Finitely presented Categories and Homology

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March 27, 1995

Key words : Rewriting systems, word problem, noetherian confluent relation, modules over preadditive categories, projective resolutions, Theory of homology, classifying space.

AMS Subject Classification : 16E30, 18A20, 18A25, 18A40, 18E05, 18E10, 18G10, 18G35, 20F10.

Abstract

Squier[1] has proved that the third homology $\mathbb{Z}$-module of a monoid presented by a finite noetherian confluent rewriting system is finitely generated. Kobayashi[2] has given a more precise result about the homology of such a monoid, by proving that it is $(FP)_\infty$.

Burroni[3] has dealt with the word problem for 2-monoids and has pointed out the word problem for monoids has a natural generalization to categories.

Our aim is precisely to deal with this question by generalizing Kobayashi's result to categories. More precisely, after briefly recalling how one can interpret a functional programming language as a category[4], we are going to prove that a category presented by a finite noetherian confluent rewriting system is $(FP)_\infty$. We can then provide such a category with finitely generated homology modules. The homology we will calculate is the singular homology of the classifying space of the category.

We end by giving a counter-example illustrating how important the hypothesis *finite*, *confluent* and *noetherian* are and a concrete example dealing with matrices showing the limits of our homological criterion.

I would like to thank Alain Prouté for all the precious suggestions he made to me and Peter Greenberg[5] for the entrancing conversations we had together. I am indebted to Gerard Vinel for all the remarks he made about this paper and his friendly help.

[1] See [?].
[2] See [?].
[3] See [?].
[4] It then gives the opportunity to deal with programme rewriting.
[5] (1956-1993)

Contents

1 Recalls about Monoids and homology.

Formally one can define the homology of a monoid $\mathcal{M}$ as the homology of the complex obtained by trivializing[6] and truncating the standard resolution over $\mathbb{Z}$:

$$... \longrightarrow L^n(\mathbb{Z}) \longrightarrow L^{n-1}(\mathbb{Z}) \longrightarrow ... \longrightarrow L^1(\mathbb{Z}) \longrightarrow L^0(\mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where L denotes the endofunctor of the cotriple associated to the adjunction, $Set \rightleftarrows \mathbb{Z}\mathcal{M}\text{-}Mod$.

1.1 The Lafont-Prouté approach.

Lafont and Prouté[7] have associated to any monoid $\mathcal{M}$ presented by a finite noetherian confluent rewriting system, $(\Sigma, \mathcal{R})$, a partial free resolution over $\mathbb{Z}$:

$$\mathbb{Z}\mathcal{M}[\mathcal{P}] \xrightarrow{\partial_3} \mathbb{Z}\mathcal{M}[\mathcal{R}] \xrightarrow{\partial_2} \mathbb{Z}\mathcal{M}[\Sigma] \xrightarrow{\partial_1} \mathbb{Z}\mathcal{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where $\mathcal{P}$ denotes the set of critical pairs associated to the presentation. Roughly speaking, a critical pair consists in a pair of nondisjoint elementary reductions of the same word[8]. Therefore, they have found again the following results.

THEOREM 1 — *If $\mathcal{M}$ is a finitely generated[9] monoid then $H_1(\mathcal{M})$ is finitely generated.*

THEOREM 2 — *If $\mathcal{M}$ is finitely related[10] then $H_2(\mathcal{M})$ is finitely generated.*

THEOREM 3 (Squier) — *If $\mathcal{M}$ is presented by a finite confluent noetherian rewriting system then $H_3(\mathcal{M})$ is finitely generated.*

Lafont and Prouté have given a geometrical interpretation of the partial free resolution smartly constructed. This allows to visualize the space the homology of which is calculated.

1.2 The Kobayashi approach.

Let us consider a unitary monoid $\mathcal{M}$ having a noetherian confluent presentation, $(\Sigma, \mathcal{R})$. Let us suppose moreover that $\mathcal{M}$ has a right rewriting system, $\xrightarrow{\mathcal{S}}$, generated by a finite set of rules $\mathcal{S}$, prefix-free, reduced, confluent and noetherian over $\mathcal{R}$. Recall that in a *right rewriting system*, the reductions on Σ^* have the following form : $x \xrightarrow{\mathcal{S}} y$, where $x = ux_1$, $y = vx_1$ and $u \longrightarrow v \in \mathcal{S}$. $\mathcal{S}$ is *prefix-free* if u_1 is not a prefix of any u_2, for all $u_1 \longrightarrow v_1 \in \mathcal{S}$ and $u_2 \longrightarrow v_2 \in \mathcal{S}$. $\mathcal{S}$ is *$\mathcal{R}$-reduced* if for any $\mathcal{S}$-reduction $u \longrightarrow v$, u and v are $\mathcal{R}$-irreducible. Finally, $\mathcal{S}$ is *confluent and noetherian* over $\mathcal{R}$, if the rewriting system obtained by union of $\xrightarrow{\mathcal{R}}$ and $\xrightarrow{\mathcal{S}}$ is confluent and noetherian.

Let $\mathcal{N}$ be the monoid defined (up to isomorphism) as the quotient of Σ^* by the congruence generated by the rewriting system obtained by the union of $\xrightarrow{\mathcal{R}}$ and $\xrightarrow{\mathcal{S}}$. k being any unitary commutative

[6] By applying the coefficient functor trivializing the action of the monoid ring $\mathbb{Z}\mathcal{M}$.
[7] See [?].
[8] For more precisions, one can have a look at [?].
[9] *i.e.* $\mathcal{M}$ is presented by a rewriting system $(\Sigma, \mathcal{R})$ such that Σ is finite.
[10] *i.e.* $\mathcal{M}$ is presented by a rewriting system $(\Sigma, \mathcal{R})$ such that $\mathcal{R}$ is finite.

ring, $k\mathcal{N}$ will denote the monoid ring associated to $\mathcal{N}$. Let . be the right action of $k\mathcal{M}$ on $k\mathcal{N}$, defined for any $\overline{\overline{x}} \in \mathcal{N}$ and any $\overline{y} \in \mathcal{M}$ by $\overline{\overline{x}}.\overline{y} = \overline{\overline{xy}}$.

In [?], Kobayashi calls V the set of reddex of $\mathcal{S}$, $irr^+(\mathcal{R}) = irr(\mathcal{R}) \setminus \{1\}$, $E = \{(u,v) \in irr^+(\mathcal{R}) \times irr^+(\mathcal{R}) | uv$ is a right minimal $\mathcal{R}$-word $\}$[11] and for all $n \geq 1$, $V^{(n)} = \{(v^1, ..., v^n) | v^1 \in V, v^i \in irr^+(\mathcal{R}), \forall i$ and $(v^i, v^{i+1}) \in E, \forall i \leq n-1\}$. He then proves the following result.

THEOREM 4 (Kobayashi) — *If a monoid $\mathcal{M}$ has a noetherian confluent presentation, $(\Sigma, \mathcal{R})$, together with a right rewriting system $\overset{\mathcal{S}}{\longrightarrow}$ which is reduced prefix-free confluent and noetherian over $\mathcal{R}$ then one has a free resolution*[12] *:*

$$... \longrightarrow [V^{(n)}]k\mathcal{M} \overset{\delta_n}{\longrightarrow} [V^{(n-1)}]k\mathcal{M} \longrightarrow ... \longrightarrow [V]k\mathcal{M} \overset{\delta_1}{\longrightarrow} k\mathcal{M} \overset{\varepsilon}{\longrightarrow} k\mathcal{N} \longrightarrow 0.$$

COROLLARY 1 (Kobayashi) — *If $\mathcal{R} \cup \mathcal{S}$ if finite then $\mathcal{M}$ is $(FP)_\infty$.*

We take an interest in the case of resolutions over $\mathbb{Z}$ by $\mathbb{Z}\mathcal{M}$-modules, since by this way we can find again the Squier's results. Kobayashi looks at this particular case in §5 of [?]. He considers a monoid $\mathcal{M}$ presented by a finite noetherian confluent system, $(\Sigma, \mathcal{R})$ and defines $\mathcal{S}$ by $\mathcal{S} = \{a \longrightarrow 1 | a \in \Sigma\}$. He claims that the right rewriting system generated by this $\mathcal{S}$ is confluent and noetherian over $\mathcal{R}$ and then applies theorem 4 and finds again the Squier theorem [13]. Take care ! this is true only if the presentation does not admit any rule whose left term is a letter. Indeed, we can only apply theorem 4, if the right rewriting system generated by $\mathcal{S}$ is prefix-free[14] and $\mathcal{R}$-reduced too, what is not the case, in general, of the set $\mathcal{S}$ suggested by Kobayashi. To illustrate this remark, it suffices to consider the following example. Let $\mathcal{M}$ be the monoid, presented by :

$$\Sigma = \{a, b, c\}, \mathcal{R} = \{a \longrightarrow bc\}.$$

This presentation is clearly noetherian confluent and finite, but if we write $\mathcal{S} = \{a \longrightarrow 1, b \longrightarrow 1, c \longrightarrow 1\}$, as suggested by Kobayashi, the right rewriting system generated by $\mathcal{S}$ is not $\mathcal{R}$-reduced because a is a common reddex to $\mathcal{R}$ and $\mathcal{S}$. To avoid this kind of problem, we can write in the general case :

$$\mathcal{S} = \{a \longrightarrow 1 | a \in \Sigma \cap irr^+(\mathcal{R})\}.$$

1.3 Comparison between Kobayashi and Lafont-Prouté resolutions.

We are going to compare the resolutions[15] constructed in [?] and [?] and see that under one additional hypothesis (which we can always boil down to) those resolutions are not only homotopic[16] but even isomorphic.

Recall that a rewriting system, $(\Sigma, \mathcal{R})$ is *minimal* if for any rule $r \overset{R}{\longrightarrow} s$, the reddex r is not contained in any other reddex of $\mathcal{R}$.

PROPOSITION 1 — *If we assume that $(\Sigma, \mathcal{R})$ is a minimal rewriting system without any reddex in Σ then the resolutions constructed by Kobayashi and Lafont-Prouté are isomorphic.*

[11] *i.e.* uv is $\mathcal{R}$-reducible but every proper prefix of uv is $\mathcal{R}$-irreducible.
[12] Precisions about this resolution will be given in the next paragraph.
[13] indeed in this case $\mathcal{N}$ is reduced to a single element monoid $\{1\}$ and we get a resolution over $\mathbb{Z}$.
[14] It is the case of the set of rules $\mathcal{S}$ proposed by Kobayashi.
[15] When they are comparable : in the case of resolutions over $\mathbb{Z}$.
[16] What is always true for free resolutions.

PROPOSITION 2 — *Any finite noetherian confluent rewriting system can always be turned into an equivalent one*[17] *verifying the assumptions of proposition 1.*

The proof of both previous propositions is left to the reader.

Thanks to those propositions and to [?], we get a geometrical interpretation of [?]. This is helpful to understand Kobayashi's resolution and to imagine the generalization we are looking for.

New definitions will be necessary to extend Kobayashi's work to categories ; for example we will generalize the notion of module over a ring to module over a preadditive category. In that case, words will be replaced by arrows and multiplication by composition ; it will then be generally impossible to erase arrows[18] without changing the source or target types.

1.4 Functional programming languages.

In this section, we are going to recall how to define a category corresponding to a functional programming language on which we have added a few innocuous hypothesis.[19]

Roughly speaking, a functional programming language consists of a language made of a set whose objects are called *primitive data types*, *constants of each type*, *operations*[20] and *constructors*. The constructors can be applied to data types and operations and then produce *derived* data types and operations of the language.

A *programme* consists in applying constructors to types, constants and functions.

A functional programming language $\mathcal{L}$ does not own *variables* nor *assignment statements* as is the case with *lambda calculus*.

The functional programming languages, $\mathcal{L}$, we are interested with verify the following assumptions :

1) there exists a type, 1, such that for each type T, there is a unique operation $T \longrightarrow 1$. Each constant c of type T is interpreted as an operation $1 \longrightarrow T$.

2) There is a *do-nothing operation* for each primitive and constructed type.

3) $\mathcal{L}$ has a composition constructor, *i.e.* doing an operation one after the other[21] is a derived operation whose input (resp. output) type is the input (resp. output) type of the first (resp. last) operation.

Under those assumptions, $\mathcal{L}$ has a category structure, *i.e.* corresponds to the category $\mathfrak{C}(\mathcal{L})$ defined by :

· the objects of $\mathfrak{C}(\mathcal{L})$ are the types of $\mathcal{L}$,

· The arrows of $\mathfrak{C}(\mathcal{L})$ are the primitive and derived operations of $\mathcal{L}$. The source and target of an arrow are respectively the input and output types of the corresponding operation.

The composition is determined by the composition constructor, the existence of which is assumed. The identity id_T corresponds to the do-nothing operation for the type T.

[17] *i.e.* presenting the same monoid.
[18] It is precisely what S does with letters in the case of monoids in [?].
[19] As for example the one consisting in assuming that the considered languages have a composition constructor.
[20] *i.e.* functions between types.
[21] Which is possible if the input type of the second operation is equal to the output type of the first one.

We are going to generalize theorem 4 to categories. More precisely, we are going to prove the following result : if a category can be presented by a finite confluent noetherian rewriting system then it is $(FP)_\infty$.

The word problem for monoids has a natural extension to categories which can be interpreted as the functional programming languages we described. An immediate consequence of the announced result is the possibility of defining an homology for those functional programming languages and a necessary homological condition for the existence of a confluent noetherian rewriting system for programmes.

