Plane Curves with C–hyperbolic Complements

G. Dethloff, M. Zaidenberg

Abstract

The general problem which initiated this work is:
What are the quasiprojective varieties which can be uniformized by means of bounded domains in $\mathbb{C}^n$ *?*
Such a variety should be, in particular, C–hyperbolic, i.e. it should have a Carathéodory hyperbolic covering. We study here the plane projective curves whose complements are C–hyperbolic. For instance, we show that most of the curves whose duals are nodal or, more generally, immersed curves, belong to this class.

Contents

1 Introduction

1.1. A complex space X is called *C–hyperbolic* if it has a (non–ramified) covering $\tilde{X}$ which is *Carathéodory hyperbolic*, i.e. the points of $\tilde{X}$ can be separated by bounded holomorphic functions [Ko1, pp. 129–130] (see also [LiZa, 1.3]). In this paper we study C–hyperbolicity of the complements of plane projective curves. In particular, we are interesting in what the minimal degree of a plane curve with C–hyperbolic complement is.

It is well known that any C–hyperbolic space is Kobayashi hyperbolic [Ko1, p. 130]. The opposite property to Carathéodory hyperbolicity is *liouvilleness*. A complex space X is called *Liouville* if it has no non-constant bounded holomorphic functions. For example, any quasi–projective variety X is Liouville, and by the theorem of V. Lin its liouvilleness is preserved by passing to a nilpotent covering over X, i.e. a Galois covering with nilpotent group of deck transformations [Li, Theorem B] (see also [LiZa], Theorem 3 at p.119). Thus, if X is a quasi–projective variety whose Poincaré group $\pi_1(X)$ is (almost) nilpotent, then any covering over X is Liouville and therefore X can not be C–hyperbolic. In particular, this is the case for $X = \mathbb{P}^2 \setminus C$, where C is a (not necessarily irreducible) nodal curve, i.e. a plane curve with normal crossing singularities only. Indeed, in this case by the Deligne-Fulton theorem [Del, Fu] the fundamental group $\pi_1(X)$ is abelian, and thus by Lin's Theorem any covering over $X = \mathbb{P}^2 \setminus C$ is a Liouville one.

The fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ for an irreducible plane curve C of degree d, which is not necessarily nodal, is known to be abelian in a number of other cases, and hence to be isomorphic to $\mathbb{Z}/d\mathbb{Z}$ (see e.g. the survey article [Lib] and the references therein). For instance, this is so for any rational or elliptic Plücker curve except those of even degree with the maximal number of cusps, and therefore also for the curves that can be specialized to such ones [Zar, pp. 267, 327-330] (cf. [DL], [Kan]). This is true as well for any irreducible curve of degree $d \le 4$ with the only exception of the three–cuspidal quartic; in the latter case $\pi_1(\mathbb{P}^2 \setminus C)$ is a finite non-abelian metacyclic group of order 12 [Zar, pp. 135, 145], and so it is almost abelian. Therefore, in all these cases any covering over $\mathbb{P}^2 \setminus C$ is a Liouville one.

1.2. At the same time, the complement of a nodal plane curve can be Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. The well known example is the complement of five lines in $\mathbb{P}^2$ in general position [Gr3; KiKo, Corollary 3 in section 4]; for further examples of reducible curves see e.g. [DSW1,2] and the literature therein. There exist even the irreducible smooth quintics with these properties [Za3]. Moreover, Y.-T. Siu and S.-K. Yeung [SY] have announced recently a proof of a long standing conjecture that generic (in Zariski sense) smooth curve in $\mathbb{P}^2$ of degree d large enough ($d \ge 1,200,000$) has the complement which is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$ (while all its coverings are Liouville).

This shows that for the complement of a curve in $\mathbb{P}^2$ the property to be C–hyperbolic is much stronger than those of Kobayashi hyperbolicity, and it can occure only for the curves with singularities worse than the ordinary double points.

1.3. However, plane curves with C–hyperbolic complements do exist. The simplest example is a reducible quintic C_5 with the ordinary triple points as singularities at worst. Namely, C_5 is the union of five lines which is given in homogeneous coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ by the equation

$$x_0x_1x_2(x_0 - x_1)(x_0 - x_2) = 0$$

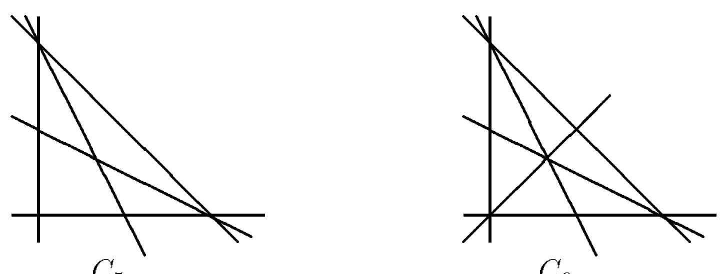


Figure 1

Indeed, $\mathbb{P}^2 \setminus C_5$ is biholomorphic to $(\mathbb{C}^{**})^2$, where $\mathbb{C}^{**} = \mathbb{P}^1 \setminus \{3\,\text{points}\}$, and thus its universal covering is the bidisk Δ^2 (hereafter $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ denotes the unit disc).

Slightly modifying the previous example, consider further the reducible sextic $C_6 \subset \mathbb{P}^2$ which is the line arrangement given by the equation

$$x_0 x_1 x_2 (x_0 - x_1)(x_0 - x_2)(x_1 - x_2) = 0 \ .$$

It is known [Kal] that the universal covering of the complement $M_2 := \mathbb{P}^2 \setminus C_6$ is biholomorphic to the Teichmüller space $T_{0,5}$ of the Riemann sphere with five punctures. Furthermore, via the Bers embedding $T_{0,5} \hookrightarrow \mathbb{C}^2$ it is biholomorphic to a bounded Bergman domain of holomorphy in $\mathbb{C}^2$, which is contractible and Kobayashi complete hyperbolic. The automorphism group of $T_{0,5}$ is discrete and isomorphic to the mapping class group, or modular group, Mod(0, 5) [Ro].

Note that 5 is the minimal degree of a plane curve whose complement is C–hyperbolic. Indeed, the complement of a quartic curve is not even Kobayashi hyperbolic [Gr2, section 6]. While there do exist irreducible plane sextics with C–hyperbolic complements (see Proposition 4.5 below), there does not exist such an irreducible plane quintic (see (4.8)-(4.10)), and so the minimal degree of an irreducible plane curve with C–hyperbolic complement is 6 (Proposition 4.10).

1.4. To obtain examples of irreducible plane curves whose complements are C–hyperbolic we can apply the method that was used by M. Green [Gr1] (see also [CaGr, GP]) for constructing curves with hyperbolic complements. In this paper we study systematically the class of curves with C–hyperbolic complements which can be obtained by this method. Let us describe briefly its main idea.

Let $S^n X$ denote the n–th symmetric power of a variety X and $R_n \subset S^n X$ be its discriminant variety, i.e. the ramification locus of the branched Galois covering $s_n : X^n \to S^n X$. For a plane curve $C \subset \mathbb{P}^2$ there is a natural embedding $\rho_C : \mathbb{P}^2 \hookrightarrow S^n C^*_{norm}$, where $C^* \subset \mathbb{P}^{2*}$ is the dual curve, $n = \deg C^*$ and C^*_{norm} is the normalization of C^*. It may happen that this gives an embedding of the complement $\mathbb{P}^2 \setminus C$ into the n–th configuration space $S^n C^* \setminus R_n$, and that either this configuration space, or some subspace of it containing the image $\rho_C(\mathbb{P}^2 \setminus C)$ has nice hyperbolic properties.

Here we give an example.

Suppose that the dual curve $C^* \subset \mathbb{P}^{2*}$ is smooth and of degree $n \geq 4$. For $z = (a : b : c) \in \mathbb{P}^2$ denote by $\rho_C(z)$ the non–ordered set of n points of intersection $l_z \cap C^*$, where l_z is the dual line $ax_0^* + bx_1^* + cx_2^* = 0$ in $\mathbb{P}^{2*}$; here an intersection point of multiplicity m is repeated m times. In this way we obtain a morphism $\rho_C : \mathbb{P}^2 \to S^n C^*$ into n–th symmetric power of C^*, which is a smooth variety (see e.g. [Zar, p.253] or [Na, (5.2.15)]). The ramified covering $s_n : C^{*n} \to S^n C^*$ has the ramification divisors $D_n := \bigcup_{1 \leq i < j \leq n} D_{ij} \subset C^{*n}$ resp. $R_n = s_n(D_n) \subset S^n C^*$, where $D_{ij} := \{x = (x_1, \dots, x_n) \in C^{*n} \,|\, x_i = x_j\}$ is a diagonal hypersurface. Following

Zariski [Zar, p.266] we call R_n *the discriminant hypersurface.* Since C^* is smooth, the preimage $\rho_C^{-1}(R_n)$ coincides with C, and so we have the commutative diagram

$$\begin{array}{ccccc} Y & \overset{\tilde{\rho}_C}{\hookrightarrow} & C^{*n} \setminus D_n & \hookrightarrow & C^{*n} \\ \tilde{s}_n \big\downarrow & & s_n \big\downarrow & & \\ \mathbb{P}^2 \setminus C = X & \overset{\rho_C}{\hookrightarrow} & S^n C^* \setminus R_n & & \end{array} \qquad (1)$$

where $\tilde{s}_n : Y \to X$ is the induced covering. The genus $g(C^*) \geq 3$, therefore C^{*n} has the polydisc Δ^n as the universal covering. Passing to the induced covering $Z \to Y$ we can extend (1) to the diagram

$$\begin{array}{ccc} Z & \hookrightarrow & \Delta^n \\ \big\downarrow & & \big\downarrow \\ Y & \overset{\tilde{\rho}_C}{\hookrightarrow} & C^{*n} \\ \tilde{s}_n \big\downarrow & & \big\downarrow s_n \\ \mathbb{P}^2 \setminus C = X & \overset{\rho_C}{\hookrightarrow} & S^n C^* \end{array} \qquad (2)$$

Being a submanifold of the polydisc, Z is Carathéodory hyperbolic, and so X is C–hyperbolic. Therefore, we have proved the following assertion.

1.5. Proposition. *Let $C \subset \mathbb{P}^2$ be an irreducible curve whose dual curve C^* is smooth and of degree at least* 4. *Then $\mathbb{P}^2 \setminus C$ is C–hyperbolic.*

1.6. Note that, furthermore, $\mathbb{P}^2 \setminus C$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. The latter is also true under the weaker assumptions that (a) the geometric genus G of C is at least two; (b) each tangent line to C^* intersects with C^* in at least two points, and (c) the following inequality is fulfilled: $2n < d$, where $d = \deg C$ and $n = \deg C^*$, or, what is equivalent,

$$\sum_{i=1}^{l} (m_i^* - 1) < 2g - 2 \ , \qquad (3)$$

where $m_1^*, \ldots, m_l^*$ are the multiplicities of the singular branches of C^* [Gr1, CaGr, GP]. Moreover, under these assumptions a stronger conclusion is valid. Namely, there exists a continuous hermitian metric on $\mathbb{P}^2 \setminus C$ whose holomorphic sectional curvature is bounded from above by a negative constant and which dominates some positive multiple of the Fubini-Study metric on $\mathbb{P}^2$ [GP].

1.7. It is clear that a subspace of a C–hyperbolic space is also C–hyperbolic. In particular, if $D \subset \mathbb{P}^n$ is a hypersurface whose complement $\mathbb{P}^n \setminus D$ is C–hyperbolic, then any plane section of D is a plane curve with C–hyperbolic complement. In this way, considering curve complements, one might at least obtain necessary conditions for $\mathbb{P}^n \setminus D$ to be C–hyperbolic (cf. [Za2]).

1.8. *Contents of the paper.* The main results are summarized at Theorem 7.12 at the very end of the paper. Besides this Introduction, the paper contains six sections. Sections 2 and 3 are preliminary. The first of them deals with some necessary facts from hyperbolic analysis, while in the second one certain generalities on plane curves are given. In section 4 we prove C–hyperbolicity of the complements of irreducible curves of genus at least 1, whose duals are immersed curves (for instance, nodal curves) (see Theorem 4.1). We give examples of such curves of any even degree $d \geq 6$.

Furthermore, we study the general case when the dual curve C^* may have cusps. Then, under the morphism $\rho_C : \mathbb{P}^2 \to S^n C^*_{norm}$, the discriminant divisor $R_n \subset S^n C^*_{norm}$, besides the curve C itself, cuts out a line configuration $L_C \subset \mathbb{P}^2$ which consists of the dual lines of cusps of $C^* \subset \mathbb{P}^{2*}$; they are the inflexional tangents and some cuspidal tangents of C. We call L_C *the artifacts of* C (see (3.3)). In Theorem 4.1 we prove C–hyperbolicity of $\mathbb{P}^2 \setminus (C \cup L_C)$, where C is an irreducible curve of genus $g \geq 1$.

The case of rational curves is studied in sections 5 - 7. In section 5 we give necessary preliminaries. If C is rational, then $S^n C^*_{norm} \cong \mathbb{P}^n$, $\rho_C : \mathbb{P}^2 \to \mathbb{P}^n \cong S^n C^*_{norm}$ is a linear embedding and the discriminant hypersurface $R_n \subset \mathbb{P}^n$ is the projective hypersurface defined by the usual discriminant of the universal polynomial of degree n. Therefore, we have the following assertion.

1.9. Lemma (cf. [Zar, p.266]). *Any rational curve $C \subset \mathbb{P}^2$ whose dual C^* is of degree n, together with its artifacts L_C is a plane section of the discriminant hypersurface $R_n \subset \mathbb{P}^n$.*

We call ρ_C *the Zariski embedding* [1]. Using this lemma as well as a duality between the Zariski embedding and a projection of the rational normal curve (see 5.4-5.5), we establish an analog of Theorem 4.1 for a rational curve whose dual has at least one cusp. This is done in Theorem 6.5, where also all exceptions are listed. A classification of the orbits of the natural $\mathbb{C}^*$–actions is an important ingredient of the proof. We give several concrete examples.

[1] In an unexplicit way it is contained already in [Ve, Ch. IY, p.208]

In section 7 we deal with the rational curves whose duals are nodal Plücker curves, i.e. with the maximal cuspidal rational curves. For such a curve of degree $d \geq 8$ we prove that its complement is almost C–hyperbolic (Corollary 7.10; see 2.4 below for the terminology). The proof is based on passing to the moduli space of the punctured Riemann sphere and on a study of the orbits of the natural representation of the group $\mathbb{P}GL(2;\mathbb{C})$ on the projectivized space of binary forms.

The second of the authors had fruitful discussions on the content of section 7 with D. Akhiezer, M. Brion, Sh. Kaliman and H. Kraft; its his pleasure to thank all of them. He also is gratefull to the SFB-170 'Geometry and Analysis' at Göttingen University for its hospitality and excellent working conditions.

2 Preliminaries in hyperbolic complex analysis

2.1. *Lin's Theorem.* Here we recall some definitions and facts from [Li]. A complex space X is called *ultra–Liouville* if any bounded plurisubharmonic function on X is constant. For instance, any quasi–projective variety is ultra–Liouville. By Lin's Theorem [Li, Theorems B and 3.5] any almost nilpotent (or even almost ω–nilpotent) Galois covering of an ultra–Liouville complex space X is Liouville. A covering is called *almost nilpotent* (resp. *almost ω–nilpotent*) if its group of deck transformations is so. Recall that a group G is *almost nilpotent* if it has a nilpotent subgroup of a finite index (G is *almost ω–nilpotent* if the union of the members of its upper central series is a subgroup of G of finite index; for a finitely generated group G the almost ω–nilpotency is equivalent to the almost nilpotency).

