,s)}$ by substituting $\mathcal{C}^\infty$ to $\mathcal{C}^{\mathbb{R}}$, $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$ and $\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}$ by taking the global sections on the fibers of the projection of $T^*\Lambda$ to the corresponding projective fiber bundle. By restriction to the zero section we recover the sheaves of section 2.3:

$$\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}{}_{(r,s)}|_\Lambda = \mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}|_\Lambda = \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$$
$$\mathcal{E}_\Lambda^{2(\infty,\mathbb{R})}{}_{(r,s)}|_\Lambda = \mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}|_\Lambda = \mathcal{D}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}$$

In fact, we will not make here full use of these sheaves which were studied more closely in [20]. The sheaf $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(r,s)$ will be used in a special case in the appendix, the sheaf $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ only in theorem 4.1.8 to calculate the characteristic variety of vanishing cycles.

The sheaf $\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s)$ will be used in section 3.2 only through its symbols which we redefine now. We use local coordinates of X as in 1.1 so that $T^*\Lambda$ is provided with coordinates (x,τ,x^*,τ^*) and consider an open subset U of $T^*\Lambda$ where they are defined. We assume that U is $\mathbb{C}$-conic in (τ,x^*) and in (x^*,τ^*). We assume also that the rational numbers (r,s) satisfy $\infty > r > s \geq 1$ (the cases $r=\infty$ or $r=s$ are not very different but not used here).

If $r>s$, the set $\Gamma(U,\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s))$ of sections of $\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s)$ on U is in bijection with the formal series

$$P = \sum_{(i,k)\in\mathbb{Z}^2} P_{ik}(x,\tau,x^*,\tau^*)$$

such that $P_{ik}(x,\tau,x^*,\tau^*)$ is homogeneous of degree i in (x^*,τ^*) and of degree $i+k$ in (τ,x^*) and satisfy the following inequalities :

$$\forall K \subset\subset U,\ \exists C>0,\ \forall\varepsilon>0,\ \exists C_\varepsilon>0,\ \forall(x,\tau,x^*,\tau^*)\in K, \tag{2.3.3}$$

$$\begin{aligned}
&(i)\quad |P_{ik}(x,\tau,x^*,\tau^*)| < C^{-i-k}(-i)!(-k)!^s && \text{if } i<0, k<0\\
&(ii)\quad |P_{ik}(x,\tau,x^*,\tau^*)| < C^{-i+k}\frac{(-i)!}{k!^r} && \text{if } i<0, k\geq 0\\
&(iii)\quad |P_{ik}(x,\tau,x^*,\tau^*)| < (C_\varepsilon)^{-k}\varepsilon^i\frac{(-k)!^s}{i!} && \text{if } i\geq 0, k<0\\
&(iv)\quad |P_{ik}(x,\tau,x^*,\tau^*)| < C_\varepsilon\varepsilon^i C^k\frac{1}{k!^r\, i!} && \text{if } i\geq 0, k\geq 0
\end{aligned}$$

The restriction of $\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s)$ to the zero section of $T^*\Lambda$ is $\mathcal{D}_\Lambda^{2(\infty,\infty)}(r,s)$ and this is clearly compatible with our definitions of the symbols.

We say that P is of finite order if there exists (i_0,k_0) such that $P_{ik}=0$ if $i+rk > i_0+rk_0$ or $i+sk > i_0+sk_0$. The sheaf of operators of finite order is denoted by $\mathcal{E}_\Lambda^2(r,s)$.

2.4 Inverse image.

Let $i: Y \hookrightarrow X$ be the canonical map. The inverse image of an $\mathcal{O}_X$-module $\mathcal{N}$ by i is

$$i^*\mathcal{N} = \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{N}$$

The definition of inverse image of left $\mathcal{D}_X$-modules is the same but to explicit its structure of left $\mathcal{D}_Y$-module, was introduced a $(\mathcal{D}_Y,\mathcal{D}_X)$-bimodule (see [31] for example) :

$$\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X$$

and then if $\mathcal{M}$ is a left $\mathcal{D}_X$-module :

$$i^*\mathcal{M} = \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{i^{-1}\mathcal{D}_X} i^{-1}\mathcal{M}$$

In this paper, we will define the vanishing cycles as the inverse image by i of a $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$-module. So, we define now $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)$ which is a $\left(\pi^{-1}\mathcal{D}_Y^\infty, \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)\right)$-bimodule on Λ (with $\pi : \Lambda \to Y$). It may be defined as the inverse image of the $\mathcal{D}_X$-module $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ by $i : Y \longrightarrow X$ that is :

$$\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s) = \pi^{-1}\mathcal{D}_{Y\to X} \otimes_{\pi^{-1}\mathcal{D}_X} \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$$

In this way, it is the quotient of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ by the right ideal generated by the equations of Y.

In local coordinates it may also be assimilated to the subsheaf of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ of operators with a symbol independent of τ^* (i.e. operators which commutes with the operators of symbols $\tau_1, \dots, \tau_d$). To make precise calculations in the next section we will need a cohomological definition of this sheaf.

Let us consider the canonical injective morphism $j : Y \times X \longrightarrow X \times X$. It defines a morphism $p : T_Y^*X \times T_Y^*X \longrightarrow Y \times T_Y^*X$ which is the canonical projection.

The morphism $j^* : j^{-1}\mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{Y\times X}$ defines a morphism [29, lem. 2.2.5. chI] :

$$j^* : \mathbb{R}p_!\, \mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X}[d] \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}$$

and it was proved in [20] that this morphism may be extended to :

$$j^* : \mathbb{R}p_!\, \mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X}(r,s)[d] \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)$$

Taking the $(n-d)$-th cohomology group we get a morphism :

$$\mathcal{H}^n_{T_Y^*X}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X}(r,s)\right) \otimes p_2^{-1}\Omega_X \longrightarrow \mathcal{H}^{n-d}_{T_Y^*X}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)\right) \otimes p_2^{-1}\Omega_X$$

The first term is $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ by definition and it is not difficult to verify that the second is equal to $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)$ [18, §2.10.].

In the same way we may define a $\left(\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s), \pi^{-1}\mathcal{D}_Y^\infty\right)$-bimodule

$$\mathcal{D}_{\Lambda\to Y}^{2(\mathbb{R},\infty)}(r,s) = \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s) \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{D}_{X\leftarrow Y} = \mathcal{H}^{n-d}_{T_Y^*X}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)\right) \otimes p_1^{-1}\Omega_Y$$

It is the quotient of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ by the left ideal generated by the equations of Y.

The sheaves $\mathcal{D}_{Y\to X}$ and $\mathcal{D}_{X\leftarrow Y}$ are duals as $\mathcal{D}_X$-modules, that is :

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\leftarrow Y}, \mathcal{D}_X)[\operatorname{codim} Y] = \mathcal{D}_{Y\to X}$$

and, $\mathcal{D}_{Y\to X}$ being flat over $\mathcal{D}_X$, we deduce immediately the corresponding result :

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)}(\mathcal{D}_{\Lambda\to Y}^{2(\mathbb{R},\infty)}(r,s), \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s))[\operatorname{codim} Y] = \mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)$$

The same definitions may be stated for the sheaves $\mathcal{E}^2$, in particular we may define the sheaf

$$\mathcal{E}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\mathbb{R})}(r,s) = \mu_{T_Y^*X}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)\right)[n-d] \otimes p_2^{-1}\Omega_X$$

and this sheaf is, in local coordinates, isomorphic to the subsheaf of $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(r,s)$ of operators commuting with the operators of symbol $\tau_1, \dots, \tau_d$. Replacing $\mathcal{C}^{\mathbb{R}}$ by $\mathcal{C}^\infty$ we get $\mathcal{E}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)$.

2.5 Symbols of holomorphic microfunctions.

The symbolic calculus for germs of the sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ was defined by Boutet de Monvel [5] and Aoki [1]. We will here to be more explicit in the case of multivalued sections of this sheaf. This point will be the key of the proof of theorem 3.1.1.

It follows from the definition of $\mathcal{C}^{\mathbb{R}}_{Y|X}$ and [29, prop.1.2.4. ch I] or [17, th 4.3.2.] that if U is an open subset of $T^*_Y X$ with convex conic fibers we have :

$$\Gamma(U, \mathcal{C}^{\mathbb{R}}_{Y|X}) = \varinjlim_{V,Z} H^d_Z(V, \mathcal{O}_X)$$

where V ranges through the family of open subsets of X such that $V \cap Y = \pi(U)$ and Z through the family of closed subsets of X such that $C_Y(Z) \subset U^o$.

Here π is the canonical projection of $T^*_Y X$ on Y, U^o is the antipodal of the polar set of U in $T_Y X$ and $C_Y(Z)$ is the tangent cone to Z along Y.

We assume now that Y is a domain of holomorphy of $\mathbb{C}^{n-d}$ and $X = Y \times \mathbb{C}^d$. The coordinates will be denoted by $(x_1, \dots, x_p, t_1, \dots, t_d)$ on X and the corresponding coordinates (x, τ) on $T^*_Y X$.

Let $G = \{ (x,\tau) \in T^*_Y X \mid \operatorname{Re}\tau_i \geq |\operatorname{Im}\tau_i|, i = 1, \dots, d \}$, we have :

$$\Gamma(G, \mathcal{C}^{\mathbb{R}}_{Y|X}) = \varinjlim_{V,\varepsilon>1} H^d_{Z_\varepsilon}(V, \mathcal{O}_X)$$

where V ranges through the family of open neighborhoods of Y in X and

$$Z_\varepsilon = \{ (x,t) \in X \mid \operatorname{Re} t_i \leq \varepsilon |\operatorname{Im} t_i|, i = 1, \dots, d \}$$

From a theorem of Siu, Y has a fundamental system of neighborhoods in X which are domains of holomorphy, so we may assume that V is a holomorphy set and define :

$$W^{(i)}_\varepsilon = \{ (x,t) \in V \mid \operatorname{Re} t_i < \varepsilon |\operatorname{Im} t_i| \}, \quad W^{[i]}_\varepsilon = \bigcap_{j \neq i} W^{(j)}_\varepsilon, \quad W_\varepsilon = \bigcap_{1 \leq i \leq d} W^{(i)}_\varepsilon$$

Then Čech cohomology gives an exact sequence :

$$\bigoplus_{1 \leq i \leq d} \Gamma(W^{[i]}_\varepsilon, \mathcal{O}_X) \longrightarrow \Gamma(W_\varepsilon, \mathcal{O}_X) \longrightarrow H^d_{Z_\varepsilon}(V, \mathcal{O}_X) \longrightarrow 0$$

A section u of $\mathcal{C}^{\mathbb{R}}_{Y|X}$ on G is thus represented by a holomorphic function f on some W_ε. Following [29] and [1] we will say that f is a defining function for u.

For any $(\alpha, \lambda) \in ((\mathbb{R}^*_+)^d \times \mathbb{C}^d$, we may replace G,$W^{(i)}_\varepsilon$, and W_ε by :

$$\begin{aligned}
G_\lambda &= \{ (x,\tau) \in T^*_Y X \mid \alpha_i \operatorname{Re} \lambda_i^{-1} \tau_i \geq |\operatorname{Im} \lambda_i^{-1} \tau_i|, i = 1, \dots, d \} \\
W^{(i)}_{\lambda_i,\varepsilon} &= \{ (x,t) \in V \mid x \in Y, \operatorname{Re} \lambda_i t_i < (\alpha_i + \varepsilon) |\operatorname{Im} \lambda_i t_i| \}, \; i = 1, \dots, d, \; \varepsilon > 0 \\
W_{\lambda,\varepsilon} &= \bigcap_{1 \leq i \leq d} W^{(i)}_{\lambda_i,\varepsilon}
\end{aligned}$$

Let $\widetilde{\mathbb{C}}$ be the universal covering of $\mathbb{C}^* = \mathbb{C} - \{0\}$ and p be the canonical projection $Y \times \widetilde{\mathbb{C}}^d \to Y \times \mathbb{C}^d$. As usual, a section u of $p^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X}$ on $Y \times \widetilde{\mathbb{C}}^d$ will be called a multivalued section in τ on $\{ (x,\tau) \in T^*_Y X \mid \tau_i \neq 0,\ i = 1, \dots, d \}$. We recall that the sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ has the property of unique continuation [29, th. 2.2.8. ch III].

For any $\alpha_i > 0, \lambda_i \in \mathbb{C}$, u has a defining function on $W_{\lambda,\varepsilon}$ for some open set V. Two such functions differ from the sum of d functions, each of them being holomorphic in one variable t_i near 0. More precisely, given λ and λ', u has defining functions φ on $W_{\lambda,\varepsilon}$, φ' on $W_{\lambda',\varepsilon}$ and $\varphi - \varphi' = \sum \varphi_i$ with φ_i holomorphic on $W^{[i]}_{\lambda,\varepsilon} \cap W^{[i]}_{\lambda',\varepsilon}$. But the union $W^{[i]}_{\lambda,\varepsilon} \cup W^{[i]}_{\lambda',\varepsilon}$ is still Stein and thus we have a decomposition $\varphi_i = \psi_i - \psi'_i$ with ψ_i holomorphic on $W^{[i]}_{\lambda,\varepsilon}$ and ψ'_i holomorphic on $W^{[i]}_{\lambda',\varepsilon}$. So a defining function for u is $\varphi - \sum \psi_i = \varphi' - \sum \psi'_i$ which is holomorphic on $W_{\lambda,\varepsilon} \cup W_{\lambda',\varepsilon}$.

We may iterate this procedure on $Y \times \widetilde{\mathbb{C}}^d$, therefore u has a defining function f which is multivalued in each variable t_i on $V \cap \{t_i \neq 0, i = 1, \dots, d\}$. Two such functions differ from the sum of functions which are holomorphic in one of the variables t_i near 0.

In fact this is not absolutely true because the open set V has to be shrinked at each turn around $t_i = 0$. This will not affect our results because we will use only a finite number of turns.

The symbol of the section u as defined in [1] is calculated from a defining function. Here we will use the same formula, but we will take a special path :

For $b \in \mathbb{C}^*$, we consider a path γ_b in $\mathbb{C}^*$ beginning at b, ending at b, and such that the index of the point $\{0\}$ is 1 relatively to γ_b. If $\varphi(t)$ is a holomorphic multivalued function defined on $D^* = \{ t \in \mathbb{C} \mid 0 < |t| < \delta \}$ then the integral of φ on γ_b is well defined if $\gamma_b \subset D^*$, it depends only on b and on the choice of the determination of φ at b.

If a is a point of $(\mathbb{C}^*)^d$, we define $Y_a(V)$, or Y_a for short,as the subset of Y of points x such that $(x, \varepsilon a) \in V$ for any $\varepsilon \in [0,1]$.

A symbol of the microfunction u with defining function f is the function :

$$F_a(x,\tau) = \int_{\gamma_{a_1} \times \dots \times \gamma_{a_d}} f(x,t) e^{-<t,\tau>}\, dt_1 \dots dt_d \tag{2.5.1}$$

It is defined on $Y_a(V) \times \mathbb{C}^2$ and satisfy :

$$\forall K \subset\subset Y_a, \forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall x \in K, \forall \tau \in \mathbb{C}^d, \operatorname{Re} a_i \tau_i \geq 0, i = 1, \dots, d,$$
$$|F_a(x,\tau)| < C_\varepsilon e^{\varepsilon|\tau_1| + \dots + \varepsilon|\tau_d|}$$

Hence F is a symbol for u on $\{ (x,\tau) \in Y_a \times \mathbb{C}^d \mid \operatorname{Re} a_i \tau_i \geq 0 \}$. The difference of two defining functions being the sum of holomorphic functions in one of t_i, the symbol is independent of the choice of the defining function and for the same reason, it is independent of the choice of the determination of f.

Usually, that is in the results of Aoki and in 2.2.1, the symbols of u differ from an exponentially decreasing holomorphic function. Here we have chosen one of them which depends only on a. Another crucial difference is that the symbol is dominated by a product $e^{\varepsilon|\tau_1| + \dots + \varepsilon|\tau_d|}$ on an open set which is itself a product. This will allow to take a punctual value as follows.

Let us fix some value σ such that $\operatorname{Re} a_d\sigma > 0$ and consider the resulting function $F_a(x,\tau',\sigma)$. It is well defined and subexponential as a function of $\tau' = (\tau_1,\dots,\tau_{d-1})$ on $\{\,(x,\tau') \in Y_a \times \mathbb{C}^{d-1} \mid \operatorname{Re} a\tau_i \geq 0\,\}$ hence it is the symbol of a section $u_{a_d}(\sigma)$ of $\mathcal{C}^{\mathbb{R}}_{Y|Y\times\mathbb{C}^{d-1}}$ on this set.

This section depends on σ and on a_d but not on $(a_1,\dots,a_{d-1})$ because moving a_i modifies F by a function which is exponentially decreasing in τ_i. So we may define $u_{a_d}(\sigma)$ as a section of $\mathcal{C}^{\mathbb{R}}_{Y|Y\times\mathbb{C}^{d-1}}$ multivalued in τ on $Y'_{a_d} \times \mathbb{C}^{d-1}$ where Y'_{a_d} is the set Y_a for $a_1 = \cdots = a_{d-1} = 0$.

Finally we proved :

Lemma 2.5.1. *Let Y be a domain of holomorphy in $\mathbb{C}^{n-d}$, $X = Y \times \mathbb{C}^d$ and $X' = Y \times \mathbb{C}^{d-1}$. We identify $T^*_Y X$ to $Y \times \mathbb{C}^d$ with coordinates (x,τ).*

(i) Any section u of $\mathcal{C}^{\mathbb{R}}_{Y|X}$ on $Y \times (\mathbb{C}^*)^d$ multivalued in τ_i for $i = 1,\dots,d$, has a defining function on $V \cap \{t_i \neq 0,\}$ multivalued in t_i for $i = 1,\dots,d$ and some neighborhood V of Y in X.

(ii) Given a point $a \in (\mathbb{C}^*)^d$, such a section has a uniquely determined symbol $F_a(x,\tau)$ on $Y_a(V) \times \mathbb{C}^d$ where $Y_a(V)$ is subset of Y of points x such that $(x,\varepsilon a) \in V$ for any $\varepsilon \in [0,1]$.

(iii) For any $\sigma \in \mathbb{C}^*$ such that $\operatorname{Re} a_d\sigma > 0$, the function $(x,\tau') \mapsto F_a(x,\tau',\sigma)$ defines a section of $\mathcal{C}^{\mathbb{R}}_{Y|X'}$ on $Y_{(0,a_d)}(V) \times (\mathbb{C}^*)^{d-1}$ multivalued in τ_i and independent of $(a_1,\dots,a_{d-1})$. This section will be denoted by $u_{a_d}(\sigma)$ and called the value of u at $\tau_d = \sigma$.

Remark 2.5.2. The lemma is still true if $d = 1$. In this case $Y = X'$ and thus $\mathcal{C}^{\mathbb{R}}_{Y|X'} = \mathcal{O}_Y$.

As we will see later, it is sometimes necessary to define $u_{a_d}(\sigma)$ as a section on the whole of $Y \times (\mathbb{C}^*)^{d-1}$. This is possible only on a subsheaf of $\mathcal{C}^{\mathbb{R}}_{Y|X}$:

Lemma 2.5.3. *Let Y be a domain of holomorphy in $\mathbb{C}^{n-d}$, $X = Y \times \mathbb{C}^d$ and $X' = Y \times \mathbb{C}^{d-1}$. We identify $T^*_Y X$ to $Y \times \mathbb{C}^d$ with coordinates (x,τ).*

If C is a positive number, let $\widetilde{\Gamma}_{[C]}(Y \times (\mathbb{C}^*)^d, \mathcal{C}^{\mathbb{R}}_{Y|X})$ be the set of multivalued sections of $\mathcal{C}^{\mathbb{R}}_{Y|X}$ on $Y \times (\mathbb{C}^*)^d$ which have a defining function on $Y \times \{\, t \in \mathbb{C}^d \mid 0 < |t_i| < 1/C\,\}$.

(i) For any (σ, a_d) such that $\operatorname{Re} a_d\sigma > 0$ and $|a_d| < 1/C$, the value $u_{a_d}(\sigma)$ is well defined as a multivalued section of $\mathcal{C}^{\mathbb{R}}_{Y|X'}$ on $Y \times (\mathbb{C}^*)^{d-1}$.

(ii) The result $u_{a_d}(\sigma)$ is an element of $\widetilde{\Gamma}_{[C]}(Y \times (\mathbb{C}^*)^{d-1}, \mathcal{C}^{\mathbb{R}}_{Y|X'})$.

(iii) Let u be a multivalued section of $\mathcal{C}^{\infty}_{Y|X}$ on $Y \times (\mathbb{C}^*)^d$ represented by a symbol

$$u(x,\tau) = \sum_{n\in\mathbb{Z}} u_k(x,\tau)$$

where each u_k is a holomorphic multivalued function on $Y \times (\mathbb{C}^*)^d$ homogeneous of degree n in τ. We assume that :

$$\exists C > 0 \forall K \subset\subset Y, \exists C_K, \forall n < 0, \forall x \in K,\ |u_n(x,\tau)| < C_K C^{-n}(-n)!$$

Then u is an element of $\widetilde{\Gamma}_{[C/n]}(Y \times (\mathbb{C}^*)^d, \mathcal{C}^{\mathbb{R}}_{Y|X})$.

Proof. Let us first remark that any multivalued section of $\mathcal{C}^{\infty}_{Y|X}$ on $Y \times (\mathbb{C}^*)^d$ has a symbol similar to the symbol of the lemma except that the constant C usually depends on K.

With the additional condition of the lemma, the defining function of u as it is calculated in [29, Proposition 1.4.3.] is defined on $V = Y \times \{\, t \in \mathbb{C}^d \mid 0 < |t| < 1/C \,\}$ and we get the result if we apply lemma 2.5.1 to this set V. □

These lemmas extends to all sheaves $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ defined in [20] :

First, we remark that $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,1)$ is the subsheaf of $\mathcal{C}^{\mathbb{R}}_{Y|X}$ of the sections whose symbols have growth $\exp(c|\tau|^{1/r})$. This means that the defining function $f(x,t)$ has growth $\exp(c|1/t|^{1/(r-1)})$. The value of u at σ is thus well defined as a section of $\mathcal{C}^{\mathbb{R}}_{Y_a|X'}(r,1)$.

If $s > 1$, we use the map $(\tau) \mapsto (\tau_1^s, \ldots, \tau_d^s)$. It defines an isomorphism between the sections of $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ outside $\{\, \tau_i = 0 \,\}$ and the sections of $\mathcal{C}^{\mathbb{R}}_{Y|X}(r/s,1)$. This proves that lemma 2.5.1 and lemma 2.5.3 extend to the sheaves $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ and $\mathcal{C}^{\infty}_{Y|X}(r,s)$ for all (r,s) such that $+\infty > r \geq s \geq 1$.

2.6 Non local operators

These non local operators will be used in the next section to apply the morphism of lemma 2.5.1. In this section X is of dimension n and Y of codimension d.

A differential operator, that is a section of $\mathcal{D}^{\infty}_X$ on some open set U of X is represented in local coordinates $(x_1, \ldots, x_n)$ as :

$$P(x, D_x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha(x) D_x^\alpha$$

where $p_\alpha(x)$ is a holomorphic function on U satisfying :

$$\forall K \subset\subset U, \forall \varepsilon > 0, \exists C_\varepsilon > 0, \forall x \in K, |p_\alpha(x)| < C_\varepsilon \varepsilon^{|\alpha|} \alpha! \tag{2.6.1}$$

and such an operator defines a endomorphism on the sheaf $\mathcal{O}_X$ of holomorphic functions.

If we replace 2.6.1 by :

$$\exists \delta > 0, \forall K \subset\subset U, \exists \mathbb{C} > 0, \forall x \in K, |p_\alpha(x)| < C \delta^{|\alpha|} \alpha! \tag{2.6.2}$$

we get an operator wich is not local on $\mathcal{O}_X$, the series $\sum p_\alpha(x) f^{(\alpha)}(x)$ is convergent only if the Taylor series of f at x converges on a polydisk of radius δ, i.e. P defines a map from $\Gamma(U, \mathcal{O}_X)$ to $\Gamma(U_\delta, \mathcal{O}_X)$ with $U_\delta = \{\, x \in U \mid \forall y \notin U, \forall j = 1, \ldots, n-d, |x_j - y_j| > \delta \,\}$.

The sheaf of operators satisfying 2.6.2 may be defined by 2.1.1 where the diagonal of $X \times X$ is replaced by $\{\, (x, x') \in X \times X \mid |x_i - x'_i| \leq \delta \,\}$. Of course, this is coordinate dependant. Our aim here is to develop the same definition for 2-microdifferential operators, that is replace $\mathcal{O}_X$ by $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$.

