A sum formula for the Casson-Walker invariant

Christine Lescop *

January 23, 1995

Abstract

We study the following question: How does the Casson-Walker invariant λ of a rational homology 3-sphere obtained by gluing two pieces along a surface depend on the two pieces? Our partial answer may be stated as follows. For a compact oriented 3-manifold A with boundary ∂A, the kernel $\mathcal{L}_A$ of the map from $H_1(\partial A;\mathbf{Q})$ to $H_1(A;\mathbf{Q})$ induced by the inclusion is called the *Lagrangian* of A. Let Σ be a closed oriented surface, and let A, A', B and B' be four rational homology handlebodies such that ∂A, $\partial A'$, $-\partial B$ and $-\partial B'$ are identified via orientation-preserving homeomorphisms with Σ. Assume that $\mathcal{L}_A = \mathcal{L}_{A'}$ and $\mathcal{L}_B = \mathcal{L}_{B'}$ inside $H_1(\Sigma;\mathbf{Q})$ and also assume that $\mathcal{L}_A$ and $\mathcal{L}_B$ are transverse. Then we express

$$\lambda(A \cup_\Sigma B) - \lambda(A' \cup_\Sigma B) - \lambda(A \cup_\Sigma B') + \lambda(A' \cup_\Sigma B')$$

in terms of the form induced on $\bigwedge^3 \mathcal{L}_A$ by the algebraic intersection on $H_2(A \cup_\Sigma -A')$ paired to the analogous form on $\bigwedge^3 \mathcal{L}_B$ via the intersection form of Σ. The simple formula that we obtain naturally extends to the extension of the Casson-Walker invariant of the author. It also extends to gluings along non-connected surfaces.

Mots-clefs: Invariant de Casson, variétés de dimension 3, torsion de Reidemeister, polynôme d'Alexander, groupe de Torelli, groupe des difféotopies d'une surface
Keywords: Casson invariant, 3-manifolds, Reidemeister torsion, Alexander polynomial, mapping class group, Torelli group
A.M.S. subject classification: 57N10, 57M05, 57N05, 57Q10

*CNRS, Institut Fourier (URA 188 du CNRS)

1 Introduction

In 1985, A. Casson defined an integral invariant of integral homology 3-spheres by introducing an appropriate way of counting the SU(2)-representations of their fundamental groups (see [A-M, M, G-M]). His invariant, and all of its interesting original properties, were extended to rational homology 3-spheres (closed 3-manifolds M such that $H_*(M;\mathbf{Q}) = H_*(S^3;\mathbf{Q})$) by K. Walker in 1988 (see [W]). In [L], I defined an extension of the Casson-Walker invariant to all closed oriented 3-manifolds from a global surgery formula. Here, we study the following question:

How does the so-extended Casson-Walker invariant λ of a closed 3-manifold, obtained by gluing two pieces along a surface, depend on the two pieces?

Our partial answer is the homogeneous sum formula of Theorem 1.11 below. Since the behaviour of the so-called quantum invariants in these circumstances is crucial, our sum formula might be used to see how the Casson-Walker invariant fits in in the framework of topological quantum fields theories. In particular, it could lead to another understanding of the Murakami relation [Mu] between the Witten-Reshetikhin-Turaev invariants and the Casson-Walker invariant. Our formula may also be applied to generalize a result of Morita [Mo] which measures how far some functions induced by the Casson invariant on the Torelli group are from being homomorphisms (see Corollary 1.6).

In order to prove our sum formula we will be led to introduce a (tautological) extension of the Reidemeister torsion to compact 3-manifolds with arbitrary boundary. The study of this extension, that we call the Alexander function, is performed in Section 3. This section may be of independent interest and can be read independently.

We begin by stating our sum formula for the Walker invariant in Theorem 1.3 (it involves less notation in this case and the general formula is its natural extension). In order to state it properly, we list our conventions.

Conventions 1.1

- Throughout this paper, all the manifolds are compact and oriented. We use the 'outward normal first' convention to orient the boundary ∂ of a manifold. The symbol $-$ in front of a manifold reverses its orientation.

- We use the following normalization for the Casson-Walker invariant λ. If λ_W denotes the Walker normalization (see [W]) of his invariant, then

$$\lambda = \frac{\lambda_W}{2}$$

  This is Casson's original normalization of his invariant used in [A-M, M, G-M].

- We define a *rational homology handlebody* (or RHH) as an (oriented, compact) 3-manifold with the same rational homology as a standard handlebody.

- We denote by $<,>_\Sigma$ the intersection form on a surface Σ.

- For a (compact) 3-manifold A with boundary, we denote by $\mathcal{L}_A$ the kernel of the map:
$$H_1(\partial A; \mathbf{Q}) \longrightarrow H_1(A; \mathbf{Q}).$$
It is a Lagrangian of $(H_1(\partial A; \mathbf{Q}), <,>_{\partial A})$, we call it the *Lagrangian* of A.

Note that, if A and B are two 3-manifolds whose connected boundaries are identified by an orientation-reversing homeomorphism, then $A \cup_{\partial A} B$ is a rational homology sphere if and only if A and B are RHH and $\mathcal{L}_A$ and $\mathcal{L}_B$ are transverse in $H_1(\partial A; \mathbf{Q})$.

Definition 1.2 Let A and A' be two connected 3-manifolds whose boundaries are identified via orientation-preserving homeomorphisms with a fixed connected surface Σ and which have the same Lagrangian $\mathcal{L}_A$ in $H_1(\Sigma; \mathbf{Q})$.

- If both A and A' are RHH, then the Mayer-Vietoris sequence provides the isomorphism
$$\partial_{AA'} : H_2(A \cup_\Sigma -A'; \mathbf{Q}) \longrightarrow \mathcal{L}_A$$
chosen so that it maps the homology class of an embedded surface S of $A \cup_\Sigma -A'$ transverse to ∂A to the class of $S \cap \partial A$ *oriented* as the boundary of $S \cap A$.

  Let $[A \cup_\Sigma -A']$ be the homology class of $A \cup_\Sigma -A'$, let $D = [A \cup_\Sigma -A'] \cap \cdot$ be the Poincaré duality isomorphism:
$$D : H^1(A \cup_\Sigma -A'; \mathbf{Q}) \longrightarrow H_2(A \cup_\Sigma -A'; \mathbf{Q})$$
  The *intersection form* $\mathcal{I}_{AA'}$ of (A, A') is defined on $\bigwedge^3 \mathcal{L}_A$ by:
$$\mathcal{I}_{AA'}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) =$$
$$(\partial_{AA'} \circ D)^{-1}(\alpha_1) \cup (\partial_{AA'} \circ D)^{-1}(\alpha_2) \cup (\partial_{AA'} \circ D)^{-1}(\alpha_3)([A \cup_\Sigma -A'])$$

- If A or A' is not an RHH, then we set $\mathcal{I}_{AA'} = 0$.

Now, we can state the main result of the paper for rational homology spheres that is for the Walker invariant:

Theorem 1.3 *Let Σ be a closed, connected surface, and let A, A', B and B' be four RHH such that ∂A, $\partial A'$, $-\partial B$ and $-\partial B'$ are identified via orientation-preserving homeomorphisms with Σ. Assume that $\mathcal{L}_A = \mathcal{L}_{A'}$ and $\mathcal{L}_B = \mathcal{L}_{B'}$ inside $H_1(\Sigma;\mathbf{Q})$ and also assume that $\mathcal{L}_A$ and $\mathcal{L}_B$ are transverse ($\mathcal{L}_A \cap \mathcal{L}_B = \{0\}$ in $H_1(\Sigma;\mathbf{Q})$). Let $(\alpha_1,\dots,\alpha_g)$ and $(\beta_1,\dots,\beta_g)$ be two bases for $\mathcal{L}_A$ and $\mathcal{L}_B$, respectively, such that $<\alpha_i,\beta_j>_\Sigma$ is equal to the Kronecker symbol δ_{ij} for any i,j in $\{1,\dots,g\}$. Then Formula $\mathcal{E}(A,A',B,B')$ below is satisfied:*

$$\lambda(A\cup_\Sigma B)-\lambda(A'\cup_\Sigma B)-\lambda(A\cup_\Sigma B')+\lambda(A'\cup_\Sigma B')$$

$$= -2\sum_{\{i,j,k\}\subset\{1,\dots,g\}} \mathcal{I}_{AA'}(\alpha_i\wedge\alpha_j\wedge\alpha_k)\mathcal{I}_{BB'}(\beta_i\wedge\beta_j\wedge\beta_k) \qquad (\mathcal{E}(A,A',B,B'))$$

Remark 1.4 Using the isomorphism $i_A:\mathcal{L}_A \longrightarrow \mathcal{L}_B^* = \mathrm{Hom}(\mathcal{L}_B;\mathbf{Q})$ which maps α to $(i_A(\alpha) =<\alpha,.>_\Sigma)$ allows us to write the right-hand side of the equality above as:

$$-2\,\mathcal{I}_{AA'}(\bigwedge^3 i_A^{-1})(\mathcal{I}_{BB'})$$

A more elegant expression will be given in Notation 1.8.

Remark 1.5 Let $(\Sigma,\mathcal{L}_A)$ be a closed, connected surface equipped with a rational Lagrangian (as above). In [S], D. Sullivan proved that any integral form on $\bigwedge^3(H_1(\Sigma;\mathbf{Z})\cap\mathcal{L}_A)$ may be realized as a $\mathcal{I}_{AA'}$ for two standard handlebodies A and A' with boundary Σ and Lagrangian $\mathcal{L}_A$.

The sum formula of Theorem 1.3 may be applied to reduce the computation of a $\lambda(A\cup B)$ to the case where A or B is a standard handlebody by choosing standard handlebodies A' and B' with the right Lagrangians. Note that the right-hand side is zero when the genus of Σ is lower than 3. In particular, in genus 0, for $A'=B'=B^3$, the equality is the additivity formula of the Casson-Walker invariant under connected sum. The genus one formula, when A' and B' are solid tori, is the splicing formula, shown by several authors (see [B-N, F-M]) for the Casson invariant, and generalized by Fujita to the Walker invariant (see [F]). In this case, there is a unique way of filling in A with a solid torus B' having the right Lagrangian, $A'\cup B$ and $A'\cup B'$ are similarly well-determined (starting from $A\cup B$), and the Walker invariant of the lens space $A'\cup B'$ is a known Dedekind sum.

This formula also evaluates how far the functions induced by the Casson-Walker invariant on some subgroups of the mapping class group are from being homomorphisms. Let us be more specific. Let a rational homology sphere be cut into two pieces by an embedding of a surface Σ, allowing us to write it as

$A \cup_\Sigma B$ where ∂A and $-\partial B$ are identified with Σ by the (orientation-preserving) homeomorphisms:

$$j_A : \Sigma \longrightarrow \partial A \quad \text{and} \quad j_B : \Sigma \longrightarrow -\partial B$$

Note that the datum (A, B) (with the underlying (j_A, j_B)) is equivalent to the datum of the rational homology sphere equipped with the embedding of Σ. For an (orientation-preserving) homomeorphism f of Σ which preserves $\mathcal{L}_A$ and $\mathcal{L}_B$, we define $A(f^{-1})$ so that $A(f^{-1})$ is equal to A as a manifold, but $\partial(A(f^{-1}))$ is identified with Σ by $j_A \circ f^{-1}$; and we set:

$$A \cup_f B = A(f^{-1}) \cup_\Sigma B (= A \cup_\Sigma B(f))$$

and

$$\lambda_{AB}(f) = \lambda(A \cup_f B) - \lambda(A \cup_\Sigma B)$$

The formula of Theorem 1.3 yields the following corollary.

Corollary 1.6 *For any two homomeorphisms f and g which preserve $\mathcal{L}_A$ and $\mathcal{L}_B$,*

$$\lambda_{AB}(g \circ f) - \lambda_{AB}(g) - \lambda_{AB}(f) = -2\,\mathcal{I}_{AA(f^{-1})}(\bigwedge^3 i_A^{-1})(\mathcal{I}_{BB(g)})$$

When (the isotopy classes of) f and g are in the Torelli group of Σ (that is when they induce the identity on $H_1(\Sigma)$), the right-hand side is independent of A and B (but depends on $\mathcal{L}_A$ and $\mathcal{L}_B$), it is a function of the evaluations of the Johnson homomorphism at f and g (see [J, Second definition, p.170]). With completely different methods (based mainly on Johnson's study of the Torelli group), S. Morita proved Corollary 1.6 for Heegaard embeddings into integral homology spheres and homomorphisms of the Torelli group [Mo, Theorem 4.3], but he did not think that it extended to general embeddings [Mo, Remark 4.7].

For integral homology spheres, reducing λ mod 2 yields the Rohlin μ-invariant (see [G-M]). When $A \cup B$ is an integral homology sphere, Corollary 1.6 proves that μ_{AB} defines a homomorphism from the Torelli group to $\mathbf{Z}/2\mathbf{Z}$. These homomorphisms were first studied by J. Birman and R. Craggs [B-C], they are the so-called Birman-Craggs homomorphisms.

To prove Theorem 1.3, we first find a sequence of simple surgeries on links transforming A into A' and staying among the RHH with Lagrangian $\mathcal{L}_A$. Then we apply the surgery formula of [L, B-L] to these surgeries and analyse how the involved formulae depend on B when B varies among the RHH with boundary $-\partial A$ and with fixed Lagrangian.

The paper is organized as follows. In Section 2, we present the special types of surgeries that we need to study, and we reduce the proof to the following question. *How do certain derivatives of Alexander polynomials of specific links of a RHH ($A \subset A \cup B$) depend on B when B varies among the RHH with boundary*

$-\partial A$ and with fixed Lagrangian? To answer this type of question, we introduce a (tautological) generalization of the Alexander polynomial to compact 3-manifolds with boundary, in Section 3. The first properties of this *Alexander function*, also given in Section 3, allow us to conclude the proof of Theorem 1.3 in Section 4.

In Section 5, we give formulae to describe gluings along non-connected surfaces.

Now, we finish this introduction with the complete statement of our sum formula for the extended Casson-Walker invariant.

1.1 Complete statement of the sum formula

In our general statement, A' and B' will be standard handlebodies. It is not hard to see that adding this hypothesis in Theorem 1.3 does not weaken its statement (see Subsection 2.1). We first introduce suitable notation.

Notation 1.7 (Notation related to handlebodies)

Let Σ be a closed surface of genus g.

- A Σ-*system* is a system $a = (a_1, a_2, ..., a_g)$ of g simple closed curves on Σ such that the curves a_i are pairwise disjoint and their union does not separate Σ.
- A Σ-system $z = (z_1, \dots, z_g)$ is said to be *geometrically dual* to the Σ-system a, if for any $i, j \in \{1, \dots, g\}$, a_i and z_j are transverse, $a_i \cap z_j$ contains exactly δ_{ij} points and $< a_i, z_j >_\Sigma = \delta_{ij}$ (as in Figure 1).

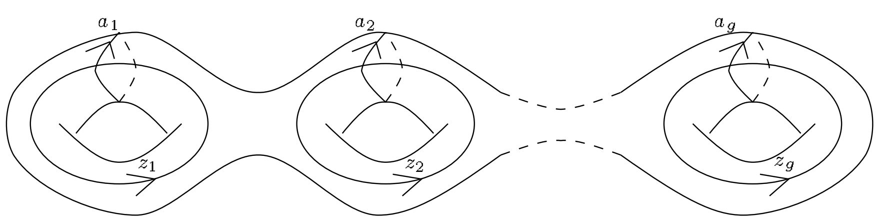

Figure 1: Σ equipped with its two geometrically dual systems a and z

- Let $\mathcal{L}$ be a Lagrangian of $(H_1(\Sigma, \mathbf{Q}); < .,. >_\Sigma)$. A $(\Sigma, \mathcal{L})$-system is a Σ-system whose components have their homology classes in $\mathcal{L}$.
- Let a be a Σ-system, Σ_a denotes the handlebody with oriented boundary Σ, where the components of a bound 2-disks.
- Let A be a compact 3-manifold with boundary Σ. Let b be a Σ-system. Then A_b denotes the 3-manifold obtained by gluing thickened 2-disks along the b_i and filling in the resulting S^2-boundary with a 3-ball (or by filling in ∂A with $-\Sigma_b$). The manifold A_b inherits the orientation of A.

- Let a and b be two Σ-sytems, then we define

$$\Sigma_{ab} = (\Sigma_a)_b$$

applying the definitions above one after the other.

- Let A be a RHH and let a be a $(\partial A, \mathcal{L}_A)$-system. Then we denote by $\mathcal{I}_{Aa}$ the form $\mathcal{I}_{A(\partial A)_a}$.

Now, we take care of the fact that the right-hand side of the equality of Theorem 1.3 does not make sense anymore when the Lagrangians are not transverse. We fix a genus g surface Σ and two Lagrangians $\mathcal{L}_A$ and $\mathcal{L}_B$ of $(H_1(\Sigma;\mathbf{Q}), <,>_\Sigma)$.

Notation 1.8 • Let $\mathcal{L}$ be a $\mathbf{Q}$-vector space of dimension g, and let $\hat{a}$ be a generator of $\bigwedge^g \mathcal{L}$. We define the isomorphism

$$\hat{a}\cap : \bigwedge^{k} \mathcal{L}^* \longrightarrow \bigwedge^{g-k} \mathcal{L} \overset{\text{can}}{=} \bigwedge^{g-k} \mathcal{L}^{**}$$

$$\mathcal{I} \mapsto \hat{a} \cap \mathcal{I}$$

where

$$(\hat{a} \cap \mathcal{I})(\mathcal{J}) = \hat{a}(\mathcal{I} \wedge \mathcal{J})$$

(Recall that $\bigwedge^k \mathcal{L}^* = (\bigwedge^k \mathcal{L})^*$, $v_1^* \wedge \cdots \wedge v_k^*(w_1 \wedge \cdots \wedge w_k) = det(v_i^*(w_j))$.)

- We also denote by $<,>_\Sigma$ the bilinear form induced on $\bigwedge^k \mathcal{L}_A \times \bigwedge^k \mathcal{L}_B$ by $<,>_\Sigma$, for any integer k. ($< a_1 \wedge \cdots \wedge a_k, b_1 \wedge \cdots \wedge b_k >_\Sigma = det[< a_i, b_j >_\Sigma]$.)

Note that the bilinear form defined on $\bigwedge^3 \mathcal{L}_A^* \times \bigwedge^3 \mathcal{L}_B^*$ by

$$(\mathcal{I}_A, \mathcal{I}_B) \mapsto < \hat{a} \cap \mathcal{I}_A, \hat{b} \cap \mathcal{I}_B >_\Sigma$$

linearly depends on $\hat{a} \otimes \hat{b} \in \bigwedge^g \mathcal{L}_A \otimes_{\mathbf{Q}} \bigwedge^g \mathcal{L}_B$. In particular, if $\mathcal{L}_A$ and $\mathcal{L}_B$ are transverse, we may rewrite the right-hand side of the equation of Theorem 1.3 as:

$$-2\,\mathcal{I}_{AA'}(\bigwedge^3 i_A^{-1})(\mathcal{I}_{BB'}) = -2\frac{< \hat{a} \cap \mathcal{I}_{AA'}, \hat{b} \cap \mathcal{I}_{BB'} >_\Sigma}{< \hat{a}, \hat{b} >_\Sigma}$$

for any generator $\hat{a} \otimes \hat{b}$ of $\bigwedge^g \mathcal{L}_A \otimes_{\mathbf{Q}} \bigwedge^g \mathcal{L}_B$. For the other cases, we will specify a generator $\hat{a} \otimes \hat{b}$ of $\bigwedge^g \mathcal{L}_A \otimes_{\mathbf{Q}} \bigwedge^g \mathcal{L}_B$.

Notation 1.9 Choose two bases $a = (a_1, ..., a_g)$ and $b = (b_1, ..., b_g)$ of $\mathcal{L}_A \cap H_1(\Sigma;\mathbf{Z})$ and of $\mathcal{L}_B \cap H_1(\Sigma;\mathbf{Z})$, respectively, satisfying $(*)$:

$(*)$ For $i = 1, \ldots, d = dim(\mathcal{L}_A \cap \mathcal{L}_B)$, $a_i = b_i$.

(Such bases exist.) Let $\hat{a}$ denote $a_1 \wedge \cdots \wedge a_g$ and let $\hat{b}$ denote $b_1 \wedge \cdots \wedge b_g$. Then our canonical generator of $\bigwedge^g \mathcal{L}_A \otimes_{\mathbf{Q}} \bigwedge^g \mathcal{L}_B$ is

$$\gamma(\Sigma, \mathcal{L}_A, \mathcal{L}_B) = sign\,(det\,([< a_i, b_j >_\Sigma]_{i,j=d+1,\dots,g}))\,\hat{a} \otimes \hat{b}$$

(where $sign(x) \stackrel{\text{def}}{=} \frac{x}{|x|}$ if $x \in \mathbf{R} \setminus \{0\}$).

Let a and b be two bases of $\mathcal{L}_A \cap H_1(\Sigma; \mathbf{Z})$ and of $\mathcal{L}_B \cap H_1(\Sigma; \mathbf{Z})$, respectively, which do not necessarily satisfy $(*)$. Then we define $sign_\Sigma(\hat{a}, \hat{b}) = \pm 1$ so that

$$\gamma(\Sigma, \mathcal{L}_A, \mathcal{L}_B) = sign_\Sigma(\hat{a}, \hat{b})\hat{a} \otimes \hat{b}$$

in any case.

Now, we can generalize Theorem 1.3 for the extension of λ described in [L] when $A \cup B$ is not necessarily a rational homology sphere anymore.

Notation 1.10

- If M is a $\mathbf{Z}$-module, we denote by $|M|$ its *order*, that is its cardinality if M is finite, and 0 otherwise.
- We denote by $\overline{\lambda}$ the extension denoted by λ in [L]. With the current notation, we have the following relation between the two normalizations:

$$\lambda = \frac{\overline{\lambda}}{|H_1(.; \mathbf{Z})|}$$

Theorem 1.11 *Let Σ be a closed connected surface. Let A and B be two connected 3-manifolds such that ∂A and $-\partial B$ are identified with Σ by orientation-preserving homeomorphisms. Let a be a $(\Sigma, \mathcal{L}_A)$-system and let b be a $(\Sigma, \mathcal{L}_B)$-system. Let z be a Σ-system geometrically dual to a and let y be a $(-\Sigma)$-system geometrically dual to b. Then $\overline{\lambda}(A \cup B)$ is given by Formula $\mathcal{F}(A, a, B, b)$ below.*

$$\overline{\lambda}(A \cup B) =$$

$$|H_1(A_z)|\,\overline{\lambda}(B_a) + |H_1(B_y)|\,\overline{\lambda}(A_b) - |H_1(A_z)|\,|H_1(B_y)|\,\overline{\lambda}(\Sigma_{ab})$$
$$-2\,|H_1(A_z)||H_1(B_y)|\,sign_\Sigma(\hat{a}, \hat{b}) < \hat{a} \cap \mathcal{I}_{Aa}, \hat{b} \cap \mathcal{I}_{Bb} >_\Sigma \qquad (\mathcal{F}(A, a, B, b))$$

Easy homological considerations performed in Subsection 2.1 show that this statement does generalize Theorem 1.3, and that it could be given in an analogous form. The theorem is easily seen as a consequence of the geometrical interpretation of $\overline{\lambda}$ when the rank of $A \cup B$ is larger than one in Subsection 2.2. The rank one case will follow from the properties of the Alexander function, its proof will be complete at the end of Section 4.

This article partially answers a question of Pierre Vogel. It also benefited from conversations with Steven Boyer, Lucien Guillou, Dieter Kotschick, Daniel Lines, Alexis Marin and Gregor Masbaum. I thank all of them.

2 Reducing the proofs to technical lemmas

In all this section, Σ denotes a closed connected genus g surface equipped with two Lagrangians $\mathcal{L}_A$ and $\mathcal{L}_B$ of $(H_1(\Sigma;\mathbf{Q}),<,>_\Sigma)$, we set $<,>=<,>_\Sigma$. We call a connected 3-manifold with boundary Σ and Lagrangian $\mathcal{L}_A$ a $(\Sigma,\mathcal{L}_A)$-manifold.