2.2. *Super–liouvilleness.* Let us say that a complex space X is *super-Liouville* if any covering over X is Liouville. Super-liouvilleness is a property which in a sense is opposite to C-hyperbolicity. It is clear that X is super-Liouville iff the universal covering U_X of X is Liouville. By Lin's theorem any ultra-Liouville complex space X which has almost ω–nilpotent fundamental group $\pi_1(X)$ is a super-Liouville one. In particular, a smooth quasi-projective curve C is super-Liouville iff the group $\pi_1(C)$ is abelian, i.e. iff C is non-hyperbolic.

Note that if any two points of X can be connected by a finite chain of Liouville subspaces (which are assumed to be connected but not necessarily closed), then X itself is Liouville. More generally, we have the following lemma.

2.3. Lemma. *Let X be a complex space (with countable topology) such that any two points of X can be connected by a finite chain of super-Liouville subspaces of X. Then X is super-Liouville. In particular, if X is a quasi-projective variety such that each pair of points of X can be connected by a finite chain of non-hyperbolic curves, then X is super-Liouville.*

Proof. Let $\pi : U_X \to X$ be the universal covering. Suppose that there exists a non-constant bounded holomorphic function f on U_X. Let $\mathcal{F}$ be the collection of all super-Liouville subspaces of X, and let $\tilde{\mathcal{F}}$ be the collection of subspaces consisting of all connected components of preimages $\pi^{-1}(A)$, where $A \in \mathcal{F}$. We define an equivalence relation on U_X as follows:
Two points in U_X are equivalent iff they can be connected by a finite chain of members of $\tilde{\mathcal{F}}$.
By the condition of the lemma it is easily seen that the union of the equivalence classes of the points of a given fibre of π coincides with the whole space U_X. Since $\pi_1(X)$ is an at most countable group, the fibre of π is at most countable, too, and therefore there exists an at most countable set of equivalence classes. If M is any of them, then clearly $f|M \equiv const$. Therefore, f takes at most countable set of values, which is impossible.

The second statement is an easy corollary of the first one. □

2.4. *Weak C–hyperbolicity.* We say that a complex space X is *almost* resp. *weakly Carathéodory hyperbolic* if for any point $p \in X$ there exist only finitely many resp. countably many points $q \in X$ which cannot be separated from p by bounded holomorphic functions (i.e. if the equivalence relation defined on X by the functions from the algebra $H^\infty(X)$ is finite resp. at most countable). It will be called *almost* resp. *weakly C–hyperbolic* if X has a covering $Y \to X$, where Y is almost resp. weakly Carathéodory hyperbolic.

These notions are meaningful due to the following reasons. It is unknown whether the universal covering space U_X of a C–hyperbolic complex space X is Carathéodory hyperbolic, or more generally, whether there is a Carathéodory hyperbolic covering $Y \to X$ which can be defined in a functorial way. In contrary, one can make the following observation.

A complex space X is weakly C–hyperbolic iff the universal covering space U_X is weakly Carathéodory hyperbolic.

Hereafter we assume X to be reduced and with countable topology. In particular, the universal covering U_X of a C–hyperbolic space X is weakly Carathéodory hyperbolic. One may consider on X the pseudo–distance which is the quotient of the Carathéodory pseudo–distance c_{U_X} on U_X (resp. the quotient of the inner Carathéodory pseudo–distance c'_{U_X} resp. of the differential Carathéodory–Reiffen pseudo–distance C_{U_X}; see [Re]). All three of these quotient pseudo–distances are contracted by holomorphic mappings. Furthermore, the deck transformations on U_X being isometries, the quotient of C_{U_X} on X is locally isometric to C_{U_X} itself, and thus it is non-degenerate iff C_{U_X} is so (for a weakly C–hyperbolic space X it is at least non–trivial).

The proof of the following lemma is easy and can be omitted.

2.5. Lemma. *Let $f : Y \to X$ be a holomorphic mapping of complex spaces. If f is injective (resp. f has finite resp. at most countable fibres) and X is C–hyperbolic (resp. almost resp. weakly C–hyperbolic), then so is Y.*

2.6. *Brody hyperbolicity.* Recall that a complex space X is *Brody hyperbolic* if it contains no entire curve, i.e. if every holomorphic mapping $\mathbb{C} \to X$ is constant. Note that sometimes by Brody hyperbolicity one means absence of *Brody entire curves* in X, i.e. entire curves whose derivatives are uniformly bounded with respect to a fixed hermitian metric on X (see e.g. [Za3]). Usually this is enough in applications. But in this paper we do not need such a precision.

It is clear that any weakly C–hyperbolic complex space is Brody hyperbolic.

2.7. *Kobayashi hyperbolicity.* For a curve $C \subset \mathbb{P}^2$ denote by $\operatorname{sing} C$ the set of all singular points of C. Put $\operatorname{reg} C = C \setminus \operatorname{sing} C$.

The next statement follows from Theorem 2.5 in [Za1].

Proposition. *Let the Riemann surface* $\operatorname{reg} C$ *be hyperbolic and $\mathbb{P}^2 \setminus C$ be Brody hyperbolic (the latter is true, in particular, if $\mathbb{P}^2 \setminus C$ is weakly C–hyperbolic). Then $\mathbb{P}^2 \setminus C$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$.*

Note that in Example 1.3 in the Introduction the first condition fails while the second one is fulfilled. It is easily seen that in this example $\mathbb{P}^2 \setminus C$ is not hyperbolically embedded into $\mathbb{P}^2$. In fact, the condition ‘ $\operatorname{reg} C$ is hyperbolic’ is necessary for $\mathbb{P}^2 \setminus C$ being hyperbolically embedded into $\mathbb{P}^2$ [Za1, Corollary 1.3].

2.8. *Relative hyperbolicities.* Let X be a complex space resp. a quasi–projective variety and $Z \subset X$ be a closed analytic subset resp. a closed algebraic subvariety. We say that X is *Brody hyperbolic modulo Z* if any (non-constant) entire curve $\mathbb{C} \to X$ is contained in Z.

For instance, this is the case if X is Kobayashi hyperbolic modulo Z (see [KiKo]). (We mention that in [Za4] the above property of relative Brody hyperbolicity was called *strong algebraic degeneracy.*)

We will say that X is *C–hyperbolic modulo* Z (resp. *almost resp. weakly C–hyperbolic modulo* Z) if there is a covering $\pi : Y \to X$ such that for each point $p \in Y$ any other point $q \in Y \setminus \pi^{-1}(Z)$ (resp. any other point $q \in Y \setminus \pi^{-1}(Z)$ besides only finitely many resp. besides only countably many such points) can be separated from p by bounded holomorphic functions.

By the monodromy theorem weak C–hyperbolicity of X modulo Z implies Brody hyperbolicity of X modulo Z.

The next lemma, which is a generalization of Lemma 2.5, easily follows from the definitions.

2.9. Lemma. *Let $f : Y \to X$ be a holomorphic mapping of complex spaces and let Z be a closed complex subspace of X. If $f\,|\,(Y \setminus f^{-1}(Z))$ is injective (resp. has finite resp. at most countable fibres) and X is C–hyperbolic (resp. almost resp. weakly C–hyperbolic) modulo Z, then Y is C–hyperbolic (resp. almost resp. weakly C–hyperbolic) modulo $f^{-1}(Z)$.*

3 Background on plane algebraic curves

3.1. *Classical singularities. Immersed curves.* We say that a curve C in $\mathbb{P}^2$ has *classical singularities* if its singular points are nodes and ordinary cusps. It is called *a Plücker curve* if both C and the dual curve C^* have classical singularities only and no flex at a node. If the normalization mapping $\nu : C^*_{norm} \to C \hookrightarrow \mathbb{P}^2$ is an immersion, or, which is equivalent, if all irreducible local analytic branches of C are smooth, then we say that C is *an immersed curve.* An immersed curve C is called *a curve with tidy* or *ordinary singularities*, or simply *a tidy curve*, if at each point $p \in C$ the local irreducible branches of C have pairwise distinct tangents. By M. Noether's Theorem [Co] any plane curve can be transformed into a tidy curve by means of Cremona transformations.

3.2. *Cusps and flexes.* In the sequel by *a cusp* we mean an irreducible plane curve singularity. In particular, an irreducible local analytic branch A of a plane curve C with centrum $p \in C$ is a cusp iff it is singular. The tangent line to a cusp A at p is called *a cuspidal tangent of C*. Recall that *the multiplicity sequence* of a plane analytic germ A at $p_0 \in A$ is the sequence of multiplicities of A at p_0 and its infinitely nearby points. Following [Na, (1.5)] A is called *a simple cusp* if its first Puiseaux pair is $(m, m+1)$, where $m = \text{mult}_p A \geq 2$, or, what is the same, if the multiplicity sequence of A is $(m, 1, 1, \ldots)$. By Lemma 1.5.7 in [Na] a cusp A is a simple cusp iff the corresponding branch A^* of the dual curve C^* is smooth. In this case A^* is a flex of order $m-1$ (see the definition below), and vice versa, if A^* is a flex of order $m-1$, then A is a simple cusp of multiplicity m. A simple cusp A of multiplicity 2 is called *an ordinary cusp* if C is locally irreducible at p.

A smooth irreducible local branch A of the curve C at a point $p \in C$ is called *a flex of order k* if $i(A, T_pA; p) = k+2 \geq 3$. The tangent line T_pA to a flex A at p is called *an inflexional tangent. An ordinary flex* is a flex of order 1 at a smooth point of C.

Thus, the dual C^* is an immersed curve iff C has no flex and all its cusps are simple.

3.3. *Artifacts.* If C^* has cusps, denote by L_C the union of their dual lines in $\mathbb{P}^2$. Clearly, L_C consists of the inflexional tangents of C and the cuspidal tangents at those cusps of C which are not simple. Due to some analogy in tomography, we call L_C *the artifacts* of C.

Note that the dual curve C^* of C is an immersed curve iff $L_C = \emptyset$. Such a curve C may have complicated reducible singularities, which correspond to multiple tangents of C^*; for instance, it may have tacnodes, etc.

3.4. *The Class Formula.* Let $C \subset \mathbb{P}^2$ be an irrducible curve of degree $d \geq 2$, of geometric genus g and of class c. Then $c = d^* = \deg C^*$ is defined by the Class Formula (see [Na, (1.5.4)])

$$c = d^* = 2(g+d-1) - \sum_{p \in \text{sing}\, C} (m_p - r_p) =$$

$$= 2(g+d-1) - \sum_{A=(A,p),\, p \in \text{sing}\, C} (m_A - 1) \ , \qquad (4)$$

where $m_p = \text{mult}_p C$, r_p is the number of irreducible analytic branches of C at p, $A = (A, p)$ is a local analytic branch of C at p and m_A is its multiplicity at p. In particular, C is an immersed curve iff $d^* = 2(g+d-1)$ (indeed, this is the case iff the last sum in (4) vanishes). Furthermore, for an immersed curve C one has $d^* \geq 2d+2 \geq 10$ if $g \geq 2$, $d^* = 2d \geq 6$ if $g = 1$, and $d^* = 2d-2 \geq 2$ if $g = 0$.

For reader's convenience we recall here also the usual Plücker formulas:

$$g = 1/2(d-1)(d-2) - \delta - \kappa = 1/2(d^*-1)(d^*-2) - b - f$$

$$d^* = d(d-1) - 2\delta - 3\kappa \quad \text{and} \quad d = d^*(d^*-1) - 2b - 3f$$

for a Plücker curve C with δ nodes, κ cusps, b bitangent lines and f flexes.

3.5. *The n–th Abel–Jacobi map.* Let M be a compact Riemann surface of genus g, and let $j : M \to J(M)$ be a fixed Abel–Jacobi embedding of M into its Jacobian variety $J(M) \cong \mathrm{Pic}^0(M)$. The n–th symmetric power S^nM may be identified with the space of effective divisors of degree n on M. Let $\varphi_n : S^nM \to J(M)$ be the n–th Abel–Jacobi map $D = p_1 + \ldots + p_n \longmapsto \varphi_n(D) := j(p_1) + \ldots + j(p_n)$, so that $j = \varphi_1$. We recall the following well known facts (see e.g. [GH, 2.2], [Zar, pp.352–353] or [Na, (5.2), (5.3)] and references therein):

i) φ_n is holomorphic;
ii) (Abel's Theorem) $\varphi_n^{-1}(\varphi_n(D)) = |D| = \mathbb{P}H^0(M, O([D])) \cong \mathbb{P}^{\dim |D|}$ is the complete linear system of D, where $D \in S^nM$ is an effective divisor of degree n on M;
iii) the natural injection $|D| \hookrightarrow S^nM$ is a holomorphic embedding, i.e. $|D| = \varphi_n^{-1}(\varphi_n(D))$ is a smooth subvariety in S^nM;
iv) if $n \le g$, then $\varphi_n : S^nM \to J(M)$ is generically one–to–one; in particular, the image $\varphi_{g-1}(S^{g-1}M) \subset J(M)$ is a translation of the theta divisor Θ on $J(M)$;
v) (Jacobi Inversion) if $n \ge g$, then $\varphi_n : S^nM \to J(M)$ is surjective. For $n > 2g - 2$ it is an algebraic projective bundle (see [Ma]); in particular, if $g = 1$, then it is a $\mathbb{P}^{n-1}$–bundle over $J(M) \cong M$.

3.6. *The Zariski embedding.* Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \ge 2$ and let $\nu : C^*_{norm} \to C^*$ be the normalization of the dual curve. Following Zariski [Zar, p.307, p.326] and M. Green [Gr1] (see also [DL]), as in (1.4) we consider the mapping $\rho_C : \mathbb{P}^2 \to S^nC^*_{norm}$ of $\mathbb{P}^2$ into the n-th symmetric power of C^*_{norm}, where $n = \deg C^*$. We put $\rho_C(z) = \nu^*(l_z) \subset S^nC^*_{norm}$, where $z \in \mathbb{P}^2$ and $l_z \subset \mathbb{P}^{2*}$ is the dual line. Clearly, $\rho_C : \mathbb{P}^2 \to S^nC^*_{norm}$ is holomorphic. We still denote by D_n the union of the diagonal divisors in $(C^*_{norm})^n$ and by $R_n = s_n(D_n)$ the discriminant divisor, i.e. the ramification locus of the branched covering $s_n : (C^*_{norm})^n \to S^nC^*_{norm}$ (cf. (1.4)).

It is clear that ρ_C is a holomorphic embedding, which we call in the sequel *the Zariski embedding.* More precisely, it is composed of two embeddings $i_1 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^{h(C)}$ and $i_2 : \mathbb{P}^{h(C)} \hookrightarrow S^nC^*_{norm}$ which are described below. Denote by H a divisor of degree n on $M := C^*_{norm}$, which is the trace of a line cut of C^* in $\mathbb{P}^{2*}$. The two dimensional linear system g^2_n of all such line cuts is naturally identified with the original projective plane $\mathbb{P}^2 = (\mathbb{P}^{2*})^*$. Let $h(C) := \dim |H|$; then $i_1 : \mathbb{P}^2 = g^2_n \hookrightarrow \mathbb{P}^{h(C)} = |H|$ is defined to be the canonical linear embedding of g^2_n into the complete linear system $|H|$. (Let us mention that g_n^2 itself might be complete; for instance, this is the case when C^* is a nodal curve with δ nodes, where $\delta < \frac{n(n-2)}{4}$ for n even or $\delta < \frac{(n-1)^2}{4}$ for n odd; see [Na, p.115].) The Abel embedding $i_2 : \mathbb{P}^{h(C)} = |H| \hookrightarrow S^n C^*_{norm} = S^n M$ identifies $|H|$ with the fibre $\varphi_n^{-1}(\varphi_n(H))$ of the n–th Abel–Jacobi map $\varphi_n : S^n M \to J(M)$ (see (3.5)).

What we really need in section 4 is that the restriction $\rho_C \mid (\mathbb{P}^2 \setminus C)$ is injective, which can also be shown as follows.