We set $\Delta_\delta = \{\, (x, \tau, x', \tau') \in T^*_Y X \times T^*_Y X \mid \forall i = 1, \ldots, n-d, |x_i - x'_i| \leq \delta,\ \tau = \tau' \,\}$ and define :

$$\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s;\delta) = \mathcal{H}^n_{\Delta_\delta}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X}(r,s)\right) \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\Omega_X$$

Lemma 2.6.1. *For all $k \neq n$ we have :*

$$\mathcal{H}^k_{\Delta_\delta}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X^{(r,s)}}\right) = 0$$

Proof. Let Ω be a Stein open subset of $T^*_Y X \times T^*_Y X$ which is $\mathbb{R}^*_+$-conic and contained in some half-space in τ, we define :

$$\begin{aligned}
\Omega_i &= \{\, (x,\tau,x',\tau') \in T^*_Y X \times T^*_Y X \mid |x_i - x'_i| > \delta \,\} &\quad& \text{if } i = 1,\dots,n-d \\
\Omega_{i+n-d} &= \{\, (x,\tau,x',\tau') \in T^*_Y X \times T^*_Y X \mid |\tau_i - \tau'_i| > 0 \,\} && \text{if } i = 1,\dots,d
\end{aligned}$$

These sets are $\mathbb{R}^*_+$-conic Stein open sets hence $H^k(\Omega_i, \mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X}) = 0$ for $i > 0$ [20, Prop 2.5.1]. As $\Omega - \Delta_\delta$ is covered by n acyclic open sets, the lemma is true by elementary Čech cohomology. □

Proposition 2.6.2. *Let V be an open set of $T^*_Y X$ and*

$$V_\delta = \{\, (x,\tau) \in V \mid \forall y \notin V,\ \forall j = 1,\dots,n-d,\ |x_j - y_j| > \delta \,\}$$

A section of $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s;\delta)$ on $V \times V$ defines a morphism

$$\Gamma(V, \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}) \longrightarrow \Gamma(V_\delta, \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}})$$

This action extends the natural action of $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$ on the sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}$.

Proof. Consider the the two projections p and q from $T^*_Y X \times T^*_Y X$ to $T^*_Y X$. There is a canonical morphism :

$$\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X^{(r,s)}} \otimes_{q^{-1}\mathcal{O}_X} q^{-1}\Omega_X\right) \otimes_{\mathbb{C}} q^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}[-n] \longrightarrow p^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}} \tag{2.6.3}$$

This morphism may be defined by inverse image under $X \times X \hookrightarrow X \times X \times X$ or as in [18, Proposition 2.1.4.] using the isomorphism :

$$\begin{aligned}
\left(\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}} \otimes_{\mathcal{O}_X} \Omega_X\right) \otimes_{\mathcal{E}^{\mathbb{R}}_X(r,s)} \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}[-n] &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}^{\mathbb{R}}_X(r,s)}(\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}, \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}) \\
&= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{C}_{Y|X}, \mathcal{C}_{Y|X}) \otimes \mathcal{E}^{\mathbb{R}}_X(r,s) \simeq \mathbb{C}_{T^*_Y X}
\end{aligned}$$

The fibers of p being of dimension n, there is a morphism of direct image [8] :

$$\mathbb{R}p_! p^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}[2n] \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}$$

hence a morphism :

$$\mathbb{R}p_! \mathbb{R}\Gamma_{\Delta_\delta} p^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}[2n] \longrightarrow \mathbb{R}p_! p^{-1}\mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}[2n] \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}$$

which composed with (2.6.3) gives :

$$\mathbb{R}p_! \left(\mathbb{R}\Gamma_{\Delta_\delta}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times X^{(r,s)}} \otimes_{q^{-1}\mathcal{O}_X} q^{-1}\Omega_X\right) \otimes_{\mathbb{C}} \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}\right)[n] \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y|X^{(r,s)}}$$

and using lemma 2.6.1 this gives :

$$p_! \left(\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s;\delta) \otimes_\mathbb{C} q^{-1}\mathcal{C}^\mathbb{R}_{Y|X}(r,s)\right) \longrightarrow \mathcal{C}^\mathbb{R}_{Y|X}(r,s) \tag{2.6.4}$$

If Δ_δ is replaced by the diagonal of $T^*_Y X \times T^*_Y X$, p is an isomorphism on the diagonal and we get the action of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ on $\mathcal{C}^\mathbb{R}_{Y|X}(r,s)$. Here we remark that the projection $p : q^{-1}(V) \cap \Delta_\delta \cap p^{-1}(U) \longrightarrow U$ is proper if U is contained in V_δ and this gives the morphism of the proposition :

$$\Gamma(V \times V, \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s;\delta)) \otimes_\mathbb{C} \Gamma(V, \mathcal{C}^\mathbb{R}_{Y|X}(r,s)) \longrightarrow \Gamma(V_\delta, \mathcal{C}^\mathbb{R}_{Y|X}(r,s))$$

□

We may also consider a sheaf $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s;\delta)$ as in section 2.4 :

$$\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s;\delta) = \mathcal{H}^{n-d}_{\Delta'_\delta}\left(\mathcal{C}^\mathbb{R}_{Y\times Y|Y\times X}(r,s)\right) \otimes p_2^{-1}\Omega_X$$

with $\Delta'_\delta = \{\, (x, x', \tau) \in Y \times T^*_Y X \mid \forall i = 1, \dots, n-d, |x_i - x'_i| \le \delta \,\}$. It is identified with the subsheaf of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s;\delta)$ of sections commuting with $\tau_1, \dots, \tau_d$.

2.7 Symbols of 2-microlocal operators.

Let $X' = \{\, (x,t) \in X \mid t_d = 0 \,\})$ and $\Lambda' = T^*_{Y\times Y}(Y \times X')$. We want now apply the results of section 2.5 to define the "value at $\tau_d = \sigma$" of a section of $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)$ as a section of $\mathcal{D}_{Y\leftarrow\Lambda'}^{2(\mathbb{R},\infty)}(r,s)$. In fact, there is no such morphism and we will have to shrink the first sheaf or to extend the second.

In this section, we fix a domain of holomorphy Ω of Y and consider the open sets $U = \{\, (x,\tau) \in T^*_Y X \mid x \in \Omega,\ \tau_i \neq 0 \,\}$ and $U' = \{\, (x,\tau) \in \Lambda' \mid x \in \Omega,\ \tau_i \neq 0 \,\}$. The set of sections of $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)$ and of $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\infty,\infty)}(r,s)$ on U and the set of sections of $\mathcal{D}_{Y\leftarrow\Lambda'}^{2(\mathbb{R},\infty)}(r,s)$ on U' which extend analytically along any path will be denoted respectively by

$$\widetilde{\Gamma}(U, \mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)),\ \widetilde{\Gamma}(U, \mathcal{D}_{Y\leftarrow\Lambda}^{2(\infty,\infty)}(r,s)) \text{ and } \widetilde{\Gamma}(U', \mathcal{D}_{Y\leftarrow\Lambda'}^{2(\mathbb{R},\infty)}(r,s))$$

An element P of $\widetilde{\Gamma}(U, \mathcal{D}_{Y\leftarrow\Lambda}^{2(\infty,\infty)}(r,s))$ has a symbol $P = \sum_{ik} P_{ik}(x,\tau,x^*)$ given by formula 2.3.2 where each function P_{ik} is multivalued on U. Let us denote by $\widetilde{\Gamma}_{[C]}(U, \mathcal{D}_{Y\leftarrow\Lambda}^{2(\infty,\infty)}(r,s))$ the subset of these sections whose symbol satisfy the stronger condition :

$$\forall K \subset\subset U,\ \forall\varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall (x,\tau) \in K, \tag{2.7.1}$$

$$\begin{aligned} &(i)\ |P_{ik}(x,\tau,x^*)| < C_\varepsilon \varepsilon^i C^{-k}\frac{(-k)!^s}{i!}|x^*|^i && \text{if } i \ge 0, k < 0 \\ &(ii) |P_{ik}(x,\tau,x^*)| < C_\varepsilon \varepsilon^i C^{k}\frac{1}{(k!)^r\, i!}|x^*|^i && \text{if } i \ge 0, k \ge 0 \end{aligned}$$

Proposition 2.7.1. *Let $C > 0$ and $(\sigma, a) \in \mathbb{C}^* \times \mathbb{C}^*$ such that* $\mathrm{Re}\, a\sigma > 0$ *and* $|a| < 1/C$. *The morphism of lemma 2.5.1 defines a morphism :*

$$\widetilde{\Gamma}_{[C]}(U, \mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s)) \longrightarrow \widetilde{\Gamma}(U', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s))$$

and a morphism :

$$\widetilde{\Gamma}(U, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)) \longrightarrow \bigcup_{\delta>0} \widetilde{\Gamma}(U', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s;\delta))$$

Given a section P of $\widetilde{\Gamma}(U, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s))$ and an arbitrary $\delta > 0$, the image $P_a(\sigma)$ of P is in $\widetilde{\Gamma}(U', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s;\delta)$ if $|a|$ is small enough.

Proof. Let Z a $\mathbb{R}^*_+$-conic subspace of $(\mathbb{C}^*)^d$ contained in some half-space $\mathrm{Re} < \lambda, \tau >> 0$, $\widetilde{\Omega}$ a Stein neighborhood of Ω in $Y \times Y$ and $V = \widetilde{\Omega} \times Z \subset Y \times T^*_Y X$. Let $U_1 = V \cap T^*_Y X$ and for $k = 1, \dots, n-d$:

$$V_k = \{ (x, x', \tau) \in V \mid x_k \neq x'_k \} \qquad W_k = \bigcap_{j\neq k} V_k \qquad W = \bigcap_{1\leq j\leq n-d} V_k$$

The sets V_k are acyclic for $\mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)$ [20, Prop 2.5.1.], thus we have an exact sequence of Čech cohomology :

$$\oplus\Gamma(W_k, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)) \longrightarrow \Gamma(W, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)) \longrightarrow \Gamma(U_1, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)) \longrightarrow 0 \qquad (2.7.2)$$

A section of $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ on U_1 is thus represented by a microfunction on W which has a symbol $u(x, x', \tau)$ on W.

In the same way we may define, for $\delta > 0$, Z' a $\mathbb{R}^*_+$-conic subspace of $(\mathbb{C}^*)^{d-1}$ contained in some half-space and $V' = \widetilde{\Omega} \times Z'$:

$$V^{(\delta)}_k = \{ (x, x', \tau) \in V' \mid |x_k - x'_k| > \delta \} \quad W^{(\delta)}_k = \bigcap_{j\neq k} V^{(\delta)}_k \quad W^{(\delta)} = \bigcap_{j=1\dots n-d} V^{(\delta)}_k$$

and we get an exact sequence :

$$\oplus\Gamma(W^{(\delta)}_k, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X'}(r,s)) \longrightarrow \Gamma(W^{(\delta)}, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)) \longrightarrow \Gamma(V', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s;\delta)) \longrightarrow 0$$

If we replace $\mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}$ by $\mathcal{C}^{\infty}_{Y\times Y|Y\times X}$, the corresponding result is true if Z is contractile. This was proved in [18, th. 1.1.3.]. So we get an exact sequence :

$$\oplus\Gamma(W_k, \mathcal{C}^{\infty}_{Y\times Y|Y\times X}(r,s)) \longrightarrow \Gamma(W, \mathcal{C}^{\infty}_{Y\times Y|Y\times X}(r,s)) \longrightarrow \Gamma(U_1, \mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s)) \longrightarrow 0$$

A section of $\Gamma(U_1, \mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s))$ is given by its symbol $P = \sum_{(i,k)\in\mathbb{N}\times\mathbb{Z}} P_{ik}(x, \tau, x^*)$. Each function P_{ik} is a homogeneous polynomial of degree i in x^* hence equal to :

$$P_{ik}(x, \tau, x^*) = \sum_{|\alpha|=i} P_{\alpha k}(x, \tau)(x^*)^\alpha$$

and by definition of this symbol (in [18] sections 1.5,1.6) the corresponding section of $\Gamma(W, \mathcal{C}^\infty_{Y\times Y|Y\times X}(r,s))$ is the microfunction of symbol $u = \sum u_k(x, x', \tau)$ with :

$$u_k(x, x', \tau) = \sum_{\alpha\in\mathbb{N}^p} P_{\alpha k}(x, \tau)\Phi_\alpha(x - x')$$

$$\text{with } \Phi_\alpha(x) = \Phi_{\alpha_1}(x_1)\ldots\Phi_{\alpha_p}(x_p) \quad \text{and} \quad \Phi_k(x_1) = \frac{(-1)^{k+1}}{2i\pi}\frac{k!}{x_1^{k+1}}$$

If P is a section of $\widetilde{\Gamma}(U, \mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s))$, the microfuntion $u = \sum u_k$ extends as a multi-valued section of $\mathcal{C}^\infty_{Y\times Y|Y\times X}(r,s)$ on $Y \times U$ because the symbol $\sum P_{ik}$ of P is unique and extends itself as a sum of multivalued functions. Now we may apply the results of the previous section in two ways :

First, if P is a section of $\widetilde{\Gamma}_{[C]}(U, \mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s))$, the conditions 2.7.1 implies that u satisfies the conditions of lemma 2.5.3, hence for a suitable (a, σ), the microfunction $u_a(\sigma)$ is defined on $W' = \widetilde{\Omega}\times\mathbb{C}^*$ as a multivalued section. This define a section of $\widetilde{\Gamma}(U', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s))$.

Second, if P is any section of $\widetilde{\Gamma}(U, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s))$, u satisfies the conditions of lemma 2.5.1, hence for a given (a, σ), the microfunction $u_a(\sigma)$ is defined on the points of $W' = \widetilde{\Omega} \times \mathbb{C}^*$ at a distance δ of the boundary for some $\delta > 0$ hence on $W^{(\delta)}$. So this define a section of $\Gamma(V', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s;\delta))$ which extends to a section of $\widetilde{\Gamma}(V', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s;\delta))$. □

When a local system of coordinate is given, $\mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s)$ is identified to the subset of $\mathcal{D}^{2(\infty,\infty)}_{\Lambda}(r,s)$ of operators whose symbol is independant of τ^* and in this way is a sheaf of rings. The same is true for $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s)$ subsheaf of $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda'}(r,s)$.

Proposition 2.7.2. *A local system of coordinates being given, the morphism*

$$\widetilde{\Gamma}_{[C]}(U, \mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s)) \longrightarrow \widetilde{\Gamma}(U', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s))$$

of proposition 2.7.1 is compatible with the ring structures.

Proof. The product of two sections of $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ is given by the formula 2.3.1. In τ it is just the ordinary product and therefore, the operation of taking the value at a point σ is compatible with the product :

$$(PQ)_a(\sigma) = P_a(\sigma)Q_a(\sigma)$$

□

To end this section, let us give some extensions of proposition 2.7.1 which will be used only in the appendix.

First we may define $\widetilde{\Gamma}_{[C]}(U, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s))$ as the subset of $\widetilde{\Gamma}(U, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s))$ which is the image of $\widetilde{\Gamma}_{[C]}(W, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s))$ in the exact sequence (2.7.2). From lemma 2.5.3, we deduce that $\widetilde{\Gamma}_{[C]}(U, \mathcal{D}^{2(\infty,\infty)}_{Y\leftarrow\Lambda}(r,s))$ is a subset of $\widetilde{\Gamma}_{[C]}(U, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s))$ and that the morphism of proposition 2.7.1 is defined from $\widetilde{\Gamma}_{[C]}(U, \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s))$ to $\widetilde{\Gamma}_{[C]}(U', \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda'}(r,s))$.

We may also extend the results to the sheaf $\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda}(r,s)$. To make easier calculations, we will assume that the dimension of Y is 1 which is the case needed in the appendix. Let α be a point $(x_0 = 0, x_0^* = 1, \tau_0)$ of $T^*_{T^*_Y X}(Y \times T^*_Y X)$. Then from the definition of the microlocalization we have :

$$\begin{aligned}\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda}(r,s)_\alpha &= \mu_{T^*_Y X}\left(\mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)\right)[1]_\alpha = \varinjlim_{\varepsilon} \mathcal{H}^1_{Z_\varepsilon}(U_\varepsilon, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)) \\ &= \varinjlim_{\varepsilon} \Gamma(U_\varepsilon - Z_\varepsilon, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s)) \Big/ \Gamma(U_\varepsilon, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s))\end{aligned}$$

with

$$\begin{aligned}U_\varepsilon &= \{\, (x, x', \tau) \in Y \times Y \times \mathbb{C}^d \mid |x - x'| < \varepsilon, \tau \in \Gamma_\varepsilon \,\} \\ Z_\varepsilon &= \{\, (x, x', \tau) \in Y \times Y \times \mathbb{C}^d \mid \operatorname{Re}(x - x') \le \varepsilon |\operatorname{Im}(x - x')| \,\}\end{aligned}$$

where Γ_ε is a fundamental system of open conic neighborhoods of τ_0.

If we define $\tilde{\mathcal{E}}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda}(r,s)[C]_\alpha$ as the subset of $\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda}(r,s)_\alpha$ of elements generated by sections of $\widetilde{\Gamma}_{[C]}(U_\varepsilon - Z_\varepsilon, \mathcal{C}^{\mathbb{R}}_{Y\times Y|Y\times X}(r,s))$, we extend proposition 2.7.1 in the following way :

Proposition 2.7.3. *(i)Let $C > 0$ and $(\sigma, a) \in \mathbb{C}^* \times \mathbb{C}^*$ such that $\operatorname{Re} a\sigma > 0$ and $|a| < 1/C$. The morphism of lemma 2.5.1 defines a* multiplicative *morphism :*

$$\tilde{\mathcal{E}}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda}(r,s)[C]_\alpha \longrightarrow \tilde{\mathcal{E}}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda'}(r,s)[C]_{\alpha'}$$

(ii) If P is an element of $\mathcal{E}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)_\alpha$ whose symbol $P = \sum P_{ik}(x, \tau, x^*)$ satisfies the conditions (2.3.3) but with (iii) replaced by 2.7.1 (ii), if the functions $P_{ik}(x, \tau, x^*)$ extend as ramified functions in τ, then P is an element of $\tilde{\mathcal{E}}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda}(r,s)[C]_\alpha$.

This result is proved exactly as proposition 2.7.1.

3 Equivalence theorem for 2-microlocal equations.

3.1 The result.

In this section, Y is a submanifold of *codimension 1* of the complex analytic manifold X and Λ is the conormal bundle $T^*_Y X$, while $\dot{\Lambda} = \Lambda - Y$.

We assume that we are given local coordinates $(x_1, \ldots, x_{n-1}, t)$ of X where $(t = 0)$ is an equation for Y. Then $\Lambda = \{\, (x, t, \tau, \xi) \in T^*X \mid t = 0, \xi = 0 \,\}$ and the local coordinates of $T^*\Lambda$ are (x, τ, x^*, τ^*).

The function t may be considered as a differential operator on X and then has a symbol $\sigma^{(r,s)}_\Lambda(t) = \tau^*$ for any (r, s).

The aim of section 3 is to prove the following theorem :

Theorem 3.1.1. *Let $(r, s) \in \mathbb{Q}^2$ such that $r > s \ge 1$ and let P be a microdifferential operator (that is a section of $\mathcal{E}_X$), such that $\sigma^{(r,s)}_\Lambda(P) = (\tau^*)^m$ in a neighborhood of a point $\alpha \in \dot{\Lambda}$. Then there is, on a neighborhood of α, an isomorphism :*

$$\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s) \Big/ \mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s) P \simeq \left(\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s) \Big/ \mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)\, t\right)^m$$

In fact the isomorphism is given by a "multivalued" matrix :

Definition 3.1.2. The sheaf $\widetilde{\mathcal{D}}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$ is the subsheaf of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$ of sections which have a continuation along any path in the fibers of $\dot{\Lambda} \to Y$.

Proposition 3.1.3. *Under the hypothesis of theorem 3.1.1, there is an isomorphism :*

$$\widetilde{\mathcal{D}}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)} \Big/ \widetilde{\mathcal{D}}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)} P \simeq \left(\widetilde{\mathcal{D}}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)} \Big/ \widetilde{\mathcal{D}}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}\, t \right)^m$$

The proof will be made in three steps. First we will prove an equivalence theorem when $\sigma_\Lambda^{(r,s)}(P) = (x_1^*)^m$. In that case, the theorem is true in $\mathcal{D}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}$.

The situation is similar to the case of microdifferential operators. An equivalence theorem is true in $\mathcal{E}_X^\infty$ for operators with principal symbol ξ_1^m [29] while the theorem is true only in $\mathcal{E}_X^\mathbb{R}$ when the principal symbol is t^m [11].

Next we will 'add a variable', perform a quantized canonical transformation and take the value of an operator at some point using 2.7.1. This will give theorem 3.1.1 but for $s' > s$. In section 3.4 we will prove the limit case $s = s'$.

The theorem is stated for a 1-codimensional manifold Y, but extension to any submanifold is easy (using a quantized canonical transformation).

3.2 Equivalence theorem for 2-microdifferential equations.

In this section, the codimension of Y is $d \geq 1$. We keep the same notations for local coordinates. The fiber bundle $T^*\Lambda$ is thus provided with coordinates (x, τ, x^*, τ^*). We fix numbers (r, s) such that $1 \leq s < r$.

Let U be an open subset of $T^*\Lambda$ which is (complex) conic in (τ, x^*) and in (x^*, τ^*). Let q be a positive integer, μ be a rational number such that $0 \leq \mu \leq 1$ and $G_q^\mu = \{\, (i,k) \in \mathbb{Z}^2 \mid i + rk \leq q,\, i + sk \leq q,\, 2i + (r+s)k \leq 2\mu q \,\}$

We define $\widehat{E}_q^\mu(U)$ as the set of formal series of holomorphic functions on U :

$$P = \sum_{(i,k)\in G_q^\mu} P_{ik}(x, \tau, x^*, \tau^*)$$

such that $P_{ik}(x, \tau, x^*, \tau^*)$ is a homogeneous function of degree i in (x^*, τ^*) and of degree $i + k$ in (τ, x^*).

For such a series we define a Boutet de Monvel formal norm as :

$$N_q^\mu(P; S, T) = \sum_{i,k,\alpha,\beta,\gamma,\delta} c_{ik\gamma\delta}^{\alpha\beta} \left| \left(\frac{\partial}{\partial x^*}\right)^\alpha \left(\frac{\partial}{\partial \tau^*}\right)^\gamma \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial \tau}\right)^\delta P_{i,k}(x, \tau, x^*, \tau^*) \right| S^{-2i+|\alpha|+|\beta|+|\gamma|+|\delta|} T^{-2k}$$

where the sum is taken over all integers $(i,k) \in G_q^\mu$, all $(\alpha, \beta) \in \mathbb{N}^{n-d} \times \mathbb{N}^{n-d}$ and all $(\gamma, \delta) \in \mathbb{N}^d \times \mathbb{N}^d$.

The coefficient $c_{ik\gamma\delta}^{\alpha\beta}$ is, by definition :

$$c_{ik\gamma\delta}^{\alpha\beta} = 2(2n)^{i+k}\frac{N!}{(N+|\alpha|+|\gamma|)!(N+|\beta|+|\delta|)!}$$

with $N = \inf(-i-rk+q, -i-sk+q, -i-\frac{(r+s)}{2}k+\mu q)$ and $N! = \Gamma(N+1)$.

When $\mu = 1$ this formal norm is the norm of [18, def. 2.4.3.] and the same proof as [18, th. 2.4.9.] gives :

Lemma 3.2.1.

$$N_q^\mu(P+Q;S,T) \ll N_q^\mu(P;S,T) + N_q^\mu(Q;S,T)$$
$$N_{q+q'}^\mu(PQ;S,T) \ll N_q^\mu(P;S,T)N_{q'}^\mu(Q;S,T)$$

In this lemma, $\ll$ means that at each point (x,τ,x^*,τ^*) the coefficient of each monomial S^kT^l of the right side is greater than the corresponding term of the left side and the product PQ is given by formula 3.2.4 of [20] (or theorem 2.3.3. of [18] but with other notations).

We will denote by $E_q^\mu(U)$ the subset of $\widehat{E}_q^\mu(U)$ of elements P such that, for each compact subset K in U, there exists some $C > 0$ such that $N_q^\mu(P,S,T)$ is convergent when $(x,\tau,x^*,\tau^*) \in K$ and $|S|^r \leq \frac{1}{C}|T|, |T| \leq \frac{1}{C}|S|^s$.

It is clear that the union of $E_q^\mu(U)$ over all $q \in \mathbb{N}$ is independent of μ, it will be denoted by $E(U)$.

Lemma 3.2.1 shows that $E(U)$ is a ring. In fact, for $\mu = 1$ the definition is exactly the same as the definition of 2-microdifferential operators of finite order in section 2.3 and thus we have $E(U) = \mathcal{E}^2_{\Lambda(r,s)}(U)$.

Lemma 3.2.2. *Let μ be a real number such that $0 \leq \mu < 1$ and for each $q \in \mathbb{N}$ let $P^{(q)} \in \widehat{E}_q^\mu(U)$. We assume that :*

$$\forall K \subset\subset U,\ \exists C, C_0, C_1 > 0,\ \forall (x,\tau,x^*,\tau^*) \in K,$$
$$\forall (S,T),\ |S|^r \leq \frac{1}{C}|T|,\ |T| \leq \frac{1}{C}|S|^s, \quad N_q^\mu(P^{(q)},S,T) < C_0\frac{C_1^q}{q!}$$

Then the series $\sum_q P^{(q)}$ converges in $\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}(U)$.