2.1 Preliminary remarks

Lemma 2.1 *Let a and b be a $(\Sigma,\mathcal{L}_A)$-system and a $(\Sigma,\mathcal{L}_B)$-system, respectively. Let z be a Σ-system geometrically dual to a and let y be a $(-\Sigma)$-system geometrically dual to b. If A is a $(\Sigma,\mathcal{L}_A)$-manifold and if B is a $(\Sigma,\mathcal{L}_B)$-manifold, then*

$$|H_1(A\cup_\Sigma B)| = |H_1(A_z)|\,|H_1(B_y)|\,|H_1(\Sigma_{ab})|$$

PROOF: If A and B are RHH,

$$\frac{|H_1(A\cup_\Sigma B)|}{|H_1(A_z)|\,|H_1(B_y)|} = \left|\frac{\wedge_{i=1}^g (z_i^A - z_i^B)\wedge\wedge_{i=1}^g (a_i^A - a_i^B)}{\wedge_{i=1}^g z_i^A \wedge \wedge_{i=1}^g y_i^B}\right|$$

$$= \left|\frac{\wedge_{i=1}^g z_i^A\wedge\wedge_{i=1}^g a_i^B}{\wedge_{i=1}^g z_i^A\wedge\wedge_{i=1}^g y_i^B}\right| = \left|\frac{det[<b_j,a_i>]}{det[<b_j,y_i>]}\right|$$

where z_i^A denotes the class of z_i in $H_1(A;\mathbf{Q})$, for example, and the exterior products are taken in $\bigwedge_{i=1}^{2g}(H_1(A;\mathbf{Q})\oplus H_1(B;\mathbf{Q}))$. □

This proves that Theorem 1.3 is a particular case of Theorem 1.11 when A' and B' are standard handlebodies.

Now, assume that $\mathcal{L}_A$ and $\mathcal{L}_B$ are transverse. Fix a $(\Sigma,\mathcal{L}_A)$-RHH A_0, and a $(-\Sigma,\mathcal{L}_B)$-RHH B_0. Assume that $\mathcal{E}(A,A_0,B,B_0)$ is true for any $(\Sigma,\mathcal{L}_A)$-RHH A and for any $(-\Sigma,\mathcal{L}_B)$-RHH) B. Then because of the identity:

$$\mathcal{I}_{AA_0} - \mathcal{I}_{A'A_0} = \mathcal{I}_{AA'}$$

$\mathcal{E}(A,A',B,B')$ is true for all the (A,A',B,B') (such that A and A' are $(\Sigma,\mathcal{L}_A)$-RHH and B and B' are $(-\Sigma,\mathcal{L}_B)$-RHH with the given $(\Sigma,\mathcal{L}_A,\mathcal{L}_B)$). We have just reduced the proof of Theorem 1.3 to the proof of the following statement:

Statement 2.2 *Let Σ be a closed connected genus g surface equipped with two transverse Lagrangians $\mathcal{L}_A$ and $\mathcal{L}_B$ of $(H_1(\Sigma;\mathbf{Q}),<,>)$. There exists a $(\Sigma,\mathcal{L}_A)$-system a and a $(\Sigma,\mathcal{L}_B)$-system b such that:*

For any $(\Sigma,\mathcal{L}_A)$-RHH A and for any $(-\Sigma,\mathcal{L}_B)$-RHH B:

$$\lambda(A\cup B) - \lambda(A_b) - \lambda(B_a) + \lambda(\Sigma_{ab})$$

$$= -2\sum_{\{i,j,k\}\subset\{1,\dots,g\}} \mathcal{I}_{A_a}(\alpha_i\wedge\alpha_j\wedge\alpha_k)\mathcal{I}_{B_b}(\beta_i\wedge\beta_j\wedge\beta_k)$$

where $(\alpha_1, \ldots, \alpha_g)$ *and* $(\beta_1, \ldots, \beta_g)$ *are two bases for* $\mathcal{L}_A$ *and* $\mathcal{L}_B$*, respectively, such that* $< \alpha_i, \beta_j >_\Sigma = \delta_{ij}$ *for any* $i, j \in \{1, \ldots, g\}$*.*

Similarly, it is sufficient to prove $\mathcal{F}(A, a, B, b)$ for a particular $(\Sigma, \mathcal{L}_A)$-system a and a particular $(\Sigma, \mathcal{L}_B)$-system b to get it in any case (for our given $(\Sigma, \mathcal{L}_A, \mathcal{L}_B)$). We could also state Theorem 1.11 for any (A, A', B, B') such that A and A' are $(\Sigma, \mathcal{L}_A)$-manifolds and B and B' are $(-\Sigma, \mathcal{L}_B)$-manifolds.

Until the end of this subsection, A denotes a $(\Sigma, \mathcal{L}_A)$-RHH equipped with a $(\Sigma, \mathcal{L}_A)$-system a and a Σ-system z geometrically dual to a. We study the homological properties of A once for all.

Definition 2.3 We define the *integral Lagrangian* $\mathcal{L}_A^{int}$ of A as the kernel of

$$i_* : H_1(\partial A; \mathbf{Z}) \longrightarrow H_1(A; \mathbf{Z})$$

Note that:

$$\mathcal{L}_A^{int} \subset \mathbf{Z}a_1 \oplus \mathbf{Z}a_2 \oplus \cdots \oplus \mathbf{Z}a_g = \mathcal{L}_A \cap H_1(\partial A; \mathbf{Z})$$

Lemma 2.4

$$\frac{|\mathrm{Torsion}(H_1(A))|}{|\mathrm{Torsion}(H_1(A_a))|} = \left| \frac{\mathcal{L}_A \cap H_1(\partial A; \mathbf{Z})}{\mathcal{L}_A^{int}} \right| = \frac{|H_1(A_z)|}{|\mathrm{Torsion}(H_1(A))|}$$

PROOF:

$$H_1(A_a) = \frac{H_1(A)}{\mathbf{Z}a_1 \oplus \mathbf{Z}a_2 \oplus \ldots \oplus \mathbf{Z}a_g}$$

Since the a_i are torsion elements of $H_1(A)$, the same equality holds for the torsion parts and we have the following short exact sequence:

$$0 \longrightarrow \frac{\mathbf{Z}a_1 \oplus \mathbf{Z}a_2 \oplus \ldots \oplus \mathbf{Z}a_g}{\mathcal{L}_A^{int}} \longrightarrow \mathrm{Torsion}(H_1(A)) \longrightarrow \mathrm{Torsion}(H_1(A_a)) \longrightarrow 0$$

which proves the first part of the equality.

For the second part, we have:

$$H_1(A_z) = \frac{H_1(A)}{\mathbf{Z}z_1 \oplus \mathbf{Z}z_2 \oplus \ldots \oplus \mathbf{Z}z_g}$$

where the rank of $H_1(A)$ is g. Hence, if $(f_1, \ldots, f_g)$ is a basis of $\mathrm{Hom}(H_1(A); \mathbf{Z}) = H^1(A)$, then:

$$|H_1(A_z)| = |\mathrm{Torsion}(H_1(A))||det[f_i(z_j)]_{i,j=1,\ldots,g}|$$

Choose a basis $(k_1, k_2, \ldots, k_g)$ of $\mathcal{L}_A^{int}$. Using the Poincaré duality isomorphism between $H^1(A)$ and $H_2(A, \partial A)$ and the boundary isomorphism from $H_2(A, \partial A)$

to $\mathcal{L}_A^{int}$ allows us to choose f_i as 'the algebraic intersection with a surface bounded by k_i'. Thus,

$$|H_1(A_z)| = |\text{Torsion}(H_1(A))||det[< k_i, z_j >_{\partial A}]_{i,j=1,\dots,g}|$$

where

$$|det[< k_i, z_j >_{\partial A}]_{i,j=1,\dots,g}| = \frac{|det[< k_i, z_j >_{\partial A}]_{i,j=1,\dots,g}|}{|det[< a_i, z_j >_{\partial A}]_{i,j=1,\dots,g}|} = \left|\frac{k_1 \wedge k_2 \wedge \dots \wedge k_g}{a_1 \wedge a_2 \wedge \dots \wedge a_g}\right|$$

$$= \left|\frac{\mathcal{L}_A \cap H_1(\partial A; \mathbf{Z})}{\mathcal{L}_A^{int}}\right|$$

□

The previous lemma implies the following one:

Lemma 2.5 *The three following assertions are equivalent:*

1. $H_1(A; \mathbf{Z}) = \mathbf{Z}^g$.
2. $H_1(A, \partial A) = \{0\}$.
3. $i_* : H_1(\partial A; \mathbf{Z}) \longrightarrow H_1(A; \mathbf{Z})$ *is onto.*

All of them imply:

$$\mathcal{L}_A^{\text{int}} = \mathcal{L}_A \cap H_1(\partial A; \mathbf{Z})$$

Definition 2.6 An RHH A satisfying the assertions of the previous lemma is an *integral homology handlebody.*

2.2 Proving $\mathcal{F}(A, a, B, b)$ when the rank of $A \cup B$ is larger than 1

This case could be (at least as quickly) studied with the methods described in the next sections. Since it does not require them, we give a geometrical proof of the formula based on the geometrical interpretation of $\overline{\lambda}$ of [L, Section 1.5] for the manifolds of rank larger than 1. This proof should help feeling the general result (but since it is almost independent of the rest of the paper, this subsection can also be skipped).

For closed 3-manifolds with positive rank (of the H_1), we interpret

$$\overline{\lambda}(M)/|\text{Torsion}(H_1(M))|$$

as the evaluation of a quadratic form q_M defined on $\bigwedge^{\beta_1(M)} H_2(M; \mathbf{Q})$ at a generator of $\bigwedge^{\beta_1(M)} H_2(M; \mathbf{Z})$. Let us associate q_M to M.

Definition 2.7 The *linking number* of two (oriented) rationally null-homologous disjoint links in an (oriented, compact) 3-manifold M is the algebraic intersection number of a rational 2-chain bounded by one of them and the other. We denote it by $lk_M(,)$.

Definition 2.8 Let M be a closed 3-manifold of positive rank. q_M is defined on $\bigwedge^{\beta_1(M)} H_2(M;\mathbf{Q})$ as follows.

If $\beta_1(M)=1$, let $\tilde{\Delta}(M)$ be the Alexander polynomial of M (the order of the first homology $\mathbf{Q}[t,t^{-1}]$-module of M with local coefficients in

$$\mathbf{Q}[H_1(M;\mathbf{Z})/\text{Torsion}] \cong \mathbf{Q}[t,t^{-1}]$$

normalized so that $\tilde{\Delta}(M)$ is symmetric and $\tilde{\Delta}(M)(1)=1$). q_M is defined by its value

$$q_M(g_M) = \frac{\tilde{\Delta}(M)''(1)}{2} - \frac{1}{12}$$

at a generator g_M of $H_2(M;\mathbf{Z})$.

If $\beta_1(M)=2$, define $q_M(\sigma_1 \wedge \sigma_2)$ so that if σ_1 and σ_2 are two elements of $H_2(M;\mathbf{Z}) \subset H_2(M;\mathbf{Q})$ represented by two surfaces S_1 and S_2 embedded in general position in M, then

$$q_M(\sigma_1 \wedge \sigma_2) = -lk(S_1 \cap S_2, S_1^+ \cap S_2)$$

where $S_1 \cap S_2$ denotes the (oriented) intersection of S_1 and S_2 and S_1^+ is a parallel copy of S_1. (It is left to the reader to verify that this provides a consistent definition.)

If $\beta_1(M)=3$, and if $D(=[M]\cap .) : H^1(M;\mathbf{Q}) \longrightarrow H_2(M;\mathbf{Q})$ denotes the Poincaré duality isomorphism, then

$$q_M(\sigma_1 \wedge \sigma_2 \wedge \sigma_3) = \Big((D^{-1}(\sigma_1) \cup D^{-1}(\sigma_2) \cup D^{-1}(\sigma_3))[M]\Big)^2$$

If $\beta_1(M) \geq 4$, then

$$q_M = 0$$

Proposition 2.9 *(see [L, Section 1.5]) Let M be a closed 3-manifold of positive rank, and let g_M denote a generator of $\bigwedge^{\beta_1(M)} H_2(M;\mathbf{Z})$, then*

$$\overline{\lambda}(M) = |\text{Torsion}(H_1(M))| q_M(g_M)$$

Let A be a $(\Sigma, \mathcal{L}_A)$-manifold and let B be a $(-\Sigma, \mathcal{L}_B)$-manifold. The Mayer-Vietoris sequence allows us to split $H_2(A \cup B;\mathbf{Z})$ as

$$H_2(A \cup B) = H_2(A) \oplus H_2(B) \oplus \mathcal{L}_A^{int} \cap \mathcal{L}_B^{int} \tag{2.10}$$

(this is not canonical).

Proposition 2.11 *$\mathcal{F}(A,a,B,b)$ is true when neither A nor B is a RHH.*

PROOF: Proposition 2.9 and Equation 2.10 make clear that $\overline{\lambda}(A\cup B)$ must be zero in this case. □

For symmetry reasons, we may assume that B is a RHH from now on. So do we.

Notation 2.12 Recall that B is a RHH and that A is a 3-manifold such that ∂A and $-\partial B$ are identified with the connected surface Σ. We denote by d the dimension of $\mathcal{L}_A \cap \mathcal{L}_B$, we set $\beta \stackrel{\text{def}}{=} \beta_1(A\cup B)$ and we assume $\beta \geq 1$. We choose a $(\Sigma, \mathcal{L}_A)$-system a and a $(\Sigma, \mathcal{L}_B)$-system b such that $a_i = b_i$ for $i = 1, \ldots, d$. We also choose a $(-\Sigma)$-system y geometrically dual to b and a Σ-system z geometrically dual to a such that $y_i = -z_i$ for $i = 1, \ldots, d$. We use the following notation:

$$< \hat{a}, \hat{b} >_{>d} = det\left([< a_i, b_j >_\Sigma]_{i,j=d+1,\ldots,g}\right)$$

Lemma 2.13 *Under the hypotheses of (2.12),*

$$|\text{Torsion}(H_1(A\cup B))| =$$

$$|\text{Torsion}(H_1(A_z))|\,|H_1(B_y)|\,|< \hat{a}, \hat{b} >_{>d}|\,\left|\frac{\mathcal{L}_A \cap \mathcal{L}_B \cap H_1(\partial A; \mathbf{Z})}{\mathcal{L}_A^{int} \cap \mathcal{L}_B^{int}}\right|^{-2}$$

PROOF: First note that the possible free part in $H_1(A, \partial A)$ may be isolated in this homology computation. So, we assume without loss, that A and B are RHH. Unglue $A\cup B$ along the regular neighborhood in Σ of the wedge of the z_i, $i = 1, \ldots, d$, joined by paths to a basepoint. Then fill in the obtained manifold with a handlebody with meridians the z_i , $i = 1, \ldots, d$, to obtain a closed 3-manifold M. As in the proof of Lemma 2.1,

$$\frac{|H_1(M)|}{|H_1(A_z)|\,|H_1(B_y)|} = \left|\frac{\wedge_{i=1}^g (z_i^A - z_i^B) \wedge \wedge_{i=1}^d z_i^A \wedge \wedge_{i=d+1}^g (a_i^A - a_i^B)}{\wedge_{i=1}^g z_i^A \wedge \wedge_{i=1}^g y_i^B}\right| = |< \hat{a}, \hat{b} >_{>d}|$$

Now, according to Lemma 2.4,

$$\frac{|H_1(M)|}{|\text{Torsion}(H_1(A\cup B))|} = \left|\frac{\mathcal{L}_A \cap \mathcal{L}_B \cap H_1(\partial A; \mathbf{Z})}{\mathcal{L}_A^{int} \cap \mathcal{L}_B^{int}}\right|^2$$

□

Lemma 2.14 *Under the hypotheses of (2.12), for $i = 1, \ldots, d$, let A_i denote the homology class of a rational 2-cycle of $A\cup B$ whose intersection with A has boundary a_i and let $(\sigma_{d+1}, \ldots, \sigma_\beta)$ denote a basis of $H_2(A; \mathbf{Z})$. Then*

$$\overline{\lambda}(A\cup B) =$$

$$|\text{Torsion}(H_1(A_z))|\,|H_1(B_y)|\,|< \hat{a}, \hat{b} >_{>d}|\, q_{A\cup B}\left(A_1 \wedge \cdots \wedge A_d \wedge \sigma_{d+1} \wedge \cdots \wedge \sigma_\beta\right)$$

PROOF:
$$\frac{\overline{\lambda}(A \cup B)}{|\mathrm{Torsion}(H_1(A \cup B))|}$$
$$= q_{A\cup B}\left(\left|\frac{\mathcal{L}_A \cap \mathcal{L}_B \cap H_1(\partial A;\mathbf{Z})}{\mathcal{L}_A^{int} \cap \mathcal{L}_B^{int}}\right| A_1 \wedge \cdots \wedge A_d \wedge \sigma_{d+1} \wedge \cdots \wedge \sigma_\beta\right)$$
□

Now, we assume $\beta > 1$. Let us get rid of the case where A is not a RHH.

Proposition 2.15 *$\mathcal{F}(A, a, B, b)$ is true when A (or B) is not a RHH and $A \cup B$ has rank larger than one.*

PROOF: In this case, the formula to be shown is

$$\overline{\lambda}(A \cup B) = |H_1(B_y)|\overline{\lambda}(A_b)$$

that is

$$q_{A\cup B}\left(A_1 \wedge \cdots \wedge A_d \wedge \sigma_{d+1} \wedge \cdots \wedge \sigma_\beta\right) = q_{A_b}\left(A_1 \wedge \cdots \wedge A_d \wedge \sigma_{d+1} \wedge \cdots \wedge \sigma_\beta\right)$$

Since A is not a RHH all the involved intersections take place inside A. This concludes the case where β is larger than 2. Thus, the only thing to see is that, when $\beta = 2$, the involved linking number does not depend on the $(-\Sigma, \mathcal{L}_B)$-RHH B. Since this linking number can be viewed as the intersection number of one of the links ($\subset A$) and a rational 2-chain of A cobounded by the other link and an element of $\mathcal{L}_B$, it is indeed independent of B. □

Proposition 2.16 *$\mathcal{F}(A, a, B, b)$ is true when A and B are RHH and $A \cup B$ has rank larger than one.*

PROOF: From now on A and B are RHH (in addition to the previous hypotheses). So, $d = \beta$. The case $d \geq 4$ is clear because all of the terms in the formula are zero, and we are left with the cases $d = 2$ and $d = 3$.

If $d = 3$,

$$\begin{aligned}\frac{\overline{\lambda}(A \cup B)}{|H_1(A_z)|\,|H_1(B_y)|\,|<\hat{a},\hat{b}>_{>3}|} &= q_{A\cup B}(A_1 \wedge A_2 \wedge A_3) \\ &= (\mathcal{I}_{A_a} - \mathcal{I}_{B_b})(a_1 \wedge a_2 \wedge a_3)^2\end{aligned}$$

and the conclusion is easy.

Here, we open a parenthesis to note that we have just proved the following lemma which will be used later:

Lemma 2.17 *Let A and B be two genus 3 RHH such that ∂A and $-\partial B$ are identified and such that $\mathcal{L}_A = \mathcal{L}_B$ inside $H_1(\partial A;\mathbf{Q})$. Let a be a $(\partial A, \mathcal{L}_A)$-system and let z be a ∂A-system geometrically dual to a. Then*

$$\overline{\lambda}(A \cup_{\partial A} B) = |H_1(A_z)|\,|H_1(B_z)|(\mathcal{I}_{A_a} - \mathcal{I}_{B_a})(a_1 \wedge a_2 \wedge a_3)^2$$

End of parenthesis.

If $d = 2$, choose a nonzero integer n such that na_i (represented as n parallel copies of a_i) bounds properly embedded surfaces S_i^A and S_i^B in A and B, respectively, for i = 1, 2. For $C = A$ or B, define J^C as the (oriented) intersection $S_1^C \cap S_2^C$ and K^C as its parallel $(S_1^C)^+ \cap S_2^C$. Then

$$\frac{\overline{\lambda}(A \cup B)}{|H_1(A_z)|\,|H_1(B_y)|\,|<\hat{a},\hat{b}>_{>2}|}$$

$$= \frac{1}{n^4} q_{A\cup B}(S_1^A \cup -S_1^B \wedge S_2^A \cup -S_2^B)$$

$$= \frac{-1}{n^4} lk_{A\cup B}(J^A \cup J^B, K^A \cup K^B)$$

$$= \frac{-1}{n^4}\Big(lk_{A\cup B}(J^A, K^A) + lk_{A\cup B}(J^B, K^B) + 2\, lk_{A\cup B}(J^A, J^B)\Big)$$

Again, note that $lk_{A\cup B}(J^A, K^A) = lk_{A_b}(J^A, K^A)$ and that $J^{-\Sigma_b}$ is (or may be chosen) empty. This proves:

$$\frac{\overline{\lambda}(A \cup B)}{|(H_1(A_z))|\,|H_1(B_y)|\,|<\hat{a},\hat{b}>_{>2}|} + \frac{\overline{\lambda}(\Sigma_{ab})}{|<\hat{a},\hat{b}>_{>2}|}$$

$$- \frac{\overline{\lambda}(A_b)}{|(H_1(A_z))|\,|<\hat{a},\hat{b}>_{>2}|} - \frac{\overline{\lambda}(B_a)}{|H_1(B_y)|\,|<\hat{a},\hat{b}>_{>2}|}$$

$$= \frac{-2}{n^4} lk_{A\cup B}(J^A, J^B)$$

We are now left with the proof of the following lemma:

Lemma 2.18

$$\frac{1}{n^4} lk_{A\cup B}(J^A, J^B) = \frac{1}{<\hat{a},\hat{b}>_{>2}} <\hat{a} \cap \mathcal{I}_{Aa}, \hat{b} \cap \mathcal{I}_{Bb}>_\Sigma$$

Proof of the lemma: First note that

$$\frac{1}{n^2}[J^A] = \sum_{i=3}^{g} \mathcal{I}_{A_a}(a_1 \wedge a_2 \wedge a_i) z_i^A$$

in $H_1(A;\mathbf{Q})$ where

$$z_i^A = \sum_{j=3}^{g} \frac{<\hat{a}, \hat{b}(\frac{z_i}{b_j})>_{>2}}{<\hat{a},\hat{b}>_{>2}} b_j^A$$

Here, for a curve x of $\Sigma \subset A \cup B$, x^A denotes its parallel inside A, and $< \hat{a}, \hat{b}(\frac{z_i}{b_j}) >_{>2}$ denotes the determinant of the matrix of the $< a_k, b_l >_{k,l=3,\dots,g}$ where b_j is replaced by z_i. This determinant is in fact the cofactor of (i,j) in $< \hat{a}, \hat{b} >_{>2}$.

In $H_1(B;\mathbf{Q})$, we have:

$$\frac{1}{n^2}[J^B] = \sum_{j=3}^{g} \mathcal{I}_{B_b}(a_1 \wedge a_2 \wedge b_j) y_j^B$$

Since, for a curve y of Σ, $lk_{A\cup B}(b_j^A, y) =< y, b_j >_\Sigma$, we have:

$$\frac{1}{n^4} lk_{A\cup B}(J^A, J^B) = \sum_{i=3}^{g}\sum_{j=3}^{g} \frac{< \hat{a}, \hat{b}(\frac{z_i}{b_j}) >_{>2}}{< \hat{a}, \hat{b} >_{>2}} \mathcal{I}_{A_a}(a_1 \wedge a_2 \wedge a_i)\mathcal{I}_{B_b}(a_1 \wedge a_2 \wedge b_j)$$

This concludes the proof of the lemma □

and the proof of the proposition. □

We have proved that Formula $\mathcal{F}(A,a,B,b)$ is true as soon as the rank of $A \cup B$ is larger than one. For the other cases, we will use surgeries.

2.3 About surgeries

Let C be a compact 3-manifold (possibly with boundary) equipped with a link $L = (K_i)_{i\in N=\{1,\dots,n\}}$ (in its interior). Assume that each component K_i of L is equipped with a parallel μ_i. Then $\mathbf{L} = (K_i, \mu_i)_{i\in N=\{1,\dots,n\}}$ is said to be an *integral surgery presentation* in C, and the manifold $\chi_C(\mathbf{L})$ obtained by surgery on $\mathbf{L} \subset C$ is defined as:

$$\chi_C(\mathbf{L}) = \overline{C \setminus T(L)} \cup_{\partial T(L)} \coprod_{i=1}^{n} D_{\mu_i} \times S^1$$

where $T(L)$ is a tubular neighborhood of L, D_{μ_i} is a 2-disk and $\partial(D_{\mu_i} \times S^1)$ is glued with $\partial(T(K_i))$ by a homeomorphism which maps $\partial(D_{\mu_i} \times \{1\})$ to μ_i.