We have to show that any projective line l in $\mathbb{P}^{2*}$ which is not tangent to C^* meets C^* in at least two distinct points, and so it is uniquely defined by its image $\rho(l) = l \cap C^*$. Suppose that there exists a line $l_0 \subset \mathbb{P}^{2*}$ which has only one point p_0 in common with C^* and which is not tangent to C^* at this point. Let l_1 be the tangent to a local analytic branch of C^* at p_0. Then we have

$$i(l_1, C^*; p_0) > i(l_0, C^*; p_0) = n = l_1 \cdot C^* \ ,$$

which is impossible. In the same elementary way it can be shown that ρ_C is a holomorphic injection.

3.7. Lemma. *The preimage $\rho_C^{-1}(R_n) \subset \mathbb{P}^2$ is the union of C with its artifacts L_C. In particular, $\rho_C^{-1}(R_n) = C$ iff the dual curve C^* is an immersed curve.*

Proof. Note that a point $z \in \mathbb{P}^2 \setminus C$ is contained in $\rho_C^{-1}(R_n)$ iff its dual line l_z passes through a cusp of C^*. The lines in L_C are just the dual lines to the cusps of C^*. $\square$

4 C-hyperbolicity of complements of plane curves of genus $g \geq 1$

In this section we keep all the notation from 3.6. The main result here is the following theorem.

4.1. Theorem. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of genus $g \geq 1$ and L_C be its artifacts. Then*
a) $\mathbb{P}^2 \setminus (C \cup L_C)$ is C–hyperbolic.
b) If the dual curve $C^* \subset \mathbb{P}^{2*}$ is an immersed curve, then $\mathbb{P}^2 \setminus C$ is C–hyperbolic, Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$.

Proof. a) Consider first the case when $g \geq 2$. In this case $S^n C^*_{norm} \setminus R_n$ is C–hyperbolic (cf. (1.4)). Indeed, its covering $(C^*_{norm})^n \setminus D_n$ is a domain in $(C^*_{norm})^n$. Since the universal covering of $(C^*_{norm})^n$ is the polydisc Δ^n, it is C–hyperbolic. Therefore, $(C^*_{norm})^n \setminus D_n$, and hence also $S^n C^*_{norm} \setminus R_n$ are C–hyperbolic, too. By Lemma 3.7 the image $\rho_C(\mathbb{P}^2 \setminus (C \cup L_C)) \subset S^n C^*_{norm}$ does not meet the discriminant variety R_n and by 3.6 $\rho_C \,|\, (\mathbb{P}^2 \setminus (C \cup L_C)) \, : \, \mathbb{P}^2 \setminus (C \cup L_C) \to S^n C^*_{norm} \setminus R_n$ is injective. Therefore, by Lemma 2.5 $\mathbb{P}^2 \setminus (C \cup L_C)$ is C–hyperbolic.

Next we consider the case when C is an elliptic curve. Denote $E = C^*_{norm}$. Note that both $E^n \setminus D_n$ and $S^n E \setminus R_n$ are not C–hyperbolic or even hyperbolic, and so we can not apply the same arguments as above.

Represent E as $E = J(E) = \mathbb{C}/\Lambda_\omega$, where Λ_ω is the lattice generated by 1 and $\omega \in \mathbb{C}_+$ (here $\mathbb{C}_+ := \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$). By Abel's Theorem we may assume this identification of E with its jacobian $J(E)$ being chosen in such a way that the image $\rho_C(\mathbb{P}^2)$ is contained in the hypersurface $s_n(H_0) = \varphi_n^{-1}(\bar{0}) \cong \mathbb{P}^{n-1} \subset S^n E$, where

$$H_0 := \{z = (z_1, \ldots, z_n) \in E^n \,|\, \sum_{i=1}^{n} z_i = 0\}$$

is an abelian subvariety in E^n (see (3.5)). The universal covering $\tilde{H}_0$ of H_0 can be identified with the hyperplane $\sum_{i=1}^{n} x_i = 0$ in $\mathbb{C}^n = \tilde{E}^n$.

Consider the countable families $\tilde{D}_{ij}$ of parallel affine hyperplanes in $\mathbb{C}^n$ given by the equations $x_i - x_j \in \Lambda_\omega\,, \;\; i, j = 1, \ldots, n, \;\; i < j$.

Claim. The domain $\tilde{H}_0 \setminus \bigcup_{i=1}^{n-1} \tilde{D}_{i,i+1}$ *is biholomorphic to* $(\mathbb{C} \setminus \Lambda_\omega)^{n-1}$.

Indeed, put $y_k := (x_k - x_{k+1}) \,|\, \tilde{H}_0\,, \; i = 1, \ldots, n-1$. It is easily seen that $(y_1, \ldots, y_{n-1}) \, : \, \tilde{H}_0 \to \mathbb{C}^{n-1}$ is a linear isomorphism whose restriction yields a biholomorphism as in the claim.

The universal covering of $(\mathbb{C} \setminus \Lambda_\omega)^{n-1}$ is the polydisc Δ^n, and so $(\mathbb{C} \setminus \Lambda_\omega)^{n-1}$ is C–hyperbolic. Put $\tilde{D}_n := \bigcup_{i,j=1,\ldots,n} \tilde{D}_{ij}$. The open subset $\tilde{H}_0 \setminus \tilde{D}_n$ of $\tilde{H}_0 \setminus \bigcup_{i=1}^{n-1} \tilde{D}_{i,i+1} \cong (\mathbb{C} \setminus \Lambda_\omega)^{n-1}$ is also C–hyperbolic.

Denote by p the universal covering map $\mathbb{C}^n \to (\mathbb{C}/\Lambda_\omega)^n$. The restriction

$$p \,|\, \tilde{H}_0 \setminus \tilde{D}_n \, : \, \tilde{H}_0 \setminus \tilde{D}_n \to H_0 \setminus D_n \subset E^n \setminus D_n$$

is also a covering mapping. Therefore, $H_0 \setminus D_n$ is C–hyperbolic, and so $s_n(H_0) \setminus R_n$ is C–hyperbolic, too. Since $\rho_C \,|\, (\mathbb{P}^2 \setminus (C \cup L_C)) \, : \, \mathbb{P}^2 \setminus (C \cup L_C) \to s_n(H_0) \setminus R_n$ is an injective holomorphic mapping, by Lemma 2.5 $\mathbb{P}^2 \setminus (C \cup L_C)$ is C–hyperbolic.

b) Assume further that C^* is an immersed curve. Then C can not be smooth. Indeed, being smooth C would have flexes at the points of intersection of C with its Hesse curve (see [Wa]), and hence C^* would have cusps. Thus, $\operatorname{reg} C$ is hyperbolic, and by that what has been proven above $\mathbb{P}^2 \setminus C$ is C–hyperbolic. Therefore, by Proposition 2.7 it is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. This completes the proof. □

4.2. Remark. The complement to a rational curve whose dual is an immersed curve is not necessarily C–hyperbolic; it even may be not Brody or Kobayashi hyperbolic. An example is a plane quartic C with three cusps. Such a quartic C is projectively equivalent to the curve given by the equation

$$4x_1^2(x_1 - 2x_0)(x_1 + x_2) - (2x_0x_2 - x_1^2)^2 = 0$$

(see [Na, (2.2.5)]). Its dual curve C^* is a nodal cubic with equation

$$x_0x_1^2 + x_1^3 - x_0x_2^2 = 0 \ .$$

Thus, $g(C) = 0$ and C^* is an immersed curve. The complement $\mathbb{P}^2 \setminus C$ is not Kobayashi hyperbolic, because its Kobayashi pseudo–distance vanishes on any of three cuspidal tangent lines of C, on any of three lines passing through two cusps of C each one and on the only bitangent line of C. Indeed, each of these seven lines meets C in only two points; but $k_{\mathbb{C}^*} = 0$, where $\mathbb{C}^* = \mathbb{P}^1 \setminus \{2\,\text{points}\}$. Therefore, $\mathbb{P}^2 \setminus C$ is not C–hyperbolic. Note that $\pi_1(\mathbb{P}^2 \setminus C)$ is a finite group of order 12 [Zar, p.145], and thus $\mathbb{P}^2 \setminus C$ is super–Liouville (see (2.2)).

Note, furthermore, that by an analogous reason the complement of any quartic curve $C \subset \mathbb{P}^2$ is neither Kobayashi hyperbolic nor Brody hyperbolic [Gr2]. The fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ for an irreducible quartic C being almost abelian (see (1.1)), by Lin's Theorem $\mathbb{P}^2 \setminus C$ is super–Liouville.

Next we give some examples, or at least computations of numerical characters of plane curves which satisfy the assumptions of Theorem 4.1. First we consider examples of curves of genus $g \geq 2$ with the dual an immersed curve.

4.3. Example. Let $C \subset \mathbb{P}^2$ be an irreducible curve whose dual C^* is a nodal curve of degree $n = d^* \geq 3$ with δ nodes. Assume that the genus $g(C) = g(C^*) = \frac{(n-1)(n-2)}{2} - \delta$ is at least 2. Such a curve does exist for any given δ with $0 \leq \delta \leq \frac{(n-1)(n-2)}{2} - 2$ (see [Se, §11, p.347]; [O, Proposition 6.7]). By the Class Formula (4) C has degree $d = n(n-1) - 2\delta$, which can be any even integer from the interval $[2(n+1), n(n-1)]$. The least possible value of n resp. d is $n = 4$ resp. $d = 10$, which corresponds to the case when C^* is a nodal quartic with one node ($\delta = 1$) (see e.g. [Na, p.130]). To be a Plücker curve (see (3.1)) such C should be a curve of degree 10 with 16 nodes and 18 ordinary cusps (cf. [Zar, p.176]).

If C is a curve of genus $g \geq 2$ whose dual is a nodal curve, then by Theorem 4.1 the complement $\mathbb{P}^2 \setminus C$ is C–hyperbolic, Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. This yields examples of such curves C of any even degree $d \geq 10$.

There are similar examples with elliptic curves.

4.4. Example. If the dual C^* of C is an immersed elliptic curve, then by the Class Formula (4) $d = \deg C = 2n \geq 6$, where $n = \deg C^* \geq 3$ (see 3.4). Thus, the least possible value of the degree d of such a curve C is $d = 6$. Let C be a sextic in $\mathbb{P}^2$ with 9 cusps. Then C is an elliptic Plücker curve whose dual C^* is a smooth cubic; vice versa, the dual curve to a smooth cubic is a sextic with 9 ordinary cusps (see e.g. [Wa]). By Theorem 4.1 the complement of such a curve is C–hyperbolic, Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. The same is true if C is the dual curve to a nodal quartic C^* with two nodes; here $d = \deg C = 8$ (see e.g. [Na, p.133]). To be Plücker such a curve C must have 8 nodes and 12 ordinary cusps.

These examples lead to the following conclusion.

4.5. Proposition. *For any even $d \geq 6$ there are irreducible plane curves of degree d and of genus $g \geq 1$ whose duals are nodal curves, and so which satisfy the assumptions of Theorem 4.1, b).*

Next we pass to examples to part a) of Theorem 4.1.

4.6. Examples. Let $C^* \subset \mathbb{P}^{2*}$ be an irreducible curve of genus $g \geq 1$ with classical singularities. If C^* has δ nodes and κ cusps, then the dual curve $C \subset \mathbb{P}^2$ has κ ordinary flexes as the only flexes, and so L_C is the union of inflexional tangents of C. By the Class Formula (4) we have $d = \deg C = 2(n+g-1)-\kappa$. Since all κ inflexional tangents of C are distinct, it follows that $\deg(C \cup L_C) = 2(n+g-1) \geq 2n \geq 6$. Assume that $\kappa > 0$ to exclude the case considered in Examples 4.3 and 4.4 above, when C^* was an immersed curve. Since $g \geq 1$, the case when C is a singular cubic has also been excluded. Thus, we have $n \geq 4$, and hence $\deg(C \cup L_C) \geq 8$.

The simplest example is a quartic C^* with an ordinary cusp and a node as the only singularities; such a quartic does exist (see [Na, p.133]). The dual curve C is an elliptic septic with the only inflexional tangent line $l = L_C$. To be a Plücker curve,

C must have 4 nodes and 10 ordinary cusps.

Another example is a quartic C^* with two ordinary cusps as the only singular points; it also does exist [Na, p.133]. Here C is an elliptic sextic and L_C is the union of two inflexional tangents of C. To be a Plücker curve, C should have one node and 8 ordinary cusps.

In all these examples the assumptions of Theorem 4.1 are fulfilled.

4.7. Remark. Of course, it may happen that the complement of the artifacts $\mathbb{P}^2 \setminus L_C$ is itself C–hyperbolic. For instance, this is so if L_C contains an arrangement of five lines with two triple points on one of them, which is projectively equivalent to those C_5 of Example 1.3 in the Introduction. But this is not the case if L_C consists only of few lines like in the examples 4.6, or if it consists of lines in general position (cf. (1.2)). For instance, if $C = F_d$ is the Fermat curve of degree $n \geq 3$ in $\mathbb{P}^2$, then it is easily checked that the inflexional tangents are in general position (note that here all the flexes are hyperflexes of high order). We suppose that for a generic smooth plane curve C of degree $d \geq 4$ its inflexional tangents are in general position, and therefore the complement $\mathbb{P}^2 \setminus L_C$ is super–Liouville.

Next we consider the problem of existence of an irreducible quintic with C–hyperbolic complement.

4.8. We have already noted that there is no plane quartic C with Kobayashi hyperbolic complement $\mathbb{P}^2 \setminus C$ [Gr2]. The obstructions are lines in $\mathbb{P}^2$ which intersect C in at most two points; e.g. cuspidal tangents, inflexional tangents, bitangents, etc. Recall that if $C \subset \mathbb{P}^2$ is a nodal quintic, then $\mathbb{P}^2 \setminus C$ is super–Liouville (see (1.1) and (2.2)). Although there are irreducible quintics whose complements are Kobayashi hyperbolic [Za3], there is no one with C–hyperbolic complement (cf. (1.3) and (6.1) for examples of reducible quintics with C–hyperbolic complements). It is shown in the next lemma that there is no one among the non-Plücker quintics; as for the Plücker ones, see Proposition 4.10 below.

4.9. Lemma. *Let $C \subset \mathbb{P}^2$ be an irreducible quintic. Suppose that C is not a Plücker curve. Then $\mathbb{P}^2 \setminus C$ is not Brody hyperbolic. Moreover, there exists a line $l_0 \subset \mathbb{P}^2$ which intersects with C in at most two points.*

Proof. Assume that C has a singular point p_0 which is not a classical one. Let l_0 be the tangent line to an irreducible local analytic branch of C at p_0. If $\operatorname{mult}_{p_0} C \geq 3$, then $i(C, l_0; p_0) \geq 4$, and so the line l_0 has at most one more intersection point with

C. If $\mathrm{mult}_{p_0} C = 2$, then either
1) C has two smooth branches at p_0 with the same tangent l_0 (e.g. $p_0 \in C$ is *a tacnode*),
or
2) C is locally irreducible in p_0 and has the multiplicity sequence $(2, 2, \ldots)$ at p_0 (see (3.2)).
In both cases we still have $i(C, l_0; p_0) \geq 4$, and the same conclusion as before holds. It holds also in the case when l_0 is the inflexional tangent to C at a point where C has a flex of order at least 2 (see (3.2)).

Therefore, from now on we may suppose that C has only classical singularities and ordinary flexes. Let q_0 be a singular point of C^* which is not classical. It can not be locally irreducible, since C has only ordinary flexes. If one of the irreducible local branches of C^* at q_0 is singular, then the dual line l_0 of q_0 is a multiple tangent line to C which is an inflexional tangent at some flex of C. Therefore, by the Bezout Theorem l_0 is a bitangent line with intersection indices 2 and 3. The remaining case to consider is the case when C^* has only smooth local branches at q_0. If two of them, say, A_0^* and A_1^*, are tangent to each other, then by duality the corresponding local branches A_0 and A_1 of C should have common center and moreover, they should be tangent to each other, too. But this is impossible since C is assumed to have only classical singularities. Thus, we are left with the case that q_0 is a tidy singularity consisting of at least three disinct irreducible local branches of C^*. But then the dual line l_0 of q_0 is tangent in at least three different points of C. Since C is of degree five, by Bezout's Theorem this is also impossible. This completes the proof. □

From this lemma, Proposition 4.5 and the computations done by A. Degtjarjov [Deg 1, 2] we obtain the following statement.