More precisely, if the symbol of $P^{(q)}$ is $\sum P_{ik}^{(q)}$, the series

$$P_{ik} = \sum_{q\geq 0} P_{ik}^{(q)}$$

are convergent series of bihomogeneous holomorphic functions on U for each (i,k) and the resulting functions P_{ik} satisfy the following inequalities for some $\nu > 0$:

$$\forall K \subset\subset U,\ \exists C_2 > 0,\ \forall (x,\tau,x^*,\tau^*) \in K,$$

$$
\begin{array}{lll}
(i) & |P_{ik}(x,\tau,x^*,\tau^*)| < C_2^{-i-k}(-i)!(-k)!^s & \text{if } i<0, k<0 \\
(ii) & |P_{ik}(x,\tau,x^*,\tau^*)| < C_2^{-i+k}\dfrac{(-i)!}{k!^r} & \text{if } i<0, k\geq 0 \\
(iii) & |P_{ik}(x,\tau,x^*,\tau^*)| < C_2^{i-k}\dfrac{(-k)!^s}{i!}\dfrac{1}{[(1-\mu)(i+(s+\nu)k)]_+!} & \text{if } i\geq 0, k<0 \\
(iv) & |P_{ik}(x,\tau,x^*,\tau^*)| < C_2^{i+k+1}\dfrac{1}{k!^r\, i!}\dfrac{1}{[(1-\mu)(i+(r-\nu)k)]_+!} & \text{if } i\geq 0, k\geq 0
\end{array}
$$

($[a]_+$ is the lowest positive integer greater than the rational number a)

Proof. First, we remark that inequalities (i)-(iv) imply immediately (2.3.3) hence series satisfying (i)-(iv) define a symbol of $\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s)$.

Applying Cauchy inequalities in (S,T) to $N_q^\mu(P^{(q)},S,T)$ we get for (x,τ,x^*,τ^*) in K and $N=\inf(-i-rk+q,-i-sk+q,-i-\frac{(r+s)}{2}k+\mu q)$:

$$|P_{ik}^{(q)}(x,\tau,x^*,\tau^*)| < C_0C_1^q\frac{N!}{q!}C^{2\frac{-i-rk-i-sk}{r-s}}$$

(We denote by $r!$ the number $\Gamma(r+1)$ for any $r\in\mathbb{Q}$).

If ν is the strictly positive number $\frac{r-s}{2(1-\mu)}$, we have :

$$
\begin{array}{ll}
N=-i-\frac{(r+s)}{2}k+\mu q & \text{if } q\geq \nu|k| \\
N=-i-sk+q & \text{if } 0\leq q\leq -\nu k \\
N=-i-rk+q & \text{if } 0\leq q\leq +\nu k
\end{array}
$$

Assume first that $i<0, k<0$, then lemma 2.4.8 of [18] gives a constant C_s depending only on s such that :

$$
\begin{aligned}
\frac{N!}{q!} &= \frac{(-i-sk+q)!}{q!} \leq C_s^{-i-sk+q}(-i)!(-k)!^s \quad \text{if } 0\leq q\leq -\nu k \\
\frac{N!}{q!} &= \frac{(-i-sk+q-(1-\mu)(q+\nu k))!}{q!} \\
&\leq C_s^{-i-sk+q}(-i)!(-k)!^s\frac{1}{[(1-\mu)(q+k\nu)]!} \quad \text{if } q\geq -\nu k
\end{aligned}
$$

This shows that the series $P_{ik}=\sum_{q\geq 0}P_{ik}^{(q)}$ is convergent and satisfy :

$$
\begin{aligned}
|P_{ik}(x,\tau,x^*,\tau^*)| &< C_0C_2^{-i-k}(-i)!(-k)!^s\left(\sum_{0\leq q<-\nu k}C_3^q+\sum_{q\geq -\nu k}C_3^q\frac{1}{(q+\nu k)!^{(1-\mu)}}\right) \\
&< C_4^{-i-k}(-i)!(-k)!^s
\end{aligned}
$$

Let us consider now the case $k \geq 0, i < 0$, then

$$\begin{aligned}
\frac{N!}{q!} &= \frac{(-i-rk+q)!}{q!} \leq 2^{-i+q}\frac{(-i)!}{k!^r} && \text{if } 0 \leq q \leq \nu k \\
\frac{N!}{q!} &= \frac{(-i-rk+q-(1-\mu)(q-\nu k))!}{q!} \\
&\leq 2^{-i+q}\frac{(-i)!}{k!^r}\frac{1}{[(1-\mu)(q-k\nu)]!} && \text{if } q \geq \nu k
\end{aligned}$$

and we get :

$$|P_{ik}(x,\tau,x^*,\tau^*)| < C_5^{-i+k}\frac{(-i)!}{k!^r}$$

Now we consider $k < 0, i \geq 0$, we have :

$$\begin{aligned}
\frac{N!}{q!} &= \frac{(-i-sk+q)!}{q!} \leq C_s^{-i-sk+q}\frac{(-k)!^s}{i!} && \text{if } 0 \leq q \leq -\nu k \\
\frac{N!}{q!} &= \frac{(-i-sk+q-(1-\mu)(q+\nu k))!}{q!} \\
&\leq C_s^{-i-sk+q}\frac{(-k)!^s}{i!}\frac{1}{[(1-\mu)(q+\nu k)]!} && \text{if } q \geq -\nu k
\end{aligned}$$

but $q \geq i + sk$ so $q \leq -k\nu$ will occur only if $i \leq -k(\nu + s)$. We get :

$$\begin{aligned}
|P_{ik}(x,\tau,x^*,\tau^*)| &< C_6^{i-k}\frac{(-k)!^s}{i!} && \text{if } i \leq -k(\nu+s) \\
|P_{ik}(x,\tau,x^*,\tau^*)| &< C_6^{i-k}\frac{(-k)!^s}{i!}\frac{1}{[(1-\mu)(i+(s+\nu)k)]!} && \text{if } i \geq -k(\nu+s)
\end{aligned}$$

The last case is $i \geq 0, k \geq 0$. The same calculation shows that :

$$\begin{aligned}
|P_{ik}(x,\tau,x^*,\tau^*)| &< C_7^{i+k}\frac{1}{k!^r\, i!} && \text{if } i \leq k(\nu-r) \\
|P_{ik}(x,\tau,x^*,\tau^*)| &< C_7^{i+k}\frac{1}{k!^r\, i!}\frac{1}{[(1-\mu)(i-(\nu-r)k)]!} && \text{if } i \geq k(\nu-r)
\end{aligned}$$

This concluded the proof of the lemma. □

An operator $P \in \mathcal{E}^2_{\Lambda}(r,s)(U)$ will be said of order (q,μ) if its symbol is in $\widehat{E}^{\mu}_q(U)$.

Let $a : \mathbb{N} \to \mathbb{N}$ be any function. A $(m \times m)$-matrix A of $\mathcal{E}^2_{\Lambda}(r,s)(U)$ will be said of order (q,μ) if, for each (i,j), the coefficient A_{ij} of A is a microdifferential operator of order at most $(q + a(i) - a(j), \mu)$. Then the order of the product of a matrix of order (q,μ) by a matrix of order (q',μ) is not greater than $(q+q',\mu)$. The usual order of matrices is given by $a = 0$ but in the proof of theorem 3.2.4 we will use $a(i) = i$.

We define the formal norm of such a matrix as:

$$N^{\mu}_q(A;S,T) = \sup_{1\leq i\leq m} \sum_{1\leq j\leq m} N^{\mu}_{q+a(i)-a(j)}(A_{ij};S,T)$$

and it is clear that lemmas 3.2.1 and 3.2.2 are still true for matrices.

The matrix A is said to be independent of D_{x_1} if the symbols of the microdifferential operators A_{ij} are independent of x_1^*. This is equivalent to the fact that A commutes with x_1. Then A may be considered as a matrix of 2-microdifferential operators in the variables $(x_2, \dots, x_{n-d}, t)$ with holomorphic parameter x_1 and we will write it as $A(x_1)$.

Such a matrix $A(x_1, x', \tau, x'^*, \tau^*)$ will be said of medium order (δ, μ) if there exists some $q \in \mathbb{N}$ such that any product of q terms

$$A(x_1^{(1)}, x', \tau, x'^*, \tau^*)A(x_1^{(2)}, x', \tau, x'^*, \tau^*) \dots A(x_1^{(q)}, x', \tau, x'^*, \tau^*)$$

is of order at most $(q\delta, \mu)$.

For example, if A_0 is a matrix of order $(1, \mu)$ and A_1 a matrix of order $(1,1)$ nilpotent of order m (i.e. $A_1^m = 0$) then $A_0 + A_1$ is of medium order $(1, 1 - \frac{1-\mu}{m})$.

We denote by I the identity $(m \times m)$-matrix.

Proposition 3.2.3. *Let U be an open subset of $T^*\Lambda$ which is (complex) conic in (τ, x^*) and in (x^*, τ^*). We assume that for each point $(x_1, x_0', \tau_0, x_0^*, \tau_0^*)$ in U the set $\{ \lambda \in \mathbb{C} \mid (\lambda, x_0', \tau_0, x_0^*, \tau_0^*) \in U \}$ is a simply connected open subset of $\mathbb{C}$ which contains the origin.*

Let A be a $m \times m$-matrix of $\mathcal{E}^2_{\Lambda}(r,s)(U)$ which is independent of D_{x_1} and of medium order $(1, \mu)$ with $\mu < 1$. Then there exists on U an invertible matrix R with coefficients in $\mathcal{E}_{\Lambda}^{2(\infty,\infty)}(r,s)(U)$ such that :

$$(D_{x_1} I - A)R = RD_{x_1}$$

Moreover we may choose R such that $R = I$ on $\{ x_1 = 0 \}$ and then R is unique and independent of D_{x_1}. The symbols of the coefficients of R satisfy the conditions (i)-(iv) of lemma 3.2.2.

Proof. The proof is similar to the proof of theorem 5.2.1. ch.II in [29] :

The equation $(D_{x_1} I - A)R = RD_{x_1}$ is equivalent to $\left(\frac{\partial}{\partial x_1}\right) R = AR$ where $\left(\frac{\partial}{\partial x_1}\right)$ is the operator of derivation acting on the symbols of the coefficients of R.

We define a sequence R_k of matrices with coefficients in $\mathcal{E}^2_{\Lambda}(r,s)$ by $R_0 = I$ and the relation :

$$\left(\frac{\partial}{\partial x_1}\right) R_k = AR_{k-1} \qquad R_k|_{x_1=0} = 0$$

and we have to prove that the series $\sum R_k$ is convergent.

The coefficients of AR_{k-1} are formal series of holomorphic function thus R_k is given by the formula :

$$R_k(x_1) = \int_0^{x_1} AR_{k-1}(\lambda) d\lambda$$

where the integration is made along any path from 0 to x_1. It is independent of the path as U is simply connected in x_1.

In a product $A(\lambda)A(\lambda')$ there is no derivation in the variables (λ, λ') so we have :

$$R_k(x_1) = \int_0^{x_1} \int_0^{\lambda_k} \cdots \int_0^{\lambda_2} A(\lambda_k)A(\lambda_{k-1}) \dots A(\lambda_1) d\lambda_k d\lambda_{k-1} \dots d\lambda_1$$

From the hypothesis, there exists some $q \in \mathbb{N}$ such that the matrix

$$B(x_1^{(1)}, \dots, x_1^{(q)}) = A(x_1^{(1)})A(x_1^{(2)}) \dots A(x_1^{(q)})$$

is of order at most (q,μ) hence for each compact $G\times K$ of U, there exists some $C, C_0>0$ such that $N_q^\mu(B;S,T)$ is bounded by C_0 when $x^{(i)}\in G$ for $i=1\dots q$, $(x',\tau,x^*,\tau^*)\in K$ and $|S|^r\leq\frac{1}{C}|T|, |T|\leq\frac{1}{C}|S|^s$.

Applying lemma 3.2.1 we get :

$$\sup_{x_1\in G} N_{qk}^\mu(R_{qk};S,T)\leq\left(\sup_{x_1\in G}N_q^\mu(B;S,T)\right)^k\int_0^{x_1}\int_0^{\lambda_{qk}}\cdots\int_0^{\lambda_2}d\lambda_{qk}\dots d\lambda_1$$
$$\leq C_0^kC_1^{qk}\frac{1}{(qk)!}$$

The same proof apply to R_{qk+l} for $l=0\dots q-1$ and this proves that the series $\sum R_k$ satisfy the hypothesis of lemma 3.2.2 hence is convergent as a $m\times m$-matrix with coefficients in $\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}(U)$.

The matrix $R=\sum R_k$ is a solution of the equation $\left(\frac{\partial}{\partial x_1}\right)R=AR$ and $R(0)=I$. To prove that R is invertible we use the method of [29], that is we define a matrix S such that $\left(\frac{\partial}{\partial x_1}\right)S=-SA$ and $S(0)=I$. The preceding proof gives a solution to this equation and we have $SR(0)=I$ with $\left(\frac{\partial}{\partial x_1}\right)SR=0$ hence $SR=I$ and R is invertible.

This proves also the uniqueness of R because if R' is another solution then for the same reason we have $SR'=I$ hence $R'=R$. And R is independent of D_{x_1} because $[x_1,R]$ satisfy the same equation as R and is 0 on $x_1=0$. □

Theorem 3.2.4. *Let P a 2-microdifferential operator of $\mathcal{E}^2_{\Lambda}{}_{(r,s)}$ such that $\sigma_\Lambda^{(r,s)}(P)=(x_1^*)^m$. Then there is, locally on $\dot\Lambda$, an isomorphism :*

$$\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}\Big/\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}P\simeq\left(\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}\Big/\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}D_{x_1}\right)^m$$

Proof. Using the division theorem 2.5.2. of [18] we can write :

$$P(x,\tau,x^*,\tau^*)=E(x,\tau,x^*,\tau^*)\left((x_1^*)^m-\sum_{j=0}^{m-1}P_j(x,\tau,x_2^*,\dots,x_{n-d}^*,\tau^*)(x_1^*)^j\right)$$

where P_j is a 2-microdifferential operator whose symbol do not depend on x_1^* and E is invertible.

As $\sigma_\Lambda^{(r,s)}((x_1^*)^m-\sum_{j=0}^{m-1}P_j(x,\tau,x_2^*,\dots,x_{n-d}^*,\tau^*)(x_1^*)^j)=(x_1^*)^m$, the symbol $P_j=\sum P_{i,k}^j$ satisfy :

$$P_{i,k}^j\neq0\Longrightarrow(i+rk\leq m, i+sk\leq m \text{ and } i<m \text{ if } k=0)$$

and P_j belongs to $\widehat{E}_{m-j}^\mu(U)$ with $\mu=\max(0,1-\frac{r-s}{2})$.

The module $\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}\Big/\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)}P$ is isomorphic to

$$(\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)})^m\Big/(\mathcal{E}_\Lambda^{2(\infty,\infty)}{}_{(r,s)})^m(D_{x_1}I-A)$$

where A is the $m \times m$-matrix $A_0 + A_1$ with :

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 \\ P_0 & P_1 & \dots & \dots & P_{m-1} \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

The matrix A_0 is of order $(1, \mu)$ for the function $a : \mathbb{N} \longrightarrow \mathbb{N}$ equal to $a(i) = i$ while the matrix A_1 is of order $(1, 1)$ for the same function. As A_1 is nilpotent of order m the matrix A is of medium order $(1, \mu')$ with $\mu' = 1 - (r - s)/2m$. So we apply proposition 3.2.3 to A and get the result. □

The symplectic structure of the manifold $T^*\Lambda$ is given first by the structure of cotangent bundle to Λ which gives a canonical 1-form ω (hence a structure of homogeneous symplectic manifold) and second by the action of $\mathbb{C}^*$ on the fibers of Λ which gives an Euler vector field on Λ whose principal symbol F is well defined on $T^*\Lambda$ up to a multiplicative constant [18].

These data are equivalent to the canonical 2-form of the symplectic manifold $T^*\Lambda$ and the two actions H_0 and H_∞ of section 1.3.

The 1-form ω and the function F define on $T^*\Lambda$ a structure of bihomogeneous symplectic manifold and to each bihomogeneous symplectic isomorphism is associated an isomorphism of $\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s)$ and of $\mathcal{E}_\Lambda^2(r,s)$ which is called a "quantized bicanonical transformation" ([18] Theorem 2.9.11.).

From this we can transform theorem 3.2.4 into the following :

Corollary 3.2.5. *Let f be a bihomogeneous holomorphic function on $T^*\Lambda$ and Q a 2-microdifferential operator whose symbol is f. Let η be a point of $T^*\Lambda$ where ω, dF and df are linearly independent.*

Let P be a 2-microdifferential operator such that $\sigma_\Lambda^{(r,s)}(P) = f^m$. Then there is an isomorphism near η :

$$\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s) \Big/ \mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s)P \simeq \left(\mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s) \Big/ \mathcal{E}_\Lambda^{2(\infty,\infty)}(r,s)Q\right)^m$$

In local coordinates we have $\omega = \sum x_i^* dx_i + \sum \tau_j^* d\tau_j$ and $F = \sum \tau_j \tau_j^*$. The corollary may be applied, for example, to $f = \tau_1^*$ and $Q = t_1$ if η is a point where $(\tau_2, \dots, \tau_d) \neq 0$.

Let us consider a manifold Z and denote $X' = X \times Z$, $Y' = Y \times Z$ and $\Lambda' = T_{Y'}^* X' = T_Y^* X \times Z$. It is proved in [29, theorem 5.3.1. ch. II] that, if a coherent $\mathcal{E}_{X'}$-module $\mathcal{M}$ has a support in $T^*X \times Z$, then $\mathcal{E}_{X'}^\infty \otimes \mathcal{M}$ is completely determined by its inverse image on X by $j : X \hookrightarrow X'$. The corresponding result is true here (the inverse image of a $\mathcal{E}_{\Lambda'}^2(r,s)$-module was defined in [18]):

Theorem 3.2.6. *If $\mathcal{M}$ is a coherent $\mathcal{E}_{\Lambda'}^2(r,s)$-module with support in $T^*\Lambda \times Z$ there is a canonical isomorphism :*

$$\mathcal{E}_{\Lambda'}^{2(\infty,\infty)}(r,s) \otimes_{\mathcal{E}_{\Lambda'}^2(r,s)} \mathcal{M} = \mathcal{E}_{\Lambda' \to \Lambda}^{2(\infty,\infty)}(r,s) \otimes_{\varrho^{-1}\mathcal{E}_\Lambda^2(r,s)} \varrho^{-1} j^* \mathcal{M}$$

with $\varrho : T^*\Lambda \times Z \longrightarrow T^*\Lambda$, $\pi : T^*\Lambda \times Z \longrightarrow X'$ *and*

$$\mathcal{E}^{2(\infty,\infty)}_{\Lambda'\to\Lambda}(r,s) = \mathcal{E}^{2(\infty,\infty)}_{\Lambda'}(r,s) \otimes_{\pi^{-1}\mathcal{E}_{X'}} \pi^{-1}\mathcal{E}_{X'\leftarrow X}$$

This theorem is a direct consequence of theorem 3.2.4 with the same proof than the corresponding result of [29]. By restriction to the zero section of Λ' we get :

Proposition 3.2.7. *If $\mathcal{M}$ is a coherent $\mathcal{D}^2_{\Lambda'}(r,s)$-module such that $Ch_{\Lambda'(r,s)}(\mathcal{M}) \subset T^*\Lambda \times Z$ there is a canonical isomorphism :*

$$\mathcal{D}^{2(\infty,\infty)}_{\Lambda'}(r,s) \otimes_{\mathcal{D}^2_{\Lambda'}(r,s)} \mathcal{M} = \mathcal{D}^{2(\infty,\infty)}_{\Lambda'\to\Lambda}(r,s) \otimes_{\mathcal{D}^2_{\Lambda}(r,s)} j^*\mathcal{M}$$

3.3 Equivalence theorem for 2-microlocal equations.

We assume now that the codimension of Y is 1. We fix local coordinates (x,t) of X such that $Y = \{\, t = 0 \,\}$ and thus coordinates (x,τ) of Λ and (x,τ,x^*,τ^*) of $T^*\Lambda$.

Then Λ is locally isomorphic to $Y \times \mathbb{C}$ and $T^*\Lambda$ to $T^*Y \times \mathbb{C}^2$.

In the following proposition we consider a matrix A of operators of $\mathcal{D}^2_{\Lambda}(r,s)$ independent of τ^*. This means that A commutes with the operator D_t or that the symbols of the elements of A are independent of τ^*.

The sheaf $\mathcal{D}^2_{\Lambda}(r,s)$ is the restriction to Λ of $\mathcal{E}^2_{\Lambda}(r,s)$ and the order (σ,μ) has the same meaning than in section 3.2.

Such a matrix $A(x,\tau,x^*)$ will be said of medium order (δ,μ) if there exists some $q \in \mathbb{N}$ such that any product of q terms

$$A(x,\tau^{(1)},x^*)A(x,\tau^{(2)},x^*)\ldots A(x,\tau^{(q)},x^*)$$

is of order at most $(q\delta,\mu)$.

Proposition 3.3.1. *Let V be an open subset of Y and $U = \{\, (x,\tau,x^*,\tau^*) \in T^*\Lambda \mid x \in V,\, \tau \neq 0 \,\}$.*

Let (r,s) be two rational numbers such that $1 \leq s < r < +\infty$ and A be a $(m \times m)$-matrix of 2-microdifferential operators in $\mathcal{D}^2_{\Lambda}(r,s)(U)$ which is independent of τ^* and of medium order $(1,\mu)$ with $\mu < 1$.

For each $s' > s$, there exists on U an invertible matrix R in $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s')$ with coefficients independent of τ^* and multivalued in τ such that :

$$(tD_tI - A)R = RtD_t$$

Proof. We first add a variable and apply Proposition 3.2.3:

We consider $X' = X \times \mathbb{C}$ and $Y' = Y \times \{0\} \subset X'$ with coordinates (x,t,z), $\Lambda' = T^*_{Y'}X' \simeq \Lambda \times \mathbb{C}$ and the open set $U' = \{\, (x,\tau,\zeta,x^*,\tau^*,\zeta^*) \in T^*\Lambda' \mid x \in V, \tau \neq 0, \zeta \neq 0 \,\}$.

The partial Legendre transform in (t,z,τ,ζ) on T^*X' is given by :

$$x = x, \xi = \xi, y_1 = \tau\zeta^{-1}, y_2 = z + t\tau\zeta^{-1}, \eta_1 = -t\zeta, \eta_2 = \zeta$$

It defines a homogeneous isomorphism between Λ' and the conormal Λ'' to the submanifold $Y'' = \{(x,y) \in X \mid y_2 = 0\}$ and a bihomogeneous canonical transformation Φ from $T^*\Lambda'$ to $T^*\Lambda''$ given by :

$$x = x, x^* = x^*, y_1 = \tau\zeta^{-1}, \eta_2 = \zeta, y_1^* = -\tau^*\zeta, \eta_2^* = \zeta^* + \tau^*\tau\zeta^{-1}$$

It is proved in [18, theorem 2.9.11.] that to any bihomogeneous canonical transformation Φ is associated a "quantized" canonical transformation, that is an isomorphism of sheaves of rings $\mathcal{E}^2_{\Lambda'}{}_{(r,s)} \to \Phi^{-1}\mathcal{E}^2_{\Lambda''}{}_{(r,s)}$ which preserves orders and principal symbols. It is shown in [20] that this transformation extend uniquely to sheaves $\mathcal{E}^{2(\infty,\infty)}_{\Lambda}{}_{(r,s)}$.

So, we can choose a quantized canonical transformation associated to Φ which exchanges tD_t and $y_1 D_{y_1}$ and now we are in the situation of proposition 3.2.3 which gives an invertible matrix R in $\mathcal{E}^{2(\infty,\infty)}_{\Lambda}{}_{(r,s)}$ such that $(D_{y_1}I - A)R = RD_{y_1}$ and $R(y_1 = 1) = I$. The matrix R is defined as a multivalued function in y_1 because proposition 3.2.3 is global on simply connected open sets.

As A commutes with z and t it commutes with D_{y_1} and y_2, so the same is true for R by uniqueness.

Coming back to variables (z,t) by the inverse quantized canonical transformation, we find a matrix $R(x,x^*,\tau,\zeta)$ with coefficients in $\mathcal{E}^{2(\infty,\infty)}_{\Lambda'}{}_{(r,s)}(U')$ (multivalued in τ/ζ) which is invertible and such that :

$$(tD_t - A)R = RtD_t \quad \text{and} \quad R|_{\tau=\zeta} = I$$

In fact A is defined near the zero section of $T^*\Lambda'$ and is thus a matrix of $\mathcal{D}^{2(\infty,\infty)}_{\Lambda'}{}_{(r,s)}$. This equation may rewritten as

$$\left(\frac{\partial}{\partial\tau}\right)\tau R(\tau,\zeta) = A(\tau)R(\tau,\zeta) \quad \text{and} \quad R(\tau,\tau) = I$$

We want now to find a solution to the initial problem as the value of $R(\tau,\zeta)$ at a point $\zeta = \zeta_0$.