If C is a rational homology sphere, the *linking matrix* of $\mathbf{L}$ is the symmetric matrix

$$E(\mathbf{L}) = [lk(K_i, \mu_j)]_{i,j=1,\dots,n}$$

Note the following standard lemma:

Lemma 2.19 *Let $\mathbf{L} = (K_i, \mu_i)_{i\in N=\{1,\dots,n\}}$ be an integral surgery presentation in a rational homology sphere R, then*

$$\frac{|H_1(\chi_R(\mathbf{L}))|}{|H_1(R)|} = |det(E(\mathbf{L}))|$$

PROOF: $H_1(R\setminus L)$ has rank n and

$$H_1(\chi_R(\mathbf{L})) = \frac{H_1(R\setminus L)}{\bigoplus_{i=1}^{n} \mathbf{Z}\mu_i}$$

Therefore, if m_i denotes the meridian of K_i,

$$\frac{|H_1(\chi_R(\mathbf{L}))|}{|H_1(R)|} = \left|\frac{\mu_1 \wedge \ldots \wedge \mu_n}{m_1 \wedge \ldots \wedge m_n}\right| = |det(E(\mathbf{L}))|$$

where the exterior products are to be taken in

$$\bigwedge^n (H_1(R\backslash L) \otimes_{\mathbf{Z}} \mathbf{Q}) = \bigwedge^n H_1(R\backslash L; \mathbf{Q})$$

(Here, we use the following definition of the linking number (equivalent to Definition 2.7): The *linking number* $lk(K_i, \mu_j)$ of K_i and μ_j is defined by the condition that μ_j is rationally homologous to $lk(K_i, \mu_j)m_i$ in $H_1(R\backslash K_i; \mathbf{Q})$ which is generated by the meridian m_i of K_i.) □

2.4 Sketching the proof of the theorem for rational homology spheres

We sketch the proof of Statement 2.2 which has been shown to be equivalent to Theorem 1.3 that is the theorem for rational homology spheres.

Hypotheses 2.20 Let Σ be a surface of genus g. Let $\mathcal{L}_A$ and $\mathcal{L}_B$ be two transverse Lagrangians of $(H_1(\Sigma; \mathbf{Q}), <,>)$. Choose a $(\Sigma, \mathcal{L}_A)$-system a and a Σ-system $z = (z_1, \ldots, z_g)$ geometrically dual to a.

Choose a $(\partial A, \mathcal{L}_B)$-system $b = (b_1, ..., b_g)$ such that:

$$i > j \implies < a_i, b_j > = 0$$

Let A be a $(\Sigma, \mathcal{L}_A)$-RHH.

Consider the oriented link $(K_i, Z_i) = (\{-1\} \times a_i, \{-2\} \times z_i)$ in a regular neighborhood $[-3, 0] \times \partial A$ of ∂A in A.

Equip K_i with a parallel μ_{K_i} lying on $\{-1\} \times \partial A$ and Z_i with a parallel μ_{Z_i} lying on $\{-2\} \times \partial A$ to obtain the framed link $\mathbf{L}_i = ((K_i, \mu_{K_i}); (Z_i, \mu_{Z_i}))$ in A. Define $A^{(i)}$ as the RHH obtained from $A^{(i-1)}$ by surgery on $\mathbf{L}_i$, inductively, starting with $A^{(0)} = A$.

Note that the surgeries on the $\mathbf{L}_i$ do not affect ∂A. Note also the easy lemma:

Lemma 2.21 *All the $A^{(i)}$ are $(\Sigma, \mathcal{L}_A)$-RHH. Furthermore, if B is a $(-\Sigma, \mathcal{L}_B)$-RHH, then $|H_1(A^{(i)} \cup B)| = |H_1(A \cup B)|$.*

Proof: It is clear that $\mathcal{L}_{A^{(i)}} = \mathcal{L}_A$. To prove that $A^{(i)}$ is a RHH, it suffices to know that $A^{(i)} \cup B = A^{(i)} \cup_\Sigma B$ is a rational homology sphere for some $(-\Sigma, \mathcal{L}_B)$-RHH B. We prove this.

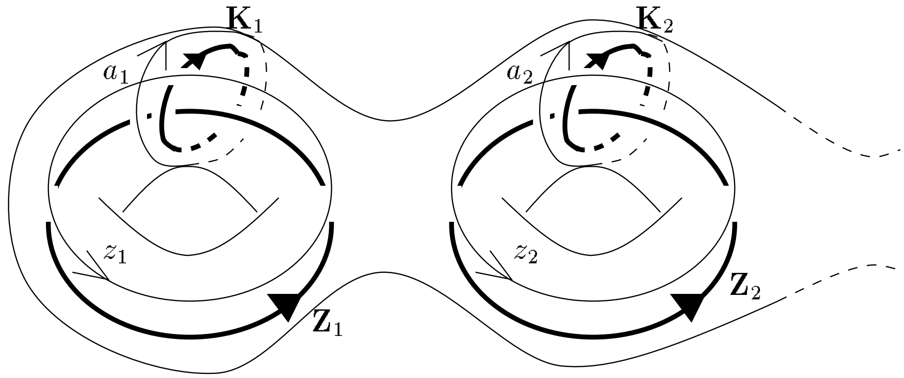

Figure 2: (K_1, Z_1), (K_2, Z_2), ...in A

Since μ_{K_i} is rationally null-homologous in $A^{(i-1)} \setminus K_i$, $lk(K_i, \mu_{K_i}) = 0$ in $A^{(i-1)} \cup B$, and since K_i is rationally homologous to the meridian of Z_i, $lk(K_i, Z_i) = 1$. Thus, the linking matrix of $\mathbf{L}_i$ has the form

$$E(\mathbf{L}_i) = \begin{pmatrix} 0 & 1 \\ 1 & lk(Z_i, \mu_{Z_i}) \end{pmatrix}$$

(in $A^{(i-1)} \cup B$) and because of Lemma 2.19,

$$|H_1(A^{(i)} \cup B)| = |H_1(A^{(i-1)} \cup B)| = |H_1(A \cup B)|$$

So, the $A^{(i)}$ satisfy the same hypotheses as A does. □

Assume the following lemma which will be proved later.

Lemma 2.22 *For any* $(\Sigma, \mathcal{L}_A)$*-RHH* A *and for any* $(-\Sigma, \mathcal{L}_B)$*-RHH* B,

$$(\lambda(A^{(1)} \cup B) - \lambda(A \cup B)) - (\lambda(A_b^{(1)}) - \lambda(A_b))$$

$$= \sum_{(j,k) \in \{1,\dots,g\}^2} \mathcal{I}_{A_a}(a_1 \wedge a_j \wedge a_k) \mathcal{I}_{B_b}(\beta_1 \wedge \beta_j \wedge \beta_k)$$

where a_j *also denotes its own homology class, and the* β_j *are the elements of* $\mathcal{L}_B$ *defined by:* $< a_j, \beta_k >_\Sigma = \delta_{jk}$ *for any* $j, k \in \{1, \dots, g\}$.

Applying Lemma 2.22 to $(A^{(i-1)}, a_i, \mathbf{L}_i)$ (instead of $(A, a_1, \mathbf{L}_1)$) yields:

$$\begin{aligned} &(\lambda(A^{(i)} \cup B) - \lambda(A^{(i-1)} \cup B)) - (\lambda(A_b^{(i)}) - \lambda(A_b^{(i-1)})) \\ &= \sum_{(j,k) \in \{1,\dots,g\}^2} \mathcal{I}_{A_a^{(i-1)}}(a_i \wedge a_j \wedge a_k) \mathcal{I}_{B_b}(\beta_i \wedge \beta_j \wedge \beta_k) \end{aligned} \quad (2.23)$$

where $\mathcal{I}_{A_a^{(i-1)}}(a_i \wedge a_j \wedge a_k)$ may be seen as the intersection number in $A^{(i-1)}$ of three rational 2-chains with respective boundaries a_i, a_j and a_k.

- On the one hand, the surgery from A to $A^{(i-1)}$ has been performed on knots parallel to ∂A, which, when projected on ∂A, do not intersect the a_j for $j \geq i$. Therefore:

  If $j \geq i$ and $k \geq i$, then:

$$\mathcal{I}_{A_a^{(i-1)}}(a_i \wedge a_j \wedge a_k) = \mathcal{I}_{A_a}(a_i \wedge a_j \wedge a_k)$$

- On the other hand, if $j < i$, a_j bounds a 2-disk in $A^{(i-1)}$ (a meridian of the solid torus which replaces the tubular neighborhood of K_j in the performed surgery). Therefore:

  If $j < i$ or if $k < i$, then:

$$\mathcal{I}_{A_a^{(i-1)}}(a_i \wedge a_j \wedge a_k) = 0$$

Thus, Equation 2.23 is equivalent to

$$\begin{aligned}&(\lambda(A^{(i)} \cup B) - \lambda(A^{(i-1)} \cup B)) - (\lambda(A_b^{(i)}) - \lambda(A_b^{(i-1)}))\\ &= \sum_{(j,k)\in\{i,i+1,\ldots,g\}^2} \mathcal{I}_{A_a}(a_i \wedge a_j \wedge a_k)\mathcal{I}_{B_b}(\beta_i \wedge \beta_j \wedge \beta_k)\end{aligned}$$

The sum of these equations for $i = 1, \ldots, g$ is equivalent to:

$$\begin{aligned}&(\lambda(A^{(g)} \cup B) - \lambda(A \cup B)) - (\lambda(A_b^{(g)}) - \lambda(A_b))\\ &= 2 \sum_{\{i,j,k\}\subset\{1,2,\ldots,g\}} \mathcal{I}_{A_a}(a_i \wedge a_j \wedge a_k)\mathcal{I}_{B_b}(\beta_i \wedge \beta_j \wedge \beta_k)\end{aligned} \tag{2.24}$$

Now, note that $A^{(g)}$ is the connected sum of Σ_a and a rational homology sphere M. (This connected sum is performed in the interior of Σ_a.) Thus, $A^{(g)} \cup B = B_a \sharp M$, $A_b^{(g)} = \Sigma_{ab} \sharp M$, and the additivity of λ under connected sum allows us to conclude the proof of Statement 2.2, and thus the proof of Theorem 1.3, assuming Lemma 2.22.

2.5 Finishing the reduction of the case of rational homology spheres to 'the first main lemma'

In this subsection, we reduce the proof of Lemma 2.22 to 'the first main lemma' (Lemma 2.28). Let us first partially state the surgery formula of [L, T2, Section 1.5].

Theorem 2.25 *For any positive integer n, there exists a function $\mathcal{F}_n$ of symmetric matrices of order n with rational coefficients and with nonzero determinant such that: For any integral n-component surgery presentation $\mathbf{L} = (K_i, \mu_i)_{i \in N=\{1,\dots,n\}}$ in a rational homology sphere R, if $\chi_R(\mathbf{L})$ is a rational homology sphere, then*

$$\lambda(\chi_R(\mathbf{L})) - \lambda(R) = \sum_{\{I, I \subset N, I \neq \emptyset\}} \frac{det(E(\mathbf{L}_{N \setminus I}))}{det(E(\mathbf{L}))} \frac{\zeta(L_I)}{|H_1(R)|} + \mathcal{F}_n(E(\mathbf{L}))$$

where ζ denotes the derivative of the several-variable Alexander polynomial described in [L, Definition 1.4.1] or in Section 3, Definition 3.19 below, and, for a subset I of N, $\mathbf{L}_I$ denotes the surgery presentation $\mathbf{L}_I = (K_i, \mu_i)_{i \in I}$.

($\mathcal{F}_n$ is explicitly described in [L].)

Let us now apply Theorem 2.25 to compute $(\lambda(A^{(1)} \cup B) - \lambda(A \cup B))$ and see how it depends on B, when B varies among the $(-\Sigma, \mathcal{L}_B)$-RHH.

We have the following lemmas:

Lemma 2.26 *If J and K are two knots in A, their linking number does not depend on B.*

Proof: Indeed, $lk(J, K)$ is the coordinate of the homology class of K in

$$H_1((A \cup B) \setminus J; \mathbf{Q}) = \mathbf{Q} m_J = \frac{H_1(A \setminus J; \mathbf{Q})}{i_*(\mathcal{L}_B)}$$

with respect of the meridian m_J of J. □

Lemma 2.27 *If the components of a link L in A are rationally null-homologous in A, then $\frac{\zeta(L)}{|H_1(A \cup B)|}$ does not depend on B.*

Proof: It is a straightforward consequence of Lemma 3.17 (with Definition 3.19 of ζ) which will be proved in Section 3. (It could also be proved without the Alexander function.) □

Recall from the proof of Lemma 2.21 that

$$E(\mathbf{L}_1) = \begin{pmatrix} 0 & 1 \\ 1 & lk(Z_1, \mu_{Z_1}) \end{pmatrix}$$

(where $lk(Z_1, \mu_{Z_1})$ does not depend on B by Lemma 2.26).

Putting all together, we obtain:

$$(\lambda(A^{(1)} \cup B) - \lambda(A \cup B)) - (\lambda(A_b^{(1)}) - \lambda(A_b))$$
$$= -\left(\frac{\zeta((K_1, Z_1) \subset A \cup B)}{|H_1(A \cup B)|} - \frac{\zeta((K_1, Z_1) \subset A_b)}{|H_1(A_b)|} \right)$$

The conclusion of the proof of Lemma 2.22 is now given by the following lemma, which is the first main lemma of the paper:

Lemma 2.28 *Under the hypotheses 2.20, for any* $(-\Sigma, \mathcal{L}_B)$*-RHH* B*,*

$$\frac{\zeta((K_1, Z_1) \subset A \cup B)}{|H_1(A \cup B)|} - \frac{\zeta((K_1, Z_1) \subset A_b)}{|H_1(A_b)|}$$

$$= - \sum_{(j,k) \in \{1,\ldots,g\}^2} \mathcal{I}_{A_a}(a_1 \wedge a_j \wedge a_k) \mathcal{I}_{B_b}(\beta_1 \wedge \beta_j \wedge \beta_k)$$

where a_j *also denotes its own homology class,* $\beta_k \in \mathcal{L}_B$*, and* $< a_j, \beta_k >= \delta_{jk}$ *for any* $j, k \in \{1, \ldots, g\}$

To prove this lemma, we will associate to each RHH B a function $\mathcal{A}_B$ such that: If B is embedded in a link exterior E, the Alexander series of the link only depends on $\overline{E \setminus B}$ and $\mathcal{A}_B$. The function $\mathcal{A}$ will be defined in Section 3 as a tautological generalization of the Alexander polynomial to RHH. It will be called the Alexander function. The properties that we will derive for the Alexander function in Section 3 will allow us to see how $\zeta(K_1, Z_1)$ depends on B, when B varies among $(-\Sigma, \mathcal{L}_B)$-RHH, and to prove our first main lemma (2.28) in Section 4.

2.6 Reducing the proof of $\mathcal{F}(A, a, B, b)$ when the rank of $A \cup B$ is one to 'the second main lemma'

This subsection finishes the reduction of the proof of Theorem 1.11 to the two main lemmas whose proofs rely on similar arguments. This reduction is very similar, too. Thus, the reader is advised to avoid reading this subsection, written for the sake of completeness.

Proposition 2.29 $\mathcal{F}(A, a, B, b)$ *is true when* $\mathcal{L}_A$ *and* $\mathcal{L}_B$ *are transverse and when the rank of* $A \cup B$ *equals one.*

PROOF: For symmetry reasons, we may assume that B is a RHH, and that A is not ($H_2(A) \cong \mathbf{Z}$). The equality to be shown is:

$$\overline{\lambda}(A \cup B) = |H_1(B_y)| \overline{\lambda}(A_b)$$

In this case, A is obtained by surgery on a null-homologous knot K in a RHH A_0, with respect to its preferred parallel. (To get A_0 and K, choose a knot C of A intersecting a closed embedded surface generating $H_2(A)$ once transversally, equip C with one of its parallels, say μ, perform surgery on (C, μ) to get A_0 from A, and call K the core of the surgery torus.) Applying the surgery formula of [L, T2, Section 1.5] yields:

$$\overline{\lambda}(A \cup B) = \zeta(K \subset A_0 \cup B) - \frac{|H_1(A_0 \cup B)|}{24}$$

Lemma 2.27 implies that:

$$\frac{\overline{\lambda}(A \cup B)}{|H_1(A_0 \cup B)|} = \frac{\overline{\lambda}(A_b)}{|H_1(A_{0b})|}$$

And since, according to Lemma 2.1,

$$|H_1(A_0 \cup B)| = |H_1(B_y)|\,|H_1(A_{0b})|$$

we are done. □

Reducing the proof of $\mathcal{F}(A, a, B, b)$ when A and B are RHH and when the rank of $A \cup B$ equals one to the 'second main lemma'

Hypotheses 2.30 *Again, we fix $(\Sigma, \mathcal{L}_A, \mathcal{L}_B)$. Here, we assume that $\mathcal{L}_A \cap \mathcal{L}_B$ has dimension one. We also fix a $(\Sigma, \mathcal{L}_A)$-system a, a $(\Sigma, \mathcal{L}_B)$-system b, a Σ-system z geometrically dual to a, and a $(-\Sigma)$-system y geometrically dual to b such that $a_1 = b_1$ and $y_1 = -z_1$. (As it has been noticed in Subsection 2.1, we do not lose generality with this choice.)*

For a $(\Sigma, \mathcal{L}_A)$-RHH A, we consider the surgery presentation $\mathbf{Z}$ in A made of the knot z_1 equipped by with one of its parallels on Σ pushed together inside A. (The underlying knot Z of $\mathbf{Z}$ is $\{-2\} \times z_1$ as in Figure 2.) We denote by A^C the manifold $\chi_A(\mathbf{Z})$ and we denote by K the core of the surgery torus.

Applying the surgery formula of [L, T2, Section 1.5] yields:

$$\frac{\overline{\lambda}(A \cup B)}{|H_1(A_z)|\,|H_1(B_y)|} = \frac{\zeta(K \subset A^C \cup B)}{|H_1(A_z)|\,|H_1(B_y)|} - \frac{|H_1(A^C \cup B)|}{24|H_1(A_z)|\,|H_1(B_y)|}$$

for any $(-\Sigma, \mathcal{L}_B)$-RHH B.

Since, according to Lemma 2.1, $|H_1(A^C \cup B)| \,/\, 24|H_1(A_z)|\,|H_1(B_y)|$ does not depend on the $(-\Sigma, \mathcal{L}_B)$-RHH B, $\mathcal{F}(A, a, B, b)$ will follow from the equality:

$$\frac{\zeta(K \subset A^C \cup B)}{|H_1(A_z)|\,|H_1(B_y)|} - \frac{\zeta(K \subset (A^C)_b)}{|H_1(A_z)|} - \frac{\zeta(K \subset \Sigma_a^C \cup B)}{|H_1(B_y)|} + \zeta(K \subset (\Sigma_a^C)_b)$$

$$= -2\, sign_\Sigma(\hat{a}, \hat{b}) \; < \hat{a} \cap \mathcal{I}_{Aa}, \hat{b} \cap \mathcal{I}_{Bb} >_\Sigma \tag{2.31}$$

For $k = 2, \ldots, g$, let $\beta_k \in \mathcal{L}_B$ be defined (up to a multiple of a_1) so that $< a_j, \beta_k > = \delta_{jk}$ for any j. Then

$$\hat{a} \otimes \hat{b} =< \hat{a}, \hat{b} >_{>1} (\hat{a} \otimes b_1 \wedge \beta_2 \wedge \cdots \wedge \beta_g)$$

where

$$< \hat{a}, \hat{b} >_{>1} \stackrel{\text{def}}{=} det[< a_i, b_j >_{i,j=2,\ldots,g}]$$

Referring to Notation 1.9, we may rewrite the right-hand side of Equation 2.31 as

$$-| < \hat{a}, \hat{b} >_{>1} | \sum_{(j,k)\in\{1,\ldots,g\}^2} \mathcal{I}_{A_a}(a_1 \wedge a_j \wedge a_k)\mathcal{I}_{B_b}(a_1 \wedge \beta_j \wedge \beta_k)$$

We have reduced the proof of Theorem 1.11 (to the proof of Theorem 1.3 and) to the proof of our second main lemma:

Lemma 2.32 *Under the hypotheses (2.30), for any* $(\Sigma, \mathcal{L}_A)$*-RHH* A *and for any* $(-\Sigma, \mathcal{L}_B)$*-RHH* B*,*

$$\frac{\zeta(K \subset A^C \cup B)}{|H_1(A_z)|\,|H_1(B_y)|\,| < \hat{a}, \hat{b} >_{>1} |} =$$

$$\frac{\zeta(K \subset (A^C)_b)}{|H_1(A_z)|\,| < \hat{a}, \hat{b} >_{>1} |} + \frac{\zeta(K \subset \Sigma_a^C \cup B)}{|H_1(B_y)|\,| < \hat{a}, \hat{b} >_{>1} |} - \frac{\zeta(K \subset (\Sigma_a^C)_b)}{| < \hat{a}, \hat{b} >_{>1} |}$$

$$- \sum_{(j,k)\in\{2,\ldots,g\}^2} \mathcal{I}_{A_a}(a_1 \wedge a_j \wedge a_k)\mathcal{I}_{B_b}(a_1 \wedge \beta_j \wedge \beta_k)$$

where $\beta_k \in \mathcal{L}_B$ *satisfies* $< a_j, \beta_k >= \delta_{jk}$ *for any* j*, for any* $k > 1$*.*

Thus, we are left with the proofs of Lemma 2.27 (easy) and with the proof of the main lemmas (2.28 and 2.32) to finish proving the results announced in the introduction.

3 The Alexander function

Definition 3.1 Define the *genus* $g(A)$ of a connected compact oriented 3-manifold A with boundary as:

$$g(A) = 1 - \chi(A) \ (= 1 - \frac{1}{2}\chi(\partial A))$$

In this section, we present a topological invariant of connected compact oriented 3-manifolds with boundary and with non-negative genus. We call this invariant *the Alexander function* owing to the fact that it is a straightforward generalization of the Alexander polynomial or equivalently the Reidemeister torsion. The main goal of this presentation is to provide a nice environment in which to investigate the properties of the normalized Alexander polynomials, and, in particular, to prove our main lemmas (2.28 and 2.32).

We define the Alexander function in Subsection 3.1. Next, we discuss its basic properties. The statements of these are often longer than their proofs, and the reader is advised to read them briefly. The first interesting property (10) of the Alexander function is given in the last subsection of this section. This property is crucial for our concerns. Its proof uses most of the previous properties.

Throughout this section, A denotes a connected compact oriented 3-manifold with non-empty boundary and with non-negative genus $g = g(A)$.

Notation 3.2 We denote by Λ_A the group ring:

$$\Lambda_A = \mathbf{Z}\left[\frac{H_1(A;\mathbf{Z})}{\text{Torsion}(H_1(A;\mathbf{Z}))}\right]$$

Recall that

$$\Lambda_A = \bigoplus_{x \in \frac{H_1(A)}{\text{Torsion}}} \mathbf{Z}\, exp(x)$$

as a $\mathbf{Z}$-module and that its $\mathbf{Z}$-bilinear multiplication law is given by:

If $x, y \in \frac{H_1(A)}{\text{Torsion}}$,

$$exp(x)exp(y) = exp(x+y)$$

We use the notation $exp(x)$ to denote x when viewed in Λ_A to remind this multiplication law. Note that the units of Λ_A are its elements of the form $exp(x \in \frac{H_1(A)}{\text{Torsion}})$ called the *positive* units, and its elements of the form $-exp(x \in \frac{H_1(A)}{\text{Torsion}})$ called the *negative* units. We use the symbol $\doteq$ to mean 'equals up to a multiplication by a positive unit of Λ_A'.

The maximal free abelian covering of A is denoted by $\tilde{A}$ and the covering map from $\tilde{A}$ to A by p_A. We fix a basepoint $\star$ in A. $H_1(\tilde{A}, p_A^{-1}(\star); \mathbf{Z})$ is a Λ_A-module. We denote it by $\mathcal{H}_A$.

3.1 Definition

The Alexander function $\mathcal{A}_A$ of A is a Λ_A-linear function

$$\mathcal{A}_A : \bigwedge^g \mathcal{H}_A \longrightarrow \Lambda_A$$

which is defined up to a multiplication by a unit of Λ_A. It is defined as follows: Take a presentation of $\mathcal{H}_A$ over Λ_A with $(r+g)$ generators $\gamma_1, \ldots, \gamma_{r+g}$ and r relators $\rho_1, \ldots, \rho_r$ (which are Λ_A-linear combinations of the γ_i). Let $\hat{u} = u_1 \wedge \ldots \wedge u_g$ be an element of $\bigwedge^g \mathcal{H}_A$.

Then $\mathcal{A}_A(\hat{u})$ is defined by the following equality:

$$\hat{\rho} \wedge \hat{u} = \mathcal{A}_A(\hat{u})\hat{\gamma} \tag{3.3}$$

where $\hat{\rho} = \rho_1 \wedge \ldots \wedge \rho_r$, $\hat{\gamma} = \gamma_1 \wedge \ldots \wedge \gamma_{r+g}$, the u_i are represented as combinations of the γ_j, and the exterior products in Equation 3.3 are to be taken in $\bigwedge^{r+g}\left(\bigoplus_{i=1}^{r+g} \Lambda_A \gamma_i\right)$.

Proof of well-definedness

That, for a fixed presentation of deficiency g of $\mathcal{H}_A$, $\hat{\rho} \wedge \hat{u}$ is a well-defined multiple of $\hat{\gamma}$ comes from the fact that two Λ_A-linear combinations of the γ_j representing u_i differ by a combination of relators.