2 Presentations and reductions.

2.1 Presentations of categories.

- *Grph* and *Cat* will respectively denote the categories of small graphs and small categories.

- Let $\mathfrak{G}$ be a graph, the objects of $\mathfrak{G}$ will be called its *types*. Its set of types (resp. of arrows) will be denoted by $\mathfrak{G}_0$ (resp. $\mathfrak{G}_1$), and we will write $\sigma : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_0$ and $\beta : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_0$ for the *source* and *target* maps.

We are going to use the same notations for categories.

- Two arrows are said to be *parallel* if they have same source and same target.

- $\mathfrak{C}_1^+$ will denote the set of non-identity arrows of a category $\mathfrak{C}$.

REMARK 1 — *There exists a forgetful functor $U-$: Cat $\longrightarrow$ Grph (see [?] p.49), to any small category $\mathfrak{C}$, it associates the graph $U\mathfrak{C}$ having the same objects and arrows, but in which we forget for each arrow whether it is a composition or an identity. This functor has a left adjoint, $C-$: Grph $\longrightarrow$ Cat, (see [?] p.50 & p.85). Up from now $\mathfrak{G}^*$ will denote $C\mathfrak{G}$.*

- let $\mathfrak{G}$ be a graph, we call *free category over* $\mathfrak{G}$, a category which is isomorphic to $\mathfrak{G}^*$.

- A *congruence on a category* $\mathfrak{C}$ is an equivalence relation $f \equiv g$, on pairs of parallel arrows of $\mathfrak{C}$ which is compatible with the composition : if $f \equiv g$ then $h \circ f \circ k \equiv h \circ g \circ k$ for all arrows such that $a \xrightarrow{k} b \overset{f,g}{\rightrightarrows} c \xrightarrow{h} d$.

- Let $\mathfrak{C}$ be a category, we call *congruence associated with a subset* $\mathcal{R}$ of $\mathfrak{C}_1^+ \times \mathfrak{C}_1$, which is made of couples of parallel arrows, the smallest congruence[22] on $\mathfrak{C}$, containing $\mathcal{R}$. Let $\equiv_{\mathcal{R}}$ be this congruence.

- Let $\equiv$ be a congruence on a category $\mathfrak{C}$, we call the *quotient of* $\mathfrak{C}$ *by* $\equiv$, the category $\mathfrak{C}/\equiv$ whose objects are those of $\mathfrak{C}$ and whose arrows are the classes of the arrows of $\mathfrak{C}$ modulo $\equiv$. If $\overline{f}$ and $\overline{g}$ are classes of composable arrows modulo $\equiv$, the composition is defined by $\overline{g} \circ \overline{f} = \overline{g \circ f}$.

[22]For the order defined by the inclusion of subsets.

• A *rewriting system* is a pair $(\mathfrak{G}, \mathcal{R})$ made of a graph $\mathfrak{G}$ and a subset $\mathcal{R}$ of $\mathfrak{G}_1^{*+} \times \mathfrak{G}_1^{*}$, constituted of couples of parallel arrows.

• Let $(\mathfrak{G}, \mathcal{R})$ be a rewriting system. The set $\mathfrak{G}_1$ is called a *set of generators* and the elements of $\mathfrak{G}^*$ are the *words* written by means of the set $\mathfrak{G}_1$. The *length* of a word written by means if $\mathfrak{G}_1$, is the number of (non-identity) arrows it is made of.

• A *presentation of a category* $\mathfrak{C}$ is a rewriting system $(\mathfrak{G}, \mathcal{R})$ such that the category $\mathfrak{G}^*/\equiv_{\mathcal{R}}$ is isomorphic to $\mathfrak{C}$. It is said *finite* if both $\mathfrak{G}$ and $\mathcal{R}$ are finite.

2.2 Reduction relation.

• Let $(\mathfrak{G}, \mathcal{R})$ be a rewriting system, up from now, pairs $(f, g) \in \mathcal{R}$ will be written $f \longrightarrow g$ and will be called the *rules*. In the case of monoids, some authors call them *atomic reductions*, so will we do in the case of categories.

• Let $f \longrightarrow g$ be an atomic reduction, we have $h \circ f \circ k \longrightarrow h \circ g \circ k$ for all arrows such that $a \xrightarrow{k} b \overset{f,g}{\rightrightarrows} c \xrightarrow{h} d$. $h \circ f \circ k \longrightarrow h \circ g \circ k$ is then called an *elementary reduction* associated with the rewriting system.

• The left term of an elementary reduction will be called a *reddex*.

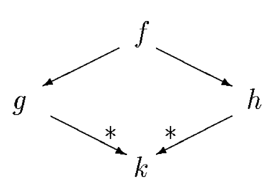
• The *reduction relation generated by* $\mathcal{R}$, denoted $\overset{*}{\longrightarrow}$, is the transitive reflexive closure relation of $\mathcal{R}$ on $\mathfrak{G}^*$, which is made compatible with composition.

• A *reduction path* from f_1 to f_{n+1} is a sequence of elementary reductions $f_1 \longrightarrow \ldots \longrightarrow f_{n+1}$ which can be written $f_1 \overset{*}{\longrightarrow} f_{n+1}$. f_1 is the *origin* and f_{n+1} the *extremity* of this reduction path. n is the *length* of the previous reduction path. n may equal zero the reduction path is then $f \overset{*}{\longrightarrow} f$ (reflexivity of $\overset{*}{\longrightarrow}$).

Note that the symmetric closure of the reduction relation generated by $\mathcal{R}$ is $\equiv_{\mathcal{R}}$.

• The reduction relation generated by $\mathcal{R}$ is said to be *noetherian*, if there is no infinite reduction path $f_1 \longrightarrow f_2 \longrightarrow \ldots \longrightarrow f_n \longrightarrow \ldots$

• The reduction relation generated by $\mathcal{R}$ is *confluent*[23] if for any pair of elementary reductions of the same word, $f \longrightarrow g$ and $f \longrightarrow h$, there are two reduction paths, (called *residual paths*), $g \overset{*}{\longrightarrow} k$ and $h \overset{*}{\longrightarrow} k$:



[23] We should rather say *locally confluent* but thanks to the Knuth-Bendix theorem, confluence and local confluence are equivalent under the assumption the reduction relation is noetherian.

• An arrow f is *reducible by the reduction relation generated by* $\mathcal{R}$ if there exists a reduction path starting from f whose length is strictly positive. Otherwise f is said to be *irreducible*.

• Let $Irr(\mathcal{R})$ denote the subset of $\mathfrak{G}_1^*$ made of the irreducible arrows for the reduction relation generated by $\mathcal{R}$.

REMARK 2 — *Let $(\mathfrak{G}, \mathcal{R})$ be a presentation whose corresponding reduction relation is confluent and noetherian. By the Knuth-Bendix theorem, we know that to any arrow $f \in \mathfrak{G}^*$ there corresponds a unique element of $Irr(\mathcal{R})$, written $\hat{f}$ and called the* canonical form *of f.*

• A rewriting system $(\mathfrak{G}, \mathcal{R})$ of a category is said to be *noetherian confluent* if the reduction relation generated by $\mathcal{R}$ is confluent and noetherian.

2.3 Finitely presented categories.

• We will say that a graph $\mathfrak{G} = (\mathfrak{G}_0, \mathfrak{G}_1)$ generates a category $\mathfrak{C}$ if $\mathfrak{G}_0 = \mathfrak{C}_0$ and if there exists a full functor from $\mathfrak{G}^*$ to $\mathfrak{C}$. $\mathfrak{G}_1$ is then called a set of generators for $\mathfrak{C}$.

PROPOSITION 3 — *The existence of a finite presentation for a category does not depend on the choice of the set of generators, provided it is finite. More precisely, if $(\mathfrak{G}_0, \mathfrak{G}_1)$ is a graph which generates a category $\mathfrak{C}$, such that $\mathfrak{G}_0$ and $\mathfrak{G}_1$ are finite, we can construct a finite set $\mathcal{R}_{\mathfrak{G}}$ (the elements of which are pairs of parallel arrows of $\mathfrak{G}^*$), such that $(\mathfrak{G}, \mathcal{R}_{\mathfrak{G}})$ is a finite presentation of $\mathfrak{C}$.*

PROOF : Let $(\Sigma, \mathcal{R})$ be a finite presentation for $\mathfrak{C}$, $\Sigma_1 = \{F_i, i \in I\}$ and $\mathcal{R} = \{(u_j(\Sigma), v_j(\Sigma)), j \in J\}$, Σ between parenthesis specifies that $u_j(\Sigma)$ and $v_j(\Sigma)$ are written by means of the set of generators Σ. There exists a full functor $\pi_\Sigma : \Sigma^* \longrightarrow \mathfrak{C}$. Let f_i be the image by π_Σ of F_i. The arrows of $\mathfrak{C}$ are then generated by $\sigma = \{f_i, i \in I\}$. Thanks to the adjunction $Grph \rightleftarrows Cat$, π_Σ is the functor generated by the morphism of graphs which maps each F_i on f_i. We then have $u_j(\sigma) = v_j(\sigma)$; σ between parenthesis specifies that $u_j(\sigma)$ and $v_j(\sigma)$ are written by means of the set σ.

Let $\mathfrak{G} = (\mathfrak{G}_0, \mathfrak{G}_1)$ be a finite graph generating $\mathfrak{C}$ and $\pi_{\mathfrak{G}} : \mathfrak{G}^* \longrightarrow \mathfrak{C}$ is a full functor (there exists such a functor since $\mathfrak{G}$ generates $\mathfrak{C}$). If we denote g_k the image of G_k by $\pi_{\mathfrak{G}}$, $\pi_{\mathfrak{G}}$ is the functor extending the morphism of graphs which maps each G_k on g_k. The set $\mathfrak{g} = \{g_k, k \in K\}$ generates $\mathfrak{C}_1$.

Let $\equiv$ be the congruence defined on $\mathfrak{G}^*$ by $\pi_{\mathfrak{G}}$, we then have a functorial isomorphism $(\mathfrak{G}^*/\equiv) \simeq \mathfrak{C}$.

Since π_Σ is full, for all $k \in K$ there exists an arrow $\psi_k(\Sigma) \in \Sigma^*$ such that $\pi_\Sigma(\psi_k(\Sigma)) = g_k$. Since $\pi_{\mathfrak{G}}$ is full, for all $i \in I$ there exists an arrow $\phi_i(\mathfrak{G}) \in \mathfrak{G}^*$ such that $\pi_{\mathfrak{G}}(\phi_i(\mathfrak{G})) = f_i$ (*i.e.* $\phi_i(\mathfrak{g}) = f_i$). If we set $\phi(\mathfrak{G}) = \{\phi_i(\mathfrak{G}), i \in I\}$, we then have $\pi_{\mathfrak{G}}(u_j(\phi(\mathfrak{G}))) = \pi_{\mathfrak{G}}(v_j(\phi(\mathfrak{G})))$ **(i)**.

Indeed, since $\pi_{\mathfrak{G}}$ is a functor, if $u_j(\Sigma) = F_{i_1} \circ ... \circ F_{i_{r(j)}}$, then $\pi_{\mathfrak{G}}(u_j(\phi(\mathfrak{G}))) = \pi_{\mathfrak{G}}(\phi_{i_1}(\mathfrak{G})) \circ ... \circ (\phi_{i_{r(j)}}(\mathfrak{G}))) = \pi_{\mathfrak{G}}(\phi_{i_1}(\mathfrak{G})) \circ ... \circ \pi_{\mathfrak{G}}(\phi_{i_{r(j)}}(\mathfrak{G})) = u_j(\pi_{\mathfrak{G}}(\phi(\mathfrak{G})))$ and $\pi_{\mathfrak{G}}(u_j(\phi(\mathfrak{G}))) = u_j(\pi_{\mathfrak{G}}(\phi(\mathfrak{G}))) = u_j(\sigma) = v_j(\sigma) = v_j(\pi_{\mathfrak{G}}(\phi(\mathfrak{G}))) = \pi_{\mathfrak{G}}(v_j(\phi(\mathfrak{G})))$.

In the same way, $\pi_{\mathfrak{G}}(\psi_k(\phi(\mathfrak{G}))) = \psi_k(\pi_{\mathfrak{G}}(\phi(\mathfrak{G}))) = \psi_k(\sigma) = g_k = \pi_{\mathfrak{G}}(G_k)$ **(ii)**.

Let $\approx$ be the congruence generated on $\mathfrak{G}^*$ by the finite set of parallel arrows :

$\mathcal{R}_{\mathfrak{G}} = \{(u_j((\phi(\mathfrak{G})), v_j((\phi(\mathfrak{G}))), j \in J\} \cup \{(G_k, \psi_k(\phi(\mathfrak{G}))), k \in K\}$.

Thanks to **(i)** and **(ii)**, the graph of the congruence $\approx$ is contained in the graph of $\equiv$, we then have a canonical morphism $\Psi : (\mathfrak{G}^*/\approx) \longrightarrow \mathfrak{C}$.