Let $s' > s$ and δ such that $1 < \delta < \dfrac{s'+\nu}{s+\nu}$, we have for all $i \geq 0$ and $k < 0$:

$$|P_{ik}(x,x',\tau)| \leq \frac{(-k)!^s}{i!}\frac{1}{[(1-\mu)(i+(s+\nu)k)]_+!} < C^{i-k}\frac{(-k)!^{s'}}{(i!)^\delta}$$

This proves that P satisfy the conditions (2.7.1) as a section of $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}{}_{(r,s')}$ and we may apply proposition 2.7.1. So, we fix some $a \in \mathbb{C}$ and ζ_0 and define the value of the matrix R at the point $\zeta = \zeta_0$. We get a matrix R_0 with coefficients in $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}{}_{(r,s')}$ and it is clear from the definition that :

$$\left(\frac{\partial}{\partial\tau}\right)\tau R_0(\tau) = \left(\frac{\partial}{\partial\tau}\right)\tau R(\tau,\sigma_0) = \left[\left(\frac{\partial}{\partial\tau}\right)\tau R\right](\tau,\sigma_0)$$

On the other hand, we know from proposition 2.7.2 that $A(\tau)R_0(\tau) = (AR)(\tau,\sigma_0)$ and also that $R_0(\tau)R^{-1}(\tau,\sigma_0) = [RR^{-1}](\tau,\sigma_0) = Id$ which proves that R_0 is an invertible solution to the initial problem. □

Remark 3.3.2. The proof of the proposition shows that the matrix R has coefficients in the subsheaf $\widetilde{\mathcal{D}}_{\Lambda}^{2(\mathbb{R},\infty)}(r,s')$ (definition 3.1.2).

Remark 3.3.3. If we replace proposition 2.7.1 by proposition 2.7.3 we get proposition 3.3.1 for a matrix A of $\mathcal{E}_{\Lambda}^{2}(r,s)$ and the solution R is in $\mathcal{E}_{\Lambda}^{2(\mathbb{R},\mathbb{R})}(r,s')$, more precisely in $\tilde{\mathcal{E}}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\mathbb{R})}(r,s')[C]$.

3.4 The limit case.

We want to prove that the result of proposition 3.3.1 is still true when $s'=s$.

Lemma 3.4.1. *Let A be a matrix satisfying the conditions of proposition 3.3.1 and $\mathcal{M}$ be the cokernel :*

$$\left(\mathcal{D}_{\Lambda}^{2}(r,s)\right)^{m} \xrightarrow{(tD_t-A)} \left(\mathcal{D}_{\Lambda}^{2}(r,s)\right)^{m} \longrightarrow \mathcal{M} \longrightarrow 0$$

Let s' such that $r>s'>s$.

Then the canonical (surjective) morphism $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s) \longrightarrow \mathcal{C}_{Y|X}^{\mathbb{R}}(r,s')$ induces an isomorphism :

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\Lambda}^{2}(r,s)}\left(\mathcal{M},\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s)\right) \xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\Lambda}^{2}(r,s)}\left(\mathcal{M},\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s')\right)$$

The result is clear if $A=0$ hence proposition 3.3.1 gives an isomorphism between solutions in $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s'')$ and in $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s')$ for any $s'>s''>s$. The problem is to get $s''=s$.

Proof. Following the beginning of the proof of prop. 3.3.1, we add a variable z. Keeping the same notations we find an invertible matrix of 2-microdifferential operators $R(x,x^*,\tau,\zeta)$ such that :

$$(tD_tI-A)R = RtD_t$$

Let u be a section of $\left(\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s')\right)^m$ in a neighborhood of a point $\alpha=(x_0,\tau_0)$ of T_Y^*X solution of $(tD_t-A)u=0$. We have :

$$tD_tR^{-1}u = R^{-1}(tD_tI-A)u = 0$$

Hence $u=Rv$ for some section v of $\left(\mathcal{C}_{Y'|X'}^{\mathbb{R}}(r,s')\right)^m$ satisfying $tD_tv=0$.

There exists some $\varepsilon>0$ such that v is defined for $|x-x_0|<\varepsilon$, $\tau\neq 0$ and $\zeta\neq 0$.

We want now to define the value of v and of R at a point ζ_0. We apply the second part of proposition 2.7.1 and choose some a so that $R_0=R(\zeta_0)$ is a section of $\mathcal{D}_{\Lambda\to Y}^{2(\mathbb{R},\infty)}(r,s;\varepsilon/4)$.

Proposition 2.6.2 shows that $u=R_0v_0$ on $|x-x_0|<\varepsilon/4$.

The microfunction v_0 is a section of $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s')$ satisfying $tD_tv_0=0$ hence of the form $f(x)\delta^{(-1)}(t)$ where f is a holomorphic function on Y and $\delta^{(-1)}(t)$ the microfunction of symbol $1/\tau$. So v_0 is in fact a section of $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s)$ and the same is true for u at α.

We proved that if u is a germ of $\left(\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s')\right)^m$ at a point α of T_Y^*X and satisfies $(tD_t-A)u=0$ then there is a unique $\tilde{u}$ solution of the same equation in $\left(\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s)\right)^m$ at α whose image in $\left(\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s')\right)^m$ is u.

In the same way, we can prove that (tD_t-A) is a surjective morphism on the germs of $\left(\mathcal{C}_{Y|X}^{\mathbb{R}}(r,s)\right)^m$. □

Corollary 3.4.2. *The conclusion of proposition 3.3.1 is true for $s' = s$.*

Proof. The canonical projection $q : X\times Y \longrightarrow X$ defines a projection $\tilde{q} : T^*_Y X\times Y \longrightarrow T^*_Y X$ and an injective morphism of sheaves of rings $\tilde{q}^{-1}\mathcal{D}^2_\Lambda(r,s) \longrightarrow \mathcal{D}^2_{\Lambda\times Y}(r,s)$. This means that we consider an operator on X as an operator on $X\times Y$ constant in the second variables.

In this way the operator $(tD_t - A)$ of proposition 2.5 may be considered as a matrix of operators of $\mathcal{D}^2_{\Lambda\times Y}(r,s)$ which satisfy the same hypothesis. Applying lemma 3.4.1 we get :

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\tilde{q}^{-1}\mathcal{D}^2_\Lambda(r,s)}\left(\mathcal{M},\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times Y}(r,s)\right) \xleftarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\tilde{q}^{-1}\mathcal{D}^2_\Lambda(r,s)}\left(\mathcal{M},\mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times Y}(r,s')\right)$$

($\mathcal{M}$ is the module defined by $tD_t - A$ as in the lemma).

Applying the functor $\mathbb{R}\Gamma_{T^*_Y X}$ to this isomorphism we get :

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\tilde{q}^{-1}\mathcal{D}^2_\Lambda(r,s)}\left(\mathcal{M},\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)\right) \xleftarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\tilde{q}^{-1}\mathcal{D}^2_\Lambda(r,s)}\left(\mathcal{M},\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s')\right)$$

The action of $\mathcal{D}^2_\Lambda(r,s)$ on $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ defined here is, by definition, the action of $\mathcal{D}^2_\Lambda(r,s)$ considered as a subring of $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$. A solution of $(tD_t - A)R = RtD_t$ is a section of $\mathcal{H}\mathrm{om}_{\tilde{q}^{-1}\mathcal{D}^2_\Lambda(r,s)}\left(\mathcal{M},\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s')\right)$ hence it is a section of $\mathcal{H}\mathrm{om}_{\tilde{q}^{-1}\mathcal{D}^2_\Lambda(r,s)}\left(\mathcal{M},\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)\right)$. (The same argument has been used in [11, §3.2]).

We have proved that for any R satisfying $(tD_t - A)R = RtD_t$ there exists R' in $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ satisfying the same equation and whose image in $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s')$ is R and now we have to prove that R' is invertible.

Let S be a section of $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ whose image in $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s')$ is R^{-1}. Then $RS = I+Q$ where I is the identity and Q is a section of $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ which vanishes in $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s')$. So, Q is very decreasing and it is easy to show that the series $\sum Q^n$ is convergent and defines an inverse to $I+Q$ in $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$.

This proves that R' is invertible. □

The proof of theorem 3.1.1 is now the same as the proof of theorem 3.2.4 using the same division theorem [18, 2.5.2.] and replacing proposition 3.2.3 by corollary 3.4.2.

Theorem 3.4.3. *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$ or $\mathcal{D}^2_\Lambda(r,s)$-module defined on an open subset U of $\dot{\Lambda}$. We assume that*

$$Ch_\Lambda(r,s)(\mathcal{M}) \subset S_\Lambda$$

and that $\mathcal{M}$ is provided with a good F_{rs}-bifiltration whose bigraded module is free of finite rank N over $\mathcal{O}_\Lambda[x^*]$.

Then $\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)\otimes\mathcal{M}$ is locally isomorphic to $\left(\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s) \big/ \mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)\,t\right)^N$.

Proof. If $\mathcal{M}$ is a $\mathcal{E}_X$-module we replace it by $\mathcal{D}^2_\Lambda(r,s)\otimes\mathcal{E}_X$, so we may assume now that $\mathcal{M}$ is a coherent $\mathcal{D}^2_\Lambda(r,s)$-module.

The bigraded ring of $\mathcal{D}^2_\Lambda(r,s)$ for the bifiltration F_{rs} is isomorphic to $\mathcal{O}_\Lambda[x^*,\tau^*]$. We denote by $\breve{\mathcal{D}}^2_\Lambda(r,s)$ the subsheaf of $\mathcal{D}^2_\Lambda(r,s)$ of operators with symbol independant of τ^*, its bigraded ring is $\mathcal{O}_\Lambda[x^*]$.

Let $u_1,\dots,u_N$ a set of local sections of $\mathcal{M}$ whose classes $\bar u_1,\dots,\bar u_N$ is a base of the associated bigraded module $gr\mathcal{M}$ as a $\mathcal{O}_\Lambda[x^*]$-module.

Theorem 2.5.3. of [18] shows that the morphism $\lambda : \left(\breve{\mathcal{D}}^2_\Lambda(r,s)\right)^N \longrightarrow \mathcal{M}$ defined by $u_1,\dots,u_N$ is an isomorphism of $\breve{\mathcal{D}}^2_\Lambda(r,s)$-modules which respects the bifiltrations (with fixed shifts due to the fact that the u_i are not of order $(0,0)$). We may modify the bifiltration so that all the u_i have order $(0,0)$.

The operator tD_t (whose symbol is $\tau\tau^*$) is an endomorphism of $\mathcal{M}$, hence there exists a matrix A of the same order than t such that $tD_tu = Au$ with $u = (u_1,\dots,u_N)$. With the notations of §3.2, this order is $(1,1)$.

The morphism $(\mathcal{D}^2_\Lambda(r,s))^m \,/(\mathcal{D}^2_\Lambda(r,s))^m\,(tDt - A) \longrightarrow \mathcal{M}$ is clearly an isomorphism.

On $\dot\Lambda$, the hypersurface S_Λ of $T^*\Lambda$ is equal to $\{\,\tau^* = 0\,\}$ and by definition $Ch_\Lambda(r,s)(\mathcal{M})$ is the support of $\mathcal{O}_{T^*\Lambda}\otimes\pi^{-1}gr\mathcal{M}$. Hence it is contained in $\tau^* = 0$ and there exists some m such that $(\tau^*)^m\bar u = 0$. This means that the principal symbol $\sigma^{(r,s)}_\Lambda(A^m)$ vanishes hence that the principal symbol of A is nilpotent.

So, as in the proof of theorem 3.2.4, the matrix A is the sum of a matrix A_0 of order $(1,1)$ whose principal symbol vanishes hence of order $(1,\mu)$ with $\mu = \max(0, 1-\frac{r-s}{2})$ and of a matrix A_1 of order $(1,1)$ which is nilpotent of order m hence A is of medium order $(1,\mu')$ with $\mu' = 1-(r-s)/2m$.

We may apply corollary 3.4.2 and find an invertible matrix R of $\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)$ such that $R^{-1}(tD_t - A)R = tD_t$. This shows that :

$$\begin{aligned}\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)\otimes\mathcal{M} &\simeq (\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s))^m \Big/ (\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s))^m\,(tDt - A)\\ &\simeq \left(\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)\Big/\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)\,tD_t\right)^N\end{aligned}$$

which proves the theorem (D_t is invertible on $\dot\Lambda$). □

4 Vanishing cycles.

4.1 Definition.

Let X be a complex analytic manifold and Y be a submanifold of *codimension 1* of X. We denote by $\Lambda = T^*_YX$ the conormal bundle to Y and set $\dot\Lambda = \dot T^*_YX = T^*_YX - Y$, $\pi : \Lambda \to Y$ and $\pi' : \dot\Lambda \to Y$.

If $\mathcal{A}$ is any sheaf of rings, we will denote by $D(\mathcal{A})$ the derived category of the category of left $\mathcal{A}$-modules and by $D_p(\mathcal{A})$ the subcategory of perfect complexes that is of right bounded complexes which are locally quasi-isomorphic to a *bounded* complex of *free* $\mathcal{A}$-modules, the bounds being uniform on the underlying space.

When $\mathcal{A}$ is a coherent sheaf of rings which is regular, that is satisfy the Hilbert syzygy theorem, $D_p(\mathcal{A})$ is equivalent to the subcategory of $D(\mathcal{A})$ of bounded complexes with coherent cohomology and is also denoted by $D^b_c(\mathcal{A})$. This is true in particular for the sheaves $\mathcal{D}_X$ and $\mathcal{E}_X$ considered here. We will denote by $D'(\mathcal{A})$ and $D'_p(\mathcal{A})$ the corresponding categories for right modules.

If $\mathcal{M}$ is a sheaf of $\mathcal{A}$-modules, it is identified with the complex $0 \to \mathcal{M} \to 0$ with $\mathcal{M}$ in degree 0 in $D(\mathcal{A})$. Conversely, if $\mathcal{M}$ is an object of $D(\mathcal{A})$ and if all its cohomology groups vanish except in degree 0, we say that $\mathcal{M}$ is "concentrated in degree 0" and identify $\mathcal{M}$ with the sheaf $\mathcal{H}^0(\mathcal{M})$.

A left $\mathcal{A}$-module which lies in $D_p(\mathcal{A})$, i.e. which admits locallly a free bounded resolutions will be said to be a perfect $\mathcal{A}$-module. If $\mathcal{A}$ is coherent than the perfect $\mathcal{A}$-modules are coherent $\mathcal{A}$-modules. The contrary is not true in general (because coherent modules do not have bounded free resolutions) but this is true for all coherent sheaves considered here.

The sheaf $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ has been defined in §2.3 for each (r,s) such that $r \geq s \geq 1$ as :

$$\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) = (\mathcal{E}_{Y\to X}|_\Lambda) \otimes_{\mathcal{E}_X|_\Lambda} \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s) = \pi^{-1}\mathcal{O}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$$

It is a $(\pi^{-1}\mathcal{D}^\infty_Y, \mathcal{E}^\infty_X|_\Lambda)$-bimodule on Λ and if t is a local equation of Y we have :

$$\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) \simeq \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s) \Big/ t\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$$

Definition 4.1.1. If $\mathcal{M}$ is an object of $D(\mathcal{E}_X)$ we set

$$\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) \otimes^{\mathbb{L}}_{(\mathcal{E}_X|_\Lambda)} (\mathcal{M}|_\Lambda)[-1]$$

$\widetilde{\Phi}_{(r,s)}(.)$ is a functor from $D(\mathcal{E}_X|_\Lambda)$ to $D(\pi^{-1}\mathcal{D}^\infty_Y)$. It is a kind of inverse image by $i : Y \longrightarrow X$.

If $\mathcal{M}$ is an object of $D(\mathcal{D}_X)$, we will define :

$$\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \widetilde{\Phi}_{(r,s)}(\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M})$$

We have defined the vanishing cycle in the derived category. In that way, this definition is always valid. In the following theorem, we show that, under suitable conditions, the complex of vanishing cycles of a $\mathcal{D}_X$-module is a single $\mathcal{D}^\infty_Y$-module.

In the following theorem, the variety S_Λ is the subvariety of $T^*\Lambda$ defined in §1.2.

Theorem 4.1.2. *Let (r_0, s_0) be a rational numbers with $r_0 > s_0 \geq 1$ and $\mathcal{M}$ be a coherent left $\mathcal{E}_X$-module (or a coherent $\mathcal{D}^2_\Lambda(r,s)$-module) such that :*

$$Ch_\Lambda(r_0, s_0)(\mathcal{M}) \subset S_\Lambda$$

Let $(r,s) \in \mathbb{Q}^2$ be such that $r_0 \geq r \geq s \geq s_0$ and $r_0 > s$.

(i) The restriction to $\dot\Lambda$ of $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is an object of $D_p(\pi'^{-1}\mathcal{D}^\infty_Y)$ which is concentrated in degree 0 and independent of (r,s) such that $r_0 \geq r \geq s \geq s_0$ and $r_0 > s$.

(ii) If $\mathcal{N}^\mathbb{R}$ is a $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$-module there is a canonical isomorphism :

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^\mathbb{R})|_{\dot\Lambda} \xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\pi'^{-1}\mathcal{D}^\infty_Y}(\widetilde{\Phi}_{(r,s)}(\mathcal{M}), \widetilde{\Phi}(\mathcal{N}^\mathbb{R}))$$

with

$$\widetilde{\Phi}(\mathcal{N}^\mathbb{R}) = \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) \otimes^{\mathbb{L}}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)} \mathcal{N}^\mathbb{R}$$

The point (i) means that the torsion groups $\mathcal{T}\mathrm{or}_k^{\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)}(\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s),\mathcal{M})$ vanishes outside the zero section of Λ if $k \neq 1$ and that $\widetilde{\Phi}(r,s)(\mathcal{M}) = \mathcal{T}\mathrm{or}_1(\dots)$ is a perfect $\pi'^{-1}\mathcal{D}_Y^\infty$-module. This module has a local presentation by free $\pi'^{-1}\mathcal{D}_Y^\infty$-modules hence is locally of the form $\pi'^{-1}\mathcal{N}$ for some perfect $\mathcal{D}_Y^\infty$-module $\mathcal{N}$. (Here π'^{-1} is the inverse image in the category of sheaves, not of $\mathcal{D}$-modules !)

If t is a local equation for Y, then $\widetilde{\Phi}(r,s)(\mathcal{M})$ is the kernel of the surjective morphism :

$$\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)\otimes\mathcal{M} \xrightarrow{t} \mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)\otimes\mathcal{M}$$

To prove the theorem we need a lemma :

Lemma 4.1.3. *Let $\mathcal{M}$ be a coherent left $\mathcal{D}_\Lambda^2(r,s)$-module such that $Ch_{\Lambda}(r,s)(\mathcal{M}) \subset S_\Lambda$. Then $\mathcal{M}$ has locally a finite resolution :*

$$0 \longrightarrow \mathcal{L}_m \longrightarrow \mathcal{L}_{m-1} \longrightarrow \dots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

by modules $\mathcal{L}_i$ satisfying the hypothesis of theorem 3.4.3.

Proof. If a module $\mathcal{M}$ is such that $Ch_{\Lambda}(r,s)(\mathcal{M}) \subset S_\Lambda$, then, for any section u of $\mathcal{M}$, the same is true for the module $\mathcal{E}_X u$. We assume $\tau \neq 0$ hence $S_\Lambda = \{\tau^* = 0\}$. Thus u is annihilated by some microdifferential operator P with $\sigma_\Lambda^{(r,s)}(P) = (\tau^*)^m$.

We take a finite set $(u_1,\dots,u_q)$ of local generators of $\mathcal{M}$ and corresponding operators $P_1,\dots,P_q$ and set $\mathcal{L} = \oplus\mathcal{E}_X/\mathcal{E}_X P_i$. The morphism $\mathcal{L} \longrightarrow \mathcal{M}$ is surjective and its kernel $\mathcal{M}'$ satisfy the same property as $\mathcal{M}$. Iterating the procedure we get long resolutions

$$\mathcal{L}_N \longrightarrow \mathcal{L}_{N-1} \longrightarrow \dots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

of $\mathcal{M}$ by modules $\mathcal{L}_i$ of the same type than $\mathcal{L}$ which satisfy the hypothesis of theorem 3.4.3.

We have now to prove that this resolution may be truncated. We assume that $N > 2n$, and replace the complex by :

$$0 \longrightarrow \mathcal{K}_{2n} \longrightarrow \mathcal{L}_{2n-1} \longrightarrow \dots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

where $\mathcal{K}_{2n}$ is the kernel of $\mathcal{L}_{2n} \longrightarrow \mathcal{L}_{2n-1}$. We get an exact sequence of $\mathcal{D}_\Lambda^2(r,s)$-module and the associated bigraded sequence is an exact sequence of $\mathcal{O}_\Lambda[x^*]$-modules hence, by the Hilbert syzygy theorem, $gr\mathcal{K}_{2n}$ is a projective of finite type hence stably free $\mathcal{O}_\Lambda[x^*]$-modules. There exists thus an integer q such that $gr\mathcal{K}_{2n} \oplus (\mathcal{O}_\Lambda[x^*])^q$ is free.

Let $\mathcal{N} = \left(\mathcal{D}_\Lambda^2(r,s)/\mathcal{D}_\Lambda^2(r,s)\ t\right)^q$ and consider the sequence :

$$0 \longrightarrow \mathcal{K}_{2n}\oplus\mathcal{N} \longrightarrow \mathcal{L}_{2n-1}\oplus\mathcal{N} \longrightarrow \dots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

This sequence is exact, all modules satisfy $Ch_{\Lambda}(r,s)(.) \subset S_\Lambda$, and by construction, the bigraded modules are free (except $gr\mathcal{M}$). □

Remark 4.1.4. This proof is very similar to the proof of the existence of finite free resolution for $\mathcal{D}_X$ or $\mathcal{E}_X$-modules (see [31] for example) and with a little more work we could prove that the maximal length of the resolution of the lemma is $n-1$ instead of $2n$.

Proof. The proof is of the theorem is a consequence of theorem 3.1.1 and is very similar to the proof of [21, theorem 2.1.1.].

First of all we have to define the morphism (ii).

As $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ is a $(\pi^{-1}\mathcal{D}^{\infty}_{Y}, \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s))$-bimodule , we have a canonical homomorphism of right $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$-modules :

$$\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s) \longrightarrow \mathcal{H}om_{\pi^{-1}\mathcal{D}^{\infty}_{Y}}\left(\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s), \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)\right)$$

which sends the unit of $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$ to the identity morphism.

Let $\mathcal{M}(r,s) = \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s) \otimes_{\mathcal{E}_X} \mathcal{M}$, then using [21, lemma 2.3.1.], we define the following morphisms :

$$\begin{aligned}
&\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbb{R}}) = \mathbb{R}\mathcal{H}om_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)}(\mathcal{M}(r,s), \mathcal{N}^{\mathbb{R}})\\
&\longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)}(\mathcal{M}(r,s),\\
&\qquad\qquad \mathbb{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}^{\infty}_{Y}}\left(\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s), \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)\right) \otimes^{\mathbb{L}}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)} \mathcal{N}^{\mathbb{R}}\Big)\\
&= \mathbb{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}^{\infty}_{Y}}\Big(\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) \otimes^{\mathbb{L}}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)} \mathcal{M}(r,s), \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) \otimes^{\mathbb{L}}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)} \mathcal{N}^{\mathbb{R}}\Big)\\
&= \mathbb{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}^{\infty}_{Y}}\Big(\widetilde{\Phi}(r,s)(\mathcal{M}), \widetilde{\Phi}(\mathcal{N}^{\mathbb{R}})\Big)
\end{aligned}$$

We have thus defined a canonical morphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}^{\mathbb{R}}) \longrightarrow \mathbb{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}^{\infty}_{Y}}\left(\widetilde{\Phi}(r,s)(\mathcal{M}), \widetilde{\Phi}(\mathcal{N}^{\mathbb{R}})\right)$$

Now the proof of the theorem is local on Y and we choose local coordinates (x,t) on X as in the previous section.

Let us first prove the theorem when $\mathcal{M} = \mathcal{E}_{X\leftarrow Y}$, we know from section 2.3 that $\mathcal{M}(r,s) = \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)$.

Let $j : Y \times Y \longrightarrow X \times Y$ be the canonical immersion, this defines an exact sequence :

$$0 \longrightarrow \mathcal{O}_{X\times Y} \xrightarrow{\ t\ } \mathcal{O}_{X\times Y} \longrightarrow j^{-1}\mathcal{O}_{Y\times Y} \longrightarrow 0$$

and by microlocalization :

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_{Y\times Y} \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times Y} \xrightarrow{\ t\ } \mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times Y} \longrightarrow 0$$

with $\pi_1 : T^*_{Y\times Y}X \times Y \longrightarrow Y \times Y$. As $\mathcal{E}_{X\times X}(r,s)$ is flat on $\mathcal{E}_{X\times X}$, we apply the tensor product by $\mathcal{E}_{X\times X}(r,s)$ and get, for all (r,s), the exact sequence :

$$0 \longrightarrow \pi_1^{-1}\mathcal{O}_{Y\times Y} \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times Y}(r,s) \xrightarrow{\ t\ } \mathcal{C}^{\mathbb{R}}_{Y\times Y|X\times Y}(r,s) \longrightarrow 0$$

and now we apply the functor $\mathcal{H}^1_{T^*_Y X}(.) \otimes \mathcal{O}^{(0,n-1)}_{Y\times Y}$ and get the exact sequence :

$$0 \longrightarrow \pi^{-1}\mathcal{D}_Y^\infty \longrightarrow \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s) \xrightarrow{\ t\ } \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s) \longrightarrow 0 \qquad (4.1.1)$$

In fact, we just need this exact sequence locally and we could get it by an elementary calculation on the symbols of section 2.3.