That the Alexander functions associated to two different presentations of $\mathcal{H}_A$ differ by a multiplication by a unit of Λ_A comes from the following facts:

- If the rank of $\mathcal{H}_A$ is larger than g, the Alexander function associated to any presentation with deficiency g of $\mathcal{H}_A$ is zero (the rank being the dimension over the field of fractions $Q(\Lambda_A)$ of Λ_A of $\mathcal{H}_A \otimes_{\Lambda_A} Q(\Lambda_A)$).

- Otherwise, two presentations with deficiency g of $\mathcal{H}_A$ are obtained one from the other by a finite number of operations of the following types. (See [L, Lemma A.2.21].)

  Renumbering the generators.

  Changing the basis of the relators. (That is performing a Λ_A-isomorphism on the system of relators.)

  Stabilizing the presentation by adding a generator, say γ_0, together with a relator which is the sum of γ_0 and a combination of the previous generators.

  Unstabilizing that is, doing the inverse operation (when possible).

  And each of these operations multiplies the Alexander function by a unit of Λ_A.

Last but not least, let us add that $\mathcal{H}_A$ has a presentation of deficiency g over Λ_A. Indeed, (a strong deformation retract of) A has a cellular decomposition with the basepoint as its only 0-cell, $(g+r)$ 1-cells and r 2-cells (such a decomposition may be obtained by Morse theory, it will be called a *good* cellular decomposition of A). Then the associated cellular Λ_A-equivariant complex provides the required presentation. □

Remark 3.4 *About the basepoint.* So far, $\mathcal{A}$ is a topological invariant of the pairs $(A,\star)$ up to homotopy equivalence of pairs (connected compact oriented 3-manifold with boundary and with non-negative genus, basepoint). (Homotopy equivalences provide identifications between the involved Λ's and $\mathcal{H}$'s.) However, changing the location of the basepoint $\star$ of A does not affect the homotopy equivalence class of $(A,\star)$. Thus, $\mathcal{A}$ may be considered as an invariant of A. But we need a reference basepoint that we may choose anywhere we want in A.

From now on, we fix a preferred lift $\star_0$ of $\star$ in $\tilde{A}$.

Notation 3.5 The choice of $\star_0$ provides a natural isomorphism between $H_0(p_A^{-1}(\star)) = \Lambda_A[\star_0]$ and Λ_A. We denote by ∂ the boundary map from $\mathcal{H}_A$ to $H_0(p_A^{-1}(\star))$ composed with this isomorphism.

3.2 First properties

The first property of $\mathcal{A}$ is that it is a straightforward generalization of the Reidemeister torsion. Namely, we have:

Property 1 (Relation to the Reidemeister torsion) *If A is a link exterior, then for any element u of $\mathcal{H}_A$,*

$$\mathcal{A}_A(u) \doteq \pm\partial(u)\tau(A)$$

PROOF: Note that $g(A) = 1$ in this case. Compare with the definition of [T]. □

It is worth noting that Property 2 below shows that the Reidemeister torsion is well-defined by Property 1 (in the field of fractions of Λ_A, up to units of Λ_A).

Notation 3.6 Let $v = (v_1, \ldots, v_s) \in \mathcal{H}_A^s$, let $u \in \mathcal{H}_A$, then $\hat{v}$ denotes $v_1 \wedge \ldots \wedge v_s$, and $\hat{v}(\frac{u}{v_i})$ denotes $\hat{v}$ where the argument v_i has been replaced by u, that is:

$$\hat{v}(\frac{u}{v_i}) = v_1 \wedge \ldots \wedge v_{i-1} \wedge u \wedge v_{i+1} \wedge \ldots \wedge v_s$$

Property 2 *Fix a normalization of $\mathcal{A}_A$, then for any $v = (v_1, \ldots, v_g) \in \mathcal{H}_A^g$, for any $u \in \mathcal{H}_A$,*

$$\sum_{i=1}^{g} \partial(v_i)\mathcal{A}_A(\hat{v}(\frac{u}{v_i})) = \mathcal{A}_A(\hat{v})\partial(u)$$

PROOF: We extend all the coefficients to the field of fractions $Q(\Lambda_A)$ of Λ_A, assume that the system of relators used to define $\mathcal{A}_A$ is free, (otherwise $\mathcal{A}_A$ is zero) and consider the linear map:

$$f_u : \bigwedge^g \mathcal{H}_A \longrightarrow Q(\Lambda_A)$$

$$v_1 \wedge \ldots \wedge v_g \mapsto \sum_{i=1}^{g} \partial(v_i)\mathcal{A}_A(\hat{v}(\frac{u}{v_i}))$$

Our goal is to compute f_u for a fixed u. We only need to evaluate f_u at a nonzero $\hat{v}$. Thus, we assume that the v_i together with the relators of the defining presentation P of $\mathcal{H}_A$ form a $Q(\Lambda_A)$-basis of the $Q(\Lambda_A)$-vector space freely generated by the generators of P. In this case, $\mathcal{A}_A(\hat{v}(\frac{u}{v_i}))/\mathcal{A}_A(\hat{v})$ is the v_i-coordinate of u with respect of this basis, and $f_u(\hat{v}) = \mathcal{A}_A(\hat{v})\partial(u)$. □

Notation 3.7 We denote by ε the ($\mathbf{Z}$-linear) *augmentation morphism:*

$$\varepsilon : \Lambda_A \longrightarrow \mathbf{Z}$$

$$exp\left(x \in \frac{H_1(A)}{\text{Torsion}}\right) \mapsto 1$$

Property 3 *For any* $\hat{u} = u_1 \wedge \ldots \wedge u_g \in \bigwedge^g \mathcal{H}_A$,

$$\varepsilon(\mathcal{A}_A(\hat{u})) = \pm \left| \frac{H_1(A;\mathbf{Z})}{\bigoplus_{i=1}^{g} \mathbf{Z} p_{A*}(u_i)} \right|$$

PROOF: Assume, for simplicity, that the presentation P used to define $\mathcal{A}_A$ comes from a good cellular decomposition of A (as in Subsection 3.1). This makes clear that mapping its coefficients from Λ_A to $\mathbf{Z}$ via ε transforms P into a presentation of $H_1(A;\mathbf{Z})$. Adding the $p_{A*}(u_i)$ as relators next yields a presentation of $H_1(A;\mathbf{Z})/\bigoplus_{i=1}^{g} \mathbf{Z} p_{A*}(u_i)$. □

Notation 3.8 Since $H_1(A;\mathbf{Z})/\text{Torsion}$ is a subgroup of $H_1(A;\mathbf{Q})$, Λ_A is a subring of

$$\tilde{\Lambda}_A \stackrel{\text{def}}{=} \mathbf{Z}[H_1(A;\mathbf{Q})]$$

If we are given a basis, say $\mathcal{X} = \{x_1, \ldots, x_k\}$, of $H_1(A;\mathbf{Q})$, we have another natural inclusion:

$$\psi_{\mathcal{X}} : \tilde{\Lambda}_A \hookrightarrow \mathbf{Q}[[x_1, \ldots, x_k]]$$

$$exp\left(\sum \lambda_i x_i\right) \mapsto exp\left(\sum \lambda_i x_i\right)$$

Here, $\mathbf{Q}[[x_1, \ldots, x_k]]$ is the ring of formal series in the x_i (seen as variables) over $\mathbf{Q}$, and we expand the exponential as usual. These inclusions associate a series to any element of Λ_A.

The order of the series associated to an element $\mathcal{S}$ of Λ_A by the process above will be called the *order* of $\mathcal{S}$. (The order of a nonzero series is the degree of its term with lowest degree and the order of 0 is $+\infty$.) It does not depend on the chosen basis. (A linear change of variables on its variables can only make it bigger or equal.) It will be denoted by $\mathcal{O}(\mathcal{S})$.

If $\mathcal{P}$ and $\mathcal{Q}$ belong to Λ_A, and if k is an integer, then we use the notation:

$$\mathcal{P} = \mathcal{Q} + O(k)$$

to say that:

$$\mathcal{O}(\mathcal{P} - \mathcal{Q}) \geq k$$

Property 4 *For any* $\hat{u} = u_1 \wedge \ldots \wedge u_g \in \bigwedge^g \mathcal{H}_A$,

$$\mathcal{O}(\mathcal{A}_A(\hat{u})) \geq dim\left(\frac{H_1(A;\mathbf{Q})}{\bigoplus_{i=1}^g \mathbf{Q}p_{A*}(u_i)}\right)$$

PROOF: Let $k = dim\left(\frac{H_1(A;\mathbf{Q})}{\bigoplus_{i=1}^g \mathbf{Q}p_{A*}(u_i)}\right)$. We may assume that the generators $\gamma_1, \ldots, \gamma_{g+r}$ of the presentation P of $\mathcal{H}_A$ used to define $\mathcal{A}_A$ satisfy:

1. $(p_{A*}(\gamma_1), \ldots, p_{A*}(\gamma_k))$ is a $\mathbf{Z}$-basis of

   $$\frac{\frac{H_1(A;\mathbf{Z})}{\bigoplus_{i=1}^g \mathbf{Z}p_{A*}(u_i)}}{\text{Torsion}}$$

   (easily obtained by stabilizations).

2. For $i > k$, $p_{A*}(\gamma_i)$ is a torsion element of

   $$\frac{H_1(A;\mathbf{Z})}{\bigoplus_{i=1}^g \mathbf{Z}p_{A*}(u_i)}$$

   (easily obtained by a Λ_A-isomorphism on the generating system).

As in the proof of Property 3, adding to P the u_i as relators and mapping all the new-relator coefficients to $\mathbf{Z}$ via ε yields a $\mathbf{Z}$-presentation of $H_1(A;\mathbf{Z})/\bigoplus_{i=1}^g \mathbf{Z}p_{A*}(u_i)$. So, if u is a relator of P or a u_i, its γ_j-coordinate is mapped to 0 by ε, for $j \leq k$, and thus has order at least 1. Now, the result follows from the standard properties of the order function. □

Convention 3.9 When c is an oriented curve in A, c will also denote the homology class it carries. If furthermore c is based at $\star$, then depending on the context, c will also denote its own class in $\pi_1(A,\star)$ or the class of the preferred lift of c starting at $\star_0$ in $\tilde{A}$.

The behaviour of $\mathcal{A}$ under gluing (thickened) 2-cells on the boundary (or adding 2-handles) may be described as follows.

Property 5 *Let $\delta = (\delta_1, \ldots, \delta_k)$ be a family of k, $(k < g)$ simple closed oriented curves pairwise disjoint on ∂A. Let A_δ be the manifold obtained by gluing (thickened) 2-disks along the δ_i. Let*

$$\phi_\delta : \Lambda_A \longrightarrow \Lambda_{A_\delta}$$

be the morphism induced by the inclusion of A into A_δ. Assume that the δ_i are equipped with paths connecting them to $\star$. Then

$$\mathcal{A}_{A_\delta} \doteq \pm\phi_\delta(\mathcal{A}_A(\delta_1 \wedge \delta_2 \wedge \ldots \wedge \delta_k \wedge .))$$

Warning: Unfortunately, when the boundary of the manifold A_δ is a sphere, by Notation 1.7, A_δ also denotes the closed manifold obtained from the current one by filling it in with a 3-ball. It should be clear though from the context whether the considered manifold must have a boundary or not. (If we compute its Alexander function, it must, and if we compute its Walker invariant, it must not.)

PROOF: Assume (without loss of generality) that the presentation P of $\mathcal{H}_A$ used in the definition of $\mathcal{A}_A$ is given by a good cellular decomposition of A. Adding the 2-cells with boundaries the δ_i to this decomposition of A yields a cellular decomposition of A_δ and thus a presentation P_δ of $\mathcal{H}_{A_\delta}$ (with the right deficiency). P_δ is obtained from P by letting ϕ_δ act on the coefficients of the relators of P and by adding the δ_i as new relators. □

Remark 3.10 *About the possible spherical components of the boundary.* If ∂A contains a sphere, then $\mathcal{A}_A$ is zero unless ∂A is a sphere. If ∂A is a sphere, $\mathcal{A}_A = \pm|H_1(A; \mathbf{Z})|$.

PROOF: Indeed, if ∂A strictly contains a sphere, this sphere is part of a good cellular decomposition of A and $\mathcal{A}_A$ is zero. Now, assume that ∂A is a sphere. $g(A) = 0$. If $H_1(A; \mathbf{Z})/\text{Torsion}$ is trivial, then the augmentation morphism ε is an isomorphism, and the statement comes from Property 3. Otherwise, filling in A with a 3-ball, and next removing a solid torus -intersecting a closed embedded surface of A exactly along one of its meridian disks- from A gives a knot exterior B. If m denotes the boundary of the meridian disk mentioned above, then B_m is homeomorphic to A. Therefore, $\mathcal{A}_A = \mathcal{A}_B(m)$ by Property 5 above, where $\mathcal{A}_B(m)$ is zero by Property 1. □

Let us now formalize the behaviour of the Alexander function under connected sum along the boundary. Let A and B be two connected compact 3-manifolds with non-negative genus equipped with basepoints on their respective boundaries. Let $D_A \subset \partial A$ be a disk containing the basepoint of A in its interior, and let $D_B \subset \partial B$

be a disk containing the basepoint of B in its interior. These data allow us to form the following connected sum $A\sharp_\partial B$ of A and B:

$$A\sharp_\partial B = A \bigcup_{D_A \sim -D_B} B$$

(gluing D_A and D_B so that the basepoints match together). This connected sum depends on the connected components of the boundaries of A and B where the basepoints were located. Note that

$$\frac{H_1(A\sharp_\partial B)}{\text{Torsion}} = \frac{H_1(A)}{\text{Torsion}} \oplus \frac{H_1(B)}{\text{Torsion}}$$

and hence that: $\Lambda_{A\sharp_\partial B} = \Lambda_A \otimes_{\mathbf{Z}} \Lambda_B$ and $\mathcal{H}_{A\sharp_\partial B} = (\mathcal{H}_A \otimes_{\mathbf{Z}} \Lambda_B) \oplus (\mathcal{H}_B \otimes_{\mathbf{Z}} \Lambda_A)$ Thus, we can see Λ_A as the subset $\Lambda_A \otimes 1$ of $\Lambda_{A\sharp_\partial B}$ and $\mathcal{H}_A$ as the subset $\mathcal{H}_A \otimes 1$ of $\mathcal{H}_{A\sharp_\partial B}$. The same is true for B.

Property 6 *Let h and k be two integers such that*

$$h + k = g(A\sharp_\partial B) = g(A) + g(B)$$

Let $\hat{u} = u_1 \wedge \ldots \wedge u_h \in \bigwedge^h \mathcal{H}_A$ and let $\hat{v} = v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \mathcal{H}_B$.
If $(h,k) \neq (g(A), g(B))$, then $\mathcal{A}_{A\sharp_\partial B}(\hat{u} \wedge \hat{v}) = 0$.
We can assume that $\mathcal{A}_{A\sharp_\partial B}$, $\mathcal{A}_A$ and $\mathcal{A}_B$ are normalized so that:
If $(h,k) = (g(A), g(B))$, then $\mathcal{A}_{A\sharp_\partial B}(\hat{u} \wedge \hat{v}) = \mathcal{A}_A(\hat{u}) \otimes \mathcal{A}_B(\hat{v})$

PROOF: For $C = A$ or B, consider a presentation

$$\left(\gamma_1^C, \ldots, \gamma_{r(C)+g(C)}^C; \rho_1^C, \ldots, \rho_{r(C)}^C\right)$$

of $\mathcal{H}_C$ over Λ_C defining

$$\mathcal{A}_C = \frac{\hat{\rho}^C \wedge .}{\hat{\gamma}^C}$$

Taking all the γ_i^C, for $C = A$ and B, as generators and all the ρ_i^C as relators (with coefficients in $\Lambda_{A\sharp_\partial B}$ containing Λ_C) yields a $\Lambda_{A\sharp_\partial B}$-presentation of $\mathcal{H}_{A\sharp_\partial B}$ which allows us to define $\mathcal{A}_{A\sharp_\partial B}$ by:

Let $\hat{w} = w_1 \wedge \ldots \wedge w_{g(A)+g(B)}$ be an element of $\bigwedge^g \mathcal{H}_{A\sharp_\partial B}$. Split $\hat{w}$ as $\hat{w} = \hat{w}_{\leq g(A)} \wedge \hat{w}_{>g(A)}$ with $\hat{w}_{\leq g(A)} = w_1 \wedge \ldots \wedge w_{g(A)}$ and $\hat{w}_{>g(A)} = w_{g(A)+1} \wedge \ldots \wedge w_{g(A)+g(B)}$. Then

$$\mathcal{A}_{A\sharp_\partial B}(\hat{w}) = \frac{\hat{\rho}^A \wedge \hat{w}_{\leq g(A)} \wedge \hat{\rho}^B \wedge \hat{w}_{>g(A)}}{\hat{\gamma}^A \wedge \hat{\gamma}^B}.$$

Applying this definition to compute $\mathcal{A}_{A\sharp_\partial B}(\hat{u} \wedge \hat{v})$ gives the result. □

The following statement asserts that replacing a RHH which is embedded in A and rationally null-homologous there by another such with the same Lagrangian multiplies the Alexander function by a rational number.

We denote by i_* any map induced by an obvious inclusion.

Property 7 *Let Σ be a closed surface equipped with a Lagrangian $\mathcal{L}_B$, with a $(\Sigma, \mathcal{L}_B)$-system b, with a Σ-system y geometrically dual to b, and with a basepoint $\star$ which will be shared with all the other mentioned manifolds. Let C be a compact 3-manifold whose boundary strictly contains Σ such that:*

1. *$E \stackrel{\text{def}}{=} C \cup_\Sigma -\Sigma_b$ has non-negative genus.*
2. *$i_* : H_1(-\Sigma_b; \mathbf{Q}) \longrightarrow H_1(E; \mathbf{Q})$ is zero.*

Let B denote any $(-\Sigma, \mathcal{L}_B)$-RHH. Define

$$E_B = C \cup_\Sigma B$$

Then

1. *$\frac{H_1(E_B)}{\text{Torsion}}$, and hence, Λ_{E_B} and $p_{E_B}^{-1}(C)$ do not depend on B.*
2. *On $\bigwedge^{g(E)} H_1(p_E^{-1}(C), p_E^{-1}(\star); \mathbf{Z})$, we have:*

$$\mathcal{A}_{E_B} \circ \bigwedge^{g(E)} i_* \doteq \pm |H_1(B_y)| \mathcal{A}_E \circ \bigwedge^{g(E)} i_*$$

PROOF: Since $H_1(E_B; \mathbf{Q}) = H_1(C; \mathbf{Q})/\mathcal{L}_B$ is independent of B, $i_*(H_1(B; \mathbf{Z}))$ is in the torsion of $H_1(E_B; \mathbf{Z})$. Thus, we get the isomorphism

$$\frac{\frac{H_1(C)}{H_1(\Sigma)}}{\text{Torsion}} \longrightarrow \frac{H_1(E_B)}{\text{Torsion}}$$

and $\frac{H_1(E_B)}{\text{Torsion}}$ is independent of B.

Remove a regular neighborhood of an arc, properly embedded in C, joining Σ to another component of ∂C, from C. Denote by C' the manifold resulting from this operation. Let $\Sigma' = \Sigma \cap C'$. (Σ' is obtained from Σ by removing a small disk.) Thus, (after a retraction)

$$E_B = C' \cup_{\Sigma'} B$$

If x is a curve of Σ, x^B will denote x viewed in B while x^C denotes x viewed in C'. We consider the family δ of curves of $\partial(C' \sharp_\partial B)$ (the connected sum identifying a disk of $\Sigma' \subset \partial C'$ with its image on $-\partial B$)

$$\delta = (b_1^C (b_1^B)^{-1}, \cdots, b_g^C (b_g^B)^{-1}, y_1^C (y_1^B)^{-1}, \cdots, y_g^C (y_g^B)^{-1})$$

where $g = g(\Sigma)$. With the notation of Property 5, we may write E_B as

$$E_B = (C' \sharp_\partial B)_\delta$$

and according to this property,

$$\mathcal{A}_{E_B} \doteq \pm \phi_\delta (\mathcal{A}_{C' \sharp_\partial B} (b_1^C - b_1^B \wedge \cdots \wedge b_g^C - b_g^B \wedge y_1^C - y_1^B \wedge \cdots \wedge y_g^C - y_g^B \wedge .))$$

Expand this expression, use Property 6 and note that, since $i_*(H_1(B;\mathbf{Z}))$ is in the torsion of $H_1(E_B;\mathbf{Z})$, the restriction of $\phi_\delta(\mathcal{A}_B)$ to

$$\bigwedge^g H_1(p_{E_B}^{-1}(\partial B), p_{E_B}^{-1}(\star)) = \bigwedge^g (H_1(\partial B) \otimes \Lambda_{E_B})$$

is

$$|H_1(B_y)| \frac{b_1 \wedge \cdots \wedge b_g \wedge .}{b_1 \wedge \cdots \wedge b_g \wedge y_1 \wedge \cdots \wedge y_g}$$

This makes clear that on $\bigwedge^{g(E)} H_1(p_E^{-1}(C), p_E^{-1}(\star); \mathbf{Z})$, we have:

$$\mathcal{A}_{E_B} \circ \bigwedge^{g(E)} i_* \doteq \pm |H_1(B_y)| \phi_\delta(\mathcal{A}_{C'}(b_1 \wedge \cdots \wedge b_g \wedge .))$$

and we are done. □

3.3 Further properties of the Alexander function of a RHH

From now on, we assume that A is a RHH equipped with two ∂A-systems, a and z, z being geometrically dual to a (as in Figure 1).

We assume that the basepoint $\star$ of A is on ∂A, and we join the a_i and the z_i to $\star$ as indicated in Figure 3. We denote by δ_i the oriented boundary of the genus 1 subsurface of ∂A containing both a_i and z_i pictured in Figure 3. We write: $\delta = (\delta_1, \delta_2, \ldots, \delta_g)$.

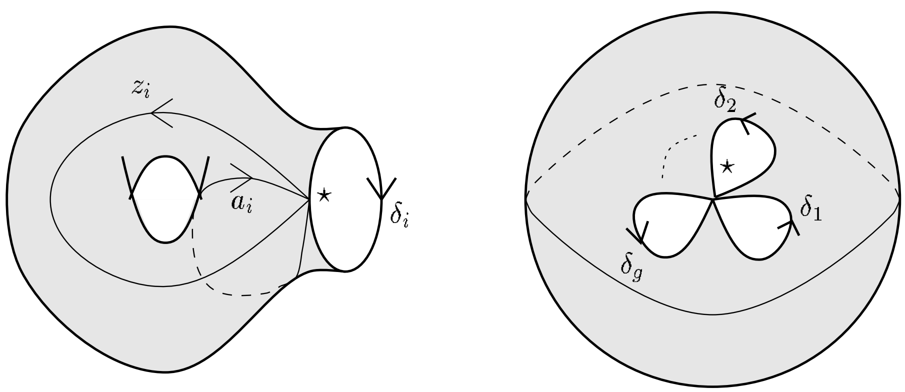

Figure 3: ∂A cut along the δ_i

Note that we have in $\pi_1(A, \star)$:

$$\delta_i = z_i a_i z_i^{-1} a_i^{-1} \tag{3.11}$$

and:

$$\delta_1 \delta_2 \ldots \delta_g = 1 \tag{3.12}$$

Hence, in $\mathcal{H}_A$,

$$\sum_{i=1}^{g}(exp(z_i)-1)a_i+\sum_{i=1}^{g}(1-exp(a_i))z_i=0 \tag{3.13}$$

Notation 3.14 To these data, we associate the oriented link L in the closed manifold A_z with meridians the z_i, with longitudes the a_i and with exterior $A_{\delta\setminus\delta_g}$. If A_z is a rational homology sphere, then

$$H_1(A;\mathbf{Q})=\bigoplus_{i=1}^{g}\mathbf{Q}z_i$$

and we denote by ψ_L the injection of $\tilde{\Lambda}_A=\mathbf{Z}[H_1(A;\mathbf{Q})]$ into $\mathbf{Q}[[z_1,\dots,z_g]]$.

The images of the positive units of $\tilde{\Lambda}_A$ under ψ_L are called the *exponential units* of $\mathbf{Q}[[z_1,\dots,z_g]]$ (because they are the elements of the form $exp(\lambda)$ for the $\mathbf{Q}$-linear combinations λ of the z_i). In $\mathbf{Q}[[z_1,\dots,z_g]]$, the notation $\doteq$ will mean 'equals up to a multiplication by an exponential unit'.