Let Φ be the unique functor $\Sigma^* \longrightarrow \mathfrak{G}^*$ such that $\Phi(F_i) = \phi_i(\mathfrak{G})$. For all pair of parallel arrows $(u_j(\Sigma), v_j(\Sigma))$ we have $\Phi(u_j(\Sigma)) = u_j(\phi(\mathfrak{G}))$ and $\Phi(v_j(\Sigma)) = v_j(\phi(\mathfrak{G}))$ so we can consider the functor $\tilde{\Phi}$ making the following diagram commute :

$$\begin{array}{ccc} \Sigma^* & \xrightarrow{\Phi} & \mathfrak{G}^* \\ {\scriptstyle \pi_\Sigma}\downarrow & & \downarrow{\scriptstyle \pi'_{\mathfrak{G}}} \\ \mathfrak{C} & \xrightarrow[\tilde{\Phi}]{} & \mathfrak{G}^*/\approx \end{array}$$

where $\pi'_{\mathfrak{G}}$ denotes the canonical projection.

It remains to prove (on generators) the two following identities $\Psi \circ \tilde{\Phi} = id_{\mathfrak{C}}$ and $\tilde{\Phi} \circ \Psi = id_{\mathfrak{G}^*/\approx}$.

On one hand, $\Psi \circ \tilde{\Phi}(f_i) = \Psi(\pi'_{\mathfrak{G}}(\Phi(F_i))) = \pi_{\mathfrak{G}}(\Phi(F_i)) = \pi_{\mathfrak{G}}(\phi_i(G)) = f_i$.

On the other hand, we notice that $\Psi(\pi'_{\mathfrak{G}}(G_k)) = \Psi(\pi'_{\mathfrak{G}}(\psi_k(\phi(\mathfrak{G}))) = \pi_{\mathfrak{G}}(\psi_k(\phi(\mathfrak{G}))) \overset{\textbf{(ii)}}{=} g_k = \pi_\Sigma(\psi_k(\Sigma))$ so we can write $\tilde{\Phi}(\Psi(\pi'_{\mathfrak{G}}(G_k))) = \tilde{\Phi}(\pi_\Sigma(\psi_k(\Sigma))) = \pi'_{\mathfrak{G}}(\Phi(\psi_k(\Sigma))) = \pi'_\Sigma(\psi_k(\phi(\mathfrak{G}))) = \pi'_{\mathfrak{G}}(G_k)$, □

2.4 A concrete example of finitely presented category.

Let $\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})$ be the graph whose objects are the integers less than or equal to r and whose arrows are all matrices

$$M = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & \dots & \dots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix}$$

where $n, m \leq r$, are respectively its source and target ; $a_{i,j} \in \mathbb{Z}/p\mathbb{Z}$, p prime number[24] for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Let $(\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z}), \mathcal{R})$ be the rewriting system such that :

· $\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})$ denotes the previous graph. It generates a free category $\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})^*$ whose arrows are juxtaposed matrices ; the juxtaposition of two matrices being allowed if the target of the right one[25] is equal to the source of the left one[26].

· $\mathcal{R}$ is the subset of $\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})^* \times \mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})^*$, made of two sets of rules $\mathcal{P}$ and $\mathfrak{G}$.

The elements of $\mathcal{P}$ are the rules $M_1\ M_2 \longrightarrow M_1 * M_2$, where $M_1,\ M_2 \in \mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})$ and $*$ denotes the usual product[27] of matrices[28]. Note that since $\mathbb{Z}/p\mathbb{Z}$ is finite so $\mathcal{P}$ is.

[24] Thus $\mathbb{Z}/p\mathbb{Z}$ is a finite field.

[25] *i.e.* its number of rows.

[26] *i.e.* its number of columns.

[27] When it makes sense.

[28] It seems that matrices appeared in a Chinese book of the first century A.D. but their product was defined last century by Hamilton and Cayley.

The elements of $\mathfrak{G}$ are the rules $M \longrightarrow M'$, where M, $M' \in \mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})$ and M' is obtained from M by "Gaussian[29] elimination". Note that $\mathbb{Z}/p\mathbb{Z}$ being finite $\mathfrak{G}$ is finite too and so $\mathcal{R}$ is.

The reduction relation generated by $\mathcal{R}$ is noetherian. Indeed, this clearly comes from the following assertions :

1) the number of juxtaposed matrices strictly decreases when we make a product.

2) by Gaussian elimination, the number of nonzero coefficients in a matrix strictly decreases.

The rewriting system $(\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z}), \mathcal{R})$ previously defined is confluent if and only if $r = 1$. Indeed, if $r = 1$, $(\mathfrak{M}_1(\mathbb{Z}/p\mathbb{Z}))$ contains only p (1×1)-matrices and $\mathcal{R} = \mathcal{P}$, *i.e.* there is no Gaussian elimination rule since p is a prime number. Confluence, in this case, exactly means that the product of matrices is associative. If $r \in \mathbb{N}^* \setminus \{1\}$, we have the following critical pair :

$$\begin{array}{ccc} & \begin{pmatrix} \bar{1} & \bar{1} \end{pmatrix} \begin{pmatrix} \bar{1} \\ \bar{1} \end{pmatrix} & \\ \swarrow & & \searrow \\ \begin{pmatrix} \bar{1} & \bar{1} \end{pmatrix} \begin{pmatrix} \bar{1} \\ \bar{0} \end{pmatrix} & & (\bar{2}) \end{array}$$

where the reduction "drawn on the left" consists in applying Gaussian elimination to the second row coefficient of the right matrix ; the reduction "drawn on the right" is the product of the matrices.

$(\bar{2})$ is irreducible and the only possible reduction for $\begin{pmatrix} \bar{1} & \bar{1} \end{pmatrix} \begin{pmatrix} \bar{1} \\ \bar{0} \end{pmatrix}$ is to make the product :

$$\begin{pmatrix} \bar{1} & \bar{1} \end{pmatrix} \begin{pmatrix} \bar{1} \\ \bar{0} \end{pmatrix} \longrightarrow (\bar{1})\,.$$

so $(\bar{1})$ being irreducible and distinct from $(\bar{2})$, this critical pair is not confluent.

3 Homology of a category.

Let X be any object of a category $\mathfrak{C}$ provided with a cotriple $\mathfrak{G}$, Barr and Beck have associated a homology theory denoted $H_*(X, K)_{\mathfrak{G}}$, where $K : \mathfrak{C} \longrightarrow \mathcal{A}b$ denotes a functor[30] into an abelian category $\mathcal{A}b$. To get more details one can have a look at [?].

In this section, we apply this definition to the cotriple associated to the adjunction of Lemma 1.

- To any category $\mathfrak{C}$, we can associate $\mathbb{Z}\mathfrak{C}$, the free preadditive category [31] generated by $\mathfrak{C}$.

3.1 Modules over a preadditive category.

- A *right (resp. left) module over a preadditive category* $\mathcal{A}$, *i.e.* a *right (resp. left)* $\mathcal{A}$*-module* is an additive functor[32] :

$$\mathcal{A}^{op} \stackrel{\alpha}{\longrightarrow} \mathbb{Z}\text{-}Mod \qquad (\text{resp. } \mathcal{A} \stackrel{\alpha}{\longrightarrow} \mathbb{Z}\text{-}Mod).$$

[29] It seems that this theory was in the Chinese book mentioned above and later on rediscovered by Gauß.

[30] Called *coefficient functor.*

[31] To get more precisions, one can have a look at [?], p.281.

[32] The $\mathcal{A}$*-action*

In other words, a right $\mathcal{A}$-module corresponds to a family $\mathfrak{M}_* = (\mathfrak{M}_x)_{x\in\mathcal{A}_0}$ (resp. ${}_*\mathfrak{M} = ({}_x\mathfrak{M})$) of $\mathbb{Z}$-modules, on which $\mathcal{A}$ *acts*. $\mathcal{A}^{op}$-*Mod* and $\mathcal{A}$-*Mod* will respectively denote the categories of right and left $\mathcal{A}$-modules.

• Let $\mathcal{A}$ and $\mathcal{B}$ be preadditive categories, we can consider the category $\mathcal{A}\otimes\mathcal{B}^{op}$ whose objects are the couples of objects (a,b), $a\in\mathcal{A}_0$, $b\in\mathcal{B}_0$ and whose arrows are the tensor products of those of $\mathcal{A}$ by those of $\mathcal{B}^{op}$. $\mathcal{A}\otimes\mathcal{B}^{op}$ is provided with the $\mathbb{Z}$-bilinear extension of the product of the compositions of $\mathcal{A}$ and $\mathcal{B}^{op}$. $\mathcal{A}\otimes\mathcal{B}^{op}$ is then a preadditive category.

• An $\mathcal{A}$-*bimodule* is an $\mathcal{A}\otimes\mathcal{A}^{op}$-module. In other words, it is a family ${}_*\mathfrak{M}_* = ({}_a\mathfrak{M}_b)_{a,b\in\mathcal{A}_0}$ of $\mathbb{Z}$-modules, one for each pair of types, provided with left and right $\mathcal{A}$-actions, which are compatible. $\mathcal{A}$-*Bimod* will denote the category of $\mathcal{A}$-bimodules.

$\mathcal{A}$ being a preadditive category, we are going to consider two examples of $\mathcal{A}$-bimodules that will be useful further.

• let ${}_*0_*$ be the $\mathcal{A}$-bimodule defined by the following functor :

$$\begin{array}{rcc} \mathcal{A}\otimes\mathcal{A}^{op} & \stackrel{\tau}{\longrightarrow} & \mathbb{Z}\text{-}Mod \\ (a,x) & \longmapsto & 0 \\ f\otimes g^{op}\downarrow & & \downarrow 0 \\ (b,y) & \longmapsto & 0 \end{array}$$

where 0 denotes the trivial $\mathbb{Z}$-module $\{0\}$. τ is additive.

• Let ${}_*\mathcal{A}_*$ be the $\mathcal{A}$-bimodule defined by :

- the family $({}_a\mathcal{A}_x)_{a,x\in\mathcal{A}_0}$, where ${}_a\mathcal{A}_x$ denote $\mathrm{Hom}_{\mathcal{A}}(x,a)$,
- the functor

$$\begin{array}{rcc} \mathcal{A}\otimes\mathcal{A}^{op} & \stackrel{\alpha}{\longrightarrow} & \mathbb{Z}\text{-}Mod \\ (a,x) & \longmapsto & {}_a\mathcal{A}_x \\ h\otimes g^{op}\downarrow & & \downarrow \alpha(h\otimes g^{op}) \\ (b,y) & \longmapsto & {}_b\mathcal{A}_y \end{array}$$

defined by $\alpha(h\otimes g^{op})(f) = h\circ f\circ g$ is additive.

• Let ${}_*\mathfrak{M}_*$ and ${}_*\mathrm{S}_*$ be two $\mathcal{A}$-bimodules, ${}_*\mathrm{S}_*$ is said to be a *sub-$\mathcal{A}$-bimodule* of ${}_*\mathfrak{M}_*$ if for all $(a,x)\in\mathcal{A}_0^2$, ${}_a\mathrm{S}_x$ is a sub-$\mathbb{Z}$-module of ${}_a\mathfrak{M}_x$ and if $({}_a\mathrm{S}_x \hookrightarrow {}_a\mathfrak{M}_x)_{(a,x)\in\mathcal{A}_0^2}$ is an $\mathcal{A}$-bimodule morphism.

REMARK 3 — *If ${}_*\mathfrak{M}_* \stackrel{\varphi}{\longrightarrow} {}_*\mathfrak{M}'_*$ is an $\mathcal{A}$-bilinear morphism then the sub-$\mathcal{A}$-bimodules ${}_*(im\,\varphi)_*$ and ${}_*(ker\,\varphi)_*$ of ${}_*\mathfrak{M}'_*$ and ${}_*\mathfrak{M}_*$ are respectively the kernel*[33] *and image*[34] *in the categorical sense*[35]

3.2 Free $\mathcal{A}$-bimodules.

• Let us consider the category functor (in the usual sense) $Set^{\mathcal{A}_0^2}$, where the set $\mathcal{A}_0^2$ is identified with the *small discrete category* whose objects are the elements of $\mathcal{A}_0^2$ and there are no arrows except the

[33] See [?], p.187.
[34] See [?], p.196.
[35] We may show that $\mathcal{A}$-*Mod* is abelian and that ${}_*0_*$ is its null object.

obligatory identity arrows. Any object S of $Set^{\mathcal{A}_0^2}$ will be identified with the bigraded set ${}_*S_*$, where for each pair $(a,b) \in \mathcal{A}_0^2$, ${}_aS_b = S(a,b)$. Note there exists a forgetful functor $\mathcal{A}\text{-}Bimod \xrightarrow{O} Set^{\mathcal{A}_0^2}$,

LEMMA 1 — *O has a left-adjoint.*

PROOF : We are going to construct a left-adjoint ${}_*\mathcal{A}[-]\mathcal{A}_* : Set^{\mathcal{A}_0^2} \longrightarrow \mathcal{A}\text{-}Bimod$ for O, *i.e.* we must have natural isomorphisms :

$$\mathcal{A}\text{-}Bimod({}_*\mathcal{A}[E]\mathcal{A}_*, {}_*\mathfrak{M}_*) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} Set^{\mathcal{A}_0^2}({}_*E_*, O({}_*\mathfrak{M}_*))$$

For any ${}_*E_* \in Set^{\mathcal{A}_0^2}$, let ${}_*\mathcal{A}[E]\mathcal{A}_*$ be the family of $\mathbb{Z}$-modules $({}_a\mathcal{A}[E]\mathcal{A}_x)_{(a,x)\in\mathcal{A}_0^2}$ such that for each $(a,x) \in \mathcal{A}_0^2$, ${}_a\mathcal{A}[E]\mathcal{A}_x = \bigoplus_{b,y} Hom_{\mathcal{A}^{op}}(a,b) \otimes \mathbb{Z}[{}_bE_y] \otimes Hom_{\mathcal{A}}(x,y)$. We will write $g[e]f$ the element $g \otimes [e] \otimes f$ of ${}_a\mathcal{A}[E]\mathcal{A}_x$. We provide the previous family of $\mathbb{Z}$-modules with an $\mathcal{A} \otimes \mathcal{A}^{op}$-action :

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A}^{op} & \xrightarrow{\lambda} & \mathbb{Z}\text{-}Mod \\ (a,x) & \longmapsto & {}_a\mathcal{A}[E]\mathcal{A}_x \\ h \otimes k^{op} \downarrow & & \downarrow \lambda(h \otimes k^{op}) \\ (b,y) & \longmapsto & {}_b\mathcal{A}[E]\mathcal{A}_y \end{array}$$

by setting $\lambda(h \otimes k^{op})(g[e]f) = (h \circ g)[e](f \circ k)$; this functor is additive.