This proves that $\widetilde{\Phi}_{(r,s)}(\mathcal{E}_{X\leftarrow Y}) = \pi^{-1}\mathcal{D}_Y^\infty$ for any (r,s) and that (i) is true in this case.

If $\mathcal{N}^{\mathbb{R}}$ is a $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$-module we have :

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_{X\leftarrow Y},\mathcal{N}^{\mathbb{R}}) &= \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_{X\leftarrow Y},\mathcal{E}_X)\otimes^{\mathbb{L}}_{\mathcal{E}_X}\mathcal{N}^{\mathbb{R}}\\ &= \mathcal{E}_{Y\to X}\otimes^{\mathbb{L}}_{\mathcal{E}_X}\mathcal{N}^{\mathbb{R}} = \widetilde{\Phi}(\mathcal{N}^{\mathbb{R}})\\ &= \mathbb{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_Y^\infty}(\pi^{-1}\mathcal{D}_Y^\infty,\widetilde{\Phi}(\mathcal{N}^{\mathbb{R}}))\end{aligned}$$

because $\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_{X\leftarrow Y},\mathcal{E}_X) = \mathcal{E}_{Y\to X}$ ([29]).

The theorem is now proved for $\mathcal{M} = \mathcal{E}_{X\leftarrow Y}$ and we assume now that $\mathcal{M}$ is any $\mathcal{D}^2_\Lambda(r,s)$-module such that $Ch_\Lambda(r_0,s_0)(\mathcal{M}) \subset S_\Lambda$. Lemma 4.1.3 gives a resolution :

$$0 \longrightarrow \mathcal{L}_N \longrightarrow \mathcal{L}_{N-1} \longrightarrow \ldots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

by modules $\mathcal{L}_i$ which, by theorem 3.4.3, are such that $\mathcal{L}_{i(r,s)}$ is locally isomorphic to some power of $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)$.

As $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$ is flat on $\mathcal{D}^2_\Lambda(r,s)$ this gives an exact sequence :

$$0 \longrightarrow \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_N} \longrightarrow \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_{N-1}} \longrightarrow \ldots \longrightarrow \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_0} \longrightarrow \mathcal{M}(r,s) \longrightarrow 0$$

We proved that

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)}\left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s),\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right) = \pi^{-1}\mathcal{D}_Y^\infty$$

hence the morphisms in this exact sequence are given by matrices of $\pi^{-1}\mathcal{D}_Y^\infty$.

We have a commutative diagram

$$\begin{array}{ccccccccc}
 & 0 & & & 0 & & 0 & & \\
 & \downarrow & & & \downarrow & & \downarrow & & \\
0 \to & \left(\pi'^{-1}\mathcal{D}_Y^\infty\right)^{p_N} & \xrightarrow{A_N} \ldots \xrightarrow{A_1} & & \left(\pi'^{-1}\mathcal{D}_Y^\infty\right)^{p_0} & \to & \widetilde{\Phi}_{(r,s)}(\mathcal{M}) & \to & 0\\
 & \downarrow & & & \downarrow & & \downarrow & & \\
0 \to & \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_N} & \xrightarrow{A_N} \ldots \xrightarrow{A_1} & & \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_0} & \to & \mathcal{M}(r,s) & \to & 0\\
 & \downarrow t & & & \downarrow t & & \downarrow t & & \\
0 \to & \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_N} & \xrightarrow{A_N} \ldots \xrightarrow{A_1} & & \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_0} & \to & \mathcal{M}(r,s) & \to & 0\\
 & \downarrow & & & \downarrow & & \downarrow & & \\
 & 0 & & & 0 & & 0 & &
\end{array}$$

which gives the theorem :

1) The middle and low lines are exact sequences hence $t : \mathcal{M}(r,s) \longrightarrow \mathcal{M}(r,s)$ is surjective.

2) The vertical lines are exact sequences, hence the higher horizontal line is exact and thus $\widetilde{\Phi}(r,s)(\mathcal{M})$ is a perfect $\pi'^{-1}\mathcal{D}_Y^\infty$-module.

3) The isomorphism $\mathcal{L}_{i(r,s)} \simeq \left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\right)^{p_i}$ is given by a matrix R from proposition 3.3.1 which by corollary 3.4.2 does not depend on (r,s) such that $r_0 \geq r \geq s \geq s_0$ and thus $\widetilde{\Phi}(r,s)(\mathcal{M})$ is independent of (r,s) as a $\pi'^{-1}\mathcal{D}_Y^\infty$-module.

4) The diagram proves that locally

$$\mathcal{M}(r,s) \simeq \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s) \otimes_{\pi'^{-1}\mathcal{D}_Y^\infty} \widetilde{\Phi}(r,s)(\mathcal{M})$$

Moreover this diagram proves that for any $i > 0$, $\mathcal{T}\mathrm{or}_i\left(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s), \widetilde{\Phi}(r,s)(\mathcal{M})\right)$ vanish hence the equality is true in the derived category :

$$\mathcal{M}(r,s) \simeq \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s) \otimes^{\mathbb{L}}_{\pi'^{-1}\mathcal{D}_Y^\infty} \widetilde{\Phi}(r,s)(\mathcal{M})$$

If $\mathcal{N}^{\mathbb{R}}$ is a $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)$-module we have thus

$$\begin{aligned}
\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M},\mathcal{N}^{\mathbb{R}}) &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)}(\mathcal{M}(r,s),\mathcal{N}^{\mathbb{R}}) \\
&= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)}(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s) \otimes^{\mathbb{L}}_{\pi'^{-1}\mathcal{D}_Y^\infty} \widetilde{\Phi}(r,s)(\mathcal{M}),\mathcal{N}^{\mathbb{R}}) \\
&= \mathbb{R}\mathcal{H}\mathrm{om}_{\pi'^{-1}\mathcal{D}_Y^\infty}\left(\widetilde{\Phi}(r,s)(\mathcal{M}), \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)}(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s),\mathcal{N}^{\mathbb{R}})\right) \\
&= \mathbb{R}\mathcal{H}\mathrm{om}_{\pi'^{-1}\mathcal{D}_Y^\infty}(\widetilde{\Phi}(r,s)(\mathcal{M}),\widetilde{\Phi}(r,s)(\mathcal{N}^{\mathbb{R}}))
\end{aligned}$$

□

If we replace theorem 3.1.1 by proposition 3.1.3 and define $\widetilde{\mathcal{D}}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ from $\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ as in definition 3.1.2 we get :

Proposition 4.1.5. *If* $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$ *then*

$$\widetilde{\Phi}(r,s)(\mathcal{M}) = \widetilde{\mathcal{D}}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) \otimes^{\mathbb{L}}_{(\mathcal{E}_X|_\Lambda)} (\mathcal{M}|_\Lambda)[-1]$$

If $\mathcal{M}$ is a bounded complex of $\mathcal{E}_X$-module we denote by $Ch_{\Lambda(r,s)}(\mathcal{M})$ the union over all i of $Ch_{\Lambda(r,s)}(\mathcal{H}^i(\mathcal{M}))$ and by $D_{(r,s)}(\mathcal{E}_X)$ the subcategory of $D^b_c(\mathcal{E}_X)$ of the complexes $\mathcal{M}$ such that $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$.

Corollary 4.1.6. $\widetilde{\Phi}(r,s)(.)$ *is an exact functor from* $D_{(r,s)}(\mathcal{E}_X|_{\dot\Lambda})$ *to* $D_p(\pi'^{-1}\mathcal{D}_Y^\infty)$.

This is a direct consequence of the theorem.

We define now a functor $\widetilde{\Phi}(r,s)(.)$ from $D^b_c(\mathcal{D}^2_\Lambda(r,s)|_{\dot\Lambda})$ to $D_p(\pi'^{-1}\mathcal{D}_Y^\infty)$ and a functor $\Theta(r,s)(.)$ from $D_p(\pi'^{-1}\mathcal{D}_Y^\infty)$ to $D_p(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s))$ by :

$$\widetilde{\Phi}(r,s)(\mathcal{M}) = \mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s) \otimes^{\mathbb{L}}_{\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s)} (\mathcal{M})[-1]$$

$$\Theta(r,s)(\mathcal{N}) = \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s) \otimes^{\mathbb{L}}_{\pi'^{-1}\mathcal{D}_Y^\infty} (\mathcal{N})$$

Proposition 4.1.7. *If $\mathcal{M}$ is an object of $D_p(\mathcal{D}^2_\Lambda(r,s)|_{\dot\Lambda})$ which is locally equal to a tensor product $\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)\otimes^{\mathbb{L}}\mathcal{L}$ for some object $\mathcal{L}$ of $D_{(r,s)}(\mathcal{E}_X|_{\dot\Lambda})$ then $\mathcal{M}=\Theta(r,s)\widetilde{\Phi}(r,s)(\mathcal{M})$.*

If $\mathcal{N}$ is an object of $D_p(\pi'^{-1}\mathcal{D}^\infty_Y)$ then $\widetilde{\Phi}(r,s)\Theta(r,s)(\mathcal{N})=\mathcal{N}$.

Proof. We know from the proof of theorem 4.1.2 that

$$\begin{aligned}\mathcal{D}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)\otimes^L_{\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)}\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s) &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)}\Big(\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s),\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda\to Y}(r,s)\Big)\\ &= \pi'^{-1}\mathcal{D}^\infty_Y[-1]\end{aligned}$$

This proves the second part of the proposition. To prove the first part, we have to define a canonical morphism from $\mathcal{M}$ to $\Theta(r,s)\widetilde{\Phi}(r,s)(\mathcal{M})$.

In fact, theorem 4.1.2 shows that :

$$\begin{aligned}&\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{2(\mathbb{R},\infty)}_\Lambda(r,s)}\Big(\mathcal{M},\Theta(r,s)\widetilde{\Phi}(r,s)(\mathcal{M})\Big)\\ &\qquad=\mathbb{R}\mathcal{H}\mathrm{om}_{\pi'^{-1}\mathcal{D}^\infty_Y}\Big(\widetilde{\Phi}(r,s)(\mathcal{M}),\widetilde{\Phi}(r,s)\Theta(r,s)\widetilde{\Phi}(r,s)(\mathcal{M})\Big)\\ &\qquad=\mathbb{R}\mathcal{H}\mathrm{om}_{\pi'^{-1}\mathcal{D}^\infty_Y}\Big(\widetilde{\Phi}(r,s)(\mathcal{M}),\widetilde{\Phi}(r,s)(\mathcal{M})\Big)\end{aligned}$$

The identity of $\widetilde{\Phi}(r,s)(\mathcal{M})$ defines a canonical morphism $\mathcal{M}\longrightarrow\Theta(r,s)\widetilde{\Phi}(r,s)(\mathcal{M})$ which is an isomorphism. □

The microlocalization of $\widetilde{\Phi}(r,s)(\mathcal{M})$ hence its characteristic variety may be calculated directly from $\mathcal{M}$, in fact we will prove a "2-microlocal" theorem 4.1.2.

The immersion $i:Y\longrightarrow X$ defines morphisms

$$T^*Y\xleftarrow{\varrho_0}(T^*X)\times_X Y\xrightarrow{\varpi_0}T^*X$$

while the projection $\pi:T^*_YX\longrightarrow Y$ gives

$$T^*Y\xleftarrow{\varpi}(T^*Y)\times_Y\Lambda\xrightarrow{\varrho}T^*\Lambda$$

We denote by $\pi_2:T^*\Lambda\longrightarrow\Lambda$ and $\pi_1:(T^*Y)\times_Y\Lambda\longrightarrow\Lambda$ the canonical projections.

As Y is of codimension 1, the variety S_Λ is the union of $\pi_2^{-1}(Y)$ and of $(T^*Y)\times_Y\Lambda$ (Y is identified to the zero section of Λ).

Let :

$$\mathcal{E}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)=\pi_1^{-1}(\mathcal{E}_{Y\to X}|_\Lambda)\otimes_{\pi_1^{-1}(\mathcal{E}_X|_\Lambda)}\varrho^{-1}\mathcal{E}^{2(\mathbb{R},\infty)}_\Lambda(r,s)$$

It is a $(\varpi^{-1}\mathcal{E}^\infty_Y,\varrho^{-1}\mathcal{E}^{2(\mathbb{R},\infty)}_\Lambda(r,s))$-bimodule.

Theorem 4.1.8. *Let $\mathcal{M}$, r_0 and s_0 satisfying the hypothesis of theorem 4.1.2.*

The cohomology groups of

$$\mathcal{E}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)\otimes^{\mathbb{L}}_{\pi_1^{-1}\mathcal{E}_X}\pi_1^{-1}\mathcal{M}$$

vanish on $(T^*Y)\times_\Lambda(\dot\Lambda)$ except in degree -1. We denote the non vanishing group by μ-$\widetilde{\Phi}(r,s)(\mathcal{M})$.

μ-$\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is a left perfect $\varpi^{-1}\mathcal{E}_Y^\infty$-module on $(T^*Y)\times_Y(\dot{\Lambda})$, hence locally isomorphic to $\varpi^{-1}\mathcal{N}$ for some perfect $\mathcal{E}_Y^\infty$-module $\mathcal{N}$ and does not depend of (r,s) such that $r_0 \geq r \geq s \geq s_0$ and $s < r_0$.

The restriction of μ-$\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ to the zero section Λ of $T^*\Lambda$ is $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ and

$$\mu\text{-}\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \varpi^{-1}\mathcal{E}_Y^\infty \otimes_{\pi'^{-1}\mathcal{D}_Y^\infty} {\pi'}^{-1}\widetilde{\Phi}_{(r,s)}(\mathcal{M})$$

Proof. Using the proof of theorem 4.1.2 we have just to prove the theorem when $\mathcal{M} = \mathcal{E}_{X\leftarrow Y}$, and thus the exactness of :

$$0 \longrightarrow \varpi^{-1}\mathcal{E}_Y^\infty \longrightarrow \mathcal{E}_{\Lambda\to Y}^{2(\mathbb{R},\infty)}(r,s) \xrightarrow{\ t\ } \mathcal{E}_{\Lambda\to Y}^{2(\mathbb{R},\infty)}(r,s) \longrightarrow 0 \tag{4.1.2}$$

The functor $\mu_{T_Y^*X}$ applied to the sequence 4.1.1 gives an exact sequence :

$$0 \longrightarrow \varpi^{-1}\mathcal{E}_Y^\mathbb{R} \longrightarrow \mathcal{E}_{\Lambda\to Y}^{2(\mathbb{R},\mathbb{R})}(r,s) \xrightarrow{\ t\ } \mathcal{E}_{\Lambda\to Y}^{2(\mathbb{R},\mathbb{R})}(r,s) \longrightarrow 0 \tag{4.1.3}$$

and the global sections of this sequence on the orbits of $\mathbb{C}^*$ give 4.1.2. □

Let q be the projection $q : T^*Y \longrightarrow Y$. The characteristic variety of a $\mathcal{D}_Y$-module $\mathcal{N}$ is equal to the support of $\mathcal{E}_Y \otimes_{q^{-1}\mathcal{D}_Y} q^{-1}\mathcal{N}$ and, as $\mathcal{E}_Y^\infty$ is faithfully flat on $\mathcal{E}_Y$, it is equal to the support of $\mathcal{E}_Y^\infty \otimes_{q^{-1}\mathcal{D}_Y} q^{-1}\mathcal{N}$.

So we can define the characteristic variety of a $\mathcal{D}_Y^\infty$-module $\mathcal{N}$ as the support of $\mathcal{E}_Y^\infty \otimes_{q^{-1}\mathcal{D}_Y^\infty} q^{-1}\mathcal{N}$. If $\mathcal{F}$ is a perfect ${\pi'}^{-1}\mathcal{D}_Y^\infty$-module it is locally constant on $\dot{\Lambda}$ and locally of the form ${\pi'}^{-1}\mathcal{N}$ with $\mathcal{N}$ a perfect $\mathcal{D}_Y^\infty$-module. The characteristic variety of $\mathcal{N}$ is independent of the choice of $\mathcal{N}$. In fact $\varpi^{-1}Char(\mathcal{N})$ is equal to the support of

$$\varpi^{-1}\mathcal{E}_Y^\infty \otimes_{{\pi_1}^{-1}{\pi'}^{-1}\mathcal{D}_Y^\infty} {\pi_1}^{-1}\mathcal{F}$$

We define this set as the characteristic variety of $\mathcal{F}$ and denote it by $Char(\mathcal{F})$.

Corollary 4.1.9. *If $Ch_\Lambda(r_0,s_0)(\mathcal{M}) \subset S_\Lambda$, the characteristic variety of $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is equal to the support of μ-$\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ that is to the microcharacteristic variety $Ch_{\Lambda(r,s)}(\mathcal{M})$ of $\mathcal{M}$.*

To prove the corollary we have just to prove that the support of μ-$\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is equal to the microcharacteristic variety.

As $Ch_{\Lambda(r,s)}(\mathcal{M})$ is the support of $\mathcal{E}_\Lambda^2(r,s) \otimes_{\pi_2^{-1}\mathcal{E}_X} \pi_2^{-1}\mathcal{M}$ according to the definition of [18], and as $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ is faithfully flat on $\mathcal{E}_\Lambda^2(r,s)$ [20], it is the support of $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s) \otimes_{\pi_2^{-1}\mathcal{E}_X} \pi_2^{-1}\mathcal{M}$.

To prove that this sheaf and μ-$\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ have the same support, we may use the same proof as theorem 4.1.2 or prove that proposition 4.1.7 is still valid with $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ replaced by $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ which is easy.

In fact, with the same proof as theorem 4.1.2 we have :

If $\mathcal{M}$ satisfy the hypothesis of theorem 4.1.2 and $\mathcal{N}^\mathbb{R}$ is a $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$-module there is a canonical isomorphism :

$$\varrho^{-1}\,\mathbb{R}\mathcal{H}om_{\pi^{-1}\mathcal{E}_X}(\pi^{-1}\mathcal{M}, \mathcal{N}^\mathbb{R}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\varpi^{-1}\mathcal{E}_Y^\infty}(\mu\text{-}\widetilde{\Phi}_{(r,s)}(\mathcal{M}), \mu\text{-}\widetilde{\Phi}(\mathcal{N}^\mathbb{R})$$

with

$$\mu\text{-}\widetilde{\Phi}(\mathcal{N}^\mathbb{R}) = \pi_1^{-1}(\mathcal{E}_{Y\to X}|_\Lambda) \otimes^\mathbb{L}_{\pi_1^{-1}(\mathcal{E}_X|_\Lambda)} \varrho^{-1}(\mathcal{N}^\mathbb{R}))$$

4.2 Monodromy.

We proved in the previous section that, if $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module with $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$ then $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is locally constant on the fibers of $\pi' : \dot{\Lambda} \longrightarrow Y$. In other words, $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is isomorphic to ${\pi'}^{-1}\mathcal{N}$ for some $\mathcal{D}_Y^\infty$-module $\mathcal{N}$ on any simply connected open subset of $\dot{\Lambda}$. As Λ is a complex fiber bundle of rank 1 over Y it is provided with a canonical orientation. This defines a canonical endomorphism of monodromy on $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ which will be denoted by T.

In order to define this endomorphism in the derived category, we use proposition 4.1.5 to define $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$.

The endomorphism of monodromy of $\widetilde{\mathcal{D}}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}(r,s)$ is well defined as this sheaf has the property of unique continuation hence it induces an endomorphism T on the object $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ of $D_p({\pi'}^{-1}\mathcal{D}_Y^\infty)$ for any $\mathcal{M}$ in $D_{(r,s)}(\mathcal{E}_X|_{\dot{\Lambda}})$.

Vanishing cycles are usually defined as a sheaf on Y instead of a sheaf on $\dot{\Lambda}$ as we did here. If an equation of Y is given the two points of vue are equivalent :

Let σ be a continuous section of $\dot{\Lambda} = \dot{T}^*_Y X$, that is a continuous map $\sigma : Y \to \dot{\Lambda}$ such that $\pi'_o\sigma = id_Y$. Such a section on Y is equivalent to a C^1-equation of Y.

As $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is locally equal to ${\pi'}^{-1}\mathcal{N}$ the sheaf $\sigma^{-1}\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is a sheaf on Y of $\mathcal{D}_Y^\infty$-modules with locally free resolutions.

Definition 4.2.1. Let σ be a continuous section of $\dot{T}^*_Y X$ and let $\mathcal{M}$ be an object of $D_{(r,s)}(\mathcal{E}_X)$. Then the complex of *vanishing cycles of type* (r,s) of $\mathcal{M}$ along Y is the object of $D_p(\mathcal{D}_Y^\infty)$ given by :

$$\Phi_{(r,s)}(\mathcal{M}) = \sigma^{-1}\widetilde{\Phi}_{(r,s)}(\mathcal{M})$$

It is provided with the endomorphism of monodromy T.

In this definition, σ^{-1} is the inverse image in the category of sheaves. If we consider two sections σ_1 and σ_2, then $\sigma_1^{-1}\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ and $\sigma_2^{-1}\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ are locally isomorphic.

Conversely, we may recover $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ from $\Phi_{(r,s)}(\mathcal{M})$ and its endomorphism T if $\mathcal{M}$ is in $D_{(r,s)}(\mathcal{E}_X)$. This is clear if $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is a single module, in the general case where it is an object of the derived category, it is a little more complicated: we have to derive the category of complexes provided with an endomorphism of monodromy instead of the category of complexes (see [4]).

Let us still remark that $\Phi_{(r,s)}(.)$ is an exact functor from $D_{(r,s)}(\mathcal{E}_X)$ to $D^b_c(\mathcal{D}_Y)$.

Example 4.2.2.

(i) $\mathcal{M} = \mathcal{C}_{Y|X}$

We have $\mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda}(r,s) \otimes \mathcal{M} = \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$ (cf 2.3) and thus, for any (r,s) :

$$\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \mathcal{E}_{Y\to X} \otimes^{\mathbb{L}}_{\mathcal{E}_X} \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)[-1] = {\pi'}^{-1}\mathcal{O}_Y$$

hence $\Phi_{(r,s)}(\mathcal{M}) = \mathcal{O}_Y$ with $T =$ Identity.

(ii) $\mathcal{M} = \mathcal{E}_{X\leftarrow Y}$

From the proof 4.1.2 we have

$$\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = {\pi'}^{-1}\mathcal{D}_Y^\infty$$

and thus $\Phi_{(r,s)}(\mathcal{M}) = \mathcal{D}_Y^\infty$ with $T =$ Identity for any (r,s).

(iii) Assume that $X = Y \times \mathbb{C}$ with a coordinate t on $\mathbb{C}$. If $\mathcal{M}$ is the $\mathcal{E}_X$module $\mathcal{E}_X / \mathcal{E}_X(tD_t - \alpha)$ with $\alpha \in \mathbb{C}$ then $\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \pi'^{-1}\mathcal{D}_Y^\infty \tau^{\alpha+1}$ hence $\Phi_{(r,s)}(\mathcal{M}) = \mathcal{D}_Y^\infty$ with $T = e^{2i\pi\alpha}$ for any (r,s).

(iv) If $\mathcal{M}$ is $\mathcal{E}_X / \mathcal{E}_X(t^2D_t - 1)$ then $\Phi_{(r,s)}(\mathcal{M}) = \mathcal{D}_Y^\infty$ with $T =$ Identity for $2 > r \geq s \geq 1$ and $r = 2 > s$ while $\Phi_{(r,s)}(\mathcal{M}) = 0$ for $r \geq s \geq 2$.

Moderate vanishing cycles of a specializable $\mathcal{E}_X$ or $\mathcal{D}_X$-module have been defined in [21] for modules which are specializable. It is a coherent $p^{-1}\mathcal{D}_Y$-module denoted by $\widetilde{\Phi}(\mathcal{M})$.

Proposition 4.2.3. *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module which is r_0-specializable for some $r_0 \geq 1$. Then for each $r \geq s \geq r_0$ there is a canonical isomorphism of $\pi^{-1}\mathcal{D}_Y^\infty$-modules :*

$$\pi^{-1}\mathcal{D}_Y^\infty \otimes_{\pi^{-1}\mathcal{D}_Y} \widetilde{\Phi}(\mathcal{M}) \xrightarrow{\sim} \widetilde{\Phi}_{(r,s)}(\mathcal{M})$$

Proof. The sheaf $\widetilde{\Phi}(\mathcal{M})$ is defined in [21] as we did here for $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ but the sheaf $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}{}_{(r,s)}$ was replaced by the subsheaf $\widetilde{\mathcal{E}}^2_{\Lambda}{}_{(\infty,1)}$.

Hence there is a canonical morphism of $p^{-1}\mathcal{D}_Y$-modules :

$$\widetilde{\Phi}(\mathcal{M}) \longrightarrow \widetilde{\Phi}_{(r,s)}(\mathcal{M})$$

which gives the morphism of the theorem.

When $\mathcal{M} = \mathcal{E}_{X \leftarrow Y}$ this morphism is an isomorphism because $\widetilde{\Phi}(\mathcal{M}) = \pi^{-1}\mathcal{D}_Y$ and $\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \pi^{-1}\mathcal{D}_Y^\infty$.