We can refine the relation of $\mathcal{A}_A$ to the Reidemeister torsion of $A_{\delta\setminus\delta_g}$ given by Properties 1 and 5 into the following signed comparison of $\mathcal{A}_A$ and the normalized Alexander series $\mathcal{D}$ of [L, Section 2.2] for several-component links (for which $\mathcal{D}(L)\doteq\pm\psi_L(\,\tau(A_z\setminus L)=\tau(A_{\delta\setminus\delta_g})\,)\,$).

Property 8 [or Definition] *With the notation above, if A_z is a rational homology sphere, and if $g>1$, then for any $j,k\in\{1,\dots,g\}$,*

$$\mathcal{D}(L)\doteq sign(\varepsilon(\mathcal{A}_A(\hat{z})))\psi_L\left(\frac{\mathcal{A}_A(\hat{\delta}(\frac{z_j}{\delta_k}))}{\partial(z_j)}\right)$$

where $\hat{\delta}=\delta_1\wedge\delta_2\wedge\ldots\wedge\delta_g$. (See also Notation 3.6.)

PROOF: The proof is nothing but a careful comparison of the two normalizations. □

Here, we take Property 8 as a definition (up to exponential units) of the Alexander series of oriented links in oriented rational homology spheres. (Note that any such link L can be obtained in this way where A is the complement of a regular neighborhood of the union of L and paths joining the components of L to a point. Note also that Property 2 applied to $A_{\delta\setminus\delta_k}$ proves that $\mathcal{D}(L)$ does not depend on j while Equation 3.12 proves that it does not depend on k.)

To conclude the normalization it suffices to make $\mathcal{D}$ satisfy its well-known symmetry property (see [L, Section A.3]), that is:

$$\mathcal{D}(L)=(-1)^g\overline{\mathcal{D}}(L) \tag{3.15}$$

where the overlining replaces z_i by $-z_i$ in $\mathbf{Q}[[z_1,\dots,z_g]]$. To make $\mathcal{D}(L)$ $\pm$-symmetric, we sometimes have to multiply it by the image under ψ_L of a positive unit of

$$\mathbf{Z}\left[\frac{1}{2}\frac{H_1(A)}{\text{Torsion}}\right] \subset \tilde{\Lambda}_A.$$

Property 8 also gives a definition in the case of knots, but does not define a genuine series anymore because it has a pole at zero. In [L], another definition had been set in this case in order to always get a genuine series and in order to fit in with the usual definition of one-variable Alexander polynomials.

Convention 3.16 *Here, we take Property 8 as the definition of $\mathcal{D}(L)$ up to exponential units in any case,* that is even if $g=1$. The normalization of $\mathcal{D}$ is next fixed by Equation 3.15. If $g=1$, then $\mathcal{D}(L)$ belongs to $z_1^{-1}\mathbf{Q}[[z_1,\dots,z_g]]$.

Lemma 3.17 *Let A be a RHH. Let L be an oriented link in A such that the components of L are rationally null-homologous in A. Let $\mathcal{L}_B$ be a Lagrangian of $(H_1(\partial A;\mathbf{Q}),<,>)$ transverse to $\mathcal{L}_A$. Then the ratio*

$$\frac{\mathcal{D}(L\subset A\cup_{\partial A} B)}{|H_1(A\cup_{\partial A} B)|}$$

is the same for all the $(-\partial A,\mathcal{L}_B)$-RHH B.

PROOF: It is a direct corollary of Property 7 together with the definition of D given by Property 8. Here are the details.

Denote by m_i the meridian of the component K_i of L. Connect the components of L together in A to see L as part of a graph $\Gamma(L)$ homotopic to the wedge of the components of L (as in Figure 4). Denote by C the exterior of $\Gamma(L)$ in A. Let δ_i be the boundary of the part of ∂C intersecting 'the' tubular neighborhood of K_i. The definition of $\mathcal{D}(L\subset A\cup B)$ given by Property 8 (where A is replaced by $C\cup B$) says:

$$\mathcal{D}(L\subset A\cup B) \doteq sign(\varepsilon(\mathcal{A}_{C\cup B}(\hat{m})))\psi_{L\subset A\cup B}\left(\frac{\mathcal{A}_{C\cup B}(\hat{\delta}(\frac{m_j}{\delta_k}))}{\partial(m_j)}\right) \tag{3.18}$$

Since $\mathcal{L}_A$ and $\mathcal{L}_B$ are transverse, $H_1(C\cup B;\mathbf{Q})$ is the $\mathbf{Q}$-vector space generated by the m_i. Since the components of L are rationally null-homologous in A, they do not algebraically link the curves of ∂A. Thus, $i_*(H_1(\partial A;\mathbf{Q}))$ is null in $H_1(C\cup B;\mathbf{Q})$ and

$$\frac{H_1(C\cup B;\mathbf{Z})}{\text{Torsion}} = \frac{\frac{H_1(C;\mathbf{Z})}{i_*(H_1(\partial A;\mathbf{Z}))}}{\text{Torsion}}$$

With this identification, $\psi_{L\subset A\cup B}:\Lambda_{C\cup B}\longrightarrow \mathbf{Q}[[m_1,\dots,m_n]]$ does not depend on B.

Property 7 implies that if B and B' are two $(-\partial A, \mathcal{L}_B)$-RHH, then

$$\mathcal{A}_{C\cup B'}(\hat{\delta}(\frac{m_j}{\delta_k})) \doteq \frac{\varepsilon(\mathcal{A}_{C\cup B'}(\hat{m}))}{\varepsilon(\mathcal{A}_{C\cup B}(\hat{m}))}\mathcal{A}_{C\cup B}(\hat{\delta}(\frac{m_j}{\delta_k}))$$

Making the $\mathcal{D}(L)$ (defined by Equation 3.18) $\pm$-symmetric and placing them into this equality concludes the proof. □

We recall the definition of the ζ-coefficient.

Definition 3.19 With the current homogeneous definition of $\mathcal{D}(L)$ (Convention 3.16), $\zeta(L)$ is the coefficient of $\prod_{i=1}^{g} z_i$ in $\mathcal{D}(L)$.

Remark 3.20 *Why this definition fits in with the definition of [L].* Definition 3.19 is exactly the same as the one given in [L, Definition 1.4.1] for several-component link. The current definition of the Alexander series $\mathcal{D}(K)$ of a knot K in a rational homology sphere M with meridian m is related to the definition used in [L, Definition 2.2.2, Bridge 2.1.1] of its Alexander polynomial $\Delta(K)$ by:

$$\mathcal{D}(K)(m) = \frac{\Delta(K)(exp(m))}{exp(\frac{1}{2O_M(K)}m) - exp(-\frac{1}{2O_M(K)}m)}$$

where $O_M(K) \stackrel{\text{def}}{=} \frac{|H_1(M)|}{|\text{Torsion}(H_1(M\setminus K))|}$. Since $\Delta(K)(1) = |\text{Torsion}(H_1(M\setminus K))|$ and $\Delta'(K)(1) = 0$, the first terms of $\mathcal{D}(K)$ may be written as:

$$\mathcal{D}(K)(m) =$$

$$\frac{O_M(K)}{m}\left(|\text{Torsion}(H_1(M\setminus K))| + \frac{\Delta\text{''}(K)(1)}{2}m^2 + O(3)\right)\left(1 - \frac{m^2}{24\,O_M(K)^2} + O(4)\right)$$

$$= \frac{|H_1(M)|}{m} + \left(\frac{1}{2}O_M(K)\Delta\text{''}(K)(1) - \frac{|H_1(M)|}{24\,O_M(K)^2}\right)m + O(2)$$

Thus, Definition 3.19 of ζ also fits in with [L, Definition 1.4.1] for knots. □

From now on, we furthermore assume that a is a $(\partial A, \mathcal{L}_A)$-system.

In this case the a_i are the preferred longitudes of the components of L which do not link each other algebraically, and Property 8 can be rewritten as follows:

Property 9 *With the notation of (3.14), for any $j, k \in \{1, \ldots, g\}$,*

$$\mathcal{D}(L) \doteq \psi_L\left(sign(\varepsilon(\mathcal{A}_A(\hat{z})))\frac{\prod_{i=1}^{g}(exp(z_i)-1)}{(exp(z_j)-1)(exp(z_k)-1)}\mathcal{A}_A(\hat{a}(\frac{z_j}{a_k}))\right)$$

□

Corollary 3.21 *If $g \geq 2$, for any $j, k \in \{1, \ldots, g\}$,*

$$\mathcal{A}_A(\hat{a}(\frac{z_j}{a_k})) = sign(\varepsilon(\mathcal{A}_A(\hat{z})))\zeta(L)z_j z_k + O(3)$$

PROOF: As a consequence of the study of the first terms of the Alexander series performed in [L, Proposition 2.5.2], we get:

$$\mathcal{D}(L) = \zeta(L)\prod_{i=1}^{g} z_i + O(n+2)$$

Compare with Property 9 above. □

Remark 3.22 This corollary together with Property 4 gives a proof that the ζ-coefficient of a rational homology unlink (that is a link whose components are pairwise algebraically unlinked in a rational homology sphere) with at least 4 components is zero.

3.4 A nice property of the Alexander function

This subsection is devoted to the proof of the following property which will be crucial for our concerns.

Property 10 *There exists $\eta = \pm 1$ such that, for any (A, a, z), where A is a RHH, a is a $(\partial A, \mathcal{L}_A)$-system and z is a ∂A-system geometrically dual to a :*

$$\mathcal{A}_A(\hat{z}(\frac{a_j}{z_k})) = \varepsilon(\mathcal{A}_A(\hat{z}))\eta\sum_{i=1}^{g}\mathcal{I}_{A_a}(a_i \wedge a_j \wedge a_k)(exp(z_i) - 1) + O(2)$$

Remark 3.23 The order one part of $\mathcal{A}_A(\hat{z}(\frac{a_j}{z_k}))$ does not depend on the way in which the curves of a and z are joined to the basepoint. Indeed, changing the way of joining a curve w to the basepoint replaces it by a curve $\tilde{w}$ such that: $\tilde{w} = twt^{-1}$ in $\pi_1(A)$, and hence $\tilde{w} = exp(t)w + (1 - exp(w))t$ in $\mathcal{H}_A$.

For $w = a_j$, $1 - exp(w) = 0$, thus, $\mathcal{A}_A(\hat{z}(\frac{a_j}{z_k}))$ is multiplied by an exponential unit and its order one part is unchanged. For $w = z_i, i \neq k$, a term of the form $(1 - exp(z_i))\mathcal{A}_A(a_j \wedge \hat{u})$ may be furthermore added to $\mathcal{A}_A(\hat{z}(\frac{a_j}{z_k}))$, but, since the order of this term is at least 2, the order one part of $\mathcal{A}_A(\hat{z}(\frac{a_j}{z_k}))$ still remains unchanged.

PROOF OF PROPERTY 10: First note that Property 10 is clear for $g = g(A) = 1$ by Property 1 and for $g = 2$ by Corollary 3.21. (In these cases, the right-hand side of the equality is zero.) Assume the following lemma (Property 10 for $g = 3$) for the moment:

Lemma 3.24 *There exists a unique $\eta = \pm 1$ such that, for any (A, a, z), where A is a genus 3 RHH, a is a $(\partial A, \mathcal{L}_A)$-system and z is a ∂A-system geometrically dual to a, we have*

$$\mathcal{A}_A(\hat{z}(\frac{a_j}{z_k})) = \varepsilon(\mathcal{A}_A(\hat{z}))\eta \sum_{i=1}^{3} \mathcal{I}_{A_a}(a_i \wedge a_j \wedge a_k)(exp(z_i) - 1) + O(2)$$

Now, let g be any integer larger than 3. Applying the restriction property (5) and the genus 3 result for $A_{z'}$, to subsets z' of $(g-3)$ curves of z (with $z_j, z_k \notin z'$) concludes the proof assuming Lemma 3.24. □

Lemma 3.24 is the consequence of the two following lemmas.

Lemma 3.25 *For any (A, a, z), where A is a genus 3 RHH, a is a $(\partial A, \mathcal{L}_A)$-system and z is a ∂A-system geometrically dual to a, there exists a rational number $\mathcal{J}(A, a, z)$ such that:*

$$\frac{\mathcal{A}_A(\hat{z}(\frac{a_j}{z_k}))}{sign(\varepsilon(\mathcal{A}_A(\hat{z})))} = \sum_{i=1}^{3} \frac{(a_i \wedge a_j \wedge a_k)}{(a_1 \wedge a_2 \wedge a_3)} \mathcal{J}(A, a, z)(exp(z_i) - 1) + O(2)$$

Lemma 3.26 *There exists a unique $\eta = \pm 1$ such that, for any (A, a, z), where A is a genus 3 RHH, a is a $(\partial A, \mathcal{L}_A)$-system and z is a ∂A-system geometrically dual to a, we have*

$$\mathcal{J}(A, a, z) = \eta \, |H_1(A_z)| \, \mathcal{I}_{A_a}(a_1 \wedge a_2 \wedge a_3)$$

where $\mathcal{J}(A, a, z)$ is defined by the lemma above.

PROOF OF LEMMA 3.25: Assume, without loss of generality, that $\mathcal{A}_A$ has been normalized so that $\varepsilon(\mathcal{A}_A(\hat{z})) > 0$.

We first prove the lemma for $j = k$, that is, for any k:

$$\mathcal{A}_A(\hat{z}(\frac{a_k}{z_k})) = O(2) \tag{3.27}$$

Assume without loss of generality that $k = 1$. Then by the restriction property 5 and by Corollary 3.21, $\phi_{z_2}(\mathcal{A}_A(\hat{z}(\frac{a_1}{z_1})))$ and $\phi_{z_3}(\mathcal{A}_A(\hat{z}(\frac{a_1}{z_1})))$ must be at least of order 2. So must be $\mathcal{A}_A(\hat{z}(\frac{a_1}{z_1}))$.

Property 2 implies that

$$(exp(z_1) - 1)\mathcal{A}_A(\hat{z}(\frac{a_j}{z_1})) + (exp(z_2) - 1)\mathcal{A}_A(\hat{z}(\frac{a_j}{z_2})) + (exp(z_3) - 1)\mathcal{A}_A(\hat{z}(\frac{a_j}{z_3})) = 0 \tag{3.28}$$

This together with Equation 3.27 yields:

$$(exp(z_1) - 1)\mathcal{A}_A(\hat{z}(\frac{a_2}{z_1})) + (exp(z_3) - 1)\mathcal{A}_A(\hat{z}(\frac{a_2}{z_3})) = O(3)$$

In particular, we can define a rational number $\mathcal{J}(A,a,z)$ such that:

$$\mathcal{A}_A(\hat{z}(\frac{a_2}{z_3})) = \mathcal{J}(A,a,z)(exp(z_1)-1)+O(2)$$

and Lemma 3.25 is true with this number when $j=2$.

Equation 3.13, which becomes

$$(exp(z_1)-1)a_1+(exp(z_2)-1)a_2+(exp(z_3)-1)a_3=0$$

here, implies:

$$(exp(z_1)-1)\mathcal{A}_A(\hat{z}(\frac{a_1}{z_k}))+(exp(z_2)-1)\mathcal{A}_A(\hat{z}(\frac{a_2}{z_k}))+(exp(z_3)-1)\mathcal{A}_A(\hat{z}(\frac{a_3}{z_k}))=0 \tag{3.29}$$

This gives the result for $\{j,k\}=\{1,3\}$. We are now left with the case $(k=2, j\in\{1,3\})$ which follows from Equation 3.28. □

Now, we start proving Lemma 3.26 by proving the following lemma.

Hypotheses 3.30 *Let (A,a,z) be a genus 3 RHH equipped with a $(\partial A,\mathcal{L}_A)$-system a and a ∂A-system z geometrically dual to a, let L be the link of A_z, with meridians the z_i as in Notation 3.14. Let (A',a',z',L') satisfy the same hypotheses as (A,a,z,L).*

Lemma 3.31 *Under the hypotheses above,*

$$|H_1(A_z)|\,|H_1(A'_{z'})|\left(\mathcal{I}_{A_a}(a_1\wedge a_2\wedge a_3)-\mathcal{I}_{A'_{a'}}(a'_1\wedge a'_2\wedge a'_3)\right)^2$$

$$=|H_1(A'_{z'})|\,\zeta(L\subset A_z)+|H_1(A_z)|\,\zeta(L'\subset A'_{z'})-2\mathcal{J}(A,a,z)\mathcal{J}(A',a',z')$$

PROOF:

Notation 3.32 Under the hypotheses (3.30), we construct the manifold $M=A\cup -A'$ by gluing A and $-A'$ along their boundaries so that after the gluing, $a_i=a'_i$ and $z_i=z'_i$. We denote by p_i the oriented parallel of the knot z_i of M lying on ∂A. Let R be the rational homology sphere obtained from M by surgery on $((z_i,p_i)_{i=1,2,3})$ and let $C_i\subset R$ be the core of the i^{th} surgery torus. Orient C_i so that p_i is its oriented meridian. Let C be the link $C=(C_1,C_2,C_3)$ in R, $(M\setminus(z_i)_{i=1,2,3}=R\setminus C)$.

Lemma 3.31 is the consequence of the following results of two different computations of $\zeta(C)$.

Sublemma 3.33

$$\zeta(C)=|H_1(A'_{z'})|\zeta(L\subset A_z)+|H_1(A_z)|\zeta(L'\subset A'_{z'})-2\mathcal{J}(A,a,z)\mathcal{J}(A',a',z')$$

Sublemma 3.34

$$\zeta(C) = |H_1(A'_{z'})|\,|H_1(A_z)|\left(\mathcal{I}_{A_a}(a_1\wedge a_2\wedge a_3) - \mathcal{I}_{A'_{a'}}(a'_1\wedge a'_2\wedge a'_3)\right)^2$$

Proof of Sublemma 3.33: We look at the exterior of C in R as the exterior E of the link of the z_i in M that we construct as follows: Start with A and A', each of them being equipped with its basepoint and its δ_i as in Figure 3. Glue a disk of ∂A intersecting the curves drawn in Figure 3 at the basepoint together with its image on - $\partial A'$ in order to obtain $A\sharp_\partial - A'$. Consider the collection κ of the curves of $A\sharp_\partial - A'$: $\kappa_4 = \delta_1'^{-1}\delta_1$, $\kappa_5 = \delta_2'^{-1}\delta_2$, $\kappa_1 = z_1 z_1'^{-1}$, $\kappa_2 = z_2 z_2'^{-1}$, $\kappa_3 = z_3 z_3'^{-1}$. With the notation of Property 5:

$$E = (A\sharp_\partial - A')_\kappa$$

Thus, according to Property 5 and Property 8 (applied to $(A\sharp_\partial - A')_{\kappa_1\kappa_2\kappa_3}$),

$$\mathcal{D}(C) \doteq sign(\varepsilon(\mathcal{A}_{A\sharp_\partial - A'}(\kappa_1\wedge\kappa_2\wedge\kappa_3\wedge\hat{z})))\psi_C\phi_{\kappa_1\kappa_2\kappa_3}\left(\frac{\mathcal{A}_{A\sharp_\partial - A'}(\hat{\kappa}\wedge z_3)}{\partial(z_3)}\right) \quad (3.35)$$

with $\hat{\kappa} = \kappa_1\wedge\ldots\wedge\kappa_5$ and $\hat{z} = z_1\wedge z_2\wedge z_3$. (Since both κ_4 and κ_5 are oriented as the boundaries of the genus one subsurfaces of $\partial((A\sharp_\partial - A')_{\kappa_1\kappa_2\kappa_3})$ corresponding with the knots C_1 and C_2, we have the right sign in Equation 3.35.)

Note that $\phi_{\kappa_1\kappa_2\kappa_3}$ identifies $exp(z_i)$ and $exp(z'_i)$ in $\Lambda_{A\sharp_\partial - A'}$. From now on, we see $\phi_{\kappa_1\kappa_2\kappa_3}(\Lambda_{A\sharp_\partial - A'})$, Λ_A and $\Lambda_{A'}$ as subsets of $\mathbf{Q}[[z_1,z_2,z_3]]$. In particular, in $(\mathcal{H}_{A\sharp_\partial - A'})\otimes_{\phi_{\kappa_1\kappa_2\kappa_3}}\mathbf{Q}[[z_1,z_2,z_3]]$, we have:

$$\begin{aligned}
\kappa_1 &= z_1 - z'_1\\
\kappa_2 &= z_2 - z'_2\\
\kappa_3 &= z_3 - z'_3\\
\kappa_4 &= (exp(z_1)-1)(a_1 - a'_1)\\
\kappa_5 &= (exp(z_2)-1)(a_2 - a'_2)
\end{aligned}$$

Thus, we can rewrite Equation 3.35 as:

$$\frac{\mathcal{D}(C)(exp(z_3)-1)}{(exp(z_1)-1)(exp(z_2)-1)} sign(\varepsilon(\mathcal{A}_{A\sharp_\partial - A'}(\kappa_1\wedge\kappa_2\wedge\kappa_3\wedge\hat{z})))$$

$$\doteq \phi_{\kappa_1\kappa_2\kappa_3}\left(\mathcal{A}_{A\sharp_\partial - A'}(z_1 - z'_1\wedge z_2 - z'_2\wedge z_3 - z'_3\wedge a_1 - a'_1\wedge a_2 - a'_2\wedge z_3)\right) \quad (3.36)$$

Now, we refer to Property 6 and assume that $\mathcal{A}_{A\sharp_\partial - A'}$ is normalized so that if $\hat{u}\in\bigwedge^3\mathcal{H}_A$ and if $\hat{v}\in\bigwedge^3\mathcal{H}_{A'}$, then

$$\mathcal{A}_{A\sharp_\partial - A'}(\hat{u}\wedge\hat{v}) = \mathcal{A}_A(\hat{u})\mathcal{A}_{A'}(\hat{v})$$

and that $\mathcal{A}_A$ and $\mathcal{A}_{A'}$ are normalized so that $\varepsilon(\mathcal{A}_A(\hat{z}))$ and $\varepsilon(\mathcal{A}_{A'}(\hat{z}'))$ are positive.

In particular, $\varepsilon(\mathcal{A}_{A\sharp_\partial -A'}(\kappa_1\wedge\kappa_2\wedge\kappa_3\wedge\hat{z}))=\varepsilon(\mathcal{A}_{A\sharp_\partial -A'}(\hat{z}\wedge\hat{z}'))$ is positive. So, expanding the exterior product with the usual rules transforms Equation 3.36 into

$$\begin{array}{rl} & \frac{\mathcal{D}(C)(exp(z_3)-1)}{(exp(z_1)-1)(exp(z_2)-1)} \\ \doteq & \mathcal{A}_A(z_1\wedge z_2\wedge z_3)\mathcal{A}_{A'}(a'_1\wedge a'_2\wedge z'_3) \\ + & \mathcal{A}_A(a_1\wedge a_2\wedge z_3)\mathcal{A}_{A'}(z'_1\wedge z'_2\wedge z'_3) \\ - & \mathcal{A}_A(z_1\wedge a_2\wedge z_3)\mathcal{A}_{A'}(a'_1\wedge z'_2\wedge z'_3) \\ - & \mathcal{A}_A(a_1\wedge z_2\wedge z_3)\mathcal{A}_{A'}(z'_1\wedge a'_2\wedge z'_3) \\ + & \mathcal{A}_A(z_1\wedge a_1\wedge z_3)\mathcal{A}_{A'}(a'_2\wedge z'_2\wedge z'_3) \\ + & \mathcal{A}_A(a_2\wedge z_2\wedge z_3)\mathcal{A}_{A'}(z'_1\wedge a'_1\wedge z'_3) \end{array} \tag{3.37}$$

Corollary 3.21 and Lemma 3.25 allow us to write the right-hand side of Equation 3.37 at order 2 as

$$[|H_1(A'_{z'})|\zeta(L\subset A_z)+|H_1(A_z)|\zeta(L'\subset A'_{z'})-2\mathcal{J}(A,a,z)\mathcal{J}(A',a',z')]\,z_3^2$$
$$+O(3)$$

and it suffices to apply Definition 3.19 of ζ to conclude the proof of Sublemma 3.33. □

Proof of Sublemma 3.34: M is obtained by surgery with null coefficients on $C=(C_1,C_2,C_3)\subset R$ and C is a rational homology unlink (the longitude of C_i which is the meridian of z_i is rationally null-homologous in $M\backslash(z_i)_{i=1,2,3}=R\backslash C$). So, according to the theorem of [L, Section 1.5],

$$\overline{\lambda}(M)=\zeta(C)$$

On the other hand, according to Lemma 2.17 (and the fact that $\mathcal{I}_{A'a'}=\mathcal{I}_{(-A')a'}$),

$$\overline{\lambda}(M)=|H_1(A'_{z'})|\,|H_1(A_z)|\left(\mathcal{I}_{A_a}(a_1\wedge a_2\wedge a_3)-\mathcal{I}_{A'_{a'}}(a'_1\wedge a'_2\wedge a'_3)\right)^2$$

□

This conclude the proof of Lemma 3.31 □

Sublemma 3.34 also yields the following lemma.