4.10. Proposition. *The minimal degree of an irreducible plane curve with C–hyperbolic complement is 6.*

Proof. Indeed, it is shown in [Deg 1, 2] that two irreducible plane quintics with the same type of singular points are isotopic in $\mathbb{P}^2$, and there are only two types of them such that the fundamental groups of the complement are not abelian. In both cases these quintics have non-classical singularities. It follows that for an irreducible Plücker quintic $C \subset \mathbb{P}^2$ the complement $\mathbb{P}^2 \setminus C$ has cyclic fundamental group, and therefore it is super-Liouville. □

5 Rational plane curves and duality

Here we precise the construction of (3.6) in the case of a rational curve.

5.1. *The Vieta covering.* The symmetric power $S^n\mathbb{P}^1$ can be identified with $\mathbb{P}^n$ in such a way that the canonical projection $s_n : (\mathbb{P}^1)^n \to S^n\mathbb{P}^1$ becomes *the Vieta ramified covering*, which is given by

$$((u_1 : v_1), \dots, (u_n : v_n)) \longmapsto$$

$$\longmapsto (\prod_{i=1}^{n} v_i)\,(1 : \sigma_1(u_1/v_1, \dots, u_n/v_n) \ : \ \dots \ : \ \sigma_n(u_1/v_1, \dots, u_n/v_n))\ ,$$

where $\sigma_i(x_1, \dots, x_n)$, $i = 1, \dots, n$, are elementary symmetric polynomials. This is a Galois covering with the Galois group being the n-th symmetric group S_n. In the case when $z_i := u_i/v_i \in \mathbb{C}$, $i = 1, \dots, n$, we have $s_n(z_1, \dots, z_n) = (a_0 : \dots : a_n)$, where the equation $a_0 z^n + \dots + a_n = 0$ has the roots $z_1, \dots, z_n$ (see [Zar, p.252] or [Na, (5.2.18)]). In general, $z_i \in \mathbb{P}^1$, $i = 1, \dots, n$, are the roots of the binary form $\sum\limits_{i=0}^{n} a_i u^{n-i} v^i$ of degree n.

5.2. *Plane cuts of the discriminant hypersurface.* If $C \subset \mathbb{P}^2$ is a rational curve of degree $d > 1$, then $C^*_{norm} \cong \mathbb{P}^1$, and so the Zariski embedding is a morphism $\rho_C : \mathbb{P}^2 \to \mathbb{P}^n \cong S^n\mathbb{P}^1$, where $n = \deg C^*$. The normalization map $\nu : \mathbb{P}^1 \to C^* \subset \mathbb{P}^2$ can be given as $\nu = (g_0 : g_1 : g_2)$, where $g_i(z_0, z_1) = \sum\limits_{j=0}^{n} b_j^{(i)} z_0^{n-j} z_1^j$, $i = 0, 1, 2$, are homogeneous polynomials of degree n without common factor.

If $x = (x_0 : x_1 : x_2) \in \mathbb{P}^2$ and $l_x \subset \mathbb{P}^{2*}$ is the dual line, then $\rho_C(x) = \nu^*(l_x) \in S^n\mathbb{P}^1 = \mathbb{P}^n$ is defined by the equation $\sum\limits_{i=0}^{2} x_i g_i(z_0 : z_1) = 0$. Thus, $\rho_C(x) = (a_0(x) : \dots : a_n(x))$, where $a_j(x) = \sum\limits_{i=0}^{2} x_i b_j^{(i)}$.

Therefore, in the case of a plane rational curve C the Zariski embedding $\rho_C : \mathbb{P}^2 \to \mathbb{P}^n$ is the linear embedding given by the $3\times(n+1)$–matrix $B_C := (b_j^{(i)})_{i=0,1,2,\, j=0,\dots,n}$. In what follows we denote by $\mathbb{P}^2_C$ the image $\rho_C(\mathbb{P}^2)$, which is a plane in $\mathbb{P}^n$. By Lemma 3.7 the curve C is an irreducible component of the plane cut of the discriminant hypersurface $R_n \subset \mathbb{P}^n$, which has degree $2n-2$, by the plane $\rho_C(\mathbb{P}^2)$; the other irreducible components come from the artifacts L_C of C. This yields Lemma 1.9 in the Introduction:

$$\mathbb{P}^2_C \cap R_n = C \cup L_C\ .$$

The embedding $C^* \hookrightarrow \mathbb{P}^{2*}$ composed with the normalization $\nu : \mathbb{P}^1 \to C^*$ is uniquely, up to projective equivalence, defined by the corresponding linear series g^2_n

on $\mathbb{P}^1$, and the embedding ρ_C is uniquely defined by C up to a choice of normalization of C^*. Thus, $\mathbb{P}^2_C$ is uniquely defined by g^2_n up to the action on $\mathbb{P}^n$ of the group $\mathbb{P}\mathrm{GL}(2,\mathbb{C}) \times \mathbb{P}\mathrm{GL}(3,\mathbb{C})$ via its natural representation in $\mathbb{P}\mathrm{GL}(n+1,\mathbb{C})$, where the second factor leaves $\mathbb{P}^2_C$ invariant.

5.3. *The rational normal curve.* The dual map ${\rho_C}^* : \mathbb{P}^{n*} \to \mathbb{P}^{2*}$, given by the transposed $(n+1)\times 3$–matrix ${}^tB_C = (b_i^{(j)})_{i=0,\dots,n,\, j=0,1,2}$, defines a linear projection with the center $N_C := \mathrm{Ker}\ {}^tB_C \subset \mathbb{P}^{n*}$ of dimension $n-3$. The curve C^* is the image under this projection of the rational normal curve $C^*_n = (z_0^n : z_0^{n-1}z_1 : \dots : z_1^n) \subset \mathbb{P}^{n*}$ (cf. [Ve, p.208]), i.e.

$$\rho^*_C(C^*_n) = C^* \ .$$

Furthermore, C^*_n is the image of $\mathbb{P}^1 \cong C^*_{norm}$ under the embedding $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{n*}$ defined by the complete linear system $|H| = |n(\infty)| \cong \mathbb{P}^n$. Therefore, $\nu = {\rho_C}^* \circ i : \mathbb{P}^1 \to C^* \subset \mathbb{P}^{2*}$ is the normalization map.

5.4. *The duality picture.* It is easily seen that the rational normal curve $C^*_n \subset \mathbb{P}^{n*}$ and the discriminant hypersurface $R_n \subset \mathbb{P}^n$ are dual to each other. This yields the following duality picture:

$$\begin{array}{ccc} (\mathbb{P}^2,\, C\cup L_C) & \overset{\rho_C}{\hookrightarrow} & (\mathbb{P}^n,\, R_n) \\ \updownarrow & & \updownarrow \\ (\mathbb{P}^{2*},\, C^*) & \overset{\rho^*_C}{\longleftarrow} & (\mathbb{P}^{n*},\, C^*_n) \end{array}$$

To describe this duality in more details, fix a point $q = (z_0^n : z_0^{n-1}z_1 : \dots : z_1^n) \in C^*_n \subset \mathbb{P}^{n*}$, and let

$$F_qC^*_n = \{T^0_qC^*_n \subset T^1_qC^*_n \subset \dots \subset T^{n-1}_qC^*_n \subset \mathbb{P}^{n*}\}$$

be the flag of the osculating subspaces to C^*_n at q, where $\dim T^k_qC^*_n = k$, $T^0_qC^*_n = \{q\}$ and $T^1_qC^*_n = T_qC^*_n$ is the tangent line to C^*_n at q (see [Na, p.110]). For instance, for $q = q_0 = (1:0:\dots:0) \in C^*_n$ we have $T^k_qC^*_n = \{x_{k+1} = \dots = x_n = 0\} \subset \mathbb{P}^{n*}$.

The dual curve $C_n \subset \mathbb{P}^n$ of C^*_n is in turn projectively equivalent to a rational normal curve; namely,

$$C_n = \{p \in \mathbb{P}^n \,|\, p = (z_1^n : -nz_0z_1^{n-1} : \dots : (-1)^k \binom{n}{k} z_0^k z_1^{n-k} : \dots : (-1)^n z_0^n)\}$$

Furthermore, the dual flag $F^\perp_q = \{\mathbb{P}^n \supset H^{n-1}_q \supset \dots \supset H^0_q\}$, where $H^{n-k}_q := (T^{k-1}_qC^*_n)^\perp$, is nothing else but the flag of the osculating subspaces $F_pC_n = \{T^{k-1}_pC_n\}^n_{k=1}$

of the dual rational normal curve $C_n \subset \mathbb{P}^n$. An easy way to see this is to look at the flags at the dual points $q_0 = (1:0:\ldots:0) \in C_n^*$ and $p_0 = (0:\ldots:0:1) \in C_n$, where all the flags consist of coordinate subspaces, and then to use the Aut $\mathbb{P}^1$-homogeneity (cf. 7.2 - 7.4 below).

The points of the osculating subspace $H_q^k = T_p^k C_n$ correspond to the binary forms of degree n for which $(z_0 : z_1) \in \mathbb{P}^1$ is a root of multiplicity at least $n-k$. In particular, $H_q^{n-2} = (T_q C_n^*)^\perp$ consists of the binary forms which have $(z_0 : z_1)$ as a multiple root. Therefore, the discriminant hypersurface R_n is the union of these linear subspaces $H_q^{n-2} \cong \mathbb{P}^{n-2}$ for all $q \in C_n^*$, and thus it is the dual hypersurface of the rational normal curve C_n^*, i.e. each of its points corresponds to a hyperplane in $\mathbb{P}^{n*}$ which contains a tangent line of C_n^*. At the same time, R_n is the developable hypersurface of the $(n-2)$–osculating subspaces $H_q^{n-2} = T_p^{n-2} C_n$ of the dual rational normal curve $C_n \subset R_n$; here $T_p^{n-2} C_n \cap C_n = \{p\}$.

If $z_0 \neq 0$, then the subspace H_q^{n-2} in $\mathbb{P}^n$ can be given as the image of the linear embedding

$$\mathbb{P}^{n-2} \ni (c_0 : \ldots : c_{n-2}) \longmapsto (a_0 : a_1 : a_2 : \ldots : a_n) =$$

$$= (\sum_{k=2}^{n} c_{k-2}(k-1) z_0^{n-k} z_1^k : -\sum_{k=2}^{n} c_{k-2} k z_0^{n-k+1} z_1^{k-1} : c_0 z_0^n : \ldots : c_{n-2} z_0^n) \in (T_q C_n^*)^* \subset \mathbb{P}^n$$

(and symmetrically for $z_0 = 0$).

Consider the decomposition $D_{ij} = d_{ij} \times (\mathbb{P}^1)^{n-2}$ of the diagonal hyperplane $D_{ij} \subset D_n$, where $d_{ij} \equiv \mathbb{P}^1$ is the diagonal line in $\mathbb{P}_i^1 \times \mathbb{P}_j^1$, as the trivial fibre bundle $D_{ij} \to \mathbb{P}^1$ with the fibre $(\mathbb{P}^1)^{n-2}$. The subspaces $H_q^{n-2} \subset R_n$ are just the images of the fibres under the Vieta map; moreover, the restriction of the Vieta map $s_n : (\mathbb{P}^1)^n \to \mathbb{P}^n$ to a fibre yields the Vieta map $s_{n-2} : (\mathbb{P}^1)^{n-2} \to \mathbb{P}^{n-2}$. The dual rational normal curve $C_n \subset \mathbb{P}^n$ is the image $s_n(d_n)$ of the diagonal line $d_n := \bigcap_{i,j} D_{ij} = \{z_1 = \ldots = z_n\} \subset (\mathbb{P}^1)^n$.

5.5. *Artifacts as linear sections and the hyperplanes dual to the cusps.* By duality we have $N_C = \operatorname{Ker} \rho_C{}^* = (\operatorname{Im} \rho_C)^\perp$, i.e. $N_C = (\mathbb{P}_C^2)^\perp$. Therefore,

$$\mathbb{P}_C^2 = N_C^\perp = \bigcap_{x^* \in N_C} \operatorname{Ker} x^* = \{x \in \mathbb{P}^n \mid < x, x^* >= 0 \text{ for all } x^* \in N_C\} \ .$$

A point q on the rational normal curve $C_n^* \subset \mathbb{P}^{n*}$ corresponds to a cusp of C^* under the projection $\rho_C{}^*$ iff the center N_C of the projection meets the tangent developable TC_n^*, which is a smooth ruled surface in $\mathbb{P}^{n*}$, in some point x_{q^*} of the tangent line $T_q C_n^*$ (see [Pi]). In this case it meets $T_q C_n^*$ at the only point x_q^*, because otherwise N_C would contain $T_q C_n^*$ and thus also the point q, which is impossible since $\deg C^* = \deg C_n^* = n$.

Let C^* have a cusp B at the point $q_0 = \rho_C{}^*(q)$, which corresponds to a local branch of C^*_n at the point $q \in C^*_n$ under the normalizing projection $\rho_C{}^* : C^*_n \to C^*$. Define $L_{B,q_0} := \mathrm{Ker}\, x^*_q \subset \mathbb{P}^n$ to be the dual hyperplane of the point $x^*_q \in N_C \cap T_q C^*_n$. Since $x^*_q \in N_C$, this hyperplane L_{B,q_0} contains the image $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2)$. This yields a correspondence between the cusps of C^* and certain hyperplanes in $\mathbb{P}^n$ containing the plane $\mathbb{P}^2_C$. From the definition it follows that L_{B,q_0} contains also the dual linear space $H^{n-2}_q = (T_q C^*_n)^\perp \subset R_n$ of dimension $n-2$. Since the plane $\mathbb{P}^2_C$ is not contained in R_n, we have $L_{B,q_0} = \mathrm{span}\,(\mathbb{P}^2_C,\, H^{n-2}_q)$. It is easily seen that the intersection $\mathbb{P}^2_C \cap H^{n-2}_q$ coincides with the tangent line $l_{q_0} \subset L_C$ of C, which is dual to the cusp q_0 of C^*. Thus, the artifacts L_C of C are the sections of $\mathbb{P}^2_C$ by those osculating linear subspaces $H^{n-2}_q \subset R_n$ for which q is a cusp of C^*; any other subspace H^{n-2}_q meets the plane $\mathbb{P}^2_C$ in one point of C only.

5.6. Lemma. *Let $C \subset \mathbb{P}^2$ be a rational curve whose dual curve $C^* \subset \mathbb{P}^{2*}$ has degree n. Let B be a cusp of C^* with center $q_0 \in C^*$, and let $L_{B,q_0} \subset \mathbb{P}^n$ be the corresponding hyperplane which contains the plane $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2)$ (see (5.5)). Then under a suitable choice of a normalization of the dual curve C^* we have $L_{B,q_0} = \bar{A}_1$, where*

$$\bar{A}_1 := \{(a_0 : \ldots : a_n) \in \mathbb{P}^n \mid a_1 = 0\}\ .$$

The preimage $\bar{H}_0 := s_n^{-1}(\bar{A}_1) \subset (\mathbb{P}^1)^n$ is the closure of the linear hyperplane in $\mathbb{C}^n$

$$H_0 := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} z_i = 0\}\ .$$

Proof. Up to a choice of coordinates in $\mathbb{P}^{2*}$, which does not affect the statement, we may assume that C^* has a cusp B at the point $q_0 = (0:0:1)$. Let $\infty = (1:0) \in \mathbb{P}^1$, and let $\nu : \mathbb{P}^1 \cong C^*_{norm} \to C^* \hookrightarrow \mathbb{P}^2$ be composition of an isomorphism $\mathbb{P}^1 \cong C^*_{norm}$ with the normalization map. This isomorphism may be chosen in such a way that the cusp B corresponds to the local branch of $\mathbb{P}^1$ at ∞, and so $\nu(\infty) = q_0$. Here as above $\nu = (g_0 : g_1 : g_2)$ is given by a triple of homogeneous polynomials $g_i(z_0, z_1) = \sum_{j=0}^{n} b_j^{(i)} z_0^{n-j} z_1^j$, $i = 0, 1, 2$, of degree n. Since $\nu(\infty) = q_0$ we have $\deg_{z_0} g_0 < n$, $\deg_{z_0} g_1 < n$, $\deg_{z_0} g_2 = n$, i.e. $b_0^{(0)} = b_0^{(1)} = 0$, $b_0^{(2)} \neq 0$. Performing Tschirnhausen transformation

$$\mathbb{P}^1 \ni (z_0 : z_1) \longmapsto (z_0 - \frac{b_1^{(2)}}{n b_0^{(2)}} z_1 : z_1) \in \mathbb{P}^1$$

we may assume, furthermore, that $b_1^{(2)} = 0$.