For any arrow ${}_*\mathcal{A}[E]\mathcal{A}_* \xrightarrow{\varphi} {}_*\mathfrak{M}_*$, define $\Phi(\varphi)$ by $\Phi(\varphi)(e) = \varphi(id_a[e]id_x)$, for all $e \in {}_aE_x$:.

For each ${}_*E_* \xrightarrow{\psi} O({}_*\mathfrak{M}_*)$ define $\Psi(\psi)$ by $\Psi(\psi)(g[e]f) = g.(\psi(e)).f^{op}$, for all $g[e]f \in {}_a\mathcal{A}[E]\mathcal{A}_x$; the points denote respectively the left and right $\mathcal{A}$-actions on ${}_*\mathfrak{M}_*$.

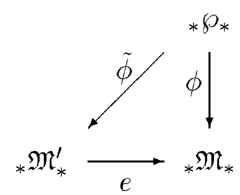
Note that $\Psi(\psi)$ is $\mathcal{A}$-bilinear and that Φ and Ψ are natural reciprocal isomorphisms. Indeed :

$\Psi(\Phi(\varphi))(g[e]f) = g.(\Phi(\varphi)(e)).f^{op} = g.\varphi(id_a[e]id_x).f^{op} = \varphi(g[e]f)$

$\Phi(\Psi(\psi))(e) = \Psi(\psi)(id_a[e]id_x) = id_a.\psi(e).id_x = \psi(e)$. □

3.3 Projective $\mathcal{A}$-bimodules.

• An $\mathcal{A}$-bimodule ${}_*\wp_*$ is *projective*, if given any morphism $\phi : {}_*\wp_* \longrightarrow {}_*\mathfrak{M}_*$ and any epi $e : {}_*\mathfrak{M}'_* \longrightarrow {}_*\mathfrak{M}_*$, there exists a morphism $\tilde{\phi} : {}_*\wp_* \longrightarrow {}_*\mathfrak{M}'_*$, such that the following diagram commute :



This is the classical definition of projectivity. We will not use it but rather an equivalent one we will give in section 9.

Note that a free $\mathcal{A}$-bimodule is projective.

3.4 Finitely generated $\mathcal{A}$-bimodules.

• An $\mathcal{A}$-bimodule ${}_*\mathfrak{M}_*$ is *finitely generated i.e. of finite type* if there exists ${}_*E_* \in Set^{\mathcal{A}_0^2}$ such that $(\coprod_{a,x\in\mathcal{A}_0} {}_aE_x)$ [36] made of elements of the various $\mathbb{Z}$-modules ${}_a\mathfrak{M}_x$ $(a, x \in \mathcal{A}_0)$ is finite and such that

[36] The set of *generators* for ${}_*\mathfrak{M}_*$.

each element of any module is a $\mathbb{Z}$-linear combination of $g.e.f$, $g, f \in \mathcal{A}_1$, e denoting a generator and the points denoting the left and right $\mathcal{A}$-actions.

3.5 $(FP)_n$ and $(FP)_\infty$ categories.

• A *complex* of $\mathcal{A}$-bimodules is an infinite sequence $\dots \xrightarrow{\partial_{n+1}} {}_*\mathcal{K}^n_* \xrightarrow{\partial_n} {}_*\mathcal{K}^{n-1}_* \xrightarrow{\partial_{n-1}} \dots$ where for each integer n, $\partial_n : {}_*\mathcal{K}^n_* \longrightarrow {}_*\mathcal{K}^{n-1}_*$ is an $\mathcal{A}$-bilinear morphism such that $\partial_{n-1}\partial_n = 0$.

• Let $\dots \xrightarrow{\partial_{n+1}} {}_*\mathcal{K}^n_* \xrightarrow{\partial_n} {}_*\mathcal{K}^{n-1}_* \xrightarrow{\partial_{n-1}} \dots$ and $\dots \xrightarrow{\partial'_{n+1}} {}_*\mathcal{K}'^n_* \xrightarrow{\partial'_n} {}_*\mathcal{K}'^{n-1}_* \xrightarrow{\partial'_{n-1}} \dots$ be two complexes. A *morphism of complexes* is a sequence $(f_n : {}_*\mathcal{K}^n_* \longrightarrow {}_*\mathcal{K}'^n_*)_n \in \mathbb{Z}$ of $\mathcal{A}$-bilinear morphisms such that $f_{n-1}\partial_n = \partial'_n f_n$, for all n, *i.e.* the following diagram commutes :

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\partial_{n+1}} & {}_*\mathcal{K}^n_* & \xrightarrow{\partial_n} & {}_*\mathcal{K}^{n-1}_* & \xrightarrow{\partial_{n-1}} & \dots \\
 & & \Big\downarrow{\scriptstyle f_n} & & \Big\downarrow{\scriptstyle f_{n-1}} & & \\
\dots & \xrightarrow[\partial'_{n+1}]{} & {}_*\mathcal{K}'^n_* & \xrightarrow[\partial'_n]{} & {}_*\mathcal{K}'^{n-1}_* & \xrightarrow[\partial'_{n-1}]{} & \dots
\end{array}$$

• Two morphisms of complexes f and g are said *homotopic* if there exists a sequence of $\mathcal{A}$-bilinear morphisms, $(h_i : {}_*\mathcal{K}^i_* \longrightarrow {}_*\mathcal{K}'^{i+1}_*)_{i \geq 0}$ such that $f_n - g_n = \partial'_{n+1}h_n + h_{n-1}\partial_n$ for all n.

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\partial_{n+1}} & {}_*\mathcal{K}^n_* & \xrightarrow{\partial_n} & {}_*\mathcal{K}^{n-1}_* & \xrightarrow{\partial_{n-1}} & \dots \\
 & \swarrow{\scriptstyle h_n} & {\scriptstyle f_n}\Big\downarrow\Big\downarrow{\scriptstyle g_n} & \swarrow{\scriptstyle h_{n-1}} & {\scriptstyle f_{n-1}}\Big\downarrow\Big\downarrow{\scriptstyle g_{n-1}} & \swarrow{\scriptstyle h_{n-2}} & \\
\dots & \xrightarrow{\partial'_{n+1}} & {}_*\mathcal{K}'^n_* & \xrightarrow[\partial'_n]{} & {}_*\mathcal{K}'^{n-1}_* & \xrightarrow{\partial'_{n-1}} & \dots
\end{array}$$

• A complex is said *positive* if ${}_*\mathcal{K}^n_* = {}_*0_*$ for all $n < 0$, we write it

$$\dots \xrightarrow{\partial_{n+1}} {}_*\mathcal{K}^n_* \xrightarrow{\partial_n} {}_*\mathcal{K}^{n-1}_* \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} {}_*\mathcal{K}^1_* \xrightarrow{\partial_1} {}_*\mathcal{K}^0_* \xrightarrow{\partial_0} {}_*0_*.$$

• A complex is *exact* if the $\mathcal{A}$-bimodule image of each morphism is the kernel of the previous one *i.e.* $im\,\partial_{i+1} = ker\,\partial_i$ for all i

• A *complex over an* $\mathcal{A}$-bimodule ${}_*\mathfrak{M}_*$ is a morphism of complexes ${}_*\mathcal{K}_* \xrightarrow{\varepsilon} {}_*\mathfrak{M}_*$ where ${}_*\mathcal{K}_*$ is a positive complex and ${}_*\mathfrak{M}_*$ is the trivial complex $\dots \xrightarrow{\partial_2 = 0} {}_*0_* \xrightarrow{\partial_1 = 0} {}_*\mathfrak{M}_* \xrightarrow{\partial_0 = 0} {}_*0_* \xrightarrow{\partial_{-1} = 0} \dots$ Since ${}_*0_*$ is the null object in the category $\mathcal{A}$-*Bimod*, for all $i \neq 0$, ε_i is a zero arrow, we can then rewrite the previous complex over ${}_*\mathfrak{M}_*$ as the following complex $\dots \xrightarrow{\partial_n} {}_*\mathcal{K}^{n-1}_* \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} {}_*\mathcal{K}^0_*$ $\xrightarrow{\varepsilon} {}_*\mathfrak{M}_* \longrightarrow {}_*0_*$.

• A *free (resp. projective) resolution* of an $\mathcal{A}$-bimodule ${}_*\mathfrak{M}_*$ is an exact complexes over ${}_*\mathfrak{M}_*$

REMARK 4 — *There always exists a free resolution of* ${}_*\mathfrak{M}_*$ *by* $\mathcal{A}$*-bimodules. Indeed, it is always possible to consider the* standard resolution [37] *:*

$$\ldots \longrightarrow (\mathcal{A}[-]\mathcal{A} \circ O)^n({}_*\mathfrak{M}_*) \longrightarrow \ldots \longrightarrow (\mathcal{A}[-]\mathcal{A} \circ O)^1({}_*\mathfrak{M}_*) \longrightarrow {}_*\mathfrak{M}_* \longrightarrow {}_*0_*,$$

where ${}_*\mathcal{A}[-]\mathcal{A}_*$ *and* O *denote the functors of lemma 1.*

- Let n be an integer, a category $\mathfrak{C}$ is $(FP)_n$, if there exists a partial free resolution :

$$ {}_*\mathfrak{R}^n_* \longrightarrow {}_*\mathfrak{R}^{n-1}_* \longrightarrow \ldots \longrightarrow {}_*\mathfrak{R}^1_* \longrightarrow {}_*\mathfrak{R}^0_* \longrightarrow {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*$$

where the ${}_*\mathfrak{R}^i_*$, for all $0 \leq i \leq n$ are finitely generated projective $\mathbb{Z}\mathfrak{C}$-bimodules.

- If for any integer n, $\mathfrak{C}$ is $(FP)_n$, it is said $(FP)_\infty$.

- A *morphism of resolutions over an* $\mathcal{A}$*-bimodule* ${}_*\mathfrak{M}_*$ is a morphism of complexes over ${}_*\mathfrak{M}_*$.

- Two morphisms of resolutions f and g from ${}_*\mathcal{K}_* \xrightarrow{\varepsilon} {}_*\mathfrak{M}_*$ to ${}_*\mathcal{K}'_* \xrightarrow{\varepsilon'} {}_*\mathfrak{M}_*$ are said *homotopic* if $f : {}_*\mathcal{K}_* \longrightarrow {}_*\mathcal{K}'_*$ and $g : {}_*\mathcal{K}_* \longrightarrow {}_*\mathcal{K}'_*$ are homotopic morphisms of positive complexes.

LEMMA 2 — *There is always a morphism between two projective resolutions and two such morphisms are homotopic.*

The proof is a straightforward generalization of the proof made in the case of modules over a ring.

3.6 Contracting homotopies.

Let $\mathcal{A}$ be a preadditive category and let

$$\ldots \xrightarrow{\partial_{n+1}} {}_*\mathfrak{R}^n_* \xrightarrow{\partial_n} {}_*\mathfrak{R}^{n-1}_* \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} {}_*\mathfrak{R}^1_* \xrightarrow{\partial_1} {}_*\mathfrak{R}^0_* \xrightarrow{\varepsilon} {}_*\mathfrak{M}_* \longrightarrow {}_*0_*$$

be a complex of $\mathcal{A}$-bimodules. In order to prove this complex is a resolution, we can exhibit a *contracting homotopy i.e.* a sequence of $\mathbb{Z}$-linear morphisms

$$\ldots \xleftarrow{s_n} {}_*\mathfrak{R}^n_* \xleftarrow{s_{n-1}} {}_*\mathfrak{R}^{n-1}_* \xleftarrow{s_{n-2}} \ldots \xleftarrow{s_1} {}_*\mathfrak{R}^1_* \xleftarrow{s_0} {}_*\mathfrak{R}^0_* \xleftarrow{\eta} {}_*\mathfrak{M}_*$$

satisfying the following properties : $\varepsilon\eta = id$, $\partial_1 s_0 + \eta\varepsilon = id$ and $\partial_{n+1}s_n + s_{n-1}\partial_n = id$ for all $n \geq 1$. Indeed, if such a sequence of $\mathbb{Z}$-linear morphisms exists we directly get that each n-cycle[38] is an n-boundary[39].

REMARK 5 — *We only need the* s_n *to be* $\mathbb{Z}$*-linear. We will explain in the next paragraph why we do not require the* s_n *to be* $\mathcal{A}$*-bilinear.*

[37] To get more precisions, one can look at [?], page 285.
[38] *i.e.* an element of $ker\,\partial_n$.
[39] *i.e.* an element of $im\,\partial_{n+1}$

3.7 Homology.

We are going to define a functor[40] which trivializes the left and right $\mathcal{A}$-actions.