Lemma 3.38 *Under Hypotheses 3.30,*

$$\zeta(L)=|H_1(A_z)|\mathcal{I}_{A_a}(a_1\wedge a_2\wedge a_3)^2$$

Proof: If A' is the standard handlebody with meridians a'_1,a'_2,a'_3, then $L=C$ and $\mathcal{I}_{A'_{a'}}(a'_1\wedge a'_2\wedge a'_3)^2=0$. □

So, we can rewrite Lemma 3.31 as:

Lemma 3.39 *Under Hypotheses 3.30,*

$$|H_1(A_z)|\,|H_1(A'_{z'})|\mathcal{I}_{A_a}(a_1\wedge a_2\wedge a_3)\mathcal{I}_{A'_{a'}}(a'_1\wedge a'_2\wedge a'_3)=\mathcal{J}(A,a,z)\mathcal{J}(A',a',z')$$

□

Now, look at the following example of a genus 3 RHH. Consider the graph $\Gamma^{\mathcal{B}}$ in S^3 made of the borromean rings R_1, R_2, R_3 each of them being attached by an edge to the basepoint as in Figure 4. Denote by $A^{\mathcal{B}}$ the exterior of (a regular neighborhood of) $\Gamma^{\mathcal{B}}$ in S^3. Denote by $a_i^{\mathcal{B}}$ the longitude of R_i and by $z_i^{\mathcal{B}}$ the meridian of R_i as in Figure 4.

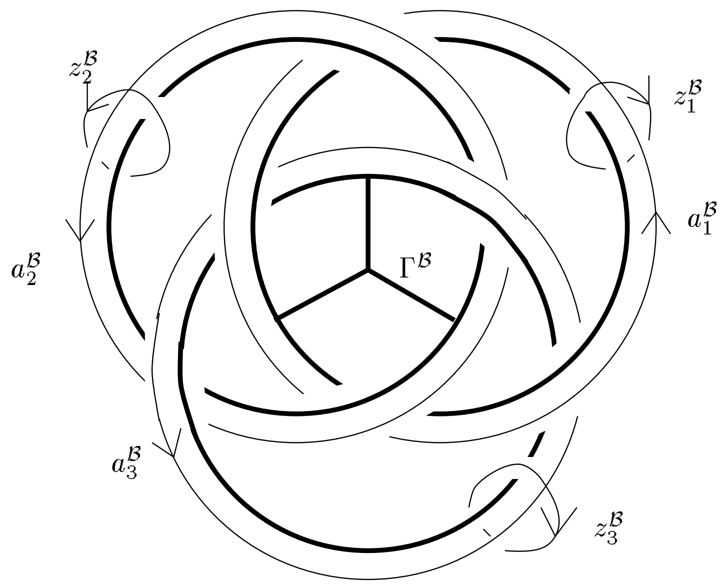

Figure 4: $\Gamma^{\mathcal{B}}$, the $a_i^{\mathcal{B}}$ and the $z_i^{\mathcal{B}}$

Lemma 3.40 *The number*

$$\eta=\frac{\mathcal{I}_{A^{\mathcal{B}}_{a^{\mathcal{B}}}}(a_1^{\mathcal{B}}\wedge a_2^{\mathcal{B}}\wedge a_3^{\mathcal{B}})}{\mathcal{J}(A^{\mathcal{B}},a^{\mathcal{B}},z^{\mathcal{B}})}$$

is well-defined and is equal to ± 1.

Proof: $A^{\mathcal{B}}$ satisfies $|H_1(A^{\mathcal{B}}_{z^{\mathcal{B}}})|=1$ and

$$\mathcal{I}_{A^{\mathcal{B}}_{a^{\mathcal{B}}}}(a_1^{\mathcal{B}}\wedge a_2^{\mathcal{B}}\wedge a_3^{\mathcal{B}})^2=1$$

(because $A^{\mathcal{B}}_{a^{\mathcal{B}}}$ is $(S^1)^3$, for example). Thus, applying Lemma 3.39 with $(A,a,z)=(A',a',z')=(A^{\mathcal{B}},a^{\mathcal{B}},z^{\mathcal{B}})$ yields

$$\mathcal{J}(A^{\mathcal{B}},a^{\mathcal{B}},z^{\mathcal{B}})=\pm 1,$$

and proves the lemma. □

Proof of Lemma 3.26: We fix the number η defined by Lemma 3.40 and apply Lemma 3.39 to any (A, a, z) as in the statement of Lemma 3.26 and to $(A', a', z') = (A^{\mathcal{B}}, a^{\mathcal{B}}, z^{\mathcal{B}})$. □

This concludes the proof of Property 10 and this subsection.

□

Remark 3.41 We could compute η explicitly from its definition given by Lemma 3.40, but it is not useful here.

4 Proof of the two main lemmas

4.1 Proving Lemma 2.28

We prove Lemma 2.28. To do it, we fix a RHH A, we fix a rational Lagrangian $\mathcal{L}_B$ transverse to $\mathcal{L}_A$ in $(H_1(\partial A;\mathbf{Q}); <.,.>_{\partial A})$, we let B vary among the $(-\partial A, \mathcal{L}_B)$-RHH, and we try to see how $\zeta((K,Z)=(K_1,Z_1)\subset (A\cup B))$ depends on B (with the link $((K_1,Z_1)\subset A)$ defined in Hypotheses 2.20).

We denote the intersection form $<,>_{\partial A}$ simply by $<,>$. We fix a $(\partial A,\mathcal{L}_A)$-system a, a ∂A-system z geometrically dual to a, and a $(\partial A,\mathcal{L}_B)$-system $b=(b_1,\dots,b_g)$ such that:

$$i>j \Longrightarrow <a_i,b_j>=0$$

We denote by $\mathcal{F}(\mathcal{L}_B)$ *any coefficient which does not depend on our fixed data.* With this additional notation, Lemma 2.28 is obviously equivalent to the following lemma that we are about to prove :

Lemma 4.1 *Under the above hypotheses,*

$$\frac{\zeta((K,Z)\subset A\cup B)}{|H_1(A\cup B)|}=$$

$$\sum_{(j,k)\in\{1,\dots,g\}^2} -\mathcal{I}_{A_a}(a_1\wedge a_j\wedge a_k)\mathcal{I}_{B_b}(\beta_1\wedge\beta_j\wedge\beta_k)+\mathcal{F}(\mathcal{L}_B)$$

where $\beta_k\in\mathcal{L}_B$ is defined so that $<a_j,\beta_k>=\delta_{jk}$ for any $j,k\in\{1,\dots,g\}$.

To prove Lemma 4.1, we first express the Alexander series of $(K,Z)=(K_1,Z_1)$ in terms of the Alexander functions of A and B. B is a subset of the exterior E of (K,Z). Retracting A onto A minus a regular neighborhood of its boundary which contains (K,Z) allows us to see A as another part of E. These inclusions provide canonical maps from Λ_A and Λ_B to Λ_E that we see as a subset of $\mathbf{Q}[[m_K,m_Z]]$ where m_K and m_Z denote the variables corresponding to the meridians of K and Z, respectively. $\overline{\mathcal{A}}_A$ (respectively $\overline{\mathcal{A}}_B$) denotes $\mathcal{A}_A$ (respectively $\mathcal{A}_B$) followed by the previous canonical morphisms. $\overline{\mathcal{A}}_A$ and $\overline{\mathcal{A}}_B$ take their values in $\mathbf{Q}[[m_K,m_Z]]$. $\mathcal{A}_A$ and $\mathcal{A}_B$ are normalized so that:

$$\overline{\mathcal{A}}_A(\hat{z})=|H_1(A_z)|+O(2) \tag{4.2}$$

and,

$$\overline{\mathcal{A}}_B(\hat{a})=|H_1(B_a)|+O(2) \tag{4.3}$$

Lemma 4.4 *Let $C\stackrel{\text{def}}{=}\{a_i,z_j;i,j\in\{1,\dots,g\}\}$. Denote by P the set of the subsets u of g elements of $C\setminus\{a_1\}$. The complement $C\setminus u$ of such an u is denoted by v. Both u and v are assumed to be equipped with an arbitrary order allowing us to write $u=\{u_1,\dots,u_g\}$, $v=\{v_1,\dots,v_g\}$, $\hat{u}=u_1\wedge u_2\ldots\wedge u_g$ and $\hat{v}=v_1\wedge v_2\ldots\wedge v_g$.*

Then the Alexander series $\mathcal{D}(K,Z)$ is given, up to exponential units, by:

$$\mathcal{D}(K,Z) \doteq \pm \sum_{u\in P} \overline{\mathcal{A}}_A(\hat{u})\overline{\mathcal{A}}_B(\hat{v})\frac{\hat{u}\wedge\hat{v}}{\hat{z}\wedge\hat{a}}$$

Proof: Denote by a_i^B and z_i^B the respective images of a_i and z_i under the gluing homeomorphism on ∂B.

Reconstruct E as follows. Start with A and B disjoint. Glue a disk of ∂A intersecting a and z at the basepoint, together with its image on ∂B in order to obtain $A\sharp_\partial B$. Consider the collection κ^0 of curves of $\partial(A\sharp_\partial B)$: $\kappa_i = z_i(z_i^B)^{-1}$, $\kappa_{i+g} = a_i^B a_i^{-1}$, for $i = 2,\ldots,g$. With the notation of Property 5, $(A\sharp_\partial B)_{\kappa^0}$ is the complement of a regular neighborhood of the subsurface Σ_1 of ∂A bounded by δ_1 containing a_1. In this manifold which is naturally embedded in E, we may see the meridians of K and Z as $m_K = (z_1^B)^{-1}z_1$ and $m_Z = a_1^B a_1^{-1}$, respectively. (See Figure 5.)

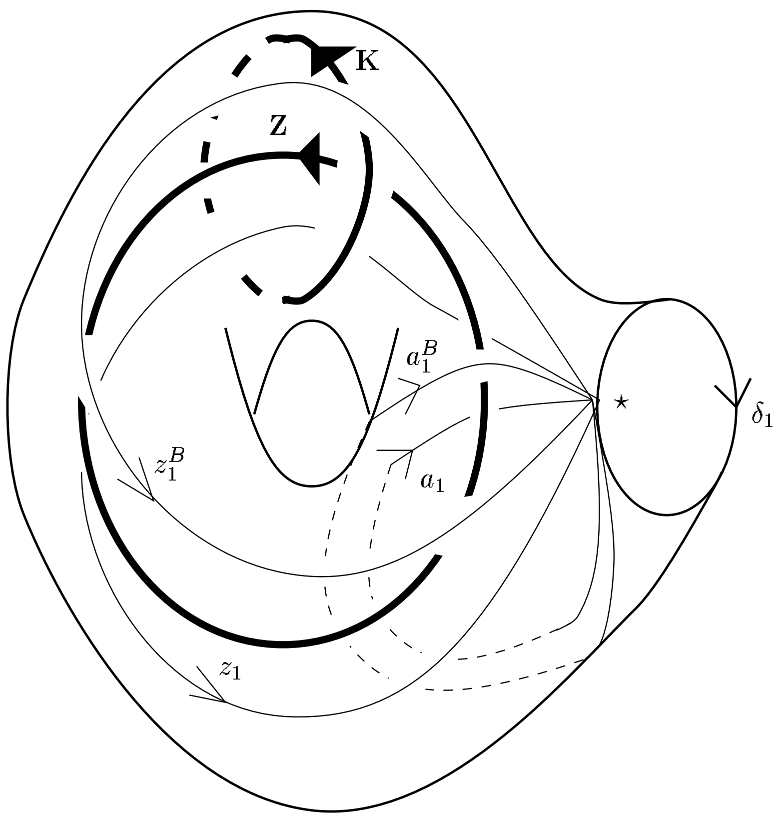

Figure 5: Around Σ_1 in $(A\sharp_\partial B)_{\kappa^0}$

Attaching to $(A\sharp_\partial B)_{\kappa^0}$ a 2-cell with boundary

$$\kappa_1 = (z_1^B)^{-1} z_1 a_1^B \left((z_1^B)^{-1} z_1\right)^{-1} (a_1^B)^{-1}$$

representing ($\pm$) the boundary of the tubular neighborhood of K in E provides a strong deformation retract of E. (Indeed, only a ring of ∂A remains unattached and the core of this ring is Z.) Let κ' be the collection of curves $\{\kappa_1 \cup \kappa^0\}$. Thus, $E = (A\sharp_\partial B)_{\kappa'}$, and, according to Properties 1 and 5 of the Alexander function:

$$\tau(E) \doteq \pm\phi_{\kappa'}\left(\frac{\mathcal{A}_{A\sharp_\partial B}(\hat{\kappa}' \wedge a_1^B)}{\partial(a_1^B)}\right)$$

Set $\kappa_{g+1} = a_1^B$ and $\hat{\kappa} = \kappa_1 \wedge \kappa_2 \wedge \ldots \wedge \kappa_{2g}$.

$$\tau(E) \doteq \pm\phi_{\kappa'}\left(\frac{\mathcal{A}_{A\sharp_\partial B}(\hat{\kappa})}{\partial(a_1^B)}\right)$$

In $\mathcal{H}_{A\sharp_\partial B} \otimes_{\phi_{\kappa'}} \Lambda_E$, $\kappa_i = z_i - z_i^B$ and $\kappa_{i+g} = a_i^B - a_i$, for $i = 2, \ldots, g$ while

$$\kappa_1 = exp(-z_1^B)(exp(a_1^B) - 1)(z_1^B - z_1) + (exp(z_1 - z_1^B) - 1)a_1^B$$

Thus, in $\bigwedge_{i=1}^{2g}(\mathcal{H}_{A\sharp_\partial B} \otimes_{\phi_{\kappa'}} \Lambda_E)$:

$$\hat{\kappa} \doteq \pm(exp(a_1^B) - 1)\bigwedge_{i=1}^{g}(z_i - z_i^B) \wedge a_1^B \wedge \bigwedge_{i=2}^{g}(a_i^B - a_i)$$

Now, Property 6 of the Alexander function allows us to write

$$\mathcal{D}(K, Z) \doteq \pm\sum_{u\in P}\overline{\mathcal{A}}_A(\hat{u})\overline{\mathcal{A}}_B(\hat{v})\frac{\hat{u} \wedge \hat{v}}{\hat{z} \wedge \hat{a}}$$

□

Notation 4.5 If Σ is a series, and if n is an integer, $\mathcal{O}_n(\Sigma)$ denotes the degree n part of Σ. Note that $\mathcal{O}_0(\overline{\mathcal{A}}) = \varepsilon(\mathcal{A})$.

Lemma 4.6 *With the chosen normalizations (4.2, 4.3) of $\overline{\mathcal{A}}_A$ and $\overline{\mathcal{A}}_B$, the normalization of $\mathcal{D}(K, Z)$ has the following form:*

$$\mathcal{D}(K, Z) = -exp(w)\sum_{u\in P}\overline{\mathcal{A}}_A(\hat{u})\overline{\mathcal{A}}_B(\hat{v})\frac{\hat{u} \wedge \hat{v}}{\hat{z} \wedge \hat{a}}$$

for a w independent of the $(-\partial A, \mathcal{L}_B)$-RHH B (so that $exp(w) = \mathcal{F}(\mathcal{L}_B)$).

Proof: The study of the first terms of the Alexander series performed in [L, Proposition 2.5.2] implies in particular that

$$\mathcal{O}_0(\mathcal{D}(K,Z)) = -|H_1(A \cup B)|$$

Thus, the normalization equations 4.2 and 4.3 allow us to replace the $\pm$ by $-$ in the statement of Lemma 4.4. So, if we set:

$$\tilde{\mathcal{D}}(K,Z) = -\sum_{u \in P} \overline{\mathcal{A}}_A(\hat{u})\overline{\mathcal{A}}_B(\hat{v})\frac{\hat{u}\wedge\hat{v}}{\hat{z}\wedge\hat{a}}$$

there exists $w \in \mathbf{Q}m_K \oplus \mathbf{Q}m_Z$ such that: $\mathcal{D}(K,Z) = exp(w)\tilde{\mathcal{D}}(K,Z)$.

Now, we compute w and we prove that it is independent of B. To be symmetric, $\mathcal{D}(K,Z)$ must satisfy $\mathcal{O}_1(\mathcal{D}(K,Z)) = 0$. Thus, w is determined by:

$$\mathcal{O}_1(\tilde{\mathcal{D}}(K,Z)) + \mathcal{O}_0(\tilde{\mathcal{D}}(K,Z))w = 0$$

where $\mathcal{O}_0(\tilde{\mathcal{D}}(K,Z)) = -|H_1(A_z)|\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{a}))$. According to Property 4 of the Alexander function, if u contains more than one element of the collection a, then:

$$\mathcal{O}(\mathcal{A}_A(\hat{u})) \geq 2$$

Since the chosen normalizations of the Alexander functions prevent $u = z$ from contributing to $\mathcal{O}_1(\tilde{\mathcal{D}}(K,Z))$, we get

$$\mathcal{O}_1(\tilde{\mathcal{D}}(K,Z)) = \sum_{(i,j)\in\{1,\ldots,g\}^2, j\neq 1} \mathcal{O}_1(\overline{\mathcal{A}}_A(\hat{z}(\frac{a_j}{z_i})))\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{a}(\frac{z_i}{a_j})))$$

where:

$$\frac{\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{a}(\frac{z_i}{a_j})))}{\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{a}))} = \frac{<\hat{a}(\frac{z_i}{a_j}),\hat{b}>}{<\hat{a},\hat{b}>}$$

(Recall that $<\hat{a},\hat{b}> = det[<a_i,b_j>]_{i,j=1,\ldots,g}$. $\hat{a}(\frac{z_i}{a_j})$ is defined as in 3.6.)
All these equations together give

$$w = \mathcal{O}_1\left(\sum_{(i,j)\in\{1,\ldots,g\}^2, j\neq 1} \frac{<\hat{a}(\frac{z_i}{a_j}),\hat{b}>}{<\hat{a},\hat{b}>}\frac{\overline{\mathcal{A}}_A(\hat{z}(\frac{a_j}{z_i}))}{|H_1(A_z)|}\right)$$

Therefore, w is independent of B. $\square$

Let u be a curve of ∂A. In $H_1(\partial A;\mathbf{Q})$, we have:

$$u = \sum_{i=1}^{g} u_{a_i}a_i + \sum_{i=1}^{g} u_{b_i}b_i$$

Let u^B be the same curve on $\partial B \subset E$. In $H_1(E;\mathbf{Q}) = \mathbf{Q}m_K \oplus \mathbf{Q}m_Z$, we have:

$$u^B = u_{a_1}m_Z = \frac{<u,b_1>}{<a_1,b_1>}m_Z \tag{4.7}$$

Now, let u^A denote u^B pushed on $\{-3\} \times \partial A$ in $A \cup B$ (through (K,Z)). In $H_1(E;\mathbf{Q}) = \mathbf{Q}m_K \oplus \mathbf{Q}m_Z$, we have:

$$u^B - u^A = < u, a_1 > m_K + < u, z_1 > m_Z$$

Thus:

$$u^A = < a_1, u > m_K + \left(\frac{< u, b_1 >}{< a_1, b_1 >} - < u, z_1 > \right) m_Z \tag{4.8}$$

We denote the partial derivatives in $\mathbf{Q}[[m_K, m_Z]]$ evaluated at $(m_K = 0, m_Z = 0)$ by $\frac{\partial}{\partial m_K}$ and $\frac{\partial}{\partial m_Z}$. With this notation:

Lemma 4.9 *Let η be the number defined in the statement of Property 10,*

$$\frac{\zeta(K,Z)}{|H_1(A \cup B)|} = \eta \sum_{(j,k)\in\{1,\ldots,g\}^2} \mathcal{I}_{A_a}(a_1 \wedge a_j \wedge a_k) \frac{\partial}{\partial m_Z} \left(\frac{\overline{\mathcal{A}}_B(\hat{a}(\frac{z_k}{a_j}))}{|H_1(B_a)|} \right) + \mathcal{F}(\mathcal{L}_B)$$

PROOF: By definition,

$$\frac{\zeta(K,Z)}{|H_1(A \cup B)|} = \frac{\partial^2}{\partial m_K \partial m_Z} \left(-exp(w) \sum_{u \in P} \frac{\overline{\mathcal{A}}_A(\hat{u})}{|H_1(A_z)|} \frac{\overline{\mathcal{A}}_B(\hat{v})}{|H_1(B_a)|} \frac{\hat{u} \wedge \hat{v}}{\hat{z} \wedge \hat{a}} \right)$$

Since Equation 4.7 implies that $\overline{\mathcal{A}}_B$ takes its values in $\mathbf{Q}[[m_Z]]$, this equation may be rewritten as:

$$\frac{\zeta(K,Z)}{|H_1(A \cup B)|} = -\sum_{u \in P} \frac{\partial}{\partial m_K} \left(exp(w) \frac{\overline{\mathcal{A}}_A(\hat{u})}{|H_1(A_z)|} \right) \frac{\partial}{\partial m_Z} \left(\frac{\overline{\mathcal{A}}_B(\hat{v})}{|H_1(B_a)|} \right) \frac{\hat{u} \wedge \hat{v}}{\hat{z} \wedge \hat{a}}$$

$$-\sum_{u \in P} \frac{\partial^2}{\partial m_K \partial m_Z} \left(exp(w) \frac{\overline{\mathcal{A}}_A(\hat{u})}{|H_1(A_z)|} \right) \frac{\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{v}))}{|H_1(B_a)|} \frac{\hat{u} \wedge \hat{v}}{\hat{z} \wedge \hat{a}} \tag{4.10}$$

Since $exp(w) = \mathcal{F}(\mathcal{L}_B)$, and, since for any $u \in P$,

$$\frac{\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{v}))}{|H_1(B_a)|} = \frac{\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{v}))}{\mathcal{O}_0(\overline{\mathcal{A}}_B(\hat{a}))} = \frac{< \hat{v}, \hat{b} >}{< \hat{a}, \hat{b} >} = \mathcal{F}(\mathcal{L}_B)$$

we can rewrite Equation 4.10 as:

$$\frac{\zeta(K,Z)}{|H_1(A \cup B)|} = -\sum_{u \in P} \frac{\partial}{\partial m_K} \left(exp(w) \frac{\overline{\mathcal{A}}_A(\hat{u})}{|H_1(A_z)|} \right) \frac{\partial}{\partial m_Z} \left(\frac{\overline{\mathcal{A}}_B(\hat{v})}{|H_1(B_a)|} \right) \frac{\hat{u} \wedge \hat{v}}{\hat{z} \wedge \hat{a}} + \mathcal{F}(\mathcal{L}_B)$$

Now, Property 4 of the Alexander function implies that the terms where u contains more than one curve of a have at least order 2 and thus may be dropped from the sum of the right hand-side. The term where $u = z$ may also be dropped because of the chosen normalization of $\overline{\mathcal{A}}_B$ (see Equation 4.3). Then the remaining

$\hat{u}$ are of the form $\hat{z}(\frac{a_j}{z_k})$. Since the order of the corresponding terms is at least one, we may drop w and find:

$$\frac{\zeta(K,Z)}{|H_1(A\cup B)|} =$$

$$\sum_{(j,k)\in\{1,\dots,g\}^2, j\neq 1} \frac{\partial}{\partial m_K}\left(\frac{\overline{\mathcal{A}}_A(\hat{z}(\frac{a_j}{z_k}))}{|H_1(A_z)|}\right)\frac{\partial}{\partial m_Z}\left(\frac{\overline{\mathcal{A}}_B(\hat{a}(\frac{z_k}{a_j}))}{|H_1(B_a)|}\right) + \mathcal{F}(\mathcal{L}_B)$$

We conclude the proof by computing $\frac{\partial}{\partial m_K}\left(\frac{\overline{\mathcal{A}}_A(\hat{z}(\frac{a_j}{z_k}))}{|H_1(A_z)|}\right)$ with the help of Equation 4.8 and Property 10 which imply:

$$\frac{\overline{\mathcal{A}}_A(\hat{z}(\frac{a_j}{z_k}))}{|H_1(A_z)|} = \eta\sum_{i=1}^{g}\mathcal{I}_{A_a}(a_i\wedge a_j\wedge a_k)(< a_1, z_i > m_K + c(z_i)m_Z) + O(2)$$

□

Thus, to prove Lemma 4.1, we are left with the proof of the following lemma:

Lemma 4.11 *For any $(j,k)\in\{1,\dots,g\}^2$,*

$$\frac{\partial}{\partial m_Z}\left(\frac{\overline{\mathcal{A}}_B(\hat{a}(\frac{z_k}{a_j}))}{|H_1(B_a)|}\right) = \eta\mathcal{I}_{B_b}(\beta_1\wedge\beta_k\wedge\beta_j) + \mathcal{F}(\mathcal{L}_B)$$

where, for any $k\in\{1,\dots,g\}$, β_k is the element of $\mathcal{L}_B$ defined so that $< a_j, \beta_k > = \delta_{jk}$ for any $j\in\{1,\dots,g\}$.