Claim 1. Under the above choice of parametrization the image $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2)$ is contained in the hyperplane $\bar{A}_1 := \{(a_0 : \ldots : a_n) \in \mathbb{P}^n \,|\, a_1 = 0\}$.

Indeed, since C^* has a cusp at q_0 we have $(g_0/g_2)'_{z_1} = (g_1/g_2)'_{z_1} = 0$ at the point $(1 : 0) \in \mathbb{P}^1$, i.e. $(g_0)'_{z_1} = (g_1)'_{z_1} = 0$ when $z_1 = 0$. This means that $\deg_{z_0} g_0 < n-1$, $\deg_{z_0} g_1 < n-1$, i.e. $b_1^{(0)} = b_1^{(1)} = 0$. And also $b_1^{(2)} = 0$, as it has been achieved above by making use of Tschirnhausen transformation.

Since $b_1^{(i)} = 0$, $i = 0, 1, 2$, we have $a_1(x) \equiv 0$. Therefore, $\rho_C(x) \in \bar{A}_1$ for any $x \in \mathbb{P}^2$, which proves the claim. □

Claim 2. The dual space H^{n-2}_q to $T_q C^*_n$ is contained in $\bar{A}_1$.

Indeed, since $\nu(\infty) = q_0$ and $\nu = \rho_C^* \circ i$ with $i : \mathbb{P}^1 \to C^*_n \subset \mathbb{P}^{n*}$ we get $q = (1 : 0 : \ldots : 0)$. Thus, by (5.4) the subspace $H^{n-2}_q = (T_q C^*_n)^\perp$ is given by the equations $\{a_0 = a_1 = 0\}$, and hence it is contained in $\bar{A}_1$. □

By (5.5) we have $L_{B,q_0} = \operatorname{span}(\mathbb{P}^2_C, H^{n-2}_q)$. Therefore, from these two claims we get $L_{B,q_0} = \bar{A}_1$.

To conclude the proof of the lemma it is enough to note that if $a_0 \neq 0$ and $a_1 = 0$, then the sum of the roots $z_1 + \ldots + z_n$ of the equation $a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$ is identically zero. Thus,

$$\bar{H}_0 = s_n^{-1}(\bar{A}_1) = \{(\,(u_1 : v_1), \ldots, (u_n : v_n)\,) \in (\mathbb{P}^1)^n \,|\, \sum_{i=1}^n u_i/v_i = 0\},$$

which is the closure of the linear hyperplane $H_0 \subset \mathbb{C}^n$ as in the lemma. □

5.7. *Monomial and quasi–monomial rational plane curves.* A rational curve $C \subset \mathbb{P}^2$, which can be normalized (up to permutation) as follows: $(x_0(t) : x_1(t) : x_2(t)) = (at^k : bt^m : g(t))$, where $a, b \in \mathbb{C}^*$, $k, m \in \mathbb{Z}_{\geq 0}$ and $g \in \mathbb{C}[t]$, will be called *a quasi–monomial curve.*

If here $g(t) = ct^l$, $c \in \mathbb{C}^*$, $l \in \mathbb{Z}_{\geq 0}$, then C is *a monomial curve*; in this case we may assume that $\min\{k, l, m\} = m = 0$ and $\gcd(k, l) = 1$, $l > k$. Note that a linear pencil of monomial curves $C_\mu = \{\alpha x_0^l + \beta x_1^{l-k} x_2^k = 0\}$, where $\mu = \alpha/\beta \in \mathbb{P}^1$, is self–dual, i.e. the dual curve of a monomial one is again monomial and belongs to the same pencil. In contrast, the dual curve to a quasi–monomial one is not necessarily projectively equivalent to a quasi–monomial curve (recall that two plane curves C, C' are *projectively equivalent* if $C' = \alpha(C)$ for some $\alpha \in \mathbb{P}\mathrm{GL}(3; \mathbb{C}) \cong \mathrm{Aut}\,\mathbb{P}^2$). The simplest example is the nodal cubic $C = \{(x_0 : x_1 : x_2) = (t : t^3 : t^2 - 1)\}$. Indeed, its dual curve is a quartic with three cusps (cf. Remark 4.2); but a quasi–monomial curve may have at most two cusps.

The statement of the next lemma is easy to check, so the proof is omitted.

5.8. Lemma. *A quasi–monomial curve* $C = (t^k : t^m : g(t))$, *where* $k < m$ *and* $g(t) = \sum_{j=0}^{n} b_j t^{n-j}$ *is a polynomial of degree* $n \geq 3$, *has no cusp iff it is one of the following curves:*

$$(t : t^{n\pm1} : g(t)),\ b_n \neq 0$$
$$(t : t^n : g(t)),\ b_1 \neq 0, b_n \neq 0$$
$$(1 : t^{n\pm1} : g(t)),\ b_{n-1} \neq 0$$
$$(1 : t^n : g(t)),\ b_1 \neq 0 \text{ and } b_{n-1} \neq 0\ .$$

In particular, a monomial curve $C = (t^k : t^l : 1)$, *where* $k < l$ *and* $gcd(k,\, l) = 1$, *has no cusp iff it is a smooth conic* $C = (t : t^2 : 1)$.

5.9. *Parametrized rational plane curves.* Note that, while the action of the projective group $PGL(3,\mathbb{C})$ on $\mathbb{P}^2$ does not affect the image $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) \subset \mathbb{P}^n = S^n\mathbb{P}^1$, the choice of the normalization $\mathbb{P}^1 \to C^*$, defined up to the action of the group $PGL(2,\mathbb{C}) = \mathrm{Aut}\mathbb{P}^1$, usually does (cf. (5.2)). This is why in the next lemma we have to fix the normalization of a rational plane curve C. This automatically fixes a normalization of its dual curve C^*, and vice versa. Indeed, recall that if $C = (g_0 : g_1 : g_2)$, where $g_i \in \mathbb{C}[t]$, $i = 0, 1, 2$, is a parametrized rational plane curve, then the dual curve C^* has, up to cancelling of the common factors, the parametrization $C^* = (M_{12} : M_{02} : M_{01})$, where M_{ij} are the 2×2–minors of the matrix

$$\begin{pmatrix} g_0 & g_1 & g_2 \\ g_0' & g_1' & g_2' \end{pmatrix}$$

Furthermore, the equation of C can be written as $\frac{1}{x_2^d}\mathrm{Res}\,(x_0g_2 - x_2g_0, x_1g_2 - x_2g_1) = 0$, where $d = \deg C$ and Res means resultant (see e.g. [Au, 3.2]).

We will use the following terminology. By *a parametrized rational plane curve* we will mean a rational curve C in $\mathbb{P}^2$ with a fixed normalization $\mathbb{P}^1 \to C$ of it. *A parametrized monomial* resp. *a parametrized quasi–monomial plane curve* is a parametrized rational plane curve such that all resp. two of its coordinate functions are monomials.

Clearly, projective equivalence between parametrized curves is a stronger relation than just projective equivalence between underlying projective curves themselves.

5.10. Lemma. *A parametrized rational plane curve* $C^* \subset \mathbb{P}^{2*}$ *of degree* n *is projectively equivalent to a parametrized monomial resp. quasi–monomial curve iff*

$\mathbb{P}^2_C \subset \mathbb{P}^n$ is a coordinate plane resp. containes a coordinate axis. This axis is unique iff C^* is projectively equivalent to a parametrized quasi–monomial curve, but not to a monomial one.

Proof. Let $\nu : t \longmapsto (at^k : bt^m : g(t))$, where $a,\, b \in \mathbb{C}^*$, $g \in \mathbb{C}[t]$ and $t = z_0/z_1 \in \mathbb{P}^1$, define a parametrized quasi–monomial curve $C^* \subset \mathbb{P}^{2*}$ of degree n. Denote $e_k = (0 : \ldots : 0 : 1_k : 0 : \ldots : 0) \in \mathbb{P}^n$. Then ρ_C is given by the matrix $B_C = (b^{(0)},\, b^{(1)},\, b^{(2)}) = (ae_{n-k},\, be_{n-m},\, b^{(2)})$, and therefore $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) = \text{span}\,(b^{(0)},\, b^{(1)},\, b^{(2)})$ contains the coordinate axis $l_{n-k,\,n-m}$, where $l_{i,j} := \text{span}\,(e_i,\, e_j) \subset \mathbb{P}^n$.

If C^* is a paramatrized monomial curve, i.e. if $g(t) = ct^r$, where $c \in \mathbb{C}^*$, then clearly $\mathbb{P}^2_C$ is the coordinate plane $\mathbb{P}_{n-k,\,n-m,\,n-r} := \text{span}\,(e_{n-k},\, e_{n-m},\, e_{n-r})$.

Since the projective equivalence of parametrized plane curves does not affect the $\mathbb{P}^2_C$, this yields the first statement of the lemma in one direction.

Vice versa, suppose that $\mathbb{P}^2_C$ coincides with the coordinate plane $\mathbb{P}_{n-k,\,n-m,\,n-r}$. Performing a suitable linear coordinate change in $\mathbb{P}^{2*}$ we may assume that $b^{(0)} = e_{n-k}$, $b^{(1)} = e_{n-m}$, $b^{(2)} = e_{n-r}$, i.e. that $\nu(t) = (t^k : t^m : t^r)$. Therefore, in this case the parametrized curve C^* is projectively equivalent to a monomial curve.

Suppose now that $\mathbb{P}^2_C$ contains the coordinate axis $l_{n-k,\,n-m}$. Performing as above a suitable linear coordinate change in $\mathbb{P}^{2*}$ we may assume that $b^{(0)} = e_{n-k}$, $b^{(1)} = e_{n-m}$, and so $\nu(t) = (t^k : t^m : g(t))$. In this case C^* is projectively equivalent to a parametrized quasi–monomial curve. This proves the first assertion of the lemma.

Let $C^* = (\text{a}t^{n-k} : bt^{n-m} : g(t))$ be a parametrized quasi–monomial curve which is not projectively equivalent to a monomial one. Then as above $\mathbb{P}^2_C \supset l_{k,m}$, and this is the only coordinate axis contained in $\mathbb{P}^2_C$. Indeed, if $l_{i,j} \subset \mathbb{P}^2_C$, where $\{i,\, j\} \neq \{k,\, m\}$, then $\mathbb{P}^2_C$ would contain at least three distinct vertices e_α, where $\alpha \in \{i,\, j,\, k,\, m\}$, and so $\mathbb{P}^2_C$ would be a coordinate plane, what has been excluded by our assumption. The opposite statement is evidently true. This concludes the proof. □

5.11. Remarks. *a.* Let $C^* = (at^k : bt^m : ct^r)$, where $a,\, b,\, c \in \mathbb{C}^*$, be a parametrized monomial curve of degree n. To be a normalization, this parametrization should be irreducible, i.e. up to permutation there should be $0 = k < m < r = n$, where $\gcd\,(m,\, n) = 1$. Thus, $\mathbb{P}^2_C = \mathbb{P}_{0,\,n-m,\,n}$ is a rather special coordinate plane.

b. Let C^* be the parametrized quasi–monomial curve $C_{k,m,g} := (at^k : bt^m : g(t))$, which is not equivalent to a monomial one. Then the only coordinate axis contained in $\mathbb{P}^2_C$ is the axis $l_{n-k,\,n-m} := \text{span}\,(e_{n-k},\, e_{n-m}) = \rho_C(l_2)$, where $l_2 := \{x_2 = 0\} \subset \mathbb{P}^2$. Furthermore, if C^* is obtained from such a curve by a permutation of the coordinates, then still the only coordinate axis contained in $\mathbb{P}^2_C$ is $l_{n-k,\,n-m}$.

5.12. *An equivariant meaning of the Vieta map.* Consider the following $\mathbb{C}^*$–actions on $(\mathbb{P}^1)^n$ resp. on $\mathbb{P}^n = S^n\mathbb{P}^1$:

$\tilde{G} : \mathbb{C}^*\times(\mathbb{P}^1)^n \ni (\lambda, ((u_1 : v_1), \ldots, (u_n : v_n))) \longmapsto ((\lambda u_1 : v_1), \ldots, (\lambda u_n : v_n)) \in (\mathbb{P}^1)^n$

resp.

$$G : \mathbb{C}^* \times \mathbb{P}^n \ni (\lambda, (a_0 : a_1 : \ldots : a_n)) \longmapsto (a_0 : \lambda a_1 : \lambda^2 a_2 : \ldots : \lambda^n a_n) \in \mathbb{P}^n$$

Note that the Vieta map $s_n : (\mathbb{P}^1)^n \to \mathbb{P}^n$ (see (5.1)) is equivariant with respect to these $\mathbb{C}^*$–actions and its branching divisors D_n resp. R_n are invariant under $\tilde{G}$ resp. G. Identifying $\mathbb{C}$ with $\mathbb{P}^1 \setminus \{(1 : 0)\}$, we fix an embedding $\mathbb{C}^n \hookrightarrow (\mathbb{P}^1)^n$; denote its image by $\mathbb{C}^n_z$. Both this Zariski open part of $(\mathbb{P}^1)^n$ and its complementary divisor are $\tilde{G}$–invariant. In turn, the hyperplane $\mathbb{P}^{n-1}_0 := \{a_0 = 0\}$ in $\mathbb{P}^n$, as well as any other coordinate linear subspace of $\mathbb{P}^n$, and its complement $\mathbb{C}^n_a := \mathbb{P}^n \setminus \mathbb{P}^{n-1}_0$ are G–invariant.

The next lemma is a usefull supplement to Lemma 5.10.

5.13. Lemma. *A parametrized rational plane curve $C^* \subset \mathbb{P}^{2*}$ is projectively equivalent to a parametrized quasi–monomial curve iff $\mathbb{P}^2_C \subset \mathbb{P}^n$ contains a one–dimensional G–orbit. This orbit is unique iff C^* is projectively equivalent to a parametrized quasi–monomial curve, but not to a monomial one.*

Proof. Let $\lambda \longmapsto (a_0 : \lambda a_1 : \ldots : \lambda^n a_n)$, where $\lambda \in \mathbb{C}^*$, be a parametrization of the G–orbit O_p throuh the point $p = (a_0 : \ldots : a_n) \in \mathbb{P}^n$. Since the non-zero coordinates here are linearly independent as functions of λ, the orbit $O_p \subset \mathbb{P}^n$ is contained in a projective plane iff all but at most three of coordinates of p vanish. If p has exactly three non–zero coordinates, then the only plane that containes $\bar{O}_p$ is a coordinate plane. If only two of the coordinates of p are not zero, then the closure $\bar{O}_p$ is a coordinate axis. Since we consider a one–dimensional orbit, the case of one non–zero coordinate is excluded. Now the lemma easily follows from Lemma 5.10. □

6 C–hyperbolicity of complements of rational curves in presence of artifacts

Before proving an analog of Theorem 4.1 for the case of a rational curve (see Theorem 6.5 below), let us consider simple examples which illustrate some ideas used in the proof.