It is proved in [21] (theorem 2.2.1.) that if $\mathcal{M}$ is r_0-specializable then $\widetilde{\mathcal{E}}^2_{\Lambda}{}_{(\infty,1)} \otimes \pi^{-1}\mathcal{M}$ has (locally) a resolution by modules which are isomorphic to a power of $\widetilde{\mathcal{E}}^2_{\Lambda}{}_{(\infty,1)} \otimes \pi^{-1}\mathcal{E}_{X \leftarrow Y}$ which proves the proposition. □

4.3 Duality and microlocal solutions.

If $\mathcal{M}$ is an object of $D'(\mathcal{E}_X)$ the derived category of the category of right $\mathcal{E}_X$-modules, the definition of vanishing cycles is :

$$\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{E}_X|_\Lambda} \mathcal{D}^{2(\mathbb{R},\infty)}_{\Lambda \to Y}{}_{(r,s)}[-1]$$

If $\mathcal{M}$ is an object of $D_c(\mathcal{E}_X)$, its dual is the object $\mathcal{M}^*$ of $D'_c(\mathcal{E}_X)$ defined as $\mathcal{M}^* = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X)[\dim X]$.

In the same way, if $\mathcal{N} \in D_p(\mathcal{D}_Y^\infty)$ then $\mathcal{N}^* = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y^\infty}(\mathcal{M}, \mathcal{D}_Y^\infty)[\dim Y]$ and if $\mathcal{N} \in D_p(\pi'^{-1}\mathcal{D}_Y^\infty)$ then $\mathcal{N}^* = \mathbb{R}\mathcal{H}\mathrm{om}_{\pi'^{-1}\mathcal{D}_Y^\infty}(\mathcal{M}, \pi'^{-1}\mathcal{D}_Y^\infty)[\dim Y]$.

Applying theorem 4.1.2 to example 4.2.2 (ii) we get that Φ commutes to duality :

Corollary 4.3.1. *If $\mathcal{M} \in D_{(r,s)}(\mathcal{E}_X)$ we have*

$$\widetilde{\Phi}_{(r,s)}(\mathcal{M}^*) \simeq \widetilde{\Phi}_{(r,s)}(\mathcal{M})^*$$

and for any continuous section σ of $\dot{\Lambda}$

$$\Phi_{(r,s)}(\mathcal{M}^*) \simeq \Phi_{(r,s)}(\mathcal{M})^*$$

If we apply now the same theorem to example (i) we get :

Corollary 4.3.2. *Let $\mathcal{M}$ be an object of $D^b_c(\mathcal{E}_X)$ such that $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$ and (r', s') with $r \geq r' \geq s' \geq s$ and $r > s'$ then :*

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X(r',s')})|_{\dot\Lambda} \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\pi'^{-1}\mathcal{D}^\infty_Y}(\widetilde{\Phi}_{(r,s)}(\mathcal{M}), \pi'^{-1}\mathcal{O}_Y)$$

and if σ is a continuous section of $\dot\Lambda$ then

$$\sigma^{-1}\,\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X(r',s')}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}^\infty_Y}(\Phi_{(r,s)}(\mathcal{M}), \mathcal{O}_Y)$$

Let us denote as in [21] by $\widetilde{\mathcal{C}}_{Y|X(r,s)}$ the subsheaf of $\mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}$ of the sections which have a continuation along any path in the fibers of $\dot\Lambda \to Y$. Then in the situation of corollary 4.3.2 the sheaf $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ has this property of continuation (proposition 4.1.5 hence the two members of the isomorphism of the corollary have it. This proves that :

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{Y|X(r,s)}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X(r,s)})$$

Let us denote by *id* the identity morphism of $\Phi_{(r,s)}(\mathcal{M})$ and by $\Phi_{0(r,s)}(\mathcal{M})$ the mapping cone of the morphism $\Phi_{(r,s)}(\mathcal{M}) \xrightarrow{T-id} \Phi_{(r,s)}(\mathcal{M})$ in the category $D^b_c(\mathcal{D}_Y)$. More precisely, we consider the complex $\mathcal{T} = \widetilde{\mathcal{D}}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}{}_{(r,s)} \xrightarrow{T-id} \widetilde{\mathcal{D}}^{2(\mathbb{R},\infty)}_{Y\leftarrow\Lambda}{}_{(r,s)}$ in $D(\mathcal{E}_X)$ and define $\Phi_{0(r,s)}(\mathcal{M}) = \mathcal{T} \otimes^{\mathbb{R}}_{\mathcal{E}_X} \mathcal{M}$

The sheaf $\mathcal{C}_{Y|X(r,s)}$ is equal to $\gamma^{-1}\gamma_*\mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}$ with $\gamma : \dot{T}^*_Y X \longrightarrow \mathbb{P}^*_Y X$ hence we have :

Corollary 4.3.3. *Under the hypothesis of the corollary 4.3.2 we have :*

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Y|X(r,s)})|_{\dot\Lambda} \xrightarrow{\sim} \pi'^{-1}\,\mathbb{R}\mathcal{H}om_{\mathcal{D}^\infty_Y}(\Phi_{0(r,s)}(\mathcal{M}), \mathcal{O}_Y)$$

The proof is the same proof as corollary 3.1.8. in [21].

4.4 Growth of solutions.

Here Y is a complex submanifold of X of any codimension and $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module defined near $\Lambda = T^*_Y X$.

Theorem 4.4.1. *Let U be an open subset of T^*X and let $r_0 > s_0 \geq 1$ be such that :*

$$Ch_{\Lambda(r_0,s_0)}(\mathcal{M}) \cap \pi^{-1}U \subset S_\Lambda$$

Then we have on U:

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}) &= \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \widetilde{\mathcal{C}}_{Y|X(r,s)} \\ &= \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X(r_0,s_0)}))\end{aligned}$$

for any (r, s) such that $r_0 \geq r \geq s \geq s_0$ and $r_0 > s$.

Proof. The proof being local on X, we first choose a point x^* of $\dot{T}^*_Y X$. Then after a quantized canonical transformation, we may assume that $\dot{T}^*_Y X$ has been transformed into the conormal to a hypersurface of X. That is, we may assume that Y has codimension 1 and apply the results of section 4.3.

If now x^* is a point of Y, we consider $\dot{\Lambda} = T^*_{Y\times\{0\}}X \times \mathbb{C}$ and $\mathcal{M}' = \mathcal{M} \otimes \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}t$ near the point $(x^*;(0,1))$ which is not on the zero section. We apply the preceding result and we conclude as usually in this situation (see [31] §2.8. for example). ☐

The restriction of $\mathcal{C}^{\mathbb{R}}_{Y|X(r,s)}$ to the zero section Y is equal to $\mathcal{B}_{Y|X(r)}$ if $r > s$ and to $\mathcal{B}_{Y|X\{r\}}$ if $r = s$ hence :

Corollary 4.4.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let $r_0 > s_0 \geq 1$ be such that :*

$$Ch_{\Lambda(r_0,s_0)}(\mathcal{M}) \subset S_\Lambda$$

Then we have :

$$\begin{aligned}\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X(r_0)}) &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X(r)}) \\ &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X\{r\}}) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X\{s_0\}})\end{aligned}$$

for any r such that $r_0 > r > s_0$.

Now we come back to codimension 1. Following [21] we consider several sheaves of holomorphic functions and formal power series with exponential growth.

First, we denote by $\mathcal{O}_{X|Y}$ the restriction to Y of the sheaf $\mathcal{O}_X$ and by $\mathcal{O}_{\widehat{X|Y}}$ its formal completion along Y. We interpolate between them with the family $\mathcal{O}_{X|Y(r)}$:

by definition, $\mathcal{O}_{X|Y(r)}$ is the subsheaf of $\mathcal{O}_{\widehat{X|Y}}$ of the elements which are written in a local chart $(x_1, \ldots, x_n, t)$ such that $Y = \{t = 0\}$ as :

$$u = \sum_{n\geq 0} a_n(x)t^n \qquad \text{with} \qquad \sum_{n\geq 0} a_n(x)t^n \,/(n!)^{r-1} < +\infty$$

When $r = +\infty$ we set $\mathcal{O}_{X|Y}(\infty) = \mathcal{O}_{\widehat{X|Y}}$ and by definition we have $\mathcal{O}_{X|Y}(1) = \mathcal{O}_{X|Y}$

On the other side we interpolate between the sheaf $\mathcal{O}_{X[*Y]}$ of meromorphic functions with poles on Y and the sheaf $\mathcal{O}_{X(*Y)} = j^{-1}j_*\mathcal{O}_X$ $(j : Y \hookrightarrow X)$ of holomorphic functions on X with essential singularities on Y with the family $\mathcal{O}_{X(*Y)((r))}$ of sections of $\mathcal{O}_{X(*Y)}$ with growth less than $\exp((1/|t|)^{1/(r-1)})$.

Such functions may be written in coordinates as

$$u = \sum_{n\in\mathbb{Z}} a_n(x)t^n \qquad \text{with} \qquad \sum_{n\leq 0} a_n(x)t^n((-n)!)^{r-1} < +\infty$$

We will also consider $\mathcal{O}_{X(*Y)(r,s)} = \mathcal{O}_{X(*Y)}((r)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X|Y(s)}$ and extend the sheaf $\mathcal{N}_{X|Y}$ of Nilsson class functions on Y to $\mathcal{N}_{X|Y(r,s)} = \mathcal{O}_{X(*Y)(r,s)} \otimes_{\mathcal{O}_X[*Y]} \mathcal{N}_{X|Y}$

Corollary 4.4.3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module such that $Ch_{\Lambda(r_0,s_0)}(\mathcal{M}) \subset S_\Lambda$ and let $\mathcal{F}(r,s)$ be one of the sheaves $\mathcal{O}_{X|Y(r)}$, $\mathcal{N}_{X|Y(r)}$, $\mathcal{O}_{X(*Y)((r))}$, $\mathcal{O}_{X(*Y)(r,s)}$ or $\mathcal{N}_{X|Y(r,s)}$.*

Then the complex of solutions $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}(r,s))$ is independent of (r,s) such that $r_0 > r > s > s_0$.

To prove this theorem, we first remark that theorem 4.4.1 is still true for the sheaf $\mathcal{C}_{Y|X}(r,s)$ which is equal to $\gamma^{-1}\gamma_*\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)$. Then we use the exact sequences of [21] prop. 3.2.10. to conclude (see the proof of corollary 5.3.7).

If we interpret theorem 4.4.1 in terms of holomorphic functions, we get the growth of holomorphic solutions in large sectors (i.e. larger then π). It would be very nice to have the corresponding results for small sectors. To do this it is necessary to translate theorem 4.4.1 through Fourier transform and get results on the specialization. This must not be very difficult with the results of Malgrange's book [24].

4.5 Non smooth hypersurfaces.

The sheaves of irregular vanishing cycles may be defined when Y is not smooth. The results are exactly the same as in [21, §3.3].

We denote by $\tilde{\Lambda} = \mathcal{O}_X(-Y)$ the vector bundle over X whose holomorphic sections are the holomorphic functions on X which vanish on Y at order 1. We denote by π the projection $\tilde{\Lambda} \to X$ and by Λ the restriction of $\tilde{\Lambda}$ to Y. On the regular part of Y there is a canonical isomorphism $\Lambda \approx T^*_Y X$.

Let U be an open subset of X and φ be a local equation of Y on U. We consider the graph map $g : U \hookrightarrow U \times \mathbb{C} = \tilde{X}$ given by $g(x) = (x, \varphi(x))$ and the subvariety $\tilde{Y} = U \times \{0\}$ of $\tilde{X}$.

The function φ defines a trivialization $\tilde{\Lambda}|_U \simeq U \times \mathbb{C}$ and therefore an isomorphism :

$$\tilde{g} : \tilde{\Lambda}|_U \longrightarrow T^*_{\tilde{Y}}\tilde{X} \simeq U \times \mathbb{C}$$

If $\mathcal{M}$ is an object of $D(\mathcal{E}_X)$, its direct image is

$$g_*\mathcal{M} = \mathbb{R}g_* \left(\mathcal{E}_{\tilde{X}\leftarrow U} \otimes^{\mathbb{L}}_{\mathcal{E}_U} \mathcal{M} \right)$$

It is an object of $D(\mathcal{E}_{\tilde{X}})$ with support in the graph $g(U)$ of φ. If $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module then $g_*\mathcal{M}$ is a coherent $\mathcal{E}_{\tilde{X}}$-module.

We denote by $D_{(r,s)}(\mathcal{E}_X)$ the subcategory of $D^b_c(\mathcal{E}_X)$ whose objects satisfy :

$$Ch_{T^*_{\tilde{Y}}\tilde{X}(r,s)}(g_*\mathcal{M}) \subset S_{T^*_{\tilde{Y}}\tilde{X}}$$

and we define :

$$\widetilde{\Phi}_{(r,s)}(\mathcal{M}) = \tilde{g}^{-1}\widetilde{\Phi}_{(r,s)}(g_*\mathcal{M})$$

where g^{-1} is the inverse image in the category of sheaves.

The same proof as [21, §3.3.] gives :

1) The definition of $D_{(r,s)}(\mathcal{E}_X)$ and $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ are independent of the choice of the equation φ.

2) $\widetilde{\Phi}_{(r,s)}(.)$ is an exact functor from $D_{(r,s)}(\mathcal{E}_X)$ to the subcategory of $D_p(\pi^{-1}\mathcal{D}^\infty_X)$ of object with cohomology supported by Λ.

3) If a continuous non vanishing section σ of $\tilde{\Lambda}$ is given (i.e. a C^1-equation for Y) than $\Phi_{(r,s)}(.) = \sigma^{-1}\widetilde{\Phi}_{(r,s)}(.)$ is an exact functor from $D_{(r,s)}(\mathcal{E}_X)$ to the subcategory of $D_p(\mathcal{D}^\infty_X)$ of object with cohomology supported by Y.

4.6 Submanifolds of higher codimension.

The theory may also be extended to smooth subvarieties Y of codimension greater than 1 and also to smooth conic lagrangian submanifolds of T^*X. The method is the same as [21, §3.4.] and we refer to it for the details.

Let X be a complex analytic manifold and Λ be an homogeneous lagrangian submanifold of $\dot{T}^*X = T^*X - X$. We denote by $\mathbb{P}\Lambda = \Lambda/\mathbb{C}^*$ the associated projective bundle and by $\gamma : \Lambda \to \mathbb{P}\Lambda$ the projection.

There exist on Λ a sheaf $\widetilde{\mathcal{D}}_\Lambda^\infty$ which is locally isomorphic to $\gamma^{-1}\mathcal{D}_{\mathbb{P}\Lambda}^\infty$ the inverse image by γ of the sheaf of differential operators of infinite order on $\mathbb{P}\Lambda$.

If $\Lambda = T_Y^*X$ then $\widetilde{\mathcal{D}}_\Lambda^\infty$ is canonically isomorphic to $\gamma^{-1}\mathcal{D}_{\mathbb{P}_Y^*X}^\infty$.

An operator adapted to Λ is a microdifferential operator Θ of order 1 whose symbol $\theta = \theta_1(x,\xi) + \theta_0(x,\xi) + \dots$ in some coordinate system (x,ξ) satisfy :

a) $\theta_1|_\Lambda = 0$ and $\theta_0|_\Lambda = 0$

b) $d\theta_1 = \omega_X$ modulo $I_\Lambda\Omega_1$ where ω_X is the canonical 1-form of $T^*\Lambda$ and I_Λ the definition ideal of Λ in T^*X.

This definition do not depend on the local coordinate system and such an operator always exists locally.

We define :

$$\widetilde{\Phi}_{\Lambda(r,s)}(\mathcal{M}) = (\mathcal{E}_X/\Theta\mathcal{E}_X) \otimes^{\mathbb{L}}_{\mathcal{E}_X} \left(\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s) \otimes^{\mathbb{L}}_{\mathcal{E}_X} \mathcal{M}\right)[-1]$$

This sheaf is independent of the choice of Θ as a $\widetilde{\mathcal{D}}_\Lambda^\infty$-module.

This definition is clearly invariant under quantized canonical transformations. In this way, we may transform Λ into the conormal of a smooth hypersurface and prove that the results of the previous sections remain valid. We get :

Theorem 4.6.1. *Let Λ be an homogeneous lagrangian submanifold of $\dot{T}^*X$ and $\mathcal{M}$ an object of $D_c^b(\mathcal{E}_X)$.*

Let (r_0, s_0) be two rational numbers such that $r_0 > s_0 \geq 1$ and

$$Ch_{\Lambda(r_0,s_0)}(\mathcal{M}) \subset S_\Lambda$$

Let (r,s) such that $r_0 \geq r \geq s \geq s_0$ and $r_0 > s$.

(i) $\widetilde{\Phi}_{\Lambda(r,s)}(\mathcal{M})$ is an object of $D_p(\widetilde{\mathcal{D}}_\Lambda^\infty)$ independent of the choice of Θ and of (r,s). It is locally of the form $\gamma^{-1}\mathcal{N}$ with $\mathcal{N}$ an object of $D_p(\mathcal{D}_{\mathbb{P}\Lambda}^\infty)$.

(ii) If $\mathcal{M}$ is an object of $D_{(r,s)}(\mathcal{E}_X)$ and $\mathcal{N}$ an object of $\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ then :

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{N}) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_\Lambda^\infty}\left(\widetilde{\Phi}_{\Lambda(r,s)}(\mathcal{M}), \widetilde{\Phi}_{\Lambda(r,s)}(\mathcal{N})\right)$$

(iii) If Y is a submanifold of X and $\Lambda = \dot{T}_Y^*X$, then $\widetilde{\Phi}_{\Lambda(r,s)}(\mathcal{M})$ is canonically an object of $D_p(\gamma^{-1}\mathcal{D}_{\mathbb{P}\Lambda})$.

(iv) If Y is a submanifold of X, then :

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)) = \mathbb{R}\mathcal{H}om_{\gamma^{-1}\mathcal{D}_{\mathbb{P}\Lambda}}(\widetilde{\Phi}_{\Lambda(r,s)}(\mathcal{M}), \gamma^{-1}\mathcal{O}_{\mathbb{P}\Lambda})$$

(v) The characteristic variety of $\widetilde{\Phi}_{\Lambda(r,s)}(\mathcal{M})$ is equal to the microcharacteristic variety $Ch_{\Lambda(r,s)}(\mathcal{M})$

5 Vanishing cycles of holonomic modules.

5.1 Critical indexes.

If $\mathcal{M}$ is a holonomic $\mathcal{E}_X$ or $\mathcal{D}_X$-module, we know (proposition 1.4.1) that $Ch_{\Lambda(r)}(\mathcal{M}) \subset S_\Lambda$ for any $r \geq 1$ and (1.4.2) that $Ch_{\Lambda(r,s)}(\mathcal{M}) \subset S_\Lambda$ if and only if there is no critical index r_0 such that $r \geq r_0 \geq s$. The sheaf of vanishing cycles $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is thus well defined as soon as there is no critical index in $[s, r]$.

In the case of holonomic modules we set the following definitions :

$$\widetilde{\Phi}_{\{r\}}(\mathcal{M}) = \widetilde{\Phi}_{(r,r)}(\mathcal{M}) \qquad \widetilde{\Phi}_{(r)}(\mathcal{M}) = \widetilde{\Phi}_{(r,r-\varepsilon)}(\mathcal{M})$$

(with $0 < \varepsilon << 1$ such that $\mathcal{M}$ has no critical index in $[r - \varepsilon, r[$).

With these notations, the previous results may be stated in the following way :

Theorem 5.1.1. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$ or $\mathcal{D}_X$-module and let $1 = r_0 < r_1 < \cdots < r_N < r_{N+1} = +\infty$ be its critical indexes.*

(i) The sheaves of vanishing cycles $\widetilde{\Phi}_{\{r\}}(\mathcal{M})$ and $\widetilde{\Phi}_{(r)}(\mathcal{M})$ are well defined as a perfect $\pi^{-1}\mathcal{D}_Y^\infty$-module for any $r \geq 1$.

(ii) If r_i and r_{i+1} are two consecutive critical indexes and $r_i < r < r_{i+1}$ then :

$$\widetilde{\Phi}_{\{r_i\}}(\mathcal{M}) = \widetilde{\Phi}_{(r)}(\mathcal{M}) = \widetilde{\Phi}_{\{r\}}(\mathcal{M}) = \widetilde{\Phi}_{(r_{i+1})}(\mathcal{M})$$

So that there is only a finite number of distinct sheaves of vanishing cycles and $\widetilde{\Phi}_{\{r\}}(\mathcal{M}) = \widetilde{\Phi}_{(r)}(\mathcal{M})$ if r is not a critical index.

(iii) The sheaf of vanishing cycles $\widetilde{\Phi}_{(r,s)}(\mathcal{M})$ is well defined for any (r, s) with no critical index in the interval $]r, s[$ and equal to $\widetilde{\Phi}_{(r)}(\mathcal{M})$.

(iv) The characteristic variety of $\widetilde{\Phi}_{\{r\}}(\mathcal{M})$ (resp. $\widetilde{\Phi}_{(r)}(\mathcal{M})$) is equal to $Ch_{\Lambda\{r\}}(\mathcal{M})$ (resp. $Ch_{\Lambda(r)}(\mathcal{M})$).

If σ is a continuous section of $\dot{\Lambda}$, we have the same results with Φ instead of $\tilde{\Phi}$, that is with sheaves of $\mathcal{D}_Y^\infty$-modules.

As the microcharacteristic varieties of holonomic $\mathcal{E}_X$-modules are lagrangian, the characteristic varieties of these modules are lagrangian.

Corollary 4.3.2 shows that the complex of solutions $\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s))$ is independent of (r, s) if $r_k \geq r > s > r_{k+1}$ for some k and is equal to the complex of solutions of $\widetilde{\Phi}_{(r)}(\mathcal{M})$ in $\pi^{-1}\mathcal{O}_Y$.

5.2 Admissibility.

A $\mathcal{D}_Y^\infty$-module $\mathcal{N}$ is said to be holonomic admissible (or simply holonomic) if there exists some holonomic $\mathcal{D}_Y$-module $\mathcal{N}_0$ such that :

$$\mathcal{N} = \mathcal{D}_Y^\infty \otimes_{\mathcal{D}_Y} \mathcal{N}_0$$

The characteristic cycle of $\mathcal{N}_0$ is independent of the choice of $\mathcal{N}_0$ [11, th. 3.2.1.] and will be the characteristic cycle of $\mathcal{N}$.

If $\mathcal{N}$ is holonomic admissible, the complex of solutions $\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y)$ is perverse and the module $\mathcal{N}_0$ may be obtained from this complex via the Riemann-Hilbert correspondence. This shows that holonomic admissibility is a local property.

Theorem 5.2.1. *If $\mathcal{M}$ is a holonomic $\mathcal{E}_X$-module then, for any $r \geq 1$, $\Phi\{r\}(\mathcal{M})$ and $\Phi(r)(\mathcal{M})$ are admissible holonomic $\mathcal{D}_Y^\infty$-modules. Their characteristic cycle are equal to the corresponding microcharacteristic cycle of $\mathcal{M}$:*

$$\varpi^{-1}\widetilde{Ch}(\Phi\{r\}(\mathcal{M})) = \widetilde{Ch}_{\Lambda\{r\}}(\mathcal{M})$$
$$\varpi^{-1}\widetilde{Ch}(\Phi(r)(\mathcal{M})) = \widetilde{Ch}_{\Lambda(r)}(\mathcal{M})$$

Proof. Let us first assume that $\mathcal{M}$ is r-specializable. This is always true for great r [21, th. 1.1.4.] and we may apply proposition 4.2.3 and [21, corollary 3.1.9.] to get the result.

Let us now assume $r = 1$ but $\mathcal{M}$ is not 1-specialisable. Theorem 5.2.1. of [15] shows that there exists a regular holonomic $\mathcal{E}_X$-module $\mathcal{M}_{\text{reg}}$ such that :

$$\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M} = \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_{\text{reg}}$$

The sheaf $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,s)$ is a right $\mathcal{E}_X^\infty$-module if $r = s = 1$ hence definition 4.1.1 shows that :

$$\Phi\{1\}(\mathcal{M}) = \Phi\{1\}(\mathcal{M}_{\text{reg}})$$

A regular holonomic $\mathcal{E}_X$-module is 1-specializable [21, th. 1.1.4.] (this result was proved by Kashiwara-Kawa in [14]) and we apply the first case to $\mathcal{M}_{\text{reg}}$.

(If $\mathcal{M}$ is regular the result is also known from [27] and [6]).

Let us consider the general case. We fix local coordinates and look at the map s_r : $(x,\tau) \mapsto (x,\tau^r)$ on the universal covering of $T_Y^*X - Y$. The inverse image by s_r is well defined on the symbols of $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,r)$ and defines a morphism $s_r^* : s_r^{-1}\mathcal{C}_{Y|X}^{\mathbb{R}}(r,r) \longrightarrow \mathcal{C}_{Y|X}^{\mathbb{R}}$.