Let $K : \mathcal{A}\text{-}Bimod \longrightarrow \mathbb{Z}\text{-}Mod$ be the morphism of graphs defined on any object ${}_*\mathfrak{M}_*$ of $\mathcal{A}$-*Bimod*, as the $\mathbb{Z}$-module :

$$K({}_*\mathfrak{M}_*) = (\bigoplus_{a,b\in\mathcal{A}_0} {}_a\mathfrak{M}_b)/\mathfrak{T},$$

where $\mathfrak{T}$ is the congruence generated by the identifications $x = f.x.g^{op}$. We will denote $\overline{x}$, the class of x modulo $\mathfrak{T}$. K associates to any $\mathcal{A}$-bilinear morphism $\phi : {}_*\mathfrak{M}_* \longrightarrow {}_*N_*$, the $\mathbb{Z}$-linear morphism $K(\phi)$, given by : $K(\phi)(\overline{x}) = \overline{\phi(x)}$.

$K(\phi)$ is well defined since ϕ is $\mathcal{A}$-bilinear. Indeed, $\phi(f.x.g^{op}) = f.\phi(x).g^{op}$ and so $\overline{\phi(x)} = \overline{\phi(f.x.g^{op})}$.

K is a functor. Indeed, $K(id_{{}_*\mathfrak{M}_*}) = id_{K({}_*\mathfrak{M}_*)}$ and :

$K(\phi\circ\psi)(\overline{x}) = \overline{(\phi\circ\psi)(x)} = \overline{\phi(\psi(x))} = K(\phi)(\overline{\psi(x)}) = K(\phi)(K(\psi)(\overline{x})) = (K(\phi)\circ K(\psi))(\overline{x})$.

• We call *homology of a category with coefficient* K, the homology of the complex obtained by trivializing and truncating the standard resolution :

$$\ldots \longrightarrow K_*(\mathcal{A}[-]\mathcal{A}\circ O)^n(\mathbb{Z}\mathfrak{C})_* \longrightarrow \ldots \longrightarrow K_*(\mathcal{A}[-]\mathcal{A}\circ O)^1(\mathbb{Z}\mathfrak{C})_* \longrightarrow K_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*,$$

where ${}_*\mathcal{A}[-]\mathcal{A}_*$ and O denote the functors of lemma 1.

REMARK 6 — *We are now in position to explain why we do not ask the maps of a contracting homotopy to be $\mathcal{A}$-bilinear but only $\mathbb{Z}$-linear. Indeed, trivialization being functorial, such a contracting homotopy would induce a contracting homotopy on the trivialized complex and the homology would be trivial.*

4 A free resolution.

Thanks to lemma 2, the homology of a category $\mathfrak{C}$ can be calculated by means of any projective resolution over ${}_*\mathbb{Z}\mathfrak{C}_*$. Let $(\mathfrak{G},\mathcal{R})$ be a finite noetherian confluent presentation of a category $\mathfrak{C}$, we can generalize [?] by constructing a free resolution of ${}_*\mathbb{Z}\mathfrak{C}_*$ by finitely generated $\mathbb{Z}\mathfrak{C}$-bimodules. For this construction, we will need a noetherian relation $>$.

In section 6, we will give a geometrical interpretation of the resolution we are going to construct. More precisely, we will exhibit a space whose homology is the constructed one. We shall particularly see how chains, boundary morphisms and the noetherian relation $>$ can be geometrically interpreted.

4.1 A noetherian relation.

• An arrow $f' \in \mathfrak{G}_1^*$ is a *prefix* (resp. a *suffix*) of $f \in \mathfrak{G}_1^*$, if there exists $f'' \in \mathfrak{G}_1^*$ such that $f = f'\circ f''$ (resp. $f = f''\circ f'$). f' is a *proper prefix* (resp. a *proper suffix*) if moreover $f'' \neq id_{\sigma(f')}$ (resp. $f'' \neq id_{\beta(f')}$).

• An arrow $f \in \mathfrak{G}_1^*$ is *right $\mathcal{R}$-minimal* if f is reducible but every proper prefix of f is $\mathcal{R}$-irreducible.

[40] Which will be the coefficient functor in the definition of homology, see [?].

- Let us denote :

  $\cdot$ $irr^+(\mathcal{R}) = irr(\mathcal{R}) \setminus \{id_x, x \in \mathfrak{C}_0\}$,

  $\cdot$ $A = \{(f,g) \in irr^+(\mathcal{R}) \times irr^+(\mathcal{R}) / f \circ g$ is right $\mathcal{R}$-minimal$\}$,

  $\cdot$ ϑ the subset of $\mathfrak{G}_1$ made of $\mathcal{R}$-irreducible arrows.

Note that $\vartheta \subset irr^+(\mathcal{R})$, otherwise $(\mathfrak{G}, \mathcal{R})$ would not be noetherian confluent.

- Let us set $\vartheta^{(n)} = \{(v_1, v_2, ..., v_n) \in (irr^+(\mathcal{R}))^n / v_1 \in \vartheta, (v_i, v_{i+1}) \in A$, for all $i = 1, ..., n-1\}$.

Note that $\vartheta^{(1)} = \vartheta$. We will agree about ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(0)}]\mathbb{Z}\mathfrak{C}_* = {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_*$ and $\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{id}_{\sigma(v_n)}$ will be preferred to $\overline{id}_{\beta(v_1)}[(v_1, ..., v_n)]\overline{id}_{\sigma(v_n)}$.

Let $X = \overline{g}[u_1|u_2|...|u_m]\overline{f}$ and $Y = \overline{k}[v_1|v_2|...|v_n]\overline{h}$ be elements of ${}_b\mathbb{Z}\mathfrak{C}[\vartheta^{(m)}]\mathbb{Z}\mathfrak{C}_a$ and ${}_b\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_a$ with $u_i : x_i \longrightarrow x_{i-1}$, $v_j : y_j \longrightarrow y_{j-1}$ and such that $\overline{f} : a \longrightarrow x_m$, $\overline{g} : x_0 \longrightarrow b$, $\overline{h} : a \longrightarrow y_n$ and $\overline{k} : y_0 \longrightarrow b$ are $\mathbb{Z}$-generators :

- We write $X \geq Y$ if there exists $p : y_0 \longrightarrow x_0$ such that $u_1 \circ ... \circ u_m \circ \hat{f} \stackrel{*}{\longrightarrow} p \circ v_1 \circ ... \circ v_n \circ \hat{h}$. If $x_0 = y_0$, we may have $p = id_{y_0}$.

- With the previous notations, we write

$$X > Y \quad \text{if} \quad X \geq Y \text{ and } \quad \begin{cases} p \neq id_{y_0} \\ \text{or} \\ u_1 \circ ... \circ u_m \circ \hat{f} \stackrel{*}{\longrightarrow} p \circ v_1 \circ ... \circ v_n \circ \hat{h} \text{ is of nonzero length.} \end{cases}$$

Note that if $X > Y$ and $Y \geq Z$, then $X > Z$. Similarly, if $X \geq Y$ and $Y > Z$, then $X > Z$.

- Let U and W be elements of ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(m)}]\mathbb{Z}\mathfrak{C}_*$ and ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$, we write $U > W$ if for any nonzero element pY of W, $p \in \mathbb{Z}$, $Y = \overline{k}[v_1|v_2|...|v_n]\overline{h}$ there is a nonzero element $p'X$ in U, $p' \in \mathbb{Z}$, $X = \overline{g}[u_1|u_2|...|u_m]\overline{f}$ such that $X > Y$.

LEMMA 3 — *The relation $>$ is noetherian on $\bigcup_{n=0}^{\infty} {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$.*

$>$ and $\geq$ are compatible with the left $\mathbb{Z}\mathfrak{C}$-action and under some conditions, compatible with the right $\mathbb{Z}\mathfrak{C}$-action. More precisely :

Let $X = \overline{g}[u_1|u_2|...|u_m]\overline{f}$ and $Y = \overline{k}[v_1|v_2|...|v_n]\overline{h}$ be elements of ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(m)}]\mathbb{Z}\mathfrak{C}_*$ and ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$,

LEMMA 4 — *if $X \geq Y$ (resp. $X > Y$) then for any arrow $\overline{l}$ of $\mathbb{Z}\mathfrak{C}$, we have :*

$$\overline{l}.X \geq \overline{l}.Y \ \ (\textit{resp. } \overline{l}.X > \overline{l}.Y).$$

LEMMA 5 — *If $X \geq Y$ (resp. $X > Y$) then for any arrow $\overline{l}$ of $\mathbb{Z}\mathfrak{C}$ such that $\hat{f} \circ \hat{l}$ is $\mathcal{R}$-irreducible :*

$$X.\overline{l} \geq Y.\overline{l} \ \ (\textit{resp. } X.\overline{l} > Y.\overline{l}).$$

The proofs of lemma 3, lemma 4 and lemma 5 are straightforward and left to the reader.

• For any arrow $f \in \vartheta$, we write $[f]$ the corresponding generator in ${}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_*$. We are going to extend this notation to any arrow $f \in irr(\mathcal{R})$. If $f = id$, let us set $[f] = 0$. Otherwise f is a unique composition of non-identity arrows $f_i \in \mathfrak{G}_1$, $f = f_1 \circ ... \circ f_n$. Since $f \in irr(\mathcal{R})^+$, we have $f_i \in \vartheta$. We set $[f] = \sum_{i=1}^{n} \overline{f_1 \circ ... \circ f_{i-1}}[f_i]\overline{f_{i+1} \circ ... \circ f_n}$.

4.2 Construction of the resolution.

Let us construct a resolution together with a contacting homotopy :

$$\cdots \underset{s_n}{\overset{\partial_{n+1}}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_* \underset{s_{n-1}}{\overset{\partial_n}{\rightleftarrows}} \cdots \underset{s_1}{\overset{\partial_2}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*.$$

Now on, any $\mathbb{Z}$-generator $\overline{g}[\beta(\overline{g})]\overline{f}$ of ${}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_*$ will simply be written $\overline{g} \otimes \overline{f}$.

• $\partial_0 = \varepsilon : {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*\mathbb{Z}\mathfrak{C}_*$ is the $\mathbb{Z}\mathfrak{C}$-bilinear morphism so that, $\varepsilon(a) = \overline{id}_a$ for all $a \in \mathfrak{C}_0$.

• $s_{-1} = \eta : {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_*$ is the left $\mathbb{Z}\mathfrak{C}$-linear morphism defined by $\eta(\overline{f}) = \overline{f} \otimes \overline{id}_{\sigma(f)}$, for all $\overline{f} \in \mathbb{Z}\mathfrak{C}$.

• $\partial_1 : {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_*$ is the $\mathbb{Z}\mathfrak{C}$-bilinear morphism so that, $\partial_1([f]) = \overline{id}_{\beta(f)} \otimes \overline{f} - \overline{f} \otimes \overline{id}_{\sigma(f)}$, for all generator $[f]$.

• $s_0 : {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_*$ is the left $\mathbb{Z}\mathfrak{C}$-linear morphism defined by $s_0(\overline{id}_{\beta(\overline{f})} \otimes \overline{f}) = \overline{id}_{\beta(\overline{f})}[\hat{f}]$, for all $\overline{f} \in \mathbb{Z}\mathfrak{C}$ and where $[\hat{f}]$ is defined by extension of the bracket notation.
Note that $\varepsilon\eta(\overline{f}) = \varepsilon(\overline{f} \otimes \overline{id}_{\sigma(f)}) = \overline{f}$.

We are going to construct [41] $\mathbb{Z}\mathfrak{C}$-bilinear morphisms $\partial_n : {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_* \to {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n-1)}]\mathbb{Z}\mathfrak{C}_*$ and $\mathbb{Z}$-linear morphisms[42] $s_{n-1} : {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n-1)}]\mathbb{Z}\mathfrak{C}_* \to {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$, in such a way that the sequence of free $\mathbb{Z}\mathfrak{C}$-bimodules

$${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_* \underset{s_{n-1}}{\overset{\partial_n}{\rightleftarrows}} \cdots \underset{s_1}{\overset{\partial_2}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*.$$

satisfies the following properties :

($\mathbf{a}_n$) $\partial_{n-1}\partial_n = 0$,

($\mathbf{b}_n$) $s_{n-2}\partial_{n-1} + \partial_n s_{n-1} = id_{{}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n-1)}]\mathbb{Z}\mathfrak{C}_*}$,

($\mathbf{c}_n$) $\partial_n(\overline{g}[v_1|...|v_n]\overline{f}) \leq \overline{g}[v_1|...|v_n]\overline{f}$ and $s_{n-1}(\overline{g}[v_1|...|v_{n-1}]\overline{f}) \leq \overline{g}[v_1|...|v_{n-1}]\overline{f}$,

($\mathbf{d}_n$) if $v_n \circ \hat{f}$ is $\mathcal{R}$-reducible then $s_{n-1}\partial_n(\overline{g}[v_1|...|v_n]\overline{f}) < \overline{g}[v_1|...|v_n]\overline{f}$, otherwise $v_n \circ \hat{f}$ is $\mathcal{R}$-irreducible and then $s_{n-1}\partial_n(\overline{g}[v_1|...|v_n]\overline{f}) = \overline{g}[v_1|...|v_n]\overline{f}$.

REMARK 7 — *Note that the s_i are left $\mathbb{Z}\mathfrak{C}$-linear but in general they are not right $\mathbb{Z}\mathfrak{C}$-linear otherwise the homology would be trivial.*

[41] By induction on n.
[42] They will even be left $\mathbb{Z}\mathfrak{C}$-linear.