In (1.3) we gave an example of a quintic $C_5 \subset \mathbb{P}^2$ (union of five lines) whose complement is C–hyperbolic. Here is another one.

6.1. Example. Let $C \subset \mathbb{P}^2$ be a smooth conic and $L = l_1 \cup l_2 \cup l_3$ be the union of three distinct tangents of C.

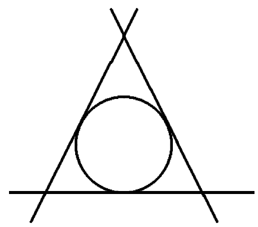


Figure 2

Claim. a) $X_1 := \mathbb{P}^2 \setminus (C \cup l_1)$ *is super–Liouville and its Kobayashi pseudo–distance* k_{X_1} *is identically zero.*

b) Put $X_2 = \mathbb{P}^2 \setminus (C \cup l_1 \cup l_2)$. *Let* (C_α), $\alpha \in \mathbb{P}^1$, *be the linear pencil of conics generated by* C *and* $l_1 + l_2$, *where* $C = C_{(1:1)}$ *and* $l_1 + l_2 = C_{(1:0)}$. *Then the image of any entire curve* $\mathbb{C} \to X_2$ *is contained in one of the conics* C_α, *and* $k_{X_2}(p, q) = 0$ *iff* $p, q \in C_\alpha$ *for some* $\alpha \in \mathbb{P}^1$. *Furthermore,* X_2 *is neither C–hyperbolic, nor super–Liouville (see (2.2)).*

c) $X = X_3 := \mathbb{P}^2 \setminus (C \cup L)$ *is C–hyperbolic, Kobayashi complete hyperbolic and hyperbolically embedded into* $\mathbb{P}^2$.

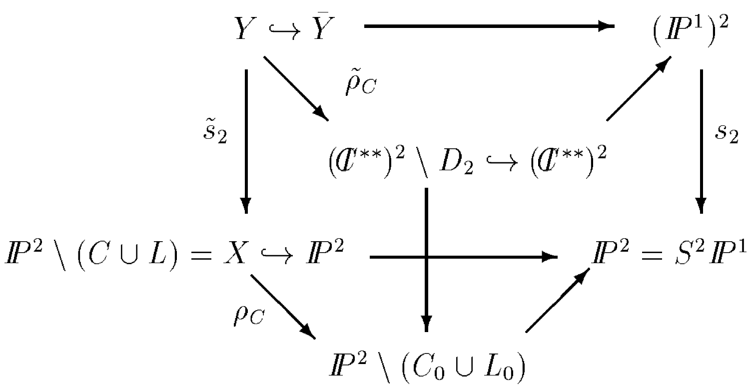
Proof. a) is easily checked by applying, for instance, Lemma 2.3. An alternative way is to note that X_1 is isomorphic to the product $\mathbb{C} \times \mathbb{C}^*$. □

b) Consider the affine chart $\mathbb{C}^2 \cong \mathbb{P}^2 \setminus l_2$ in $\mathbb{P}^2$. We have $X_2 \cong \mathbb{C}^2 \setminus \Gamma$, where the affine curve $\Gamma := (C \cup l_1) \setminus l_2$ can be given in appropriate coordinates by the equation $y(x^2 - y) = 0$. Let the double covering $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ branched over the axis l_1 be given as $(x, y) = \pi(x, z) := (x, z^2)$. It yields the non–ramified double covering $Y \to X_2$, where $Y := \mathbb{C}^2 \setminus \pi^{-1}(\Gamma)$. Here $\pi^{-1}(\Gamma)$ is union of three affine lines $m_0 = \{z = 0\}$, $m_1 = \{x = z\}$, $m_{-1} = \{x = -z\}$, which are level sets of the rational function $\varphi(x, z) := z/x$. It defines a holomorphic mapping $\varphi | Y : Y \to \mathbb{P}^1 \setminus \{0, 1, -1\}$. Therefore, for any entire curve $f : \mathbb{C} \to X_2$ its covering curve $\tilde{f} : \mathbb{C} \to Y$ has the image contained in an affine line l_{β_0} from the linear pencil $l_\beta := \{x = \beta z\}$, $\beta \in \mathbb{C}$. Thus, the image $f(\mathbb{C})$ is contained in the conic C_{α_0} from the linear pencil $C_\alpha = \{x^2 = \alpha y\}$, where $\alpha = \beta^2$. This proves the first assertion in b). The second one easily follows from the inequality $k_Y \geq \varphi^* k_{\mathbb{P}^1 \setminus \{3\,\text{points}\}}$ and the equality $k_{X_2} = \pi_* k_Y$. Finally, since the tautological line bundle $\varphi : \mathbb{C}^2 \setminus \{\bar{0}\} \to \mathbb{P}^1$ is trivial over $\mathbb{P}^1 \setminus \{\text{a point}\}$, there is an isomorphism $Y \cong \mathbb{C}^* \times (\mathbb{P}^1 \setminus \{3\,\text{points}\})$. Therefore, the universal covering $U_Y \cong U_{X_2}$ of Y resp. of X_2 is biholomorphic to $\mathbb{C} \times \Delta$. Hence, X_2 is neither C–hyperbolic, nor super–Liouville. □

c) We can treat the dual curve of $C \cup L$ as the dual conic $C^* \subset \mathbb{P}^{2*}$ with three distinguished points q_1, q_2, q_3 on it, whose dual lines are, respectively, l_1, l_2, l_3. Choose an isomorphism $C^* \cong \mathbb{P}^1$ in such a way that $q_1, q_2, q_3 \in C^*$ correspond, respectively, to the points $(0:1), (1:0), (1:1) \in \mathbb{P}^1$. The Vieta map $s_2 : (\mathbb{P}^1)^2 \to \mathbb{P}^2 = S^2\mathbb{P}^1$ is given by the formula

$$s_2 : ((u_1 : v_1), (u_2 : v_2)) \longmapsto (v_1 v_2 : -(u_1 v_2 + u_2 v_1) : u_1 u_2)\ .$$

To the distinguished points $(0:1), (1:0), (1:1) \in \mathbb{P}^1$ there correspond six generators of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, three vertical ones and three horizontal ones. Their images under the Vieta map s_2 is the union L_0 of three lines $x_0 = 0,\ x_2 = 0,\ x_0 + x_1 + x_2 = 0$ in $\mathbb{P}^2 = S^2\mathbb{P}^1$, which are tangent to the conic $C_0 := s_2(\bar{D}_2) = \{x_1^2 - 4x_0x_2 = 0\} \subset \mathbb{P}^2$, where $\bar{D}_2 = \bar{D}_{1,2}$ is the diagonal of $(\mathbb{P}^1)^2$. Thus, we have the commutative diagram:



(5)

where $\tilde{s_2} : Y \to X$ is the induced covering, $\mathbb{C}^{**} = \mathbb{P}^1 \setminus \{3\,\text{points}\}$ and the horizontal arrows are isomorphisms. It follows that $Y \cong (\mathbb{C}^{**})^2 \setminus D_2 \subset (\mathbb{C}^{**})^2$ is C–hyperbolic, and therefore, X is C–hyperbolic, too.

It is easily seen that $\text{reg}\,(C \cup L) = (C \cup L) \setminus \text{sing}\,(C \cup L)$ is hyperbolic. Therefore, by Proposition 2.7 $X = \mathbb{P}^2 \setminus (C \cup L)$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. □

Here is one more example of a curve with properties as in Claim c) above.

6.2. Example. Let the things be as in the previous example. Performing the Cremona transformation σ of $\mathbb{P}^2$ with center at the points of intersections of the lines l_1, l_2, l_3, we obtain a 3-cuspidal quartic $C' := \sigma(C)$ together with three new lines m_1, m_2, m_3, passing each one through a pair of cusps of C' (they are images of the exceptional curves of the blow-ups by σ at the above three points; see Fig. 3).

Figure 3

Put $L' = m_1 \cup m_2 \cup m_3$ and $X' = \mathbb{P}^2 \setminus (C' \cup L')$. Since $\sigma \,|\, X : X \to X'$ is an isomorphism and X is C–hyperbolic, we have that X' is also C–hyperbolic. The same reasoning as above ensures that X' is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$.

The next lemma will be used in the proof of Theorem 6.5. From now on 'bar' over a letter will denote a projective object, in contrast with the affine ones.

6.3. Lemma. *Let $\bar{H}_0$ be the hyperplane in $\mathbb{P}^{n-1}$ given by the equation $\sum_{i=1}^{n} x_i = 0$, and let $\bar{D}_{n-1} = \bigcup_{1 \le i < j \le n} \bar{D}_{ij}$ be the union of the diagonal hyperplanes, where $\bar{D}_{ij} \subset \mathbb{P}^{n-1}$ is given by the equation $x_i - x_j = 0$. Then $\bar{H}_0 \setminus \bar{D}_{n-1}$ is C–hyperbolic, Kobayashi complete hyperbolic and hyperbolically embedded into $\bar{H}_0 \cong \mathbb{P}^{n-2}$.*

Proof. Put $y_i = x_1 - x_{i+1}$, $i = 1, \dots, n-1$. Then $z_i = y_i / y_{n-1}$, $i = 1, \dots, n-2$, are coordinates in the affine chart $\bar{H}_0 \setminus \bar{D}_{1,n} \cong \mathbb{C}^{n-2}$. In these coordinates $\bar{D}_{1,i+1} \cap \bar{H}_0$ resp. $\bar{D}_{i+1,n} \cap \bar{H}_0$ is given by the equation $z_i = 0$ resp. $z_i = 1$, $i = 1, \dots, n-2$. Thus, $\bar{H}_0 \setminus \bar{D}_{n-1} \hookrightarrow (\mathbb{C}^{**})^{n-2}$, where $\mathbb{C}^{**} := \mathbb{P}^1 \setminus \{3 \ \textit{points}\}$. By Lemma 2.5 it follows that $\bar{H}_0 \setminus \bar{D}_{n-1}$ is C–hyperbolic.

To prove Kobayashi complete hyperbolicity and hyperbolic embededdness we may use the following criterion [Za1, Theorem 3.4] :

The complement of a finite set of hyperplanes $L_1, \dots, L_N$ in $\mathbb{P}^n$ is hyperbolically embedded into $\mathbb{P}^n$ iff (*) for any two distinct points p, q in $\mathbb{P}^n$ there is a hyperplane L_i , $i \in \{1, \dots, N\}$, which does not contain any of them.

Note that the complement of a hypersurface is locally complete hyperbolic [KiKo, Proposition 1], and therefore its hyperbolic embededdness implies the complete hyperbolicity (see [Ki] or [KiKo, Theorem 4]). Therefore, it is enough to check that the union of hyperplanes $\bar{H}_0 \cap \bar{D}_{n-1}$ in $\bar{H}_0 \cong \mathbb{P}^{n-1}$ satisfies the above condition (*).

Supposing the contrary we would have that there exists a pair of points $p, q \in \bar{H}_0$, $p \neq q$, such that each of the diagonal hyperplanes $\bar{D}_{ij}$ contains at least one of these points. Put $p = (x_1' : \dots : x_n')$ and $q = (x_1'' : \dots : x_n'')$. Since $(\bigcap_{i,j} \bar{D}_{ij}) \cap \bar{H}_0 = \emptyset$, we may assume that up to permutation $x_1' = \dots = x_k'$ and $x_{k+1}'' = \dots = x_n''$, where $2 \leq k \leq n-1$, and moreover, that $x_l' \neq x_i'$ for each $i \leq k < l \leq n$. The latter means that $p \notin \bar{D}_{il}$ for such i , l. Therefore, we must have $q \in \bar{D}_{il}$ for $i \leq k < l$. In particular, $q \in \bar{D}_{i,k+1}$, $i = 1, \dots, k$, and so $x_1'' = \dots = x_k'' = x_{k+1}'' = \dots = x_n''$, which is impossible, since $q \in \bar{H}_0$. □

6.4. Remark. If $n = 4$, so that $\bar{H}_0 \cong \mathbb{P}^2$, it is easily seen that $\bar{D}_3 \cap \bar{H}_0$ is a complete quadruple in $\mathbb{P}^2$, i.e. the union of six lines defined by four points in general position.

Now we are ready to extend Theorem 4.1, under certain additional restrictions, to the case of a rational curve.

6.5. Theorem. *Let $C \subset \mathbb{P}^2$ be a rational curve whose dual curve C^* has at least one cusp, so that C has the artifacts $L_C \neq \emptyset$. Let $X := \mathbb{P}^2 \setminus (C \cup L_C)$. Then the following statements hold.*

a) If the dual curve C^* is not projectively equivalent to a quasi–monomial one, then X is almost C–hyperbolic.

b) X is still almost C–hyperbolic if C^* is projectively equivalent to a quasi–monomial curve $C_{k,m,g} := \{(t^k : t^m : g(t))\}$ of degree n, but not to a monomial one, except the cases when, up to a choice of normalization, $C_{k,m,g}$ is one of the curves $\{(1 : t^n : g(t))\}$ or $\{(t : t^n : g(t))\}$, where $g \in \mathbb{C}[t]$ and $\deg g \leq n-2$. In the latter cases X is almost C–hyperbolic modulo the line $l_2 := \{x_2 = 0\} \subset \mathbb{P}^2$ in the coordinates where $C^* = C_{k,m,g}$.

c) Let $C = C_{\mu_0}$ be a monomial curve [2] *from the linear pencil $C_\mu = \{\alpha x_0^n +$*

[2]The case when C^* is projectively equivalent to a monomial curve is easily deduced to this one.

$\beta x_1^k x_2^{n-k} = 0\}$, *where* $\mu = (\alpha : \beta) \in \mathbb{P}^1$. *Then* $k_X(p, q) = 0$ *iff* $p, q \in C_\mu$ *for some* $\mu \in \mathbb{P}^1 \setminus \{\mu_0\}$. *In particular, any entire curve* $\mathbb{C} \to X$ *is contained in one of the curves of the linear pencil* (C_μ).

Proof. The proof will be done in several steps. We will start with the main construction used in the proof.

Basic construction. Fix a cusp q_0 of C^*, and let $q_0^* \subset \mathbb{P}^2$ be the dual line of q_0. Clearly, $q_0^* \subset L_C$. Choosing an appropriate isomorphism $\mathbb{P}^1 \cong C^*_{norm}$ and coordinates in $\mathbb{P}^2$ as in the proof of Lemma 5.6, by this lemma we may assume that $\nu(\infty) = q_0 = (0 : 0 : 1) \in \mathbb{P}^{2*}$, $q_0^* = l_2 = \{x_2 = 0\} \subset \mathbb{P}^2$ and $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) \subset \bar{A}_1 \subset \mathbb{P}^n = S^n\mathbb{P}^1$, where $n = \deg C^*$ and $\bar{A}_1 = \{(a_0 : \ldots : a_n) \in \mathbb{P}^n \,|\, a_1 = 0\}$. Let $\mathbb{C}^n_z \subset (\mathbb{P}_1)^n$ and $\mathbb{C}^n_{(a)} \subset \mathbb{P}^n$ be as in (5.12). Then, as it is easily seen, $\rho_C(X) \subset \rho_C(\mathbb{P}^2 \setminus l_2) \subset s_n(\mathbb{C}^n_{(z)}) \cong \mathbb{C}^n_{(a)} \subset \mathbb{P}^n$, where $s_n : \mathbb{C}^n_{(z)} \to \mathbb{C}^n_{(a)}$ is the restriction of the Vieta map (see (5.1)).

By (5.12) this affine Vieta map yields the non–ramified covering $s_n : H_0 \setminus D_n \to A_1 \setminus R_n$, where as in Lemma 5.6 above $H_0 = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, \sum_{i=1}^n z_i = 0\}$, D_n is the union of the affine diagonal hyperplanes $D_{ij} = \{z \in \mathbb{C}^n \,|\, z_i = z_j\}$, $1 \le i < j \le n$, $A_1 := \{a = (a_1, \ldots, a_n) \in \mathbb{C}^n_{(a)} \,|\, a_1 = 0\} \cong \mathbb{C}^{n-1}$ and $R_n \subset \mathbb{C}^n_{(a)}$ is the affine discriminant hypersurface.