The sheaf $\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,r)$ has been defined cohomologically from $\mathcal{C}_{Y|X}^{\mathbb{R}}(r,r)$ hence we get a morphism of rings $s_r^* : s_r^{-1}\mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(r,r) \longrightarrow \mathcal{D}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\infty)}(1,1)$. When restricted to $\mathcal{E}_X$ this morphism gives a morphism $s_r^* : s_r^{-1}\mathcal{E}_X \longrightarrow \mathcal{E}_X[1/q]$ where q is the denominator of r and $\mathcal{E}_X[1/q]$ is the sheaf of microdifferential operators with fractional order. It is clear on the definition that for any P in $\mathcal{E}_X$ we have $\sigma^{(1)}(s_r^*P) = \sigma^{(r)}(P)$.

If $\mathcal{M}$ is a holonomic $\mathcal{E}_X$-module than $s_r^*\mathcal{M}$ is a coherent $\mathcal{E}_X[1/q]$-module and we have :

$$\widetilde{Ch}_{\Lambda(1)}(s_r^*\mathcal{M}) = \widetilde{Ch}_{\Lambda(r)}(\mathcal{M})$$

In particular $Ch_{\Lambda(1)}(s_r^*\mathcal{M})$ is lagrangian and as this variety is the tangent cone along Λ to the characteristic variety of $s_r^*\mathcal{M}$, this shows that $s_r^*\mathcal{M}$ is holonomic.

The theory of holonomic $\mathcal{E}_X[1/q]$ is the same than that of holonomic $\mathcal{E}_X$-module (see [30] for the details) and in particular there exists a regular holonomic $\mathcal{E}_X$-module $\mathcal{M}_{\text{reg}}$ such that :

$$\mathcal{E}_X^{\mathbb{R}} \otimes_{\mathcal{E}_X} \mathcal{M}_{\text{reg}} = \mathcal{E}_X^{\mathbb{R}} \otimes_{\mathcal{E}_X[1/q]} s_r^*\mathcal{M}$$

We may now conclude as before because :

$$\Phi\{r\}(\mathcal{M}) = \Phi(1)(s_r^*\mathcal{M}) = \Phi(1)(\mathcal{M}_{\text{reg}})$$

As $\Phi(r)(\mathcal{M}) = \Phi\{r-\varepsilon\}(\mathcal{M})$ and $\widetilde{Ch}_{\Lambda(r)}(\mathcal{M}) = \widetilde{Ch}_{\Lambda\{r-\varepsilon\}}(\mathcal{M})$ if $\varepsilon << 1$, the same result is true with $\Phi(r)(\mathcal{M})$.

□

The proof of theorem 5.2.1 is rather simple but uses deep results of Kashiwara and Kawaï [14][15]. We give another proof in appendix.

5.3 Index theorems.

Let X be a complex analytic manifold, T^*X its cotangent bundle and Σ be a lagrangian homogeneous analytic subset of T^*X. Then Σ is a union of sets $T^*_{X_j}X$ where X_j is a subvariety of X and $T^*_{X_j}X$ the closure of the conormal bundle to the regular part of X_j.

Any positive cycle $\widetilde{\Sigma}$ with support Σ may thus be written :

$$\widetilde{\Sigma} = \sum_j m_j \left[T^*_{X_j}X\right] \qquad \text{with } m_j \in \mathbb{N}$$

The *Local Euler Obstruction* of $\widetilde{\Sigma}$ at a point $x \in X$ is defined as :

$$E_{\widetilde{\Sigma}}(x) = \sum_j m_j (-1)^{\operatorname{codim} X_j} E_{X_j}(x)$$

where $E_{X_j}(x)$ is the local Euler obstruction of X_j at x. (see [3] for more details). $E_{\widetilde{\Sigma}}(x)$ is a constructible function on X.

If $\mathcal{M}$ and $\mathcal{F}$ are two left $\mathcal{D}_X$-modules we will say that $(\mathcal{M}, \mathcal{F})$ has finite index at x if all $\mathcal{E}\mathrm{xt}^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})_x$ are finite dimensional $\mathbb{C}$-vector space and then we will set :

$$\chi(\mathcal{M}, \mathcal{F})_x = \sum_j (-1)^j dim_{\mathbb{C}}\, \mathcal{E}\mathrm{xt}^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})_x$$

It was proved by Kashiwara in [11] that if $\mathcal{M}$ is a complex of $\mathcal{D}_X$-modules with holonomic cohomology, then $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is a complex of $\mathbb{C}$-vector spaces with constructible cohomology and if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, it is a perverse sheaf. $(\mathcal{M}, \mathcal{F})$ has finite index at each $x \in X$ and :

$$\chi(\mathcal{M}, \mathcal{O}_X)_x = E_{\widetilde{Ch}(\mathcal{M})}(x)$$

where $\widetilde{Ch}(\mathcal{M})$ is the characteristic cycle of $\mathcal{M}$.

Theorem 5.3.1. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module and Y be a submanifold of X (of any codimension).*

i) For any $r \in [1, +\infty]$, $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,r))$ is a complex of $\mathbb{C}$-vector spaces with constructible cohomology on $\Lambda = T^*_Y X$ and a perverse sheaf on $\dot{\Lambda} = T^*_Y X - Y$. At each point $x \in \Lambda$ we have :

$$\chi(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,r))_x = E_{\widetilde{Ch}_\Lambda\{r\}(\mathcal{M})}(x)$$

ii)If r_k and r_{k+1} are two consecutive slopes of $\mathcal{M}$, then for any (r, s) such that $r_k \geq r > s > r_{k-1}$, $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s))$ is a complex of $\mathbb{C}$-vector spaces with constructible cohomology on Λ and a perverse sheaf on $\dot{\Lambda}$. It is independent of (r, s) and at each point $x \in \Lambda$ we have :

$$\chi(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s))_x = E_{\widetilde{Ch}_\Lambda(r)(\mathcal{M})}(x)$$

Remark 5.3.2. If $r = s = 1$ this theorem has been proved by Kashiwara and Schapira in [17], their method is purely geometric.

Proof. Assume first that $x \in \dot{\Lambda}$. After a quantized canonical transformation, we may assume that Y has codimension 1.

Corollary 4.3.2 shows that $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s))$ is independent of (r,s) and equal to $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^\infty_Y}(\Phi_{(r,s)}(\mathcal{M}), \mathcal{O}_Y)$. Then we may use theorem 5.2.1 and apply the result of Kashiwara to $\Phi_{(r,s)}(\mathcal{M})$.

We may also use directly theorem 6.2.1 which shows that the characteristic cycle of the sheaf $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s))$ as defined in [13] is equal to the microcharacteristic cycle $\widetilde{Ch}_{\Lambda(r,s)}(\mathcal{M})$. Then we apply the result of [13] which calculates the index of a perverse sheaf from its characteristic cycle.

On the zero section we get the result by adding a variable. The proof is the same as the proof of [21, theorem 4.4.1.] and we refer to it for the details. □

On the zero section Y of $T^*_Y X$ we have $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)|_Y = \mathcal{B}_{Y|X}(r)$ if $r > s$ and $\mathcal{C}^{\mathbb{R}}_{Y|X}(r,r)|_Y = \mathcal{B}_{Y|X}\{r\}$ hence :

Corollary 5.3.3. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and Y be a submanifold of X.*

For any $r \in [1,+\infty]$, $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X}(r))$ is a complex of $\mathbb{C}$-vector spaces with constructible cohomology on Y.

If r_k and r_{k-1} are two consecutive slopes of $\mathcal{M}$ then for any r such that $r_k > r > r_{k-1}$ we have :

$$\begin{aligned}\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X}(r_k)) &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X}(r)) \\ &= \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X}\{r\}) = \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X}\{r_{k-1}\})\end{aligned}$$

At any point $x \in Y$ we have :

$$\chi(\mathcal{M}, \mathcal{B}_{Y|X}(r))_x = E_{\widetilde{Ch}_{\Lambda}(r)(\mathcal{M})}(x)$$
$$\chi(\mathcal{M}, \mathcal{B}_{Y|X}\{r\})_x = E_{\widetilde{Ch}_{\Lambda}\{r\}(\mathcal{M})}(x)$$

In this formula, we calculate the index of $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{Y|X}(r))$ which a complex on Y through $\widetilde{Ch}_{\Lambda(r)}(\mathcal{M})$ which is a positive analytic cycle on $T^*_Y X$. This is related to the fact that this complex has constructible cohomology but is not a perverse sheaf.

We assume now that Y is a submanifold of codimension 1 in X and we will calculate the index as the Euler function of the difference of two positive analytic cycles on Y.

The projection $\Lambda = T^*_Y X \longrightarrow Y$ and the embedding $Y \longrightarrow \Lambda$ define maps :

$$T^*Y \xleftarrow{p_1} (T^*Y) \times_Y \Lambda \xrightarrow{j_1} T^*\Lambda$$
$$T^*Y \xleftarrow{p_2} (T^*\Lambda) \times_\Lambda Y \xrightarrow{j_2} T^*\Lambda$$

The maps p_1 and p_2 are submersions while j_1 and j_2 are immersions. With local coordinates (x,t) of X such that $Y = \{t = 0\}$ and $\Lambda = \{(x,t,\xi,\tau) \in T^*X/\ t = 0, \xi = 0\}$ and the corresponding coordinates (x,x^*) of Y and (x,τ,x^*,τ^*) of Λ, we get :

$$p_1(x,x^*,\tau) = (x,x^*) \qquad j_1(x,x^*,\tau) = (x,\tau,x^*,0)$$
$$p_2(x,x^*,\tau^*) = (x,x^*) \qquad j_2(x,x^*,\tau^*) = (x,0,x^*,\tau^*)$$

This proves that the union of $j_1((T^*Y)\times_Y\Lambda)$ and of $j_2((T^*\Lambda)\times_\Lambda Y)$ is the hypersurface S_Λ of $T^*\Lambda$.

Proposition 5.3.4. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module.*
The microcharacteristic cycle $\widetilde{Ch}_{\Lambda(r)}(\mathcal{M})$ has a unique decomposition :

$$\widetilde{Ch}_{\Lambda(r)}(\mathcal{M}) = j_1 p_1^{-1}\widetilde{S_1}_{(r)}(\mathcal{M}) + j_2 p_2^{-1}\widetilde{S_2}_{(r)}(\mathcal{M})$$

where $\widetilde{S_1}$ and $\widetilde{S_2}$ are positive lagrangian cycles of T^*Y. Moreover $\widetilde{S_1}_{(r)}(\mathcal{M})$ is the characteristic cycle of $\Phi_{(r)}(\mathcal{M})$.

The proof of this result is the same as [21, theorem 4.5.2.].

As a direct consequence of the definition, we can see that the Euler obstruction of $\widetilde{Ch}_{\Lambda(r)}(\mathcal{M})$ is the difference of the Euler Obstructions of $\widetilde{S_1}_{(r)}$ and $\widetilde{S_2}_{(r)}$. We get :

Corollary 5.3.5. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and Y be a submanifold of X. Then at any point $x\in Y$ we have :*

$$\chi(\mathcal{M},\mathcal{B}_{Y|X(r)})_x = E_{\widetilde{S_1}(r)(\mathcal{M})}(x) - E_{\widetilde{S_2}(r)(\mathcal{M})}(x)$$

The index $\chi(\mathcal{M},\mathcal{B}_{Y|X(r)})_x$ is therefore the Euler function of the (non-positive) analytic cycle $\widetilde{S_1}_{(r)} - \widetilde{S_2}_{(r)}$. The same result is true with $\{r\}$ replacing (r).

Lemma 5.3.6. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module and (r,s) such that $+\infty\geq r\geq s\geq 1$.*
We assume that no slope of $\mathcal{M}$ is in $]s,r[$, then $\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{Y|X(r,s)})$ is a complex of $\mathbb{C}$-vector spaces with constructible cohomology on Y and $\chi(\mathcal{M},\mathcal{C}_{Y|X(r,s)})=0$.

Proof. Corollary 4.3.3 shows that $\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_{Y|X(r,s)})$ is equal to the complex of holomorphic solutions of a complex of $\mathcal{D}_X$-module with holonomic cohomology and a characteristic cycle equal to 0. □

Let x be a point of Y and x^* be a point of $\dot\Lambda$ whose projection is x. We denote by $\chi(r)$ the index $\chi(\mathcal{M},\mathcal{B}_{Y|X(r)})_x$ given by corollary 5.3.5 that is $E_{\widetilde{S_1}(r)(\mathcal{M})}(x) - E_{\widetilde{S_2}(r)(\mathcal{M})}(x)$ and $\chi\{r\} = \chi(\mathcal{M},\mathcal{B}_{Y|X\{r\}})_x$.

Corollary 5.3.7. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and Y a submanifold of X of codimension 1. We have the following index theorems :*

$$\begin{aligned}
\chi(\mathcal{M},\mathcal{O}_{X|Y(r)})_x &= -\chi\{r\}\\
\chi(\mathcal{M},\mathcal{O}_{X[*Y](r)})_x &= \chi(\infty)-\chi\{r\}\\
\chi(\mathcal{M},\mathcal{O}_{X(*Y)((r))})_x &= \chi(r)-\chi(1)\\
\chi(\mathcal{M},\mathcal{O}_{X(*Y)(r,s)})_x &= \chi(r)-\chi\{s\}\\
\chi(\mathcal{M},\mathcal{C}_{Y|X(r,s)})_{x^*} &= \chi(r)-\chi\{s\}
\end{aligned}$$

Proof. They are direct consequences of [21, proposition 3.2.10.] and the previous results. More precisely, the exact sequence

$$0\longrightarrow \mathcal{B}_{Y|X\{r\}}\longrightarrow {\pi'}_*\mathcal{C}_{Y|X(r,r)}\longrightarrow \mathcal{O}_{X|Y(r)}\longrightarrow 0$$

and lemma 5.3.6 give the first equality while

$$0 \longrightarrow \mathcal{B}_{Y|X}(r) \longrightarrow \mathcal{C}_{Y|X}(r,s) \longrightarrow \mathcal{O}_{X|Y}(s) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{X|Y}(s) \longrightarrow \mathcal{O}_{X}(*Y)(r,s) \longrightarrow \mathcal{B}_{Y|X}(r) \longrightarrow 0$$

give the others. □

We may extend the result on $\mathcal{C}_{Y|X}(r,s)$ to a submanifold Y of codimension greater than 1 and to $\mathcal{E}_X$-modules :

Proposition 5.3.8. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module and (r,s) such that $+\infty \geq r > s \geq 1$.*

Then $\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{Y|X}(r,s))$ is a complex of $\mathbb{C}$-vector spaces with constructible cohomology on $T^*_Y X$ and

$$\chi(\mathcal{M}, \mathcal{C}_{Y|X}(r,s))_x = E_{\widetilde{Ch_\Lambda\{s\}(\mathcal{M})}}(x) - E_{\widetilde{Ch_\Lambda(r)(\mathcal{M})}}(x)$$

Proof. On the zero section of T^*X, the result is in corollary 5.3.3, hence we have to prove the result out of the zero section.

Corollary 1.6.4. of [15] shows that there exists some quantized canonical transformation φ such that $\varphi(T^*_Y X)$ is the conormal bundle to a smooth hypersurface of X and $\varphi(Char(\mathcal{M}))$ is in generic position.

So using a quantization of φ, we may assume that Y has codimension 1 and $Char(\mathcal{M})$ is in generic position. A result of Björk [2, theorem 8.6.3.] shows that there exists a holonomic $\mathcal{D}_X$-module $\mathcal{N}$ such that $\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{N}$.

So, we may assume that $\mathcal{M}$ is a $\mathcal{D}_X$-module and apply the corollary. □

Let us remark that the exact sequence (iii) of [21, prop. 3.2.10.]

$$0 \longrightarrow \mathcal{C}_{Y|X}(r,s) \longrightarrow \widetilde{\mathcal{C}}_{Y|X}(r,s) \longrightarrow \widetilde{\mathcal{C}}_{Y|X}(r,s) \longrightarrow 0$$

shows that $\mathbb{R}\mathcal{H}om(\mathcal{M}, \widetilde{\mathcal{C}}_{Y|X}(r,s))$ is not constructible if $\mathcal{M}$ has a slope in $]s,r[$. Indeed, if it were constructible, then the index in $\mathcal{C}_{Y|X}(r,s)$ would be 0 and this contradicts proposition 5.3.8. The same is true for $\mathcal{N}_{X|Y}(r,s)$.

If there is no slope of the $\mathcal{D}_X$-module $\mathcal{M}$ between r and s they have both constructible cohomology. The index of $\widetilde{\mathcal{C}}_{Y|X}(r,s)$ is given by theorem 5.3.1 that is $E_{\widetilde{S_1}(r)(\mathcal{M})}(x)$ while the index in $\mathcal{N}_{X|Y}(r,s)$ is given by the exact sequence (iv) of proposition 3.2.10. in [21] and is equal to $E_{\widetilde{S_2}(r)(\mathcal{M})}(x)$.

5.4 Reconstruction.

Theorem 5.4.1. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module or a complex of $D_c(\mathcal{E}_X)$ with holonomic cohomology. Let r_{k-1} and r_k be two consecutive critical indexes of $\mathcal{M}$. Then, for each (r,s) with $r_k \geq r \geq s \geq r_{k-1}$ and $r_k > s$:*

$$\mathcal{D}_\Lambda^{2(\mathbb{R},\infty)}(r,s) \otimes_{\mathcal{E}_X|_\Lambda} \mathcal{M}|_\Lambda \approx \mathbb{R}\mathcal{H}om_{\mathbb{C}}\left(\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)), \mathcal{C}^{\mathbb{R}}_{Y|X}(r,s)\right)$$

This theorem is a generalization of [21, theorem 4.3.1.] and is deduced from the previous results in the same way.

6 Appendix.

We give here a proof of the results of section 5.2 which does not use the difficult results of Kashiwara-Kawaï [14] and is more general as it is true for 2-microdifferential equations.

6.1 A new equivalence theorem.

Theorems of the same kind that proposition 3.2.3 and 3.3.1 may be proved when D_{x_1} or t is replaced by D_t and x_1. At points where the condition of corollary 3.2.5 is not satisfied these theorems are not true in the sheaf $\mathcal{E}_\Lambda^{2(\mathbb{R},\infty)}(r,s)$ but only in $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(r,s)$ (see §2.3 for its definition).

As in the previous sections, we consider a complex manifold X of dimension n, a submanifold Y of codimension d and denote by $\Lambda = T_Y^*X$ the conormal bundle to Y.

We consider a matrix A with coefficients in the sheaf $\mathcal{E}_\Lambda^2(r,s)$ of 2-microdifferential operators of finite order. We assume that A commutes with D_{x_1}, which is equivalent to the fact that it has a symbol independent of x_1. We keep the order of section 3.2 and say that $A(x',\tau,x_1^*,x'^*,\tau^*)$ is of medium order (δ,μ) if there exists some $q\in\mathbb{N}$ such that any product of q terms

$$A(x',\tau,\lambda_1,x'^*,\tau^*)A(x',\tau,\lambda_2,x'^*,\tau^*)\dots A(x',\tau,\lambda_q,x'^*,\tau^*)$$

is of order at most $(q\delta,\mu)$.

Proposition 6.1.1. *Let A be a $m\times m$-matrix of the sheaf $\mathcal{E}_\Lambda^2(r,s)$ of 2-microdifferential operators which is independent of x_1 and of medium order $(0,\mu)$ with $\mu<1$.*

There exists an invertible matrix R with coefficients in $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(r,s)$ commuting with D_{x_1} and such that :

$$(x_1I-A)R = Rx_1$$

Proof. We use the same proof than proposition 3.3.1:

We add a variable y, that is we consider $X' = X\times\mathbb{C}$, $Y' = Y\times\mathbb{C}$ and $\Lambda' = T_{Y'}^*X'$. Then we make a "quantized bicanonical transformation" and apply proposition 3.2.3. In this way, denoting $x' = (x_2,\dots,x_{n-d})$, we find a matrix $R_1(x',x^*,y^*,\tau,\tau^*)$ with coefficients in $\mathcal{E}_{\Lambda'}^{2(\infty,\infty)}(r,s)$ solution of :

$$\left(\frac{\partial}{\partial x_1^*}\right)R_1 = A(x',x^*,\tau,\tau^*)R_1$$

A solution to the initial problem is the value R of R_1 at $y^*=y_0^*$. To show that such a value does exist we use the same proof than lemma 2.5.1. Indeed, we have defined

$$\mathcal{E}_\Lambda^{2(\infty,\mathbb{R})}(r,s) = \mu_{T_Y^*X}^{n-d}\left(\mathcal{C}^\infty_{Y\times Y|X\times X}(r,s)\right)$$

and we may copy the proof of the lemma, replacing X by $T_Y^*X\times T_Y^*X$, Y by T_Y^*X and $\mathcal{O}_X$ by $\mathcal{C}^\infty_{Y\times Y|X\times X}(r,s)$.

In this way, we find a solution in $\mathcal{E}_\Lambda^{2(\infty,\mathbb{R})}(r,s)$ which is a subsheaf of $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}(r,s)$. □

Assume now that $n-d=1$. The matrix $R_1(x^*,y^*,\tau,\tau^*)$ is given by proposition 3.2.3 hence satisfy the conditions of lemma 3.2.2, hence for any $s'>s$, R_1 is in $\tilde{\mathcal{E}}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda'}{}_{(r,s')}[C]$ (see the proof of proposition 3.3.1) and thus its value at $y^*=y^*_0$, that is R is in $\tilde{\mathcal{E}}^{2(\mathbb{R},\mathbb{R})}_{Y\leftarrow\Lambda}{}_{(r,s')}[C]$.

In the next proposition, we assume that $X=\mathbb{C}^2$, $Y=\{\,(x,t)\in X\mid t=0\,\}$.

Proposition 6.1.2. *Let P and Q be two operators of $\mathcal{E}^2_{\Lambda}{}_{(r,s)}$ such that $\sigma^{(r,s)}_{\Lambda}(P)=(\tau^*)^N$ and $\sigma^{(r,s)}_{\Lambda}(Q)=x^M$. Let $\mathcal{L}=\mathcal{E}^2_{\Lambda}{}_{(r,s)}\big/\mathcal{E}^2_{\Lambda}{}_{(r,s)}P+\mathcal{E}^2_{\Lambda}{}_{(r,s)}Q$ and assume that $\mathcal{L}\neq 0$. Then there exists some integer $L>0$ such that :*

$$\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{\Lambda}{}_{(r,s)}\otimes_{\mathcal{E}^2_{\Lambda}{}_{(r,s)}}\mathcal{L}\simeq\left(\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{\Lambda}{}_{(r,s)}\Big/\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{\Lambda}{}_{(r,s)}t+\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{\Lambda}{}_{(r,s)}x\right)^L$$

Proof. We will denote in this proof by $\mathcal{E}$ the sheaf $\mathcal{E}^2_{\Lambda}{}_{(r,s)}$ and by $\mathcal{E}^{\mathbb{R}}$ the sheaf $\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{\Lambda}{}_{(r,s)}$. We will also denote by $\mathcal{E}[i,j]$ the subsheaf of $\mathcal{E}$ of operators of order i for the F_r-filtration and j for the F_s-filtration. In particular, the operator t of symbol τ^* is in $\mathcal{E}[q-p,q'-p']$ if $r=p/q$ and $s=p'/q'$ while the operator x of symbol x is in $\mathcal{E}[0,0]$. The hypothesis implies that P is in $\mathcal{E}[N(q-p),N(q'-p')]$ while Q is in $\mathcal{E}[0,0]$.

Using the preparation theorem [18, theorem 2.7.2] we may write

$$P=E\left((\tau^*)^N-\sum_{0\le k<N}P_k(x,\tau,x^*)(\tau^*)^k\right)$$

where E is invertible in $\mathcal{E}$, each P_k commutes with D_t and $\sigma^{(r,s)}_{\Lambda}((\tau^*)^N-\sum P_k(\tau^*)^k)=(\tau^*)^N$ which means that the order of $P_k(x,\tau,x^*)(\tau^*)^k$ is at most $(N(q-p),N(q'-p'))$.

We may also divide Q by P using [18, theorem 2.7.1] :

$$Q=AP+\sum_{0\le k<N}Q_k(x,\tau,x^*)(\tau^*)^k$$

the orders of A and the Q_k's being such that $\sigma^{(r,s)}_{\Lambda}(Q)=\sigma^{(r,s)}_{\Lambda}(AP)+\sigma^{(r,s)}_{\Lambda}(\sum Q_k(\tau^*)^k)$. As $\sigma^{(r,s)}_{\Lambda}(Q)=x^M$ we have $\sigma^{(r,s)}_{\Lambda}(Q_0)=x^M$ while the symbol of order $(0,0)$ of $Q_k(\tau^*)^k$ is 0 if $k>0$. Finally we may assume that Q is equal to

$$Q=x^M+\sum_{0\le k<N}Q_k(x,\tau,x^*)(\tau^*)^k$$

where $Q_k(\tau^*)^k$ is in $\mathcal{E}[0,0]$ and its symbol of order $(0,0)$ is 0 for $k=0,\dots,N-1$.

The module $\mathcal{E}/\mathcal{E}P$ is isomorphic to $\mathcal{E}^N\big/\mathcal{E}^N(tI_N-A)$ where I_N is the identity matrix and A is the matrix :

$$A=\begin{pmatrix}0&1&0&\dots&0\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ \vdots&&\ddots&\ddots&0\\ 0&\dots&\dots&0&1\\ P_0&P_1&\dots&\dots&P_{N-1}\end{pmatrix}\qquad(6.1.1)$$

This isomorphism is induced by $\varphi : \mathcal{E}^N \longrightarrow \mathcal{E}$ defined by $\varphi(\alpha_0, \dots, \alpha_{N-1}) = \sum \alpha_i t^i$ while its inverse is induced by $\psi : \mathcal{E} \longrightarrow \mathcal{E}^N$ defined by $\psi(\beta) = (\beta, 0, \dots, 0)$.