Lemma 4.11 will be proved once we have proved the two following sublemmas:

Sublemma 4.12

$$\frac{\partial}{\partial m_Z}\left(\frac{\overline{\mathcal{A}}_B(\hat{a}(\frac{z_k}{a_j}))}{|H_1(B_a)|}\right) = \eta\sum_{(h,i)\in\{1,\dots,g\}^2}\frac{< \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >}{< \hat{a}, \hat{b} >< a_1, b_1 >}\mathcal{I}_{B_b}(b_1\wedge b_h\wedge b_i)$$

$$+\mathcal{F}(\mathcal{L}_B)$$

Sublemma 4.13 *In $\bigwedge^3\mathcal{L}_B$,*

$$\sum_{(h,i)\in\{1,\dots,g\}^2}\frac{< \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >}{< \hat{a}, \hat{b} >< a_1, b_1 >} b_1\wedge b_h\wedge b_i = \beta_1\wedge\beta_k\wedge\beta_j$$

□

PROOF OF SUBLEMMA 4.12: We equip b with a geometrically dual ∂B-system $y = (y_1,\dots,y_g)$ (independent of B):

$$< b_i, y_j >_{\partial B} = < y_j, b_i > = \delta_{ij}$$

Computing $\mathcal{A}_B(\hat{y}(\frac{b_h}{y_i}))$

Property 10 implies that:

$$\mathcal{A}_B(\hat{y}(\frac{b_h}{y_i})) = \varepsilon(\mathcal{A}_B(\hat{y}))\eta \sum_{s=1}^{g} \mathcal{I}_{B_b}(b_s \wedge b_h \wedge b_i)(exp(y_s) - 1) + O(2)$$

Here:

$$\varepsilon(\mathcal{A}_B(\hat{y})) = \varepsilon(\mathcal{A}_B(\hat{a}))\frac{< \hat{y}, \hat{b} >}{< \hat{a}, \hat{b} >} = \frac{|H_1(B_a)|}{< \hat{a}, \hat{b} >}$$

and the image of $exp(y_s)$ in Λ_E is $exp\left(\frac{<y_s,b_1>}{<a_1,b_1>}m_Z = \frac{\delta_{1s}}{<a_1,b_1>}m_Z\right)$. (See Equation 4.7.) Thus, we have the following equation

$$\overline{\mathcal{A}}_B(\hat{y}(\frac{b_h}{y_i})) = \frac{|H_1(B_a)|\eta}{< a_1, b_1 >< \hat{a}, \hat{b} >}\mathcal{I}_{B_b}(b_1 \wedge b_h \wedge b_i)m_Z + O(2) \qquad (4.14)$$

Introducing more notation for the proof

To compute $\mathcal{O}_1\left(\overline{\mathcal{A}}_B(\hat{a}(\frac{z_k}{a_j}))\right)$ with the help of Equation 4.14, we first express the a_i and the z_i, as $\mathbf{Q}[[m_Z]]$-combinations of the b_j and the y_j in $\mathcal{H}_E$. We write, for a curve t of ∂B based at $\star$, in $\mathcal{H}_E$:

$$t = \sum_{j=1}^{g} << t, -y_j >> b_j + \sum_{j=1}^{g} << t, b_j >> y_j$$

We denote the Λ_E-coordinates of t by $<< t, . >>$ to remind that

$$\varepsilon(<< t, . >>) =< t, . >_{\partial A} =< t, . >$$

Note that these coordinates (when written in $\mathbf{Q}[[m_Z]]$) do not depend on B. If $u = \{u_1, \dots, u_g\}$ is a subset of $\{a_i, z_i\}$, and if $v = \{v_1, \dots, v_g\}$ is a subset of $\{b_i, (-y_i)\}$, $<< \hat{u}, \hat{v} >>$ denotes the determinant:

$$<< \hat{u}, \hat{v} >>= det[<< u_i, v_j >>]_{i,j=1,\dots,g}$$

Computing $\mathcal{O}_1(\overline{\mathcal{A}}_B(\hat{y}))$

With the chosen normalization of $\overline{\mathcal{A}}_B$ (see Equation 4.3), we have:

$$0 = \mathcal{O}_1(\overline{\mathcal{A}}_B(\hat{a}))$$

$$= \mathcal{O}_1\left(<< \hat{a}, \hat{b} >> \overline{\mathcal{A}}_B(\hat{y})\right) + \sum_{(h,i)\in\{1,\dots,g\}^2} \mathcal{O}_1\left(<< \hat{a}, \hat{b}(\frac{-y_h}{b_i}) >> \overline{\mathcal{A}}_B(\hat{y}(\frac{b_h}{y_i}))\right)$$

So,

$$\mathcal{O}_1(\overline{\mathcal{A}}_B(\hat{y})) = -\sum_{(h,i)\in\{1,\dots,g\}^2} \frac{< \hat{a}, \hat{b}(\frac{-y_h}{b_i}) >}{< \hat{a}, \hat{b} >}\mathcal{O}_1\left(\overline{\mathcal{A}}_B(\hat{y}(\frac{b_h}{y_i}))\right) + |H_1(B_a)|\mathcal{F}(\mathcal{L}_B)$$

Computing $\mathcal{O}_1(\overline{\mathcal{A}}_B(\hat{a}(\frac{z_k}{a_j})))$

$$\begin{aligned}
\mathcal{O}_1(\overline{\mathcal{A}}_B(\hat{a}(\frac{z_k}{a_j}))) &= \mathcal{O}_1\left(<< \hat{a}(\frac{z_k}{a_j}), \hat{b} >> \overline{\mathcal{A}}_B(\hat{y})\right) \\
&+ \sum_{(h,i)\in\{1,\ldots,g\}^2} < \hat{a}(\frac{z_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) > \mathcal{O}_1\left(\overline{\mathcal{A}}_B(\hat{y}(\frac{b_h}{y_i}))\right) \\
&= - < \hat{a}(\frac{z_k}{a_j}), \hat{b} > \sum_{(h,i)\in\{1,\ldots,g\}^2} \frac{< \hat{a}, \hat{b}(\frac{-y_h}{b_i}) >}{< \hat{a}, \hat{b} >} \mathcal{O}_1\left(\overline{\mathcal{A}}_B(\hat{y}(\frac{b_h}{y_i}))\right) \\
&+ \sum_{(h,i)} < \hat{a}(\frac{z_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) > \mathcal{O}_1\left(\overline{\mathcal{A}}_B(\hat{y}(\frac{b_h}{y_i}))\right) + |H_1(B_a)|\mathcal{F}(\mathcal{L}_B) \\
&= \eta|H_1(B_a)| \sum_{(h,i)\in\{1,\ldots,g\}^2} \frac{c(h,i,j,k)}{< \hat{a}, \hat{b} >< a_1, b_1 >} \mathcal{I}_{B_b}(b_1 \wedge b_h \wedge b_i) m_Z \\
&+ |H_1(B_a)|\mathcal{F}(\mathcal{L}_B)
\end{aligned}$$

with:

$$\begin{aligned}
c(h,i,j,k) &= < \hat{a}(\frac{z_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) > - \frac{< \hat{a}(\frac{z_k}{a_j}), \hat{b} >}{< \hat{a}, \hat{b} >} < \hat{a}, \hat{b}(\frac{-y_h}{b_i}) > \\
&= < \hat{a}\left(\frac{z_k - \frac{< \hat{a}(\frac{z_k}{a_j}), \hat{b} > a_j}{< \hat{a}, \hat{b} >}}{a_j}\right), \hat{b}(\frac{-y_h}{b_i}) > \\
&= < \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >
\end{aligned}$$

To justify the last equality, observe that:

$$z_k - \beta_k = \sum_{j=1}^{g} \frac{< \hat{a}(\frac{z_k}{a_j}), \hat{b} >}{< \hat{a}, \hat{b} >} a_j$$

and that $< \hat{a}(\frac{a_i}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >= 0$ if $i \neq j$. □

Proof of Sublemma 4.13: Since the two forms $< ., z_k >$ and $< ., \beta_k >$ are the same on $\mathcal{L}_A$, we have:

$$\beta_k = \sum_{s=1}^{g} \frac{< \hat{a}, \hat{b}(\frac{z_k}{b_s}) >}{< \hat{a}, \hat{b} >} b_s \tag{4.15}$$

To compute $< \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >$, we replace β_k by this expression, and next drop all the b_s for $s \neq h$, because for them $< \hat{a}(\frac{b_s}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >$ is zero. (The j^{th}

row of the underlying matrix is zero.) Thus, we get:

$$< \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >= \frac{< \hat{a}, \hat{b}(\frac{z_k}{b_h}) >}{< \hat{a}, \hat{b} >} < \hat{a}(\frac{b_h}{a_j}), \hat{b}(\frac{-y_h}{b_i}) > \tag{4.16}$$

Observe that

$$< \hat{a}(\frac{b_h}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >=< \hat{a}, \hat{b}(\frac{z_j}{b_i}) >$$

because both sides are the cofactor of the term $(s,t) = (j,i)$ in the matrix $[< a_s, b_t >]_{s,t=1,\dots,g}$. Thus, we can rewrite Equation 4.16 as follows:

$$< \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >= \frac{< \hat{a}, \hat{b}(\frac{z_k}{b_h}) >< \hat{a}, \hat{b}(\frac{z_j}{b_i}) >}{< \hat{a}, \hat{b} >}$$

So, in $\bigwedge^2 \mathcal{L}_B$, we have:

$$\sum_{(h,i)\in\{1,\dots,g\}^2} \frac{< \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >}{< \hat{a}, \hat{b} >}(b_h \wedge b_i)$$

$$= \left(\sum_{h=1}^{g} \frac{< \hat{a}, \hat{b}(\frac{z_k}{b_h}) >}{< \hat{a}, \hat{b} >} b_h\right) \wedge \left(\sum_{i=1}^{g} \frac{< \hat{a}, \hat{b}(\frac{z_j}{b_i}) >}{< \hat{a}, \hat{b} >} b_i\right)$$

Hence, according to Equation 4.15:

$$\sum_{(h,i)\in\{1,\dots,g\}^2} \frac{< \hat{a}(\frac{\beta_k}{a_j}), \hat{b}(\frac{-y_h}{b_i}) >}{< \hat{a}, \hat{b} >}(b_h \wedge b_i) = \beta_k \wedge \beta_j \tag{4.17}$$

In order to conclude the proof of Sublemma 4.13, recall that we have chosen b so that $< a_i, b_j >= 0$, if $i > j$, thus:

$$\beta_1 = \frac{1}{< a_1, b_1 >} b_1$$

□

This concludes the proof of Lemma 2.28 and the proof of Theorem 1.3.

4.2 Proving Lemma 2.32

The proof of Lemma 2.32 will be very similar to the previous one. In particular, we will need Lemma 4.11 that we restate to isolate its own hypotheses.

Lemma 4.18 *Let Σ be a closed, connected surface equipped with a Lagrangian $\mathcal{L}_B$. Let b be a $(\Sigma, \mathcal{L}_B)$-system. Let c be a Σ-system which generates a Lagrangian transverse to $\mathcal{L}_B$ and let x be a Σ-system dual to c. Assume that x and c are equipped with paths joining them to the basepoint of Σ. Let $(\beta_1, \dots, \beta_g)$ be the basis* *of* $\mathcal{L}_B$ *defined by* $< c_j, \beta_k >_\Sigma = \delta_{jk}$ *for any* $j,k \in \{1,\dots,g\}$*. For a* $(-\Sigma,\mathcal{L}_B)$*-RHH* B*, we define the ring morphism*

$$\phi_{\beta_1} : \Lambda_B \longrightarrow \mathbf{Q}[[t]]$$

so that, if $u \in H_1(\partial B)$*, then* $\phi_{\beta_1}(exp(u)) = exp(< u, \beta_1 >_\Sigma t)$*, and we denote:*

$$\overline{\mathcal{A}}_B \stackrel{\rm def}{=} \phi_{\beta_1} \circ \mathcal{A}_B$$

Then there exists a number $\mathcal{F}(\mathcal{L}_B)$ *such that, for any* $(-\Sigma,\mathcal{L}_B)$*-RHH* B*, if* $\overline{\mathcal{A}}_B$ *is normalized so that*

$$\overline{\mathcal{A}}_B(\hat{c}) = |H_1(B_c)| + \mathcal{O}(2),$$

then

$$\frac{\partial}{\partial t}\left(\frac{\overline{\mathcal{A}}_B(\hat{c}(\frac{x_k}{c_j}))}{|H_1(B_c)|}\right)_{t=0} = \eta \mathcal{I}_{B_b}(\beta_1 \wedge \beta_k \wedge \beta_j) + \mathcal{F}(\mathcal{L}_B)$$

for any $(j,k) \in \{1,\dots,g\}^2$*, with the number* η *defined in Property 10.*

□

We assume that we are under the hypotheses (2.30). So, Σ, $\mathcal{L}_A$ $\mathcal{L}_B$, a, b, y and z are fixed, $\mathcal{L}_A \cap \mathcal{L}_B = \mathbf{Q}a_1$, $a_1 = b_1$, $y_1 = -z_1$. For a $(\Sigma,\mathcal{L}_A)$-RHH A and a $(-\Sigma,\mathcal{L}_B)$-RHH B, $K \subset A^C \cup B$ is the knot whose exterior is $A \cup_\Sigma B$ unglued along z_1 and whose meridian is a parallel of z_1 on Σ.

We let A vary among the $(\Sigma,\mathcal{L}_A)$-RHH and we let B vary among the $(-\Sigma,\mathcal{L}_B)$-RHH. We denote by $\mathcal{F}(A,\mathcal{L}_B)$ *any coefficient which does not depend on the* $(-\Sigma,\mathcal{L}_B)$*-RHH* B (but may depend on A) and we denote by $\mathcal{F}(B,\mathcal{L}_A)$ *any coefficient which does not depend on the* $(\Sigma,\mathcal{L}_A)$*-RHH* A.

We prove Lemma 2.32 in the following equivalent form:

Lemma 4.19 *Under the above hypotheses,*

$$\frac{\zeta(K \subset A^C \cup B)}{|H_1(A_z)|\,|H_1(B_y)|\,|< \hat{a}, \hat{b} >_{>1}|} =$$

$$-\sum_{(j,k)\in\{2,\dots,g\}^2} \mathcal{I}_{A_a}(a_1 \wedge a_j \wedge a_k)\mathcal{I}_{B_b}(a_1 \wedge \beta_j \wedge \beta_k) + \mathcal{F}(A,\mathcal{L}_B) + \mathcal{F}(B,\mathcal{L}_A)$$

where $\beta_k \in \mathcal{L}_B$ *satisfies* $< a_j, \beta_k > = \delta_{jk}$ *for any* j*, for any* $k > 1$*.*

Proof: (Details which can be found in Subsection 4.1 are often omitted.)

Again, we first express the Alexander series of K in terms of the Alexander functions of A and B. We view the exterior E of K as $A \cup_\Sigma B$ unglued along z_1. Thus, E contains both A and B. This provides canonical maps from Λ_A and

Λ_B to Λ_E that we see as a subset of $\mathbf{Q}[[z_1]]$. $\overline{\mathcal{A}}_A$ (respectively $\overline{\mathcal{A}}_B$) denotes $\mathcal{A}_A$ (respectively $\mathcal{A}_B$) followed by the previous canonical morphisms. $\overline{\mathcal{A}}_A$ and $\overline{\mathcal{A}}_B$ take their values in $\mathbf{Q}[[z_1]]$. We normalize them so that:

$$\overline{\mathcal{A}}_A(\hat{z}) = |H_1(A_z)| + O(2) \tag{4.20}$$

and,

$$\overline{\mathcal{A}}_B(\hat{c}) = |H_1(B_c)| + O(2) \tag{4.21}$$

where c is the Σ-system $(z_1, a_2, a_3, \ldots, a_g)$.

Consider the collection κ of curves of $\partial(A\sharp_\partial B)$: $\kappa_i = z_i(z_i^B)^{-1}$, for $i = 1, \ldots, g$, $\kappa_{i+g} = a_i^B a_i^{-1}$, for $i = 2, \ldots, g$. E is homeomorphic to $(A\sharp_\partial B)_\kappa$. Thus,

$$\tau(E) \doteq \pm\phi_\kappa \left(\frac{\mathcal{A}_{A\sharp_\partial B}(\bigwedge_{i=1}^{g}(z_i - z_i^B) \wedge z_1 \wedge \bigwedge_{i=2}^{g}(a_i^B - a_i))}{\partial(z_1)} \right)$$

and since $(exp(z_1/2) - exp(-z_1/2))\mathcal{D}(K)$ must be symmetric and positive at 0,

$$\frac{exp(\frac{z_1}{2}) - exp(-\frac{z_1}{2})}{|H_1(B_c)|}\mathcal{D}(K) =$$

$$exp(w)\Big(\overline{\mathcal{A}}_A(\hat{z})\frac{\overline{\mathcal{A}}_B(\hat{c})}{|H_1(B_c)|} - \sum_{(j,k)\in\{2,\ldots,g\}^2} \overline{\mathcal{A}}_A(\hat{z}(\frac{a_j}{z_k}))\frac{\overline{\mathcal{A}}_B(\hat{c}(\frac{z_k}{a_j}))}{|H_1(B_c)|} + \mathcal{F}(A, \mathcal{L}_B) + \mathcal{O}(3) \Big)$$

where $w \in \mathbf{Q}z_1$ is independent of the $(-\Sigma, \mathcal{L}_B)$-RHH B.

Now, recall from Definition 3.19 that:

$$\zeta(K) = \frac{1}{2}\frac{\partial^2}{\partial z_1^2}(z_1\mathcal{D}(K))$$

Therefore

$$\zeta(K) = \frac{1}{2}\frac{\partial^2}{\partial z_1^2}\left(\left(exp(\frac{z_1}{2}) - exp(\frac{-z_1}{2})\right)\mathcal{D}(K)\right) - \frac{|H_1(A_z)|\,|H_1(B_c)|}{24}$$

and

$$\frac{\zeta(K)}{|H_1(B_c)|} = \frac{1}{2}\frac{\partial^2}{\partial z_1^2}\left(\frac{exp(\frac{z_1}{2}) - exp(\frac{-z_1}{2})}{|H_1(B_c)|}\mathcal{D}(K)\right) + \mathcal{F}(A, \mathcal{L}_B)$$

(Again, the derivatives with respect of z_1 are evaluated at $(z_1 = 0)$.)

With the chosen normalizations of $\overline{\mathcal{A}}_A$ and $\overline{\mathcal{A}}_B$, we get:

$$\frac{\zeta(K \subset A^C \cup B)}{|H_1(A_z)|\,|H_1(B_c)|} =$$

$$-\sum_{(j,k)\in\{2,\ldots,g\}^2} \frac{\partial}{\partial z_1}\left(\frac{\overline{\mathcal{A}}_A(\hat{z}(\frac{a_j}{z_k}))}{|H_1(A_z)|}\right)\frac{\partial}{\partial z_1}\left(\frac{\overline{\mathcal{A}}_B(\hat{c}(\frac{z_k}{a_j}))}{|H_1(B_c)|}\right) + \mathcal{F}(A,\mathcal{L}_B) + \mathcal{F}(B,\mathcal{L}_A)$$

Applying Property 10 of the Alexander function to compute $\frac{\partial}{\partial z_1}(\overline{\mathcal{A}}_A)$ and Lemma 4.18 with

$$x = (-a_1, z_2, \ldots, z_g) \qquad \text{and} \qquad \beta_1 = -a_1$$

to compute $\frac{\partial}{\partial z_1}(\overline{\mathcal{A}}_B)$ now yields:

$$\frac{\zeta(K \subset A^C \cup B)}{|H_1(A_z)|\,|H_1(B_c)|} =$$

$$-\sum_{(j,k)\in\{2,\ldots,g\}^2} \mathcal{I}_{A_a}(a_1 \wedge a_j \wedge a_k)\mathcal{I}_{B_b}(-a_1 \wedge \beta_k \wedge \beta_j) + \mathcal{F}(A,\mathcal{L}_B) + \mathcal{F}(B,\mathcal{L}_A)$$

□

Thus, Lemma 2.32 is true and Theorem 1.11 follows.

5 The case of non-connected surfaces

Definition 5.1 Let M be a connected oriented closed 3-manifold decomposed along disjoint orientable embedded surfaces into connected pieces A_i which are 3-manifolds. (M is the union of the A_i, and for $i \neq j$, $A_i \cap A_j$ is made of connected components of ∂A_i.) We define the *gluing graph* Γ of this decomposition as follows: The vertices of Γ are in one-to-one correspondence with the A_i and they are labelled with the same symbols while the edges e of Γ correspond to the connected surfaces Σ_e of the decomposition: the endpoints of e are A_i and A_j if and only if $\Sigma_e \subset A_i \cap A_j$.

In this easy and boring section, we generalize the previous formulas to compute $\overline{\lambda}(M)$ in terms of invariants of the A_i.

Remark 5.2 Since M is connected, Γ is connected. Furthermore, $H^1(\Gamma)$ injects into $H^1(M)$. Indeed, let e_1, ... $e_{\beta_1(\Gamma)}$ be $\beta_1(\Gamma)$ edges whose interiors can be removed from Γ without disconnecting Γ. After orienting Γ, mapping the algebraic intersection with an interior point of e_i to the algebraic intersection with Σ_{e_i} for $i = 1, \cdots, \beta_1(\Gamma)$, yields the desired injection. In particular, if $\beta_1(\Gamma)$ is larger than 1, then $\overline{\lambda}(M)$ is zero because of the interpretation of $\overline{\lambda}$ for manifolds of rank larger than 1 (see Proposition 2.9), and, if M is a rational homology sphere, then Γ must be a tree.

We first deal with the case where Γ is a tree.

5.1 The formula when Γ is a tree

We fix Γ, we denote by $V = \{1, \ldots, v\}$ the set of the vertex indices of Γ, and by E the set of the edges of Γ. We identify each edge of E with the non-ordered pair $\{i, j\}$ of its endpoints.

After the introduction of notation and conventions that the reader may guess, the formula will be:

Proposition 5.3 *For any collection a of systems associated to the decomposition of M,*

$$\overline{\lambda}(M) = \sum_{j \in V} \frac{|H_1(M)|}{|H_1(A_j(a))|} \overline{\lambda}(A_j(a)) - \sum_{\{i,j\} \in E} \frac{|H_1(M)|}{|H_1(\Sigma_{ij}(a))|} \overline{\lambda}(\Sigma_{ij}(a)) - 2 \sum_{\{i,l\} \subset V} \overline{T}_{il}(a)$$

Now, for the notation. To avoid subscripts of subscripts of..., we sometimes put what should be a subscript inside parentheses (e.g. $\Sigma(a,b) = \Sigma_{ab}$).

Let $\{i, j\}$ be an edge of E.

Notation 5.4 We denote by Σ_{ij} the surface corresponding to $\{i, j\}$ oriented as part of ∂A_i. Cutting M along Σ_{ij} splits M into two parts. We denote by $M_{j[i}$

the closure of the part containing A_i and by $M_{i[j}$ the other. We denote by $\mathcal{L}_{ij}$ the Lagrangian of $M_{i[j}$. Note that

$$\mathcal{L}_{ij} = \mathrm{Ker}\left(H_1(\Sigma_{ij}) \xrightarrow{i_*} \frac{H_1(A_j;\mathbf{Q})}{\oplus_{k\in S(j)\setminus\{i\}} i_*(\mathcal{L}_{jk})}\right)$$

where, for $j \in V$,

$$S(j) \stackrel{\text{def}}{=} \{k | k \neq j, A_k \cap A_j \neq \emptyset\}$$

We equip each edge $\{i,j\}$ of Γ with a $(\Sigma_{ij}, \mathcal{L}_{ij})$-system a_{ij}, and a $(\Sigma_{ji}, \mathcal{L}_{ji})$-system a_{ji}. We use this collection a of systems to transform the pieces of the decomposition into closed 3-manifolds.

Notation 5.5

$$\Sigma_{ij}(a) \stackrel{\text{def}}{=} \Sigma_{ij}(a_{ji}, a_{ij})$$

$$A_j(a) \stackrel{\text{def}}{=} A_j \cup_{\partial A_j} \left(\coprod_{k\in S(j)} \Sigma_{kj}(a_{jk})\right)$$

(The a in the notation reminds of the dependence on the choices of the systems a_{ij}.) For an edge $\{i,j\}$ of Γ, $\mathcal{I}_{ij}(A_j, a)$ is defined on $\bigwedge^3(\mathcal{L}_{ij})$ as follows:

$$\mathcal{I}_{ij}(A_j, a) = \mathcal{I}\left(\left(A_j(a) \setminus \Sigma_{ij}(a_{ji})\right), a_{ij}\right)$$

Remark 5.6 Even if $|H_1(A_j(a))| = 0$, the ratio $\frac{|H_1(M)|}{|H_1(A_j(a))|}$ makes sense in a canonical way.