The Zariski map gives the linear embedding $\rho_C \,|\, X : X \to A_1 \setminus R_n$. Let $\tilde{s}_n : Y \to X$ be the non–ramified covering induced by the Vieta covering via this embedding.

Denote by π the canonical projection $\mathbb{C}^n_{(z)} \setminus \{\bar{0}\} \to \mathbb{P}^{n-1}$. Put $\bar{H}_0 := \pi(H_0) \cong \mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$ and $\bar{D}_{ij} := \pi(D_{ij})$, $\bar{D}_{n-1} := \pi(D_n) = \bigcup_{1 \le i < j \le n} \bar{D}_{ij}$. By Lemma 6.3 $\bar{H}_0 \setminus \bar{D}_{n-1}$ is C–hyperbolic, Kobayashi complete hyperbolic and hyperbolically embedded into $\bar{H}_0 \cong \mathbb{P}^{n-2}$.

Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{\tilde{\rho}_C} & H_0 \setminus D_n & \xrightarrow{\pi} & \bar{H}_0 \setminus \bar{D}_{n-1} \hookrightarrow \mathbb{P}^{n-2} \\ {\scriptstyle \tilde{s}_n} \downarrow & & \downarrow {\scriptstyle s_n} & & \\ \mathbb{P}^2 \setminus (C \cup L_C) = X & \xrightarrow{\rho_C} & A_1 \setminus R_n & & \end{array} \qquad (6)$$

where $\tilde{\rho}_C$ is an injective holomorphic mapping. Note that here the Vieta map s_n is equivariant with respect to the $\mathbb{C}^*$–actions $\tilde{G}$ on $H_0 \setminus D_n$ and G on $A_1 \setminus R_n$, respectively, and all the fibres of the projection π are one–dimensional $\tilde{G}$–orbits (see(5.12)).

Proof of a). Under the assumption of a) C^* is not projectively equivalent to a quasi–

monomial curve. Then we have the following assertion.

Claim. The mapping $\pi \circ \tilde{\rho}_C : Y \to \bar{H}_0 \setminus \bar{D}_{n-1}$ *has finite fibres.*

Indeed, since the fibres of π are $\tilde{G}$–orbits, it is enough to show that any $\tilde{G}$–orbit in $H_0 \subset \mathbb{C}^n_{(z)}$ has a finite intersection with $\tilde{\rho}_C(Y)$. Or, what is equivalent, that any G–orbit in $A_1 \subset \mathbb{C}^n_{(a)}$ has a finite intersection with $\rho_C(X) \subset \mathbb{P}^2_C$. We have shown in Lemma 5.13 above that if the latter fails, i.e. if $\mathbb{P}^2_C$ contains a one-dimensional G–orbit, then C^* (paramatrized as above) is projectively equivalent to a (paramatrized) quasi–monomial curve, which is assumed not to be the case. This yields the claim.

Since $\bar{H}_0 \setminus \bar{D}_{n-1}$ is C–hyperbolic, by Lemma 2.5 this implies that X is almost C–hyperbolic. □

Proof of b). We still fix a parametrization of C^* as in the basic construction above, and so we fix the $\mathbb{P}^2_C$ in $\mathbb{P}^n$. If $\mathbb{P}^2_C$ does not contain any coordinate line, we can finish up the proof like in a) and conclude that X is almost C–hyperbolic. So, assume further that $\mathbb{P}^2_C$ does contain a coordinate line. By Lemma 5.10 this means that C^* as a parametrized curve is projectively equivalent to a quasi–monomial curve. Since by our assumption it is not equivalent to a monomial one, the plane $\mathbb{P}^2_C$ is not a coordinate one. After an appropriate change of coordinates in $\mathbb{P}^2$ which does not affect $\mathbb{P}^2_C$ we may assume that $C^* = C_{f,g,h} := (f : g : h)$, where $f, g, h \in \mathbb{C}[t]$ and two of them are the monomials t^k, t^m. We have that $l_{n-k,\,n-m} \subset \mathbb{P}^2_C$ is the only coordinate axis contained in $\mathbb{P}^2_C$ (see Lemma 5.10 and Remark 5.11, b)). By Lemma 5.13 it is the closure of the only one-dimensional G–orbit O_p contained in $\mathbb{P}^2_C$. Now we have to distinguish between two cases:

i) $\rho_C^{-1}(l_{n-k,\,n-m}) \subset L_C$ and ii) $\rho_C^{-1}(l_{n-k,\,n-m}) \not\subset L_C$.

In case i) we have, as in the Claim above, that $\pi \circ \tilde{\rho}_C : Y \to \bar{H}_0 \setminus \bar{D}_{n-1}$ has finite fibres, and therefore X is almost C–hyperbolic. In case ii) we have $O_p \subset \rho_C(X)$; the preimage $\tilde{s}_n^{-1}(O_p)$ is the union of $n!$ distinct $\tilde{G}$–orbits, which are π–fibres, and all the others π–fibres in Y are finite. Thus, by Lemma 2.9 it follows that Y is almost C–hyperbolic modulo $\tilde{s}_n^{-1}(O_p)$, and hence X is almost C–hyperbolic modulo O_p.

Next we show that ii) corresponds exactly to the two exceptional cases mentioned in b), which proves b).

By the assumption of the theorem C^* has a cusp, and we suppose as above this cusp being at the point $q_0 = (0 : 0 : 1)$ and corresponding to the value $t = \infty$. This means that $\deg f \le n-2$, $\deg g \le n-2$ and $\deg h = n$ (see the proof of Lemma 5.6). Thus, the dual line $l_2 = q_0^* \subset \mathbb{P}^2$ belongs to L_C. If $f = t^k$ and $g = t^m$ are monomials, then $\rho_C^{-1}(l_{n-k,\,n-m}) = l_2$ and we have case i). Therefore, up

to the transposition of f and g we may suppose further that $f = t^k$ and $h = t^m$ are monomials, while $g(t)$ is not. In that case $k \le n-2$, $\deg g \le n-2$, $m = n$ and $\rho_C^{-1}(l_{n-k,\,n-m}) = l_1 := \{x_1 = 0\} \subset \mathbb{P}^2$. The dual point $q_1 = (0:1:0) = l_1^* \in \mathbb{P}^{2*}$ is a cusp of C^* iff $k \ge 2$. Hence, ii) occurs iff here $k \le 1$, i.e. iff C^* was projectively equivalent to one of the curves $(1 : t^n : g(t))$ or $(t : t^n : g(t))$, where $\deg g \le n-2$. If C^* is one of these curves, then $X = \mathbb{P}^2 \backslash (C \cup L_C)$ is C–hyperbolic modulo $l_2 = \rho_C^{-1}(l_{n-k,\,0})$.
□

Proof of c). Let $C = C_{\mu_0}$ be a monomial curve from the linear pencil $C_\mu = \{\alpha x_0^n + \beta x_1^k x_2^{n-k} = 0\}$, where $\mu = (\alpha : \beta) \in \mathbb{P}^1$. The pencil (C_μ) is self–dual, i.e. $C_\mu^* = C_{\mu^*}$, where μ^* depends on μ (see 5.7, 5.9), and so without loss of generality we may assume that $C^* = C_\mu^* = C_{(1:-1)}$. Thus, C^* has the parametrizations $C^* = (\tau^k : \tau^n : 1) = (t^{n-k} : 1 : t^n)$, where $\tau = t^{-1}$. Since C^* has a cusp, we have $n = \deg C^* \ge 3$ and $\max(k, n-k) \ge 2$. By permuting coordinates, if necessary, we may assume that $k \ge 2$. In this case the second parametrization, which we denote by ν, fits in with the basic construction, i.e. $\nu(\infty) = q_0 = (0:0:1)$ is a cusp of C^* and $b_1^{(i)} = 0$, $i = 0, 1, 2$.

The parametrization ν being fixed as above, the Zariski embedding ρ_C is given by the matrix $B_C = (e_k,\, e_n,\, e_0)$. Therefore, $\rho_C : \mathbb{P}^2 \to \mathbb{P}^2_C = \mathbb{P}_{0,k,n} \subset \bar{A}_1 \subset \mathbb{P}^n$ is coordinatewise (cf. Remark 5.11, a):

$$\rho_C(x_0 : x_1 : x_2) = (a_0 : \ldots : a_n) = (x_2 : 0 : \ldots : \underbrace{x_0}_{k} : 0 : \ldots : 0 : x_1)$$

The $\mathbb{C}^*$–action G on $\mathbb{P}^n$ induces the $\mathbb{C}^*$–action G' on $\mathbb{P}^2$, where

$$G' : (\lambda, (x_0 : x_1 : x_2)) \longmapsto (\lambda^k x_0 : \lambda^n x_1 : x_2) = (x_0/\lambda^{n-k} : x_1 : x_2/\lambda^n)$$

It is easily seen that the closure of a one-dimensional G'–orbit is an irreducible component of a member of the linear pencil (C_μ).

In what follows we identify X resp. Y with its image under ρ_C resp. $\tilde{\rho}_C$. Let $f : \mathbb{C} \to X$ be an entire curve and $\tilde{f} : \mathbb{C} \to Y$ be its covering curve. From Lemma 6.3 it follows that the map $\pi \circ \tilde{\rho}_C \circ \tilde{f}$ is constant. This means that $\tilde{f}(\mathbb{C})$ is contained in an orbit of $\tilde{G}$, and so $f(\mathbb{C})$ is contained in a G–orbit, which in turn is contained in one of the curves C_μ, as it is stated in c).

Furthermore, $\bar{H}_0 \setminus \bar{D}_{n-1}$ being Kobayashi hyperbolic, the k_Y–distance between any two distinct $\tilde{G}$–orbits in Y is positive. Over each G–orbit O in X there is $n!$ $\tilde{G}$–orbits in Y, and each of them is maped by $\tilde{s}_n$ isomorphically onto O. Therefore, the k_X–distance between two different G–orbits in X, which is equal to the k_Y–distance between their preimages in Y, is positive, too. This proves c). Now the proof of Theorem 6.5 is complete. □

6.6. Remark. In general, b) is not true for a plane curve whose dual is a quasi-monomial curve without cusps. Indeed, if C is a three–cuspidal plane quartic, then

C^* is a nodal cubic, which is projectively equivalent to a quasi–monomial curve $t \longmapsto (t : t^3 : t^2 - 1)$, where the node corresponds to $t = \pm 1$. The Kobayashi pseudo–distance of $\mathbb{P}^2 \setminus C$ is degenerate on at least seven lines (see Remark 4.2), and thus $\mathbb{P}^2 \setminus C$ is not C–hyperbolic modulo a line.

The next examples illustrate Theorem 6.5.

6.7. Example. Let $C \subset \mathbb{P}^2$ be the cuspidal cubic $4x_0^3 - 27x_1^2x_2 = 0$. Its dual curve $C^* \subset \mathbb{P}^{2*}$ is the cuspidal cubic with the equation $y_0^3 + y_1^2y_2 = 0$. The cusp of C^* at the point $q_0 = (0 : 0 : 1)$ corresponds to the only flex of C at the point $p_0 = (0 : 1 : 0)$, with the inflexional tangent $l_2 = \{x_2 = 0\} \subset \mathbb{P}^2$, so that $L_C = l_2$. Consider the curve $C \cup l_2$. Its complement $X := \mathbb{P}^2 \setminus (C \cup l_2)$ is neither C–hyperbolic nor Kobayashi hyperbolic. Indeed, C is a member of the linear pencil of cubics $C_\mu = \{\alpha x_0^3 - \beta x_1^2 x_2 = 0\}$, where $\mu = (\alpha : \beta) \in \mathbb{P}^1$ (here $C = C_{\mu_0}$, where $\mu_0 = (4 : 27)$). This pencil is generated by its only non–reduced members $C_{(1:0)} = 3l_0$ and $C_{(0:1)} = 2l_1 + l_2$, where $l_i = \{x_i = 0\}$, $i = 0, 1, 2$. The Kobayashi pseudo-distance k_X is identically zero along any of the cubics $C_\mu, \mu \neq \mu_0$, because $C_\mu \cap X = C_\mu \setminus (C \cup l_2) \cong \mathbb{C}^*$ and $k_{\mathbb{C}^*} \equiv 0$.

Nevertheless, by Theorem 6.5, c) any entire curve $\mathbb{C} \to X = \mathbb{P}^2 \setminus (C \cup l_2)$ is contained in one of the cubics C_μ, where $\mu \in \mathbb{P}^1 \setminus \{\mu_0\}$. Moreover, $k_X(p, q) = 0$ iff $p, q \in C_\mu$ for some $\mu \in \mathbb{P}^1 \setminus \{\mu_0\}$.

6.8. Example. Let $C \subset \mathbb{P}^2$ be the nodal cubic $x_1^2x_2 = x_0^3 + x_0^2x_2$, and let l_1, l_2, l_3 be the three inflexional tangents of C. They correspond to the cusps of the dual curve $C^* \subset \mathbb{P}^{2*}$, which is the 3-cuspidal quartic $(2y_1y_2 + y_0^2)^2 = 4y_0^2(y_0 - 2y_2)(y_0 + y_2)$ (see Remark 4.2). Thus, $L_C = l_1 \cup l_2 \cup l_3$. By Theorem 6.5, a) we have that $X := \mathbb{P}^2 \setminus (C \cup L_C)$ is almost C–hyperbolic. Hence, it is also Brody hyperbolic (see (2.6)).

By Bezout Theorem three cusps of C^* are not at the same line in $\mathbb{P}^{2*}$. Therefore, their dual lines, which are inflexional tangents l_1, l_2, l_3 of C, are not passing through the same point. From this it easily follows that $\operatorname{reg}(C \cup l_1 \cup l_2 \cup l_3)$ is hyperbolic. Thus, by Proposition 2.7 X is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$.

6.9. Example. Let $C \subset \mathbb{P}^2$ be the rational quintic $t \longmapsto (2t^5 - t^2 : -(4t^3 + 1) : 2t)$ with a cusp at the only singular point $(1 : 0 : 0)$. The dual curve $C^* \subset \mathbb{P}^{2*}$ is the quasi–monomial quartic $t \longmapsto (1 : t^2 : t^4 + t)$ given by the equation $(y_0y_2 - y_1^2)^2 = y_0^3y_1$. It has the only singular point $q_0 = (0 : 0 : 1)$, which is a ramphoid cusp, i.e. it has the multiplicity sequence $(2, 2, 2, 1, \ldots)$ and $\delta = \mu/2 = 3$, where μ is the Milnor number. Any rational quartic with a ramphoid cusp is projectively equivalent to C^* (see [Na, 2.2.5(a)]). The artifacts L_C consist of the only cuspidal tangent line $l_2 = \{x_2 = 0\}$ of C. By Theorem 6.5, b) the complement $\mathbb{P}^2 \setminus (C \cup l_2)$ is almost C–hyperbolic. Note that $\Gamma := C \setminus l_2$ is a smooth rational affine curve in $\mathbb{C}^2 \cong \mathbb{P}^2 \setminus l_2$, which is isomorphic to $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Thus, $X := \mathbb{C}^2 \setminus \Gamma$ is almost C–hyperbolic.