REMARK 8 — *The left $\mathbb{Z}\mathfrak{C}$-linearity of the s_i and ∂_i together with lemma 4 show that conditions* $(\mathbf{c}_n)$ *and* $(\mathbf{d}_n)$ *would be equivalent to those we would obtain if* $\overline{g}$ *was replaced by* $\overline{id}_{\beta(v_1)}$.

Let us start the induction :

LEMMA 6 — *The sequence of free $\mathbb{Z}\mathfrak{C}$-bimodules*

$$_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*.$$

verifies $(\mathbf{a}_1)$, $(\mathbf{b}_1)$, $(\mathbf{c}_1)$ *and* $(\mathbf{d}_1)$.

PROOF : $(\mathbf{a}_1)$: $\varepsilon\partial_1([f]) = \varepsilon(\overline{id}_{\beta(f)} \otimes \overline{f}) - \varepsilon(\overline{f} \otimes \overline{id}_{\sigma(f)}) = \overline{f} - \overline{f} = 0$.

$(\mathbf{b}_1)$: $(\partial_1 s_0 + \eta\varepsilon)(\overline{id}_{\beta(\overline{f})} \otimes \overline{f}) = \partial_1(\overline{id}_{\beta(\overline{f})}[\hat{f}]) + \eta(\overline{f}) = \overline{id}_{\beta(\overline{f})} \otimes \overline{f} - \overline{f} \otimes \overline{id}_{\sigma(\overline{f})} + \overline{f} \otimes \overline{id}_{\sigma(\overline{f})} = \overline{id}_{\beta(\overline{f})} \otimes \overline{f}$.

$(\mathbf{c}_1)$: $\partial_1(\overline{g}[v]\overline{f}) = \overline{g} \otimes \overline{v \circ f} - \overline{g \circ v} \otimes \overline{f}$. Now, we have a reduction path $v \circ \hat{f} \overset{*}{\longrightarrow} (v \circ \hat{f})\hat{}$ so $\overline{g} \otimes \overline{v \circ f} \leq \overline{g}[v]\overline{f}$. Moreover, $v \neq id$ (since $v \in \vartheta$) then $\overline{g \circ v} \otimes \overline{f} \leq \overline{g}[v]\overline{f}$ and finally $\partial_1(\overline{g}[v]\overline{f}) \leq \overline{g}[v]\overline{f}$.

$(\mathbf{d}_1)$: $s_0\partial_1(\overline{g}[v]\overline{f}) = s_0(\overline{g} \otimes \overline{v \circ f}) - s_0(\overline{g \circ v} \otimes \overline{f}) = \overline{g}[(v \circ \hat{f})\hat{}\,] - \overline{g \circ v}[\hat{f}]$.

· If $v \circ \hat{f}$ is $\mathcal{R}$-reducible then assuming $(v \circ \hat{f})\hat{}$ and $\hat{f}$ are written : $(v \circ \hat{f})\hat{} = u_1 \circ ... \circ u_k$ and $\hat{f} = f_1 \circ ... \circ f_r$, we have, for each $1 \leq i \leq k$ and $1 \leq j \leq r$, reduction paths : $v \circ \hat{f} \overset{*}{\longrightarrow} (v \circ \hat{f})\hat{}$ and $v \circ \hat{f} \overset{*}{\longrightarrow} v \circ f_1 \circ ... \circ f_r$. Since $v \circ \hat{f} \neq (v \circ \hat{f})\hat{}$ this allows us to write, for each $1 \leq i \leq k$:

$\overline{g \circ u_1 \circ ... \circ u_{i-1}}[u_i]\overline{u_{i+1} \circ ... \circ u_k} < \overline{g}[v]\overline{f}$ and since for each $1 \leq j \leq r$, $v \circ f_1 \circ ... \circ f_{j-1} \neq id$: $\overline{g \circ v \circ f_1 \circ ... \circ f_{j-1}}[f_j]\overline{f_{j+1} \circ ... \circ f_r} < \overline{g}[v]\overline{f}$ which implies $s_0\partial_1(\overline{g}[v]\overline{f}) < \overline{g}[v]\overline{f}$.

· If $v \circ \hat{f}$ is $\mathcal{R}$-irreducible then $\overline{g}[(v \circ f)\hat{}\,] = \overline{g}[v]\overline{f} + \overline{g \circ v}[\hat{f}]$ thus $s_0\partial_1(\overline{g}[v]\overline{f}) = \overline{g}[v]\overline{f}$. □

Let us suppose there exists $\mathbb{Z}\mathfrak{C}$-bilinear morphisms ∂_i and left $\mathbb{Z}\mathfrak{C}$-linear morphisms s_{i-1}, $1 \leq i \leq n$, such that the sequence of free $\mathbb{Z}\mathfrak{C}$-bimodules

$$_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_* \underset{s_{n-1}}{\overset{\partial_n}{\rightleftarrows}} \cdots \underset{s_1}{\overset{\partial_2}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*.$$

verify $(\mathbf{a}_i)$, $(\mathbf{b}_i)$, $(\mathbf{c}_i)$ and $(\mathbf{d}_i)$, for all $1 \leq i \leq n$.

Let us construct a $\mathbb{Z}\mathfrak{C}$-bilinear morphism $\partial_{n+1} : {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n+1)}]\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$ and a left $\mathbb{Z}\mathfrak{C}$-linear morphism $s_n : {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n+1)}]\mathbb{Z}\mathfrak{C}_*$ such that properties $(\mathbf{a}_{n+1})$, $(\mathbf{b}_{n+1})$, $(\mathbf{c}_{n+1})$ and $(\mathbf{d}_{n+1})$ hold.

$(\mathbf{a}_n)$ and $(\mathbf{b}_n)$ allow us to write :

$$\partial_n s_{n-1} \partial_n = \partial_n. \tag{1}$$

- Set

$$\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma(v_{n+1})}) = \\ \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}} - s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}). \tag{2}$$

We get ∂_{n+1} on ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n+1)}]\mathbb{Z}\mathfrak{C}_*$ by $\mathbb{Z}\mathfrak{C}$-bilinear extension.

- Let us define s_n by noetherian induction with respect to $>$, as follows :

  · If $v_n \circ \hat{f}$ is $\mathcal{R}$-irreducible, set

$$s_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) = 0.$$

  · If $v_n \circ \hat{f}$ is $\mathcal{R}$-reducible then $v_n \circ \hat{f} = v_n \circ v_{n+1} \circ g$, with $v_{n+1} \in \mathfrak{G}_1$, $g \in \mathfrak{G}_1^*$ and $v_n \circ v_{n+1}$ is right $\mathcal{R}$-minimal.

Since $v_n \circ v_{n+1}$ is $\mathcal{R}$-reducible, $(\mathbf{d}_n)$ implies $s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}) < \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}$, since $v_{n+1} \circ g$ is $\mathcal{R}$-irreducible, lemma 5 gives :

$$s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g} < \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}, \tag{3}$$

where . denotes the right $\mathbb{Z}\mathfrak{C}$-action.

By hypothesis $s_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g})$ is already defined, hence we may set

$$s_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) = \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{g} + s_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g}) \tag{4}$$

which defines s_n on ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$ by left $\mathbb{Z}\mathfrak{C}$-linear extension.

LEMMA 7 — *If there exist $\mathbb{Z}\mathfrak{C}$-bilinear morphisms ∂_i and left $\mathbb{Z}\mathfrak{C}$-linear morphisms s_{i-1}, $1 \leq i \leq n$ such that the sequence of free $\mathbb{Z}\mathfrak{C}$-bimodules*

$${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_* \underset{s_{n-1}}{\overset{\partial_n}{\rightleftarrows}} \cdots \underset{s_1}{\overset{\partial_2}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*.$$

verifies $(\mathbf{a}_i)$, $(\mathbf{b}_i)$, $(\mathbf{c}_i)$ and $(\mathbf{d}_i)$ then for all $1 \leq i \leq n$, ∂_{n+1} and s_n defined respectively by (2) and (4) are such that the sequence of free $\mathbb{Z}\mathfrak{C}$-bimodules

$${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n+1)}]\mathbb{Z}\mathfrak{C}_* \underset{s_n}{\overset{\partial_{n+1}}{\rightleftarrows}} \cdots \underset{s_1}{\overset{\partial_2}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} {}_*\mathbb{Z}\mathfrak{C}_* \longrightarrow {}_*0_*.$$

verifies $(\mathbf{a}_{n+1})$, $(\mathbf{b}_{n+1})$, $(\mathbf{c}_{n+1})$ and $(\mathbf{d}_{n+1})$.

PROOF : $(\mathbf{a}_{n+1})$: (1) implies $\partial_n\partial_{n+1} = 0$. Indeed, for any $\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma_{v_{n+1}}}$ generating the $\mathbb{Z}\mathfrak{C}$-bimodule ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n+1)}]\mathbb{Z}\mathfrak{C}_*$, $\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}) - \partial_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}})) = 0$ thanks to (1) *i.e.* $\partial_n\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma_{v_{n+1}}}) = 0$.

$(\mathbf{b}_{n+1})$: Let $\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}$ be any element of ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$.

  · If $v_n \circ \hat{f}$ is $\mathcal{R}$-reducible then $v_n \circ \hat{f}$ is written $v_n \circ \hat{f} = v_n \circ v_{n+1} \circ g$ with $v_{n+1} \in \mathfrak{G}_1$, $g \in \mathfrak{G}_1^*$ and $v_n \circ v_{n+1}$ right $\mathcal{R}$-minimal. By (4) :

$$\begin{aligned}\partial_{n+1}s_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) = \\ \partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{g}) + \partial_{n+1}s_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}})).\overline{g}).\end{aligned} \tag{5}$$

By (3) and the induction hypothesis, let us rewrite the last term (since in particular $(\mathbf{b}_{n+1})$ is satisfied for $s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g})$:

$$\begin{aligned}\partial_{n+1}s_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g}) = s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g} \\ -s_{n-1}\partial_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g})\end{aligned} \tag{6}$$

Since ∂_n is right $\mathbb{Z}\mathfrak{C}$-linear, the last term of (6) may be written :
$s_{n-1}\partial_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g}) = s_{n-1}(\partial_n s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g})$ thanks to (1) : $s_{n-1}\partial_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g}) = s_{n-1}(\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g})$ and then

$$s_{n-1}\partial_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g}) = s_{n-1}(\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) \tag{7}$$

Moreover, thanks to (2) :

$$\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{g}) = \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f} - s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g} \tag{8}$$

From (5), (6), (7) and (8), we get :

$$\partial_{n+1}s_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) + s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) = \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}. \tag{9}$$

· If $v_n \circ \hat{f}$ is $\mathcal{R}$-irreducible then $s_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) = 0$ and $(\mathbf{d}_n)$ gives :
$s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) = \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}$ and in every cases (9) is verified.

$(\mathbf{c}_{n+1})$: On one hand, we have $\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}} \leq \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma(v_{n+1})}$. On the other hand, since $v_n \circ v_{n+1}$ is $\mathcal{R}$-reducible, $(\mathbf{d}_n)$ and the previous inequality imply :
$s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}) < \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma(v_{n+1})}$ and so :
$\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma(v_{n+1})}) \leq \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma(v_{n+1})}$. Moreover, using the notations from the definition of s_n, we have : $\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{g} \leq \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}$.

By induction hypothesis and (3), we may write :
$s_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g}) \leq s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{g} \leq \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}$.
therefore $s_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}) \leq \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{f}$.

$(\mathbf{d}_{n+1})$: Let $\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}$ be an element of ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n+1)}]\mathbb{Z}\mathfrak{C}_*$ such that $v_{n+1} \circ \hat{f}$ is $\mathcal{R}$-reducible. By (2) :
$\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}) = \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1} \circ f} - s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{f}$
and since $v_n \circ v_{n+1}$ is $\mathcal{R}$-reducible, by $(\mathbf{d}_n)$ we can write :
$s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}) < \overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}} \leq \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{id}_{\sigma(v_{n+1})}$,
and so, by lemma 5 $s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{f} < \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}$. Moreover, since $v_{n+1} \circ \hat{f}$ is $\mathcal{R}$-reducible, we have a reduction path $v_1 \circ ... \circ v_n \circ v_{n+1} \circ \hat{f} \xrightarrow{*} v_1 \circ ... \circ v_n \circ (v_{n+1} \circ \hat{f})\hat{}$, the length of which is nonzero, so $\overline{id}_{\beta(v_1)}[v_1|...|v_n]cl v_{n+1} \circ f < \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}$. We then get $\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}) < \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}$.

Applying s_n to the previous term and using $(\mathbf{c}_{n+1})$, we have :

$$s_n\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}) \leq \partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f})) < \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}.$$

Now, let $\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}$ be in ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n+1)}]\mathbb{Z}\mathfrak{C}_*$ with $v_{n+1} \circ \hat{f}$ $\mathcal{R}$-irreducible. By (2) :

$$\begin{aligned} s_n\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}) &= s_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1} \circ f}) \\ &\quad - s_n(s_{n-1}\partial_n(\overline{id}_{\beta(v_1)}[v_1|...|v_n]\overline{v_{n+1}}).\overline{f}) \end{aligned}$$

and thanks to (4) we get $s_n\partial_{n+1}(\overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}) = \overline{id}_{\beta(v_1)}[v_1|...|v_{n+1}]\overline{f}$. □

We have proved the following result :

THEOREM 5 — *If a category has a finite noetherian confluent presentation, it is* $(FP)_\infty$.