Indeed, if z_{ij} denotes a Σ_{ij}-system geometrically dual to a_{ij}, for any a_{ij}, then the following equality always holds:

$$|H_1(M)| = |H_1(A_j(a))| \prod_{i\in S(j)} |H_1(M_{j[i}(z_{ji}))|$$

(see Lemma 2.1). Thus, we set

$$\frac{|H_1(M)|}{|H_1(A_j(a))|} \stackrel{\text{def}}{=} \prod_{i\in S(j)} |H_1(M_{j[i}(z_{ji}))|$$

in any case. Similarly, in any case,

$$\frac{|H_1(M)|}{|H_1(\Sigma_{ij}(a))|} \stackrel{\text{def}}{=} |H_1(M_{j[i}(z_{ji}))|\,|H_1(M_{i[j}(z_{ij}))|$$

Let $\{i,l\} \subset V$. We define $\overline{T}_{il}(a)$.

Notation 5.7 We denote by $[i,l]$ the image of an injective path that joins i and l in Γ. Let j be the unique element of $S(i)\cap[i,l]$ and let k be the unique element of $S(l)\cap[i,l]$. We are about to pair $\mathcal{I}_{ji}(A_i,a)$ and $\mathcal{I}_{kl}(A_l,a)$.

Let $t_{il}:\mathcal{L}_{ij}\longrightarrow\mathcal{L}_{kl}$ be defined as follows. Let $M_{il}\stackrel{\rm def}{=}M_{i[j}\cap M_{l[k}$.

- If $\mathcal{L}(M_{il})\cap\mathcal{L}_{kl}\neq\{0\}$, then $t_{il}=0$.
- Otherwise, for $x\in\mathcal{L}_{ij}$, $t_{il}(x)$ is the unique element of $\mathcal{L}_{kl}$ such that $t_{il}(x)-x=0$ in $H_1(M_{il};\mathbf{Q})$.

If $\{i,l\}$ is an edge, then $M_{il}=\Sigma_{il}$ and $t_{il}=\mathrm{Id}$. (t_{il} may be thought of as an abbreviation for $t_{ij,kl}$.)

$$\overline{T}_{il}(a)\stackrel{\rm def}{=}\frac{|H_1(M)|}{|H_1(\Sigma_{ij}(a))|}\,sign_{\Sigma(i,j)}(\hat{a}_{ji},\hat{a}_{ij})<\hat{a}_{ji}\cap\mathcal{I}_{ji}(A_i,a),\hat{a}_{ij}\cap(\mathcal{I}_{kl}(A_l,a)\circ\bigwedge^3 t_{il})>_{\Sigma(i,j)}$$

Note the natural behaviour of the t_{il} under composition of cobordisms:

Lemma 5.8 *If Γ is the graph Γ_4 in Figure 6, then*

$$\frac{|H_1(M)|}{|H_1(A_1(a))|}t_{14}=\frac{|H_1(M)|}{|H_1(A_1(a))|}t_{24}\circ t_{13}$$

Figure 6: Γ_4

The proof is immediate. □

The following lemma (which can also be proved directly) will be obvious in a moment.

Lemma 5.9 *For any $\{i,l\}\subset V$, $\overline{T}_{il}(a)=\overline{T}_{li}(a)$.*

(Note only that it is already obvious if $\{i,l\}$ is an edge.)

Of course, Proposition 5.3 is true if $v=1$. If $v=2$, it is nothing but Theorem 1.11. Let us now prove it for $v=3$. We may assume that Γ is the graph Γ_3 in Figure 7.

In this case, it suffices to apply Theorem 1.11 to compute $\overline{\lambda}(M)$ from A_1 and $(A_2\cup_{\Sigma(2,3)}A_3)$, first, and $\overline{\lambda}((A_2\cup_{\Sigma(2,3)}A_3)_{a(2,1)})$ from A_2 and A_3, next, and to use the following easy lemma (left to the reader).

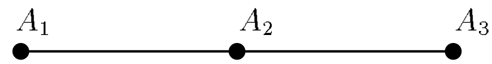

Figure 7: Γ_3

Lemma 5.10

$$\frac{|H_1(M)|}{|H_1(A_1(a))|}\mathcal{I}((A_2 \cup_{\Sigma(2,3)} A_3), a_{12}) = \frac{|H_1(M)|}{|H_1(A_1(a))|}\left(\mathcal{I}_{12}(A_2, a) + \mathcal{I}_{23}(A_3, a) \circ \bigwedge^3 t_{13}\right)$$

This gives rise to the formula of Proposition 5.3 with the number $\overline{T}_{13}(a)$ associated to the edge $\{1, 3\}$. Exchanging the roles of 1 and 3 in the process yields the same formula with $\overline{T}_{31}(a)$ instead of $\overline{T}_{13}(a)$. This proves:

$$\overline{T}_{31}(a) = \overline{T}_{13}(a)$$

and this is enough to prove Lemma 5.9 in general.

Now, the formula of Proposition 5.3 actually makes sense and it suffices to prove it by induction on v using Lemma 5.8. This is left to the reader.

□

Remark 5.11 For the Casson-Walker invariant, the formula simply reads:

$$\lambda(M) = \sum_{j \in V} \lambda(A_j(a)) - \sum_{\{i,j\} \in E} \lambda(\Sigma_{ij}(a)) - 2 \sum_{\{i,l\} \subset V} \frac{\overline{T}_{il}(a)}{|H_1(M)|}$$

where

$$\frac{\overline{T}_{il}(a)}{|H_1(M)|} = \frac{< \hat{a}_{ji} \cap \mathcal{I}_{ji}(A_i, a), \hat{a}_{ij} \cap (\mathcal{I}_{kl}(A_l, a) \circ \bigwedge^3 t_{il}) >_{\Sigma(i,j)}}{< \hat{a}_{ji}, \hat{a}_{ij} >_{\Sigma(i,j)}}$$

As a conclusion, if M is a rational homology sphere, to compute $\lambda(M)$ the only things that we need to know about an A_j are $\lambda(A_j(a))$, the $\mathcal{I}_{ij}(A_j, a)$, for $i \in S(j)$, and the transfer morphisms t_{ik}, for $\{i, k\} \subset S(j)$, (these are given by $\mathcal{L}_{A_j}$), for a collection a of systems associated to the decomposition of M.

5.2 The formula when Γ is a simple cycle

In this case, M may be written as

$$M = \frac{A}{\Sigma^+ \sim \Sigma^-}$$

where A is a 3-manifold with boundary $\partial A = \Sigma^+ \cup -\Sigma^-$ where Σ^+ and Σ^- are identified with a genus g surface Σ via the (orientation-preserving) homeomorphisms ϕ^+ and ϕ^-.

$$\Sigma^+ \overset{\phi^+}{\longleftarrow} \Sigma \overset{\phi^-}{\longrightarrow} \Sigma^-$$

The homogeneous formula below expresses $\overline{\lambda}(M)$ in terms of $\mathcal{L}_A$ ($\subset H_1(\partial A;\mathbf{Q})$) and $|H_1(A_c)|$ for a ∂A-system c transverse to $\mathcal{L}_A$ (that is the union of a Σ^+-system and a Σ^--system which generate a vector space transverse to $\mathcal{L}_A$ inside $H_1(\partial A,\mathbf{Q})$).

Proposition 5.12 *Under the hypotheses above, let $\alpha = (\alpha_1,\dots,\alpha_{2g})$ be a basis of $\mathcal{L}_A$, let $\sigma = (\sigma_1,\dots,\sigma_{2g})$ be a basis of $H_1(\Sigma;\mathbf{Z})$ and let $b_i \stackrel{\text{def}}{=} (\phi^- - \phi^+)(\sigma_i)$ be the corresponding basis of $\mathcal{L}_B \stackrel{\text{def}}{=} (\phi^- - \phi^+)(H_1(\Sigma;\mathbf{Q}))$. Then*

$$\overline{\lambda}(M) =$$

$$\frac{|H_1(A_c)|}{|<\hat{\alpha},\hat{c}>_{\partial A}|} sign_{\partial A}(\hat{\alpha},\hat{b}) \left[\left(\frac{\partial^2}{\partial t^2}\right)_{t=1} \frac{<\hat{\alpha},(t^{1/2}\phi^- - t^{-1/2}\phi^+)(\hat{\sigma})>_{\partial A}}{2} - \frac{<\hat{\alpha},\hat{b}>_{\partial A}}{12} \right]$$

where the intersection form on ∂A is linearly extended over $\bigwedge^2(H_1(\partial A;\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}[t^{1/2},t^{-1/2}])$, the hats have their usual signification (e.g. $\hat{\alpha} = \alpha_1 \wedge \cdots \wedge \alpha_{2g}$), and $sign_{\partial A}(\hat{\alpha},\hat{b}) = \pm 1$ is defined so that:

1. *$sign_{\partial A}(\hat{\alpha},\hat{b})(\hat{\alpha}\otimes\hat{b}) \in \bigwedge^{2g}\mathcal{L}_A \otimes_{\mathbf{Q}} \bigwedge^{2g}\mathcal{L}_B$ is independent of the bases α and σ up to a multiplication by a positive number.*

2. *If $\alpha_i = b_i$, for $i = 1,\dots d = \dim(\mathcal{L}_A \cap \mathcal{L}_B)$, then*

$$sign_{\partial A}(\hat{\alpha},\hat{b}) = sign(<\hat{\alpha},\hat{b}>_{\partial A,>d})$$

$$(<\hat{\alpha},\hat{b}>_{\partial A,>d} \stackrel{\text{def}}{=} det([<\alpha_i,b_j>_{\partial A}]_{i,j=d+1,\dots,2g}))$$

PROOF: First note that the proposition is true if $|H_1(A_c)| = 0$. Indeed, in this case, there exists a closed surface, say S, in A such that the homology classes of S and Σ^+ are independent in $H_2(A)$. Thus, since S and Σ^+ do not algebraically intersect, $\overline{\lambda}(M) = 0$.

Hypotheses 5.13 From now on, we assume that $|H_1(A_c)| \neq \{0\}$. Without loss, we assume that α and b satisfy $\alpha_i = b_i$, for $i = 1,\dots,d = \dim(\mathcal{L}_A \cap \mathcal{L}_B)$. For $i = 1,\dots,d$, we denote by A_i the class in $H_2(M)$ of a rational 2-chain of A with boundary α_i.

The proposition will be proved once we have proved the two following lemmas:

Lemma 5.14 *Under the hypotheses of 5.13,*

$$\overline{\lambda}(M) = \frac{|H_1(A_c)|}{|<\hat{\alpha},\hat{c}>_{\partial A}|} \,|<\hat{\alpha},\hat{b}>_{>d}|\, q_M([\Sigma^+]\wedge A_1 \wedge\cdots\wedge A_d)$$

where q_M is described in Definition 2.8.

Lemma 5.15 *Under the hypotheses of 5.13,*

$$< \hat{\alpha}, \hat{b} >_{>d} q_M([\Sigma^+] \wedge A_1 \wedge \cdots \wedge A_d) =$$

$$\left(\frac{\partial^2}{\partial t^2}\right)_{t=1} \frac{< \hat{\alpha}, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\hat{\sigma}) >_{\partial A}}{2} - \frac{< \hat{\alpha}, \hat{b} >_{\partial A}}{12}$$

Proof of Lemma 5.14: In order to use our previous work on connected surfaces, we remove the interior of the tubular neighborhood T of a path connecting Σ^+ and Σ^- from A to transform A into $\check{A}$ with $\partial\check{A} = \Sigma^+ \sharp - \Sigma^-$. See Figure 8. ($H_1(\check{A}; \mathbf{Q}) = H_1(A; \mathbf{Q})$ and $\mathcal{L}_{\check{A}} = \mathcal{L}_A$.)

We assume that $(\phi^+)^{-1}(\partial T \cap \Sigma^+)$ and $(\phi^-)^{-1}(\partial T \cap \Sigma^-)$ are equal to the same disk D of Σ so that the image of T in M is an embedded solid torus. Let k be a meridian of T inside $\check{A}$. Equip it with its preferred parallel $\mu(k)$ (which bounds in $\check{A} \setminus k$, and which may be supposed to lie on $\partial\check{A}$). Let $\hat{A}$ denote the manifold obtained from $\check{A}$ by surgery on $(k, \mu(k))$. Observe that:

$$\hat{A} = \frac{A}{\phi^+(D) \sim \phi^-(D)}$$

Thus, setting

$$B \stackrel{\text{def}}{=} (\overline{\Sigma \setminus D}) \times I$$

we may see M as

$$M = \hat{A} \cup_{\partial B} B$$

where $\overline{\Sigma \setminus D} \times \{0\}$, $\overline{\Sigma \setminus D} \times \{1\}$ are naturally identified with $\phi^+(\overline{\Sigma \setminus D})$ and $\phi^-(\overline{\Sigma \setminus D})$, respectively.

Note that the Lagrangian of B is actually $\mathcal{L}_B = (\phi^- - \phi^+)(H_1(\Sigma; \mathbf{Q}))$. This allows us to use Lemma 2.14 to conclude the proof of the lemma. □

Lemma 5.16 *Lemma 5.15 is true when $d = 0$.*

Proof: It suffices to prove the following equality (using the notation of Definition 2.8):

$$\tilde{\Delta}(M) = \frac{< \hat{\alpha}, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\hat{\sigma}) >_{\partial A}}{< \hat{\alpha}, \hat{b} >}$$

Applying the Mayer-Vietoris sequence to compute the homology of the maximal abelian covering of M (decomposed as the union of the trivial coverings of a neighborhood of Σ^+ and M minus a smaller neighborhood of Σ^+) proves this equality up to units of $\mathbf{Q}[t^{\pm 1/2}]$. Thus, it suffices to verify the symmetry of the right-hand side of the equality.

To do this, it suffices to find a basis σ of $H_1(\Sigma; \mathbf{Z})$ and a basis α of $\mathcal{L}_A$ such that

$$< \hat{\alpha}, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\hat{\sigma}) >_{\partial A}$$

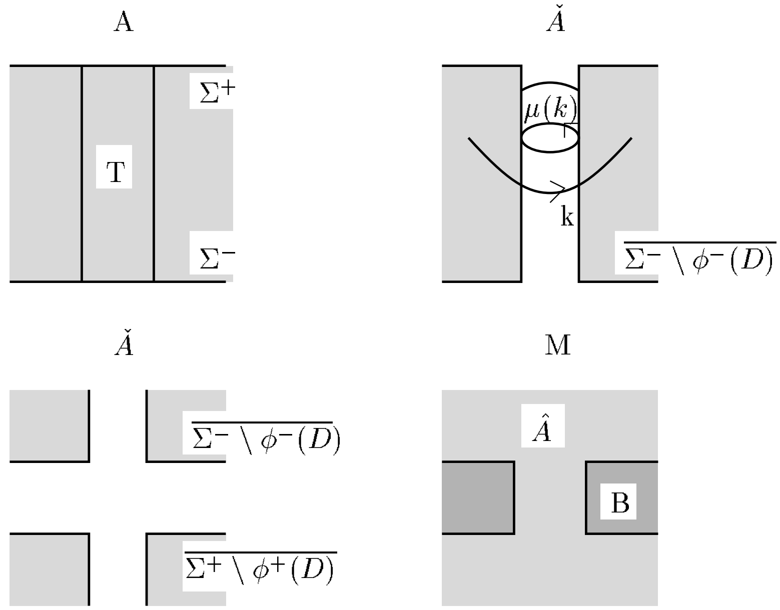

Figure 8: Local slices of A, $\check{A}$, $\hat{A}$ and M

is symmetric. Let us show such bases which make the symmetry clear. For σ, we take a symplectic basis ordered in the following way (to simplify notation):

$$< \sigma_j, \sigma_{2g+1-k} >_\Sigma = \begin{cases} \delta_{jk} & \text{if } j \leq g \\ -\delta_{jk} & \text{if } j > g \end{cases}$$

Now, let α be the basis of $\mathcal{L}_A$ defined by:

$$< \alpha_j, b_{2g+1-k} >_{\partial A} = \begin{cases} -\delta_{jk} & \text{if } j \leq g \\ \delta_{jk} & \text{if } j > g \end{cases}$$

Let y be a curve of Σ. In $H_1(A;\mathbf{Q}) \cong \frac{H_1(\partial A;\mathbf{Q})}{\mathcal{L}_A}$, we have:

$$\phi^-(y) = \sum_{i=1}^{g} - < \alpha_i, \phi^-(y) >_{\partial A} b_{2g+1-i} + \sum_{i=g+1}^{2g} < \alpha_i, \phi^-(y) >_{\partial A} b_{2g+1-i}$$

Let $\phi^-(y)$ also denote $\phi^-(y)$ pushed inside A. With this notation, since b_{2g+1-i} may be viewed as a meridian of $\pm\sigma_i$ in M (see Figure 9), we have:

$$lk_M(\phi^-(y), \sigma_j) = < \alpha_j, \phi^-(y) >_{\partial A}$$

$M \setminus \sigma_i$

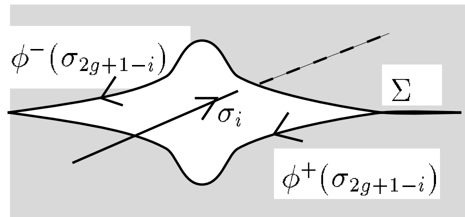

Figure 9: $M \setminus \sigma_i$ for $i \leq g$

Similarly,

$$lk_M(\phi^+(y), \sigma_j) =< \alpha_j, \phi^+(y) >_{\partial A}$$

Thus,

$$\begin{aligned} < \alpha_j, \phi^-(\sigma_i) >_{\partial A} &= lk_M(\phi^-(\sigma_i), \sigma_j) \\ &= lk_M(\sigma_i, \phi^+(\sigma_j)) \\ &= < \alpha_i, \phi^+(\sigma_j) >_{\partial A} \end{aligned}$$

This proves the symmetry and concludes the proof of the lemma. □

In the other cases, the equation of Lemma 5.15 becomes:

$$< \hat{\alpha}, \hat{b} >_{>d} \; q_M([\Sigma^+] \wedge A_1 \wedge \cdots \wedge A_d) =$$
$$\left(\frac{\partial^2}{\partial t^2}\right)_{t=1} \frac{< \hat{\alpha}, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\hat{\sigma}) >_{\partial A}}{2}$$

Lemma 5.17 *Lemma 5.15 is true when $d = 1$.*

Proof: According to the hypotheses (5.13), $(\phi^- - \phi^+)(\sigma_1) = b_1 = \alpha_1$ belongs to $\mathcal{L}_A$. In particular, for any i, $< \alpha_i, \phi^+(\sigma_1) >_{\partial A} =< \alpha_i, \phi^-(\sigma_1) >_{\partial A}$.

Thus, if $\phi^+(\sigma_1) \in \mathcal{L}_A$, the right-hand side of the equality of Lemma 5.15 is zero; since $\phi^+(\sigma_1)$ bounds in A (rationally), its linking number with its parallel along Σ^+ is also zero and the conclusion is easy.

So, we may assume that $\phi^+(\sigma_1) \notin \mathcal{L}_A$ and that, for any i, $< \alpha_i, \phi^+(\sigma_1) >_{\Sigma^+}= \delta_{i2}$. We may also assume that $< \sigma_1, \sigma_i >_\Sigma = \delta_{i2}$. We do. Thus,

$$< \alpha_1, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\sigma_j) >_{\partial A} =< (\phi^- - \phi^+)(\sigma_1), (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\sigma_j) >_{\partial A}$$
$$= -(t^{1/2} - t^{-1/2})\delta_{j2}$$

and

$$< \alpha_i, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\sigma_1) >_{\partial A} = (t^{1/2} - t^{-1/2})\delta_{i2}$$

Hence, we have:

$$< \hat{\alpha}, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\hat{\sigma}) >_{\partial A}=$$
$$(t^{1/2} - t^{-1/2})^2 < \hat{\alpha}, \hat{b} >_{>2} + O((t-1)^3)$$

We are left with the proof that:

$$q_M([\Sigma^+] \wedge A_1) = \frac{< \hat{\alpha}, \hat{b} >_{>2}}{< \hat{\alpha}, \hat{b} >_{>1}}$$

$(-q_M([\Sigma^+] \wedge A_1))$ is the linking number of $\phi^+(\sigma_1)$ with its parallel on Σ^+. Since $\phi^+(\sigma_1)$ is zero in $H_1(M; \mathbf{Q})$, $\phi^+(\sigma_1)$ may be written as a combination of the b_i, $i \geq 2$, and of the α_j in $H_1(\partial A; \mathbf{Q})$; and its linking number with its given parallel is the opposite of its b_2-coordinate. Thus,

$$q_M([\Sigma^+] \wedge A_1) = \frac{< \hat{\alpha}, \hat{b}(\frac{\phi^+(\sigma_1)}{b_2}) >_{>1}}{< \hat{\alpha}, \hat{b} >_{>1}} = \frac{< \hat{\alpha}, \hat{b} >_{>2}}{< \hat{\alpha}, \hat{b} >_{>1}}$$

□

Lemma 5.18 *Lemma 5.15 is true when $d = 2$.*

PROOF: In this case, $q_M([\Sigma^+] \wedge A_1 \wedge A_2) =< \sigma_1, \sigma_2 >^2_{\Sigma}$. On the other hand, if i or $j \in \{1, 2\}$,

$$< \alpha_i, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\sigma_j) >_{\partial A}= (t^{1/2} - t^{-1/2}) < \alpha_i, \phi^+(\sigma_j) >_{\partial A}$$

Thus, dividing the first two columns in the matrix $< \alpha_i, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\sigma_j) >_{\partial A}$ by $(t^{1/2} - t^{-1/2})$ shows that:

$$< \hat{\alpha}, (t^{1/2}\phi^- - t^{-1/2}\phi^+)(\hat{\sigma}) >_{\partial A}=$$
$$(t^{1/2} - t^{-1/2})^2 < \sigma_1, \sigma_2 >^2_{\Sigma} < \hat{\alpha}, \hat{b} >_{>2} + O((t-1)^3)$$

□

Lemma 5.19 *Lemma 5.15 is true when $d > 2$.*

PROOF: Here, we may factorize $(t^{1/2} - t^{-1/2})^3$ in the underlying determinant of the right-hand side of the equation of Lemma 5.15. Thus, this right-hand side is zero. □

This concludes the proof of Lemma 5.15 and therefore the proof of Proposition 5.12. □

References

[A-M] S. Akbulut and J. McCarthy. *Casson's invariant for oriented homology 3-spheres, an exposition.* Mathematical Notes **36**, Princeton University Press, 1990

[B-C] J. Birman and R. Craggs. *The μ-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold.* Trans. Amer. Math. Soc. **237**, 283-309, 1978

[B-L] S. Boyer and D. Lines. *Surgery formulae for Casson's invariant and extensions to homology lens spaces.* J. Reine Angew. Math. **405**, 181-220, 1990

[B-N] S. Boyer and A. Nicas. *Varieties of group representations and Casson's invariant for rational homology 3-spheres.* Trans. Amer. Math. Soc. **322**, (No.2), 507-522, 1990

[F] G. Fujita. *A splicing formula for Casson-Walker's invariant.* Math. Ann. **296**, 327-338, 1993

[F-M] S. Fukuhara and N. Maruyama. *A Sum Formula for Casson's λ-Invariant.* Tokyo J. Math. **11** (No. 2), 281-287, 1988

[G-M] L. Guillou and A. Marin. *Notes sur l'invariant de Casson des sphères d'homologie de dimension 3.* L'Enseignement Math. **38**, 233-290, 1992

[J] D. Johnson. *A survey of the Torelli group.* Contemporary Mathematics **20**, 165-179, 1983

[L] C. Lescop. *Global surgery formula for the Casson-Walker invariant.* Prépub. ENS Lyon **101**, May 1993

[M] A. Marin. *Un nouvel invariant pour les sphères d'homologie de dimension 3 (d'après Casson).* Séminaire Bourbaki **693**, 1988

[Mo] S. Morita. *On the structure of the Torelli group and the Casson invariant.* Topology **30**, 603-621, 1991

[Mu] H. Murakami. *Quantum SO(3)-invariants dominate the SU(2)-invariant of Casson and Walker.* Preprint, 1993

[S] D. Sullivan. *On the intersection ring of compact three manifolds.* Topology **14**, 275-277, 1975

[T] V. G. Turaev. *Reidemeister torsion in knot theory.* Russian Math. Surveys, 41:1, 119-182, 1986

[W] K. Walker. *An Extension of Casson's Invariant.* Annals of Mathematics Studies, **126**, Princeton University Press, 1992

Christine Lescop
Institut Fourier, URA 188 du CNRS, B.P.74
38402 Saint-Martin-d'Hères cédex
FRANCE
e-mail: lescop@fourier.grenet.fr