6.10. Example. Let $C' \subset \mathbb{P}^2$ be the rational quartic $t \longmapsto (t^3(2t+1) : -t(4t+3) : -2)$. It has two singular points, a double cusp at the point $(0:0:1)$ (i.e. a cusp with the multiplicity sequence $(2, 2, 1, \ldots)$ and $\delta = 2$) and another one, which is an ordinary cusp. The dual curve $C'^* \subset \mathbb{P}^{2*}$ is the quasi–monomial quartic $t \longmapsto (1 : t^2 : t^4 + t^3)$ given by the equation $(y_0y_2 - y_1^2)^2 = y_0y_1^3$. It has the same type of singularities as C', namely a double cusp at the point $q_0 = (0:0:1)$ and an ordinary cusp at the point $(1:0:0)$. Therefore, $L_{C'} = l_0 \cup l_2$, where $l_0 = \{x_0 = 0\}$ and $l_2 = \{x_2 = 0\}$. By Theorem 6.5, b) the complement $X := \mathbb{P}^2 \setminus (C' \cup L_{C'})$ is almost C–hyperbolic.

7 C–hyperbolicity of complements of maximal cuspidal rational curves

In Corollary 7.10 below we show that the complement of a maximal cuspidal rational curve of degree $d \geq 8$ in $\mathbb{P}^2$ is almost C–hyperbolic. In a sense, this completes the study on C–hyperbolicity of $\mathbb{P}^2 \setminus (C \cup L_C)$. The deep reason of this fact, which actually does not appear in the proof, is that the Teichmüller space $T_{0,n}$ of the Riemann sphere with n punctures is a bounded domain in $\mathbb{C}^n$ (cf. [Kal]).

Let us start with necessary preliminaries.

7.1. *Maximal cuspidal rational curves as generic plane sections of the discriminant.* Let $C \subset \mathbb{P}^2$ be a rational curve of degree $d > 1$. By the Class Formula (4) its dual curve $C^* \subset \mathbb{P}^{2*}$ is an immersed curve (or, equivalently, $L_C = \emptyset$) iff $d = 2(n-1)$, where $n = \deg C^*$ (cf. 3.4). If in addition C is a Plücker curve, then it has the maximal possible number of ordinary cusps, which is equal to $3(n-2)$, and besides this it has also $2(n-2)(n-3)$ nodes. Such a curve C is called *a maximal cuspidal rational curve* [Zar, p. 267]. Note that the dual C^* of such a curve C is a rational nodal curve of degree n in $\mathbb{P}^{2*}$. In particular, a generic maximal cuspidal rational curve C naturally appears via the Zariski embedding $\rho_C : \mathbb{P}^2 \to \mathbb{P}^2_C \hookrightarrow \mathbb{P}^n$ as a generic plane section of the discriminant hypersurface $R_n \subset \mathbb{P}^n$ (see 3.6-3.7, 5.4).

7.2. *The moduli space of the n–punctured sphere as an orbit space.* Note, first of all, that the Vieta map $s_n : (\mathbb{P}^1)^n \to S^n\mathbb{P}^1 = \mathbb{P}^n$ is equivariant with respect to the natural actions of the group $\mathbb{P}GL(2,\mathbb{C}) = \mathrm{Aut}\,\mathbb{P}^1$ on $(\mathbb{P}^1)^n$ and on $\mathbb{P}^n$, respectively. The branching divisors D_n (the union of the diagonals) resp. R_n (the discriminant divisor), as well as their complements are invariant under the corresponding actions. It is easily seen that for $n \ge 3$ the orbit space of the $\mathbb{P}GL(2,\mathbb{C})$–action on $\mathbb{P}^n \setminus R_n$ is naturally isomorphic to the moduli space $M_{0,n}$ of the Riemann sphere with n punctures. Denote by $\tilde{M}_{0,n}$ the quotient $((\mathbb{P}^1)^n \setminus D_n)\,/\,\mathbb{P}GL(2,\mathbb{C})$. We have the following commutative diagram of equivariant morphisms

$$\begin{array}{ccc} (\mathbb{P}^1)^n \setminus D_n & \xrightarrow{\tilde{\pi}_n} & \tilde{M}_{0,n} \\ {\scriptstyle s_n}\big\downarrow & & \big\downarrow \\ \mathbb{P}^n \setminus R_n & \xrightarrow{\pi_n} & M_{0,n} \end{array} \qquad (7)$$

7.3. *Description of $\tilde{M}_{0,n}$.* The cross–ratios $\sigma_i(z) = (z_1, z_2; z_3, z_i)$, where $z = (z_1, \ldots, z_n) \in (\mathbb{P}^1)^n$ and $4 \le i \le n$, define a morphism

$$\sigma^{(n)} = (\sigma_4, \ldots, \sigma_n) : (\mathbb{P}^1)^n \setminus D_n \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$$

(here as before $\mathbb{C}^{**} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$). By the invariance of cross–ratio $\sigma^{(n)}$ is constant along the orbits of the action of $\mathbb{P}GL(2,\mathbb{C})$ on $(\mathbb{P}^1)^n \setminus D_n$. Therefore, it factorizes through a mapping of the orbit space $\tilde{M}_{0,n} \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$. On the other hand, for each point $z \in (\mathbb{P}^1)^n \setminus D_n$ its $\mathbb{P}GL(2,\mathbb{C})$–orbit O_z contains the unique point z' of the form $z' = (0, 1, \infty, z'_4, \ldots, z'_n)$. This defines a regular section $\tilde{M}_{0,n} \to (\mathbb{P}^1)^n \setminus D_n$, and its image coincides with the image of the biregular embedding

$$(\mathbb{C}^{**})^{n-3} \setminus D_{n-3} \ni u = (u_4, \ldots, u_n) \longmapsto (0, 1, \infty, u_4, \ldots, u_n) \in (\mathbb{P}^1)^n \setminus D_n \ .$$

This shows that the above mapping $\tilde{M}_{0,n} \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$ is an isomorphism.

7.4. *$\mathbb{P}GL(2,\mathbb{C})$–orbits.* Here as before we treat $\mathbb{P}^n$ as the projectivized space of the binary forms of degree n in u and v. For instance, $e_k = (0 : \ldots : 0 : 1_k : 0 : \ldots : 0) \in \mathbb{P}^n$ corresponds to the forms $cu^{n-k}v^k$, where $c \in \mathbb{C}^*$. Denote by O_q the $\mathbb{P}GL(2,\mathbb{C})$–orbit of a point $q \in \mathbb{P}^n$. Clearly, $O_{e_i} = O_{e_{n-i}}$, $i = 0, \ldots, n$; O_{e_0} is the only one–dimensional orbit and, at the same time, the only closed orbit; O_{e_i}, $i = 1, \ldots, [n/2]$, are the only two-dimensional orbits, and any other orbit has dimension 3. Note that $O_{e_0} = C_n$ is the dual rational normal curve, and $S := O_{e_0} \cup O_{e_1}$ is its developable tangent surface (see 5.4).

If O_q is an orbit of dimension 3, then its closure $\bar{O}_q$ is the union of the orbits O_q, O_{e_0} and those of the orbits $O_{e_i}, i = 1, \dots, n-1$, for which the form q has a root of multiplicity i [AlFa, Proposition 2.1]. Furthermore, for any point $q \in \mathbb{P}^n \setminus R_n$, i.e. for any binary form q without multiple roots, its orbit O_q is closed in $\mathbb{P}^n \setminus R_n$ and $\bar{O}_q = O_q \cup S$, where $S = \bar{O}_q \cap R_n$. Therefore, any Zariski closed subvariety Z of $\mathbb{P}^n$ such that $\dim(O_q \cap Z) > 0$ must meet the surface S. These observations yield the following lemma. [3]

7.5. Lemma. *If a linear subspace L in $\mathbb{P}^n$ does not meet the surface $S = \bar{O}_{e_1} \subset R_n$, then it has at most finite intersection with any of the orbits O_q, where $q \in \mathbb{P}^n \setminus R_n$. In particular, this is so for a generic linear subspace L in $\mathbb{P}^n$ of codimension at least* 3.

7.6. Remark. Fix k distinct points $z_1, \dots, z_k \in \mathbb{P}^1$, where $3 \le k \le n$. Let g_0 be a binary form of degree k with the roots $z_1, \dots, z_k$. Consider the projectivized linear subspace $L_0 \subset \mathbb{P}^n$ of codimension k consisting of the binary forms of degree n divisible by g_0. It is easily seen that $L_0 \cap S = \emptyset$. This gives a concrete example of such a subspace.

The next tautological lemma is used below in the proof of Theorem 7.9.

7.7. Lemma. *Let $C \subset \mathbb{P}^2$ be a rational curve. Put $n = \deg C^*$, and let as before $\mathbb{P}^2_C = \rho_C(\mathbb{P}^2) \hookrightarrow \mathbb{P}^n$ be the image under the Zariski embedding.*

a) The plane $\mathbb{P}^2_C$ meets the surface $S = \bar{O}_{e_1}$ iff there exists a local irreducible analytic branch (A^*, p^*) of the dual curve C^* such that $i(T_{p^*}A^*,\, A^*;\, p^*) \ge n-1$.

b) Furthermore, if C^* has a cusp (A^*, p^*) of multiplicity $n-1$, then $\rho_C(l_{p^*}) \subset \mathbb{P}^2_C \cap S$, where $l_{p^*} \subset L_C \subset \mathbb{P}^2$ is the dual line of the point $p^* \in \mathbb{P}^{2*}$.

c) If the dual curve C^* has only ordinary cusps and flexes and $n = \deg C^* \ge 5$, then $\mathbb{P}^2_C \cap S = \emptyset$.

Proof. a) By the definition of the Zariski embedding $q \in \mathbb{P}^2_C \cap S$ iff, after passing to the normalization $\nu : \mathbb{P}^1 \to C^*$ and identifying $\mathbb{P}^2$ with its image $\mathbb{P}^2_C$ under the

[3] We are gratefull to H. Kraft who pointed out to us an approach which is based on the notion of the associated cone of an orbit [Kr] (here we have used a simplified version of it), and to M. Brion for mentioning to us of the paper [AlFa].

Zariski embedding ρ_C, the dual line $l_q \subset \mathbb{P}^2$ cuts out on C^* a divisor of the form $(n-1)a+b$, where $a, b \in \mathbb{P}^1$. Then $p^* := \nu(a) \in C^*$ is the center of a local branch A^* of C^* which satisfies the condition in a). The converse is evidently true. □

b) For any point $q \in l_{p^*}$ its dual line $l_q \subset \mathbb{P}^{2*}$ passes through p^*, and hence by the above consideration we have $\rho_C(q) \in \mathbb{P}^2_C \cap S$. □

c) By the condition we have that $i(T_{p^*}A^*,\, A^*;\, p^*) \le 3 < n-1$ for any local analytic branch (A^*, p^*) of C^*. Now the result follows from a). □

7.8. Lemma. *Let $C^* \subset \mathbb{P}^{2*}$ be a rational curve of degree n. Then the complement $X = \mathbb{P}^2 \setminus (C \cup L_C)$ is almost C–hyperbolic, whenever $\mathbb{P}^2_C \cap S = \emptyset$.*

Proof. Consider the following commutative diagram of morphisms:

$$\begin{array}{ccccccc}
 & Y & \xrightarrow{\tilde\rho_C} & (\mathbb{P}^1)^n \setminus D_n & \xrightarrow{\tilde\pi_n} & (\mathbb{C}^{**})^{n-3} \setminus D_{n-3} \hookrightarrow (\mathbb{C}^{**})^{n-3} \\
 & \tilde s_n \downarrow & & s_n \downarrow & & \downarrow \\
\mathbb{P}^2 \setminus (C \cup L_C) = & X & \xrightarrow{\rho_C} & \mathbb{P}^n \setminus R_n & \xrightarrow{\pi_n} & M_{0,n}
\end{array} \qquad (8)$$

where $\tilde s_n : Y \to X$ is the induced covering (cf. (7) and 7.2–7.3 above).

From Lemma 7.5 it follows that the mapping $\pi_n \circ \rho_C : X \to M_{0,n}$ has finite fibres. Hence, the same is valid for the mapping $\tilde\pi_n \circ \tilde\rho_C : Y \to (\mathbb{C}^{**})^{n-3} \setminus D_{n-3}$. By Lemma 2.5 Y, and thus also X, are almost C–hyperbolic. □

From this lemma and Lemma 7.7 we have the following theorem, which is a useful supplement to Theorem 6.5.

7.9. Theorem. *Let $C^* \subset \mathbb{P}^{2*}$ be a rational curve of degree n such that $i(T_{p^*}A^*,\, A^*;\, p^*) \le n-2$ for any local analytic branch $(A^*,\, p^*)$ of C^*. Let $C = (C^*)^* \subset \mathbb{P}^2$ be the dual curve. Then the complement $X = \mathbb{P}^2 \setminus (C \cup L_C)$ is almost C–hyperbolic. In particular, this is so if $n \ge 5$ and C^* has only ordinary cusps and flexes.*

The next corollary is an addition to Theorem 4.1, b).

7.10. Corollary. *Let $C \subset \mathbb{P}^2$ be a maximal cuspidal rational curve of degree $d = 2(n-1) \ge 8$. Then $X = \mathbb{P}^2 \setminus C$ is almost C–hyperbolic, Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$. In particular, this is the case if the dual curve C^* is a generic rational nodal curve of degree $n \ge 5$ in $\mathbb{P}^{2*}$.*

Proof. The first statement immediately follows from Theorem 7.9, while the second one follows from Proposition 2.7. Indeed, under our assumptions we have $n \geq 5$, and therefore the curve C has at least 9 cusps. Hence $\operatorname{reg} C$ is a hyperbolic curve. The last statement is evident. □

The next example shows that our method is available not for all rational curves whose dual curves are nodal.

7.11. Example. Let $C^* = (p(t) : q(t) : 1)$ be a parametrized plane rational curve, where $p, q \in \mathbb{C}[t]$ are generic polynomials of degrees n and $n-1$, respectively. Then C^* is a nodal curve of degree n which is the projective closure of an affine plane polynomial curve with one place at infinity, at the point $(1 : 0 : 0)$, and this is a smooth point of C^*. Thus, the line $l_2 = \{x_2 = 0\}$ is an inflexional tangent of order $n - 2$ of C^*, and so by Lemma 7.7, a) $\mathbb{P}^2_C \cap S \neq \emptyset$. Therefore, we can not apply in this case the same approach as above.

At last, we can summarize the main results of the paper (cf. Theorems 4.1, 6.5 and 7.9).

7.12. Theorem. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of genus g. Put $n = \deg C^*$ and $X = \mathbb{P}^2 \setminus (C \cup L_C)$.*

a) If $g \geq 1$, then X is C–hyperbolic. If $g = 0$, then X is almost C–hyperbolic if at least one of the following conditions is fulfilled:

i) $i(T_{p^*}A^*, A^*; p^*) \leq n - 2$ for any local analytic branch (A^*, p^*) of C^*;

ii) C^* has a cusp and it is not projectively equivalent to one of the curves $(1 : g(t) : t^n)$, $(t : g(t) : t^n)$, where $g \in \mathbb{C}[t], \deg g \leq n - 2$. [4]

b) Let, furthermore, C^* be an immersed curve. If $g \geq 1$, then $\mathbb{P}^2 \setminus C$ is C–hyperbolic. If $g = 0$ and i) is fulfilled, then $\mathbb{P}^2 \setminus C$ is almost C–hyperbolic; in particular, this is so if C^* is a generic rational nodal curve of degree $n \geq 5$. In both cases $\mathbb{P}^2 \setminus C$ is Kobayashi complete hyperbolic and hyperbolically embedded into $\mathbb{P}^2$.

[4] The monomial curves correspond here to $g(t) = t^k$, $k \leq n-2$. Note that the curves $(1 : t^{n-1} : t^n)$ and $(1 : t : t^n)$, being considered as non–parametrized ones, are projectively equivalent, and therefore all monomial curves have been excluded.

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Gerd Dethloff
Mathematisches Institut der Universität Göttingen
Bunsenstrasse 3-5
3400 Göttingen
Germany
e-mail: DETHLOFF@CFGAUSS.UNI-MATH.GWDG.DE

Mikhail Zaidenberg
Université Grenoble I
Laboratoire de Mathématiques
associé au CNRS
BP 74
38402 St–Martin–d'Hères Cédex
France
e-mail: ZAIDENBE@FOURIER.GRENET.FR