5 The particular case of monoids.

In the particular case of monoids, we are going to show that the constructed homology is the one defined in [?] and [?]. Note first that a monoid can be viewed as a one-object category, the composition of which is the product of the monoid. The notion of free preadditive category (over a category) generalizes the notion of monoid ring ; in other words, if $\mathfrak{C}$ is a one-object category then $\mathbb{Z}\mathfrak{C}$ corresponds to the monoid ring associated with the "monoid" $\mathfrak{C}$.

We note that the coefficient functor K (previously defined) may be seen as the composition $K_d \circ K_g$ (K_d and K_g functors), where $K_g : \mathcal{A}\text{-}Bimod \longrightarrow \mathcal{A}^{op}\text{-}Mod$ and $K_d : \mathcal{A}^{op}\text{-}Mod \longrightarrow \mathbb{Z}\text{-}Mod$, respectively denote the functors which trivialize the left and right $\mathcal{A}$-actions.

We set

$$K_g({}_*\mathfrak{M}_*) = (\bigoplus_{a,b\in\mathcal{A}_0} {}_a\mathfrak{M}_b)/\mathfrak{T}_g \text{ and } K_d(\mathfrak{M}_*) = (\bigoplus_{a\in\mathcal{A}_0} \mathfrak{M}_a)/\mathfrak{T}_d,$$

where $\mathfrak{T}_g$ and $\mathfrak{T}_d$ are the congruences generated respectively by the identifications $x = f.x$ and $x = x.g^{op}$. We will respectively denote ${}^g\overline{x}$ and $\overline{x}^d$, the classes of x modulo $\mathfrak{T}_g$ and $\mathfrak{T}_d$.

K_g assigns to each $\mathcal{A}$-bilinear morphism $\phi : {}_*\mathfrak{M}_* \longrightarrow {}_*\mathcal{N}_*$ the right $\mathcal{A}$-linear morphism $K_g(\phi)$, defined by : $K_g(\phi)({}^g\overline{x}) = {}^g\overline{\phi(x)}$. $K_g(\phi)$ is well defined since ϕ is $\mathcal{A}$-bilinear.

K_d assigns to each right $\mathcal{A}$-linear morphism $\phi : {}_*\mathfrak{M}_* \longrightarrow {}_*\mathcal{N}_*$ the $\mathbb{Z}$-linear morphism $K_d(\phi)$ defined by $K_d(\phi)(\overline{x}^d) = \overline{\phi(x)}^d$. $K_d(\phi)$ is well defined since ϕ is right $\mathcal{A}$-linear.

If $(\Sigma, \mathcal{R})$ is a finite noetherian confluent rewriting system which presents a monoid $\mathcal{M}$, $(\mathfrak{G}, \mathcal{R})$, where $\mathfrak{G}$ is the graph such that $\mathfrak{G}_0 = \{*\}$ and $\mathfrak{G}_1 = \Sigma$, is a finite noetherian confluent presentation of $\mathcal{M}$ interpreted as a category.

Theorem 4, gives a free resolution of $\mathbb{Z}\mathcal{M}$-bimodules (in the usual sense) :

$$\ldots \xrightarrow{\partial_n} \mathbb{Z}\mathcal{M}[\vartheta^{(n-1)}]\mathbb{Z}\mathcal{M} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} \mathbb{Z}\mathcal{M}[\vartheta]\mathbb{Z}\mathcal{M} \xrightarrow{\partial_1} \mathbb{Z}\mathcal{M}[\mathfrak{C}_0]\mathbb{Z}\mathcal{M} \xrightarrow{\varepsilon} \mathbb{Z}\mathcal{M} \longrightarrow 0.$$

then, applying $K_g : \mathbb{Z}\mathcal{M}\text{-}Bimod \longrightarrow \mathbb{Z}\mathcal{M}^{op}\text{-}Mod$, the contacting homotopy being left $\mathbb{Z}\mathcal{M}$-linear, we get a resolution isomorphic to the following free resolution of right $\mathbb{Z}\mathcal{M}$-modules :

$$\ldots \longrightarrow [\vartheta^{(n)}]\mathbb{Z}\mathcal{M} \longrightarrow [\vartheta^{(n-1)}]\mathbb{Z}\mathcal{M} \longrightarrow \ldots \longrightarrow [\vartheta]\mathbb{Z}\mathcal{M} \longrightarrow \mathbb{Z}\mathcal{M} \longrightarrow \mathbb{Z} \longrightarrow 0$$

which is the resolution constructed in the particular case of §.5 in [?].

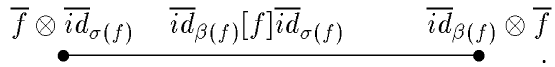
6 Geometrical interpretation, classifying space.

Let $\mathfrak{C}$ be a category of finite noetherian confluent presentation. We assume that $\mathfrak{C}$ owns such a presentation which is moreover minimal and without any reddex of length 1. Those two last assumptions are innocuous because any finite noetherian confluent rewriting system can always be modified in such a way it verifies those hypothesis[43].

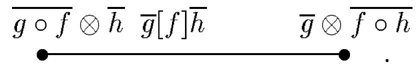
[43] There is a straightforward generalization of proposition 2 to rewriting systems for categories.

From a geometric viewpoint, we can assign to the resolution constructed in Theorem 5, a space $\mathcal{E}$, on which the arrows of $\mathfrak{C}$ act bilaterally. This space will admit a path component for each arrow of $\mathfrak{C}$.

In the first dimensions, let us construct $\mathcal{E}$ the following way : first, we consider the 0-cells, $\overline{id_a} \otimes \overline{id_a}$, one for each type a of $\mathfrak{C}$. Because of the bilateral action of the arrows, the 0-cells of $\mathcal{E}$ are tensors, $\overline{f} \otimes \overline{g}$, such that $\beta(\overline{g}) = \sigma(\overline{f})$. Thus $\mathcal{E}$ admits a path component for each arrow of $\mathfrak{C}$. For each $f \in \mathfrak{G}_1$ (thanks to the hypothesis made on the presentation) we connect $\overline{f} \otimes \overline{id_{\sigma(f)}}$ to $\overline{id_{\beta(f)}} \otimes \overline{f}$ by an edge, namely the 1-cell :



. The element $\overline{g \circ f} \otimes \overline{h}$, $f \in \mathfrak{G}_1$, is then connected to $\overline{g} \otimes \overline{f \circ h}$ by the 1-cell :



.

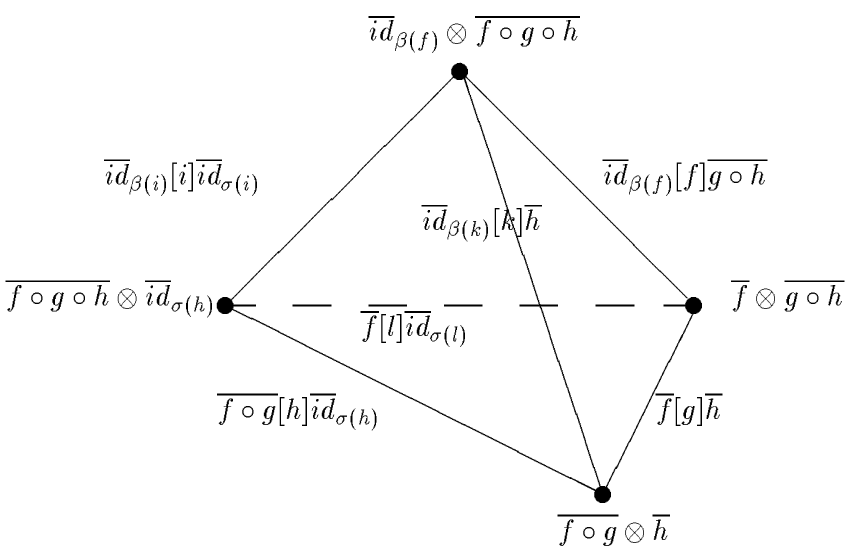
The boundaries of those 1-cells are the formal differences of their vertices $\partial_1(\overline{id_{\beta(f)}}[f]\overline{id_{\sigma(f)}}) = \overline{f} \otimes \overline{id_{\sigma(f)}} - \overline{id_{\beta(f)}} \otimes \overline{f}$ and $\partial_1(\overline{g} \otimes [f]\overline{h}) = \overline{g} \otimes \overline{f \circ h} - \overline{g \circ f} \otimes \overline{h}$.

We then get a geometrical interpretation of the $\mathbb{Z}\mathfrak{C}$-bimodules ${}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_*$ and ${}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_*$ together with the boundary ∂_1. The augmentation ε maps each $\overline{f} \otimes \overline{g}$ on the composition $\overline{f} \circ \overline{g}$; this allows to determine the path component of $\overline{f} \otimes \overline{g}$. In the next dimensions, the $\mathbb{Z}\mathfrak{C}$-bimodules ${}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(n)}]\mathbb{Z}\mathfrak{C}_*$ are interpreted as n-cells of the space.

For example, in dimension 2, we are going to visualize the noetherian relation $>$ used in the construction of the free resolution and defined in paragraph 4.1 :

let $\mathfrak{C}$ be a category with a finite noetherian confluent presentation which is moreover minimal and without any reddex of length 1. To make this example as simple as possible, let us assume that $f : z \longrightarrow t$, $g : y \longrightarrow z$, $h : x \longrightarrow y$, $l : x \longrightarrow z$, $k : y \longrightarrow t$ and $i : x \longrightarrow t$ are in $\mathfrak{G}_1$ and that the following rules : $f \circ g \longrightarrow k$, $g \circ h \longrightarrow l$, $k \circ h \longrightarrow i$ and $f \circ l \longrightarrow i$ are in $\mathcal{R}$.

We then get a 3-cell, $\overline{id_{\beta(f)}}[f|g|h]\overline{id_{\sigma(h)}}$ corresponding to the previous critical pair, represented by the "full tetrahedron", whose vertice is $\overline{id_{\beta(f)}} \otimes \overline{f \circ g \circ h} = id_{\beta(f)} \otimes \overline{i}$ and whose base is the 2-cell $\overline{f}[g|h]\overline{id_{\sigma(h)}}$ (corresponding to the reduction $f \circ g \circ h \longrightarrow f \circ l$) :



The 3-cells represented by the sides of the tetrahedron, $\overline{f}[g|h]\overline{id_{\sigma(h)}}$, $\overline{id_{\beta(f)}}[f|g]\overline{h}$, $\overline{id_{\beta(k)}}[k|h]\overline{id_{\sigma(h)}}$ and $\overline{id_{\beta(f)}}[f|l]\overline{id_{\sigma(l)}}$ correspond to the four reductions appearing in the confluence diagram drawn above.

Those are examples of ordered cells :

$$\overline{id}_{\beta(f)}[f|g|h]\overline{id}_{\sigma(h)} \geq \overline{id}_{\beta(f)}[f|g]\overline{h} > \overline{id}_{\beta(f)}[f]\overline{g \circ h} > \overline{id}_{\beta(f)} \otimes \overline{f \circ g \circ h},$$

$$\overline{id}_{\beta(f)}[f|l]\overline{id}_{\sigma(l)} \geq \overline{id}_{\beta(f)}[f]\overline{g \circ h} > \overline{id}_{\beta(f)} \otimes \overline{f \circ g \circ h},$$

$$\overline{f}[g|h]\overline{id}_{\sigma(h)} > \overline{f} \otimes \overline{g \circ h} > \overline{id}_{\beta(f)} \otimes \overline{f \circ g \circ h},$$

$$\overline{id}_{\beta(k)}[k|h]\overline{id}_{\sigma(h)} > \overline{id}_{\beta(f)} \otimes \overline{f \circ g \circ h},$$

$$\overline{f}[g]\overline{h} > \overline{f} \otimes \overline{g \circ h} > \overline{id}_{\beta(f)} \otimes \overline{f \circ g \circ h},$$

etc.

Note that $>$ is not only ordering cells belonging to the same path component. Nevertheless, we only need this order between cells belonging to the same path component. Indeed, we only use $>$ to build ∂_i and s_i and the path components are "invariant subspaces" of ∂_i and s_i.

After trivializing both right and left $\mathbb{Z}\mathfrak{C}$-actions, we get the space whose homology has been calculated, it is called the *classifying space* of the category $\mathfrak{C}$.

Note that in the particular case of a one-object category, the initial space (*i.e.* before trivialization) is not the one constructed in [?], but after trivializing the left action (this action is superfluous in the case of a one-object category), we recover precisely the contactible space of [?][44]. The trivialization (of the left action) allowed the identification of the 0-cells $\overline{f} \otimes \overline{id}_*$ with $\overline{id}_*$, for each $\overline{f}$ of $\mathfrak{M}$.

7 A counter-example.

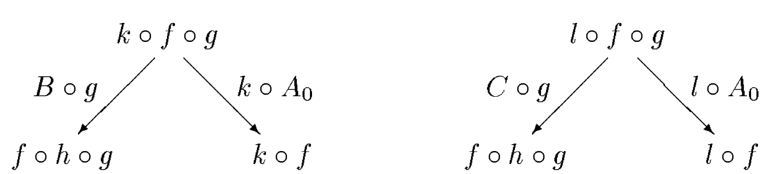
In this section, we give a counter-example of a category $\mathfrak{C}$ having two presentations : a finite one and another one which is noetherian confluent and we are going to show that $\mathfrak{C}$ does not own any presentation which is both finite and noetherian confluent. To prove this, we need only show the third homology module of $\mathfrak{C}$ is not finitely generated.