Let $\mathcal{E}'$ be the subsheaf of $\mathcal{E}$ of operators commuting with D_t, that is of operators having a symbol independent of τ^*. Thanks to the division theorem, the modules $\mathcal{E}/\mathcal{E}P$ and $\mathcal{E}^N/\mathcal{E}^N(tI_N - A)$ are both isomorphic to $\mathcal{E}'^N$ hence the image of an operator R of $\mathcal{E}$ in $\mathcal{E}^N/\mathcal{E}^N(tI_N - A)$ has a unique representation in $\mathcal{E}'^N$.

This applies in particular to $Q = x^M + \sum_{0\leq k<N} Q_k t^k$ (recall that τ^* is the symbol of t hence they represent the same operator) whose image by ψ is $(Q, 0, \dots, 0)$ which is equal to $(x^M + Q_0, Q_1, \dots, Q_{N-1})$ modulo $(tI_N - A)$.

We may also calculate the image of tQ: we have $tQ = Qt - [Q,t]$ and the symbol of $[Q,t]$ is $\frac{\partial}{\partial\tau}Q$ while $Qt = Q_{N-1}t^N + x^M t + \sum_{0\leq k<N-1} Q_k t^{k+1}$ has an image by ψ equal to $(Q_{N-1}P_0, x^M + Q_0 + Q_N - 1P_1, \dots, Q_{N-2} + Q_N - 1P_{N-1})$. So, the image of tQ by ψ may be written (modulo $(tI_N - A)$) in the form

$$(0, x^M, 0, \dots, 0) + (Q^1_0, Q^1_1, \dots, Q^1_{N-1})$$

where the order of Q^1_i is $((1-i)(q-p), (1-i)(q'-p'))$ (that is $Q^1_i t^{i-1}$ is of order $(0,0)$) with principal symbol $\sigma^{(r,s)}_\Lambda$ equal to 0.

The same calculation shows that the image of t^kQ for $0 \leq k \leq N-1$ under ψ is of the form $(0, \dots, x^M, 0, \dots, 0) + (Q^k_0, Q^k_1, \dots, Q^k_{N-1})$ where Q^k_i is of order $((k-i)(q-p), (k-i)(q'-p'))$ with principal symbol 0.

The module $\mathcal{L} = \mathcal{E}/\mathcal{E}P + \mathcal{E}Q$ is thus isomorphic to $\mathcal{E}^N/\mathcal{E}^N(tI_N - A) + \mathcal{E}^N(x^M I_N - B)$ where each element B_{ij} of B is in $\mathcal{E}'$ and of order $((i-j)(q-p), (i-j)(q'-p'))$ with principal symbol 0.

The preparation theorem [18, theorem 2.7.2] shows that the matrix $x^M I_N - B$ may be written :

$$x^M I_N - B(x, \tau, x^*) = E\left(x^M I_N - \sum_{0\leq k<M-1} B_k(\tau, x^*) x^k\right)$$

where E is invertible and the elements of each B_k satisfy the same conditions about the order that B. (In fact the preparation theorem was proved for single operators and not for matrices but its proof extends immediately to the case of matrices.)

Let $K = NM$. If we consider the morphism $(\alpha_{0,0}, \dots, \alpha_{N-1,M-1}) \mapsto \sum \alpha_{ij} t^i x^j$ from $\mathcal{E}^K$ to $\mathcal{E}$ we get an isomorphism between $\mathcal{L} = \mathcal{E}/\mathcal{E}P + \mathcal{E}Q$ and

$$\mathcal{E}^K/\mathcal{E}^K(tI - A') + \mathcal{E}^K(xI - B')$$

where I is the (K,K)-identity matrix, A' is the (K,K) square matrix given by M diagonal blocks equal to the (N,N) matrix A and B' is the matrix given by the same formula that A in (6.1.1) but with each P_j replaced by a matrix B_j.

The matrix A satisfy the conditions of proposition 3.2.3, hence there is an invertible matrix $R_0(x, \tau, x^*)$ with coefficients in $\mathcal{E}^{\mathbb{R}}$ independent of τ^*, i.e. commuting with D_t, such that :

$$R_0(tI_N - A)R_0^{-1} = tI_N$$

If R is is the (K,K) square matrix given by M diagonal blocks equal to R_0, we get :

$$R(tI - A')R^{-1} = tI$$

The matrix B' satisfies the conditions of remark 3.3.3, hence there is an invertible matrix $S(x^*,\tau)$ with coefficients in $\mathcal{E}^{\mathbb{R}}$ such that $S(xI - B')S^{-1} = xI$. As R commutes with t, we get an isomorphism :

$$\mathcal{L}^{\mathbb{R}} \simeq {\mathcal{E}^{\mathbb{R}}}^K \big/ ({\mathcal{E}^{\mathbb{R}}}^K tI + {\mathcal{E}^{\mathbb{R}}}^K xU(x,\tau,x^*))$$

where $U = SR^{-1}$ is invertible and commutes with D_t, i.e. is independent of τ^* and $\mathcal{L}^{\mathbb{R}} = \mathcal{E}^{\mathbb{R}} \otimes \mathcal{L}$.

If U commutes with t, $\mathcal{L}^{\mathbb{R}}$ is isomorphic by U^{-1} to ${\mathcal{E}^{\mathbb{R}}}^K \big/ ({\mathcal{E}^{\mathbb{R}}}^K tI + {\mathcal{E}^{\mathbb{R}}}^K xI)$ and the proposition is proved but this is not the case in general, so we will now prove that $\mathcal{L}^{\mathbb{R}}$ has another representation where xU is replaced by a matrix commuting with t.

The kernel of $\mathcal{E}/\mathcal{E}P \longrightarrow \mathcal{L}$ is $\mathcal{E}Q/(\mathcal{E}P \cap \mathcal{E}Q)$ and as a subsheaf of $\mathcal{E}/\mathcal{E}P$ it is supported by $\tau^* = 0$, hence there exits some P_1 with $\sigma_\Lambda^{(r,s)}(P_1) = (\tau^*)^{N_1}$ and some C such that $P_1 Q = CP$. We have an exact sequence :

$$\mathcal{E}/\mathcal{E}P_1 \xrightarrow{Q} \mathcal{E}/\mathcal{E}P \longrightarrow \mathcal{L} \longrightarrow 0$$

The matrix R_0 defines an isomorphism between $\mathcal{E}^{\mathbb{R}}/\mathcal{E}^{\mathbb{R}}P$ and ${\mathcal{E}^{\mathbb{R}}}^N/{\mathcal{E}^{\mathbb{R}}}^N tI$ and for the same reason, there is an isomorphism $\mathcal{E}^{\mathbb{R}}/\mathcal{E}^{\mathbb{R}}P_1 \xrightarrow{\sim} {\mathcal{E}^{\mathbb{R}}}^{N_1}/{\mathcal{E}^{\mathbb{R}}}^{N_1} tI$ hence there is an exact sequence :

$${\mathcal{E}^{\mathbb{R}}}^{N_1}/{\mathcal{E}^{\mathbb{R}}}^{N_1} tI \xrightarrow{\lambda} {\mathcal{E}^{\mathbb{R}}}^N/{\mathcal{E}^{\mathbb{R}}}^N tI \longrightarrow \mathcal{L}^{\mathbb{R}} \longrightarrow 0$$

Let F be any (N_1,N)-matrix with coefficients in $\mathcal{E}^{\mathbb{R}}$ such that the morphism λ is equal to the right multiplication by F. Dividing F by t, we may assume that F is independent of τ^*, but tF is in the module ${\mathcal{E}^{\mathbb{R}}}^N tI$ and thus we have $[t,F] = \frac{\partial}{\partial\tau}F = Zt$. As F is independent of t this means $\frac{\partial}{\partial\tau}F = 0$ and so F is independent of τ,τ^* that is commutes with t and D_t.

Remark : The sequence 4.1.3 shows that the subsheaf of operators of $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}{}_{(r,s)}$ commuting with t and D_t is isomorphic to the sheaf $\mathcal{E}_Y^{\mathbb{R}}$ of microlocal operators on Y. What we did is an elementary proof of the fact that $\mathcal{H}om_{\mathcal{E}^{\mathbb{R}}}(\mathcal{E}^{\mathbb{R}}/\mathcal{E}^{\mathbb{R}}t, \mathcal{E}^{\mathbb{R}}/\mathcal{E}^{\mathbb{R}}t) = \mathcal{E}_Y^{\mathbb{R}}$.

So we have found an exact sequence

$${\mathcal{E}^{\mathbb{R}}}^{K_1}/{\mathcal{E}^{\mathbb{R}}}^{K_1} tI \xrightarrow{F(x,x^*)} {\mathcal{E}^{\mathbb{R}}}^K/{\mathcal{E}^{\mathbb{R}}}^K tI \longrightarrow {\mathcal{E}^{\mathbb{R}}}^K \big/ ({\mathcal{E}^{\mathbb{R}}}^K tI + {\mathcal{E}^{\mathbb{R}}}^K xU(x,\tau,x^*)) \longrightarrow 0 \quad (6.1.2)$$

The second morphism being induced by the identity of ${\mathcal{E}^{\mathbb{R}}}^K$, this proves that there exists some matrix H such that $xU(x,\tau,x^*) = H(x,\tau,x^*)F(x,x^*)$.

Let s' such that $r > s' > s$, $C > 0$ and let us denote by $\mathcal{E}^{\mathbb{R}}[C]$ the sheaf $\tilde{\mathcal{E}}_{Y\leftarrow\Lambda}^{2(\mathbb{R},\mathbb{R})}{}_{(r,s')}[C]$.

Remark after proposition 6.1.1 shows that (for any s') there exists some $C > 0$ such that S is a matrix with coefficients in $\mathcal{E}^{\mathbb{R}}[C]$ and remark 3.3.3 shows that the matrix R has coefficients in $\mathcal{E}^{\mathbb{R}}[C]$ for suitable C, hence U has the same property. For the same reason, the morphism $\mathcal{E}^{\mathbb{R}}/\mathcal{E}^{\mathbb{R}}P_1 \xrightarrow{\sim} {\mathcal{E}^{\mathbb{R}}}^{N_1}/{\mathcal{E}^{\mathbb{R}}}^{N_1} tI$ is given by a matrix R_1 with coefficients in $\mathcal{E}^{\mathbb{R}}[C]$ for some C. So, the exact sequence (6.1.2) is still defined and exact when the sheaf $\mathcal{E}^{\mathbb{R}}$ is replaced by the sheaf $\mathcal{E}^{\mathbb{R}}[C]$ and the equality $xU(x,\tau,x^*) = H(x,\tau,x^*)F(x,x^*)$ is an equality of operators of $\mathcal{E}^{\mathbb{R}}[C]$.

Proposition 2.7.3 shows that U and H have a value at $\tau = 1$ which is well defined in $\mathcal{E}^{\mathbb{R}}_Y$, the morphism $U(x,\tau,x^*) \longrightarrow U(x,\tau_0,x^*)$ being multiplicative. As U is invertible we have :

$$xU(x,1,x^*) = H(x,1,x^*)F(x,x^*) \text{ and } U(x,1,x^*)U^{-1}(x,1,x^*) = I$$

Hence $U_0(x,x^*) = U(x,1,x^*)$ is invertible, and $xU(x,1,x^*) = H(x,1,x^*)F(x,x^*)$ belongs to $\mathcal{E}^{\mathbb{R}}_Y F(x,x^*)$. Therefore $xU_0(x,x^*)$ belongs to the module $\mathcal{E}^{\mathbb{R}^K} tI + \mathcal{E}^{\mathbb{R}^{K_1}} F(x,x^*)$.

Let us remark that operators of $\mathcal{E}^{\mathbb{R}}_Y$ are identified with operators of $\mathcal{E}^{\mathbb{R}}$ commuting with t and D_t and that such operators are in $\mathcal{E}^{2(\mathbb{R},\mathbb{R})}_{\Lambda}(r,s)$ for any (r,s), hence the equality $xU(x,1,x^*) = H(x,1,x^*)F(x,x^*)$ which was proved for some $s' > s$ is true for $s' = s$.

As U_0 commutes with t, this shows that :

$$\mathcal{L}^{\mathbb{R}} \xrightarrow{\ \sim\ } \mathcal{E}^{\mathbb{R}^K} \Big/ \left(\mathcal{E}^{\mathbb{R}^K} tI + \mathcal{E}^{\mathbb{R}^K}\, xU_0(x,x^*) + \mathcal{E}^{\mathbb{R}^{K_1}} F(x,x^*)\right)$$
$$\xrightarrow{U_0^{-1}(x,x^*)} \mathcal{E}^{\mathbb{R}^K} \Big/ \left(\mathcal{E}^{\mathbb{R}^K} tI + \mathcal{E}^{\mathbb{R}^K} xI + \mathcal{E}^{\mathbb{R}^{K_1}} F_1(x^*)\right)$$

(We may divide F_1 by x and assume that it depends only on x^*).

Our problem is now to show that F_1 is the constant matrix. As we do not know how to prove conjecture 6.1.3, we remark that the previous proof is still valid if we reverse the variables x and t. This shows that $\mathcal{L}^{\mathbb{R}}$ is isomorphic to $\mathcal{E}^{\mathbb{R}^J} \Big/ \left(\mathcal{E}^{\mathbb{R}^J} tI + \mathcal{E}^{\mathbb{R}^J} xI + \mathcal{E}^{\mathbb{R}^I} G(\tau)\right)$. So, we have an isomorphism :

$$\mathcal{E}^{\mathbb{R}^K} \Big/ \left(\mathcal{E}^{\mathbb{R}^K} tI + \mathcal{E}^{\mathbb{R}^K} xI + \mathcal{E}^{\mathbb{R}^{K_1}} F_1(x^*)\right) \xrightarrow{Z} \mathcal{E}^{\mathbb{R}^J} \Big/ \left(\mathcal{E}^{\mathbb{R}^J} tI + \mathcal{E}^{\mathbb{R}^J} xI + \mathcal{E}^{\mathbb{R}^I} G(\tau)\right)$$

This isomorphism is given by the multiplication by a matrix Z. We may divide Z by tI and xI so that its symbol is independent of t and x, and then, exactly as it was for F, the symbol of Z is also independent of τ and x^*, that is Z is a constant matrix and therefore F and G are constant, this shows that $\mathcal{L}^{\mathbb{R}}$ is 0 or a power of $\mathcal{E}^{\mathbb{R}}/(\mathcal{E}^{\mathbb{R}}t + \mathcal{E}^{\mathbb{R}}x)$. □

Conjecture 6.1.3. The module $\mathcal{E}^{\mathbb{R}}_{\mathbb{C}}/\mathcal{E}^{\mathbb{R}}_{\mathbb{C}} x$ has no proper submodule.

(The corresponding result for $\mathcal{E}_{\mathbb{C}}$ is easy to show.)

6.2 Micro-constructibility.

The sheaf $\mathcal{E}^2_{\Lambda}(r,s)$ is provided with two filtrations which extend the filtrations F_r and F_s of $\mathcal{E}_X$ [18]. If $\mathcal{M}$ is a coherent $\mathcal{E}^2_{\Lambda}(r,s)$-module, the multiplicity of $\mathcal{M}$ along an irreducible component of its support are defined with these filtration exactly as in section 1.3. If $\mathcal{M}$ is a coherent $\mathcal{E}_X$-module, the multiplicity of $\mathcal{E}^2_{\Lambda}(r,s) \otimes \mathcal{M}$ is the same than the multiplicity of the microcharacteristic cycle $\widetilde{Ch}_{\Lambda(r,s)}(\mathcal{M})$.

Theorem 6.2.1. *Let Σ be a smooth lagrangian bihomogeneous submanifold of $T^*\Lambda$ and $\mathcal{M}_0$ be a coherent $\mathcal{E}^2_{\Lambda}(r,s)$-module with support Σ and multiplicity 1. Let $\mathcal{M}$ be a coherent $\mathcal{E}^2_{\Lambda}(r,s)$-module with support Σ and multiplicity m.*

Let $\mathcal{M}_0^{\mathbb{R}} = \mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}{}_{(r,s)} \otimes_{\mathcal{E}^2_\Lambda(r,s)} \mathcal{M}_0$ *and* $\mathcal{M}^{\mathbb{R}} = \mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}{}_{(r,s)} \otimes_{\mathcal{E}^2_\Lambda(r,s)} \mathcal{M}$.

Then $\mathcal{H}om_{\mathcal{E}^2_\Lambda(r,s)}(\mathcal{M}_0, \mathcal{M}^{\mathbb{R}})$ *is locally isomorphic to* $(\mathbb{C}_\Sigma)^m$ *and* $\mathcal{E}xt^j_{\mathcal{E}^2_\Lambda(r,s)}(\mathcal{M}_0, \mathcal{M}^{\mathbb{R}})$ *is equal to* 0 *if* $j \neq 0$. *The canonical morphism :*

$$\mathcal{M}_0^{\mathbb{R}} \otimes_{\mathbb{C}} \mathcal{H}om_{\mathcal{E}^2_\Lambda(r,s)}(\mathcal{M}_0, \mathcal{M}^{\mathbb{R}}) \longrightarrow \mathcal{M}^{\mathbb{R}}$$

is an isomorphism.

Proof. A similar result was proved by Kashiwara in [11] for $\mathcal{E}_X^{\mathbb{R}}$-module and we follow Kashiwara's proof.

We first remark that the theorem is invariant under "quantized bihomogeneous canonical transformation" of [18, Theorem 2.9.11] hence we may assume that Σ is given by :

$$\Sigma = \{\, (x, \tau, x^*, \tau^*) \in T^*\Lambda \mid x_1 = x_2^* = \cdots = x_n^* = \tau^* = 0\,\}$$

Let X' be the submanifold of X given by $x_2 = \cdots = x_n = 0$ and $j : X' \hookrightarrow X$. Let $Y' = X' \cap Y$, $\Lambda' = T^*_{Y'}X'$ and $\Sigma = \{\, (x_1, \tau, x_1^*, \tau^*) \in T^*\Lambda' \mid x_1 = \tau^* = 0\,\}$. The module $j^*\mathcal{M}$ is a coherent $\mathcal{E}^2_{\Lambda'}(r,s)$-module with support in Σ' [18, th. 2.10.4.] and its multiplicity is m. In the same way $j^*\mathcal{M}_0$ has multiplicity 1.

If the theorem is true for $j^*\mathcal{M}$ and $j^*\mathcal{M}_0$ we use proposition 3.2.6 to prove the result for $\mathcal{M}$ and $\mathcal{M}_0$.

So, in this proof, we may now assume that $X = \mathbb{C}^2$ and $\Sigma = \{\, (x, \tau, x^*, \tau^*) \in T^*\Lambda \mid x = \tau^* = 0\,\}$. We denote by $\mathcal{E}$ the sheaf $\mathcal{E}^2_\Lambda(r,s)$ and by $\mathcal{E}^{\mathbb{R}}$ the sheaf $\mathcal{E}_\Lambda^{2(\mathbb{R},\mathbb{R})}{}_{(r,s)}$.

Proposition 6.1.2 shows that $\mathcal{M}_0^{\mathbb{R}}$ is isomorphic to $\mathcal{E}^{\mathbb{R}} / \mathcal{E}^{\mathbb{R}}t + \mathcal{E}^{\mathbb{R}}x$ and we may assume that $\mathcal{M}_0$ is equal to $\mathcal{E} / \mathcal{E}t + \mathcal{E}x$.

A section u of $\mathcal{M}$ is supported in $\tau^* = x = 0$ hence there exists P and Q in $\mathcal{E}$ such that $Pu = Qu = 0$, $\sigma_\Lambda^{(r,s)}(P) = (\tau^*)^N$ and $\sigma_\Lambda^{(r,s)}(Q) = x^M$.

Let $(u_1, \ldots, u_l)$ be a local set of generators of $\mathcal{M}$, there exists, for $j = 1, \ldots, l$, an operator P_j such that $\sigma_\Lambda^{(r,s)}(P_j) = (\tau^*)^{N_j}$ and an operator Q_j such that $\sigma_\Lambda^{(r,s)}(Q_j) = x^{M_j}$. Let us denote by $\mathcal{M}_1$ the direct sum :

$$\mathcal{M}_1 = \oplus \mathcal{E} / \mathcal{E}P_j + \mathcal{E}Q_j$$

The kernel of the morphism $\mathcal{M}_1 \longrightarrow \mathcal{M}$ satisfy the same properties than $\mathcal{M}$ hence $\mathcal{M}$ is the cokernel of a morphism $\mathcal{M}_2 \longrightarrow \mathcal{M}_1$ where $\mathcal{M}_2$ is of the same type than $\mathcal{M}_1$.

By proposition 6.1.2, $\mathcal{M}_k^{\mathbb{R}}$ $(k = 1, 2)$, is isomorphic to $(\mathcal{M}_0^{\mathbb{R}})^{m_k}$ where m_k is the multiplicity of $\mathcal{M}_k$ and we have an exact sequence :

$$(\mathcal{M}_0^{\mathbb{R}})^{m_2} \longrightarrow (\mathcal{M}_0^{\mathbb{R}})^{m_1} \longrightarrow \mathcal{M}^{\mathbb{R}} \longrightarrow 0$$

As $\mathbb{R}\mathcal{H}om_{\mathcal{E}^{\mathbb{R}}}(\mathcal{M}_0^{\mathbb{R}}, \mathcal{M}_0^{\mathbb{R}}) = \mathbb{C}$, the morphism $(\mathcal{M}_0^{\mathbb{R}})^{m_2} \longrightarrow (\mathcal{M}_0^{\mathbb{R}})^{m_1}$ is given by a constant matrix hence we have :

$$\mathcal{M}^{\mathbb{R}} \simeq (\mathcal{M}_0^{\mathbb{R}})^{m'}$$

We have to prove now that m' is the multiplicity m of $\mathcal{M}$.

By definition of the multiplicity, the graded module of $\mathcal{M}$ is, generically on Σ, a free $\mathcal{O}_{[\Sigma]}$-module of rank m. Then, using the division theorem of [18, 2.7.1] we conclude that generically $\mathcal{M}$ is a free $\widetilde{\mathcal{E}}$-module of rank m (where $\widetilde{\mathcal{E}}$ is the subsheaf of $\mathcal{E}$ of operators commuting with D_t and D_x). Then if we extend the coefficients to $\mathcal{E}^{\mathbb{R}}$ we get $(\mathcal{M}_0^{\mathbb{R}})^m$ which shows that $m = m'$. □

Let us say that a $\mathcal{D}_X^\infty$-module $\mathcal{M}$ is a holonomic $\mathcal{D}_X^\infty$-module if it satisfy :

1) The support Σ of $\mathcal{E}_X^\infty \otimes_{\pi^{-1}\mathcal{D}_X^\infty} \pi^{-1}\mathcal{M}$ is a complex lagrangian subvariety of T^*X.

2) For any point ξ of the smooth part of Σ and any holonomic $\mathcal{E}_X$-module $\mathcal{M}_0$ with support Σ near ξ and multiplicity 1, there is near ξ an isomorphism between $\mathcal{E}_X^{\mathbb{R}} \otimes_{\pi^{-1}\mathcal{D}_X^\infty} \pi^{-1}\mathcal{M}$ and $(\mathcal{E}_X^{\mathbb{R}} \otimes \mathcal{M}_0)^m$.

The number m is the multiplicity of $\mathcal{M}$ on Σ at ξ. It is constant on the irreducible components of the variety Σ. In this way we define an analytic cycle with support Σ which we will call the characteristic cycle of $\mathcal{M}$.

Kashiwara proved in [11] that any holonomic $\mathcal{D}_X$-module satisfy these properties, that is for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$, the $\mathcal{D}_X^\infty$-module $\mathcal{D}_X^\infty \otimes \mathcal{M}$ is holonomic according to this definition.

Applying the morphism of vanishing cycles to theorem 6.2.1 we get :

Corollary 6.2.2. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module. The $\mathcal{D}_Y^\infty$-module $\Phi\{r\}(\mathcal{M})$ is holonomic for any $r \geq 1$ and its characteristic cycle is (with $\varpi : (T^*Y) \times_Y \Lambda \to T^*Y$) :*

$$\varpi^{-1}\widetilde{Ch}(\Phi\{r\}(\mathcal{M})) = \widetilde{Ch}_{\Lambda\{r\}}(\mathcal{M})$$

Moreover, the proof of the index theorem in [11] is still valid with no modification for holonomic $\mathcal{D}_X^\infty$-modules (one step of that proof being precisely to show that what we call here holonomic $\mathcal{D}_X^\infty$-modules satisfy the index theorem). So we recover all the results of section 5.3.

In fact, this proves that the complex of holomorphic solutions of these modules is a perverse sheaf, so the Riemann-Hilbert correspondence shows that they are admissible. This definition of holonomic $\mathcal{D}_X^\infty$ -modules is therefore equivalent to the definition of §5.2 but what we wanted to point here, is that the index theorems may be proved independently of Riemann-Hilbert.

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