Let $\mathfrak{C}$ be the category presented by :

$\mathfrak{G}_0 = \{x, y\}$, $\mathfrak{G}_1 = \{x \xrightarrow{f} y, y \xrightarrow{g} y, y \xrightarrow{h} y, x \xrightarrow{k} x, x \xrightarrow{l} x\}$

and $\mathcal{R}_f = \{h \circ f \xrightarrow{A_0} f, f \circ k \xrightarrow{B} g \circ f, f \circ l \xrightarrow{C} g \circ f\}$.

This presentation is clearly finite but not confluent. Indeed, the two critical pics :

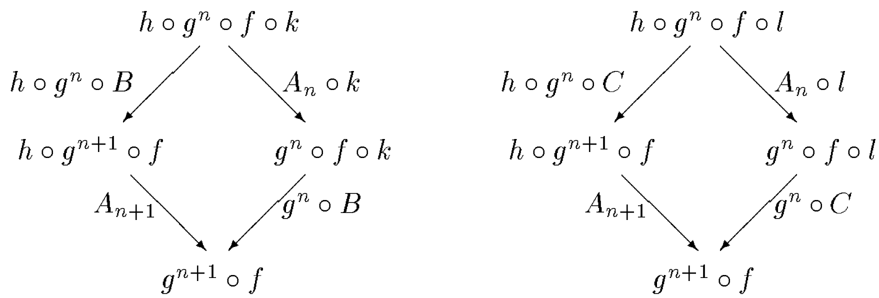


are not confluent.

Let us consider another presentation of $\mathfrak{C}$ given by :

[44]Except the fact that the remaining action is a right one when in [?] it is a left one ; nevertheless this is only a detail relative to the decision concerning the side on which we want to apply the action of the monoid.

$\mathfrak{G}_0 = \{x, y\}$, $\mathfrak{G}_1 = \{x \xrightarrow{f} y, y \xrightarrow{g} y, y \xrightarrow{h} y, x \xrightarrow{k} x, x \xrightarrow{l} x\}$
and $\mathcal{R}_c = \{h \circ g^n \circ f \xrightarrow{A_n} g^n \circ f, n \in \mathcal{N}\} \bigcup \{f \circ k \xrightarrow{B} g \circ f, f \circ l \xrightarrow{C} g \circ f\}$
This presentation is infinite noetherian and generates an infinite number of critical pairs (of 2 forms), which are confluent :



For the second presentation, we have $\vartheta = \{f, g, h, k, l\}$, $\vartheta^{(2)} = \{(h, g^n \circ f), n \in \mathcal{N}\} \bigcup \{(f, k), (f, l)\}$ and $\vartheta^{(3)} = \{(h, g^n \circ f, k), n \in \mathcal{N}\} \bigcup \{(h, g^n \circ f, l), n \in \mathcal{N}\}$.

$ker\, \partial_3 = 0_*$. Indeed $\partial_3([h|g^n \circ f|k]) = [h|g^n \circ f]\overline{k} - s_1\partial_2([h|g^n \circ f]\overline{k})$ with $\partial_2([h|g^n \circ f]\overline{k}) = [h]\overline{g^n \circ f \circ k} - s_0\partial_1([h]\overline{g^n \circ f}).\overline{k}$ and $\partial_1([h]\overline{g^n \circ f}) = \overline{id_y} \otimes \overline{h \circ g^n \circ f} - \overline{h} \otimes \overline{g^n \circ f} = \overline{id_y} \otimes \overline{g^n \circ f} - \overline{h} \otimes \overline{g^n \circ f}$

so

$$\begin{aligned} s_0\partial_1([h]\overline{g^n \circ f}).\overline{k} &= \overline{id_y}[g^n \circ f]\overline{k} - \overline{h}[g^n \circ f]\overline{k} \\ &= \overline{id_y}[g]\overline{g^{n-1} \circ f \circ k} + \ldots + \overline{g^{n-1}}[g]\overline{f \circ k} + \overline{g^n}[f]\overline{k} \\ &\quad -\overline{h}[g]\overline{g^{n-1} \circ f \circ k} - \ldots - \overline{h \circ g^n}[f]\overline{k}. \end{aligned}$$

Then :

$$\begin{aligned} \partial_2([h|g^n \circ f]\overline{k}) &= [h]\overline{g^{n+1} \circ f} - \overline{id_y}[g]\overline{g^n \circ f} - \ldots - \overline{g^{n-1}}[g]\overline{g \circ f} \\ &\quad -\overline{g^n}[f]\overline{k} + \overline{h}[g]\overline{g^n \circ f} + \ldots + \overline{h \circ g^n}[f]\overline{k}. \end{aligned}$$

So, we have :

$$\begin{aligned} s_1\partial_2([h|g^n \circ f]\overline{k}) &= [h|g^{n+1} \circ f] + s_1(s_0\partial_1([h]\overline{g^n \circ f})) - \overline{g}[f|k]\overline{id_x} \\ &\quad + s_1(s_0\partial_1(\overline{g}[f]\overline{k})) + \overline{h \circ g}[f|k]\overline{id_x} + s_1(s_0\partial_1(\overline{h \circ g}[f]\overline{k})). \end{aligned}$$

but $s_1(s_0\partial_1([h]\overline{g^n \circ f})) = s_1(s_0\partial_1(\overline{g}[f]\overline{k} = s_1(s_0\partial_1(\overline{h \circ g}[f]\overline{k})) = 0$, which implies :

$$\partial_3([h|g^n \circ f|k]) = [h|g^n \circ f]\overline{k} - [h|g^{n+1} \circ f] + \overline{g}[f|k]\overline{id_x} - \overline{h \circ g}[f|k]\overline{id_x}.$$

Replacing k by l, we get :

$$\partial_3([h|g^n \circ f|l]) = [h|g^n \circ f]\overline{l} - [h|g^{n+1} \circ f] + \overline{g}[f|l]\overline{id_x} - \overline{h \circ g}[f|l]\overline{id_x}.$$

those elements are $\mathbb{Z}\mathfrak{C}$-linearly independent and so we have the free resolution :

$$\ldots \xrightarrow{0} {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(3)}]\mathbb{Z}\mathfrak{C}_* \xrightarrow{\partial_3} {}_*\mathbb{Z}\mathfrak{C}[\vartheta^{(2)}]\mathbb{Z}\mathfrak{C}_* \xrightarrow{\partial_2} {}_*\mathbb{Z}\mathfrak{C}[\vartheta]\mathbb{Z}\mathfrak{C}_* \xrightarrow{\partial_1} {}_*\mathbb{Z}\mathfrak{C}[\mathfrak{C}_0]\mathbb{Z}\mathfrak{C}_* \xrightarrow{\varepsilon} {}_*0_*.$$

Applying K, we get a free complex. We can calculate :

$K\partial_3([h|g^n \circ f|k]) = [h|g^n \circ f] - [h|g^{n+1} \circ f] + [f|k] - [f|k] = [h|g^n \circ f] - [h|g^{n+1} \circ f]$ and $K\partial_3([h|g^n \circ f|l]) = [h|g^n \circ f] - [h|g^{n+1} \circ f]$

which proves $ker\, K\partial_3$ is generated by the infinite family : $\{[h|g^n \circ f|k] - [h|g^n \circ f|l], n \in \mathcal{N}\}$ in which none generator can be suppressed and so $H_3(\mathfrak{C})$ is not finitely generated.

8 The homological criterion is not sufficient.

We can see Theorem 5 as a necessary homological criterion for the existence of finite noetherian confluent presentations of categories. We already know that this criterion is not sufficient[45] ; which means there exists categories whose third homology module is finitely generated but nevertheless those categories do not admit any finite noetherian confluent presentation.

We are going to show, in the particular case of the example of paragraph 2.4, that our homological criterion cannot help us to decide whether $\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z})^*$ admits a finite noetherian confluent presentation.

Let us try our homological criterion and so let us have a look at the homology of the category[46] presented by $(\mathfrak{M}_r(\mathbb{Z}/p\mathbb{Z}), \mathcal{R})$. As we do not know precisely if $\mathfrak{M}$ has a finite noetherian confluent presentation, we cannot use Theorem 5. Nevertheless, we can use the standard resolution associated to $\mathfrak{M}$ (see remark 4). As $\mathbb{Z}/p\mathbb{Z}$ is finite, the standard resolution is made of finitely generated free $\mathbb{Z}\mathfrak{M}$-modules and so the homology modules of $\mathfrak{M}$ are finitely generated. In this case, the homological criterion fails : after its use we are not able to tell if $\mathfrak{M}$ can or cannot admit a finite noetherian confluent presentation.

9 An equivalent definition of projective modules.

To be able to give an equivalent definition of projective $\mathcal{A}$-modules[47], we will need to previously define :

• O denoting the forgetful functor, $\mathcal{A}$-*Bimod* $\longrightarrow Set^{\mathcal{A}_0^2}$, an $\mathcal{A}$-bilateral morphism s is said to be *surjective*, if $O(s)$ is a family of surjective maps.

LEMMA 8 — *The forgetful functor* $O : \mathcal{A}$-*Bimod* $\longrightarrow Set^{\mathcal{A}_0^2}$ *is faithful,*

PROOF : Thanks to theorem 1 p.88 in [?], it is sufficient to show that every component of the counit of the adjunction ${}_*\mathcal{A}[M]\mathcal{A}_* \xrightarrow{\varepsilon_{*\mathfrak{M}*}} id_{*\mathfrak{M}*}$ is epi. Recall that for every ${}_*\mathfrak{M}_*$, the component of the counit is defined by $\varepsilon_{*\mathfrak{M}*} = \Psi(id_{O(*\mathfrak{M}*)})$ and so for any $\mathbb{Z}$-generator $g[m]f$ in ${}_a\mathcal{A}[M]\mathcal{A}_x$, $a, x \in \mathcal{A}_0$, we have $\varepsilon_{*\mathfrak{M}*}(g[m]f) = g.id_{O(*\mathfrak{M}*)}([m]).f = g.m.f$. $\varepsilon_{*\mathfrak{M}*}$ has a right inverse $\eta_{*\mathfrak{M}*}$ defined by $\eta_{*\mathfrak{M}*}(m) = id_{\beta(m)}[m]id_{\sigma(m)}$. $\varepsilon_{*\mathfrak{M}*}$ is then epi. □

LEMMA 9 — *In* $\mathcal{A}$-*Bimod, epi arrows and* $\mathcal{A}$-*bilateral surjective morphisms coincide.*

[45] It is not already sufficient in the particular case of monoids. In [?], Squier gives examples of monoids, so single-object categories, verifying those properties.

[46] Denoted $\mathfrak{M}$ up from now.

[47] $\mathcal{A}$ being a preadditive category.

PROOF : We begin with showing an epi arrow is surjective. Let ${}_*\mathfrak{M}_* \xrightarrow{e} {}_*\mathfrak{M}'_*$ be epi, we then considers the following diagram :

$$ {}_*\mathfrak{M}_* \xrightarrow{e} {}_*\mathfrak{M}'_* \overset{\pi,\,0}{\rightrightarrows} {}_*\mathfrak{M}'_*/e({}_*\mathfrak{M}_*). $$

where :

· ${}_*\mathfrak{M}'_* \xrightarrow{\pi} {}_*\mathfrak{M}'_*/e({}_*\mathfrak{M}_*)$ denotes the canonical projection of the elements of ${}_*\mathfrak{M}'_*$ on their class modulo the sub-$\mathcal{A}$-bimodule $e({}_*\mathfrak{M}_*)$,

· 0 is the zero arrow.

If we assume e is not surjective, then there existes $m' \in {}_aM'_x$, $a, x \in \mathcal{A}_0$, $m' \neq 0$ such that $\pi(m') \neq 0$ and π is not the zero arrow. But, since $\pi \circ e = 0 = 0 \circ e$ and e is epi, then $\pi = 0$, which is impossible.

Conversely, assume that ${}_*\mathfrak{M}_* \xrightarrow{e} {}_*\mathfrak{M}'_*$ is a surjective arrow and let us show it is epi. Let ${}_*\mathfrak{M}'_* \xrightarrow{\varphi,\,\psi} {}_*\mathfrak{M}''_*$ be $\mathcal{A}$-bilateral morphisms such that $\varphi \circ e = \psi \circ e$. Applying the forgetful functor O, we get $O(\varphi) \circ O(e) = O(\psi) \circ O(e)$. But, since $O(e)$ is surjective, $O(\varphi) = O(\psi)$. O being faithful, we can conclude thanks to **(i)**, that $\varphi = \psi$. □

Here comes the following equivalent definition of $\mathcal{A}$-projective modules :

- An $\mathcal{A}$-bimodule ${}_*\wp_*$ is *projective*, if given any morphism $\phi : {}_*\wp_* \longrightarrow {}_*\mathfrak{M}_*$ and any surjective arrow $e : {}_*\mathfrak{M}'_* \longrightarrow {}_*\mathfrak{M}_*$, there exists a morphism $\tilde{\phi} : {}_*\wp_* \longrightarrow {}_*\mathfrak{M}'_*$, such that the following diagram commute :

$$ \begin{array}{ccc} & & {}_*\wp_* \\ & \swarrow^{\tilde{\phi}} & \downarrow^{\phi} \\ {}_*\mathfrak{M}'_* & \xrightarrow[e]{} & {}_*\mathfrak{M}_* \end{array} $$

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