IV. EVENTUAL ALGEBRAIC INVARIANTS FOR (NONNEGATIVE) MATRICES

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Contents

1. Introduction

To get to the overview, parts of the lecture were a bit telegraphic. There is some additional detail in these notes. A short expository paper with basic definitions for shifts of finite type and so on is my old paper "Symbolic dynamics and matrices". The Lind-Marcus book "Symbolic Dynamics" provides an excellent introduction (with a caveat: there have been significant advances since its publication). My expository-plus paper "Positive K-theory and symbolic dynamics" and the joint paper with Jack Wagoner "Positive algebraic K-theory and shifts of finite type" cover advances around polynomial matrix presentations and topological conjugacies induced by multiplication by basic elementary polynomial matrices. My old papers are generally available on my home page.

THIS IS A WRITEUP, WITH SOME ADDITIONAL MATERIAL, OF THE FOURTH AND LAST LECTURE IN THE LECTURE SERIES "NONNEGATIVE MATRICES: PERRON-FROBENIUS THEORY AND RELATED ALGEBRA" AS PART OF THE SUMMER SCHOOL "NUMBER THEORY AND DYNAMICS" HELD IN JUNE 2013 AT INSTITUT FOURIER IN GRENOBLE.

2. Definitions

Let $\mathbb{S}$ denote a semiring, always assumed to contain $\{0,1\}$. Let A, B be square matrices with entries in $\mathbb{S}$. We give some definitions.

Definition 2.1. A and B are elementary strong shift equivalent over $\mathbb{S}$ (ESSE-$\mathbb{S}$) if there exist matrices U, V over $\mathbb{S}$ such that $A = UV$ and $B = VU$.

Note: the matrices A and B do not have to be of the same size. In general, the relation ESSE-$\mathbb{S}$ is reflexive and symmetric but not transitive.

SSE-$\mathbb{S}$ is the equivalence relation which is the transitive closure of ESSE-$\mathbb{S}$.

Definition 2.2. A and B are strong shift equivalent over $\mathbb{S}$ (SSE-$\mathbb{S}$) if there is a finite chain of matrices $A_0, A_1, \dots, A_\ell$ such that $A = A_0$, $A_\ell = B$ and for $0 \le i < \ell$ the matrices A_i and A_{i+1} are ESSE over $\mathbb{S}$.

Definition 2.3. If $\mathbb{S}$ is a ring, with A and B $n \times n$, then A and B are similar over $\mathbb{S}$ (SIM-$\mathbb{S}$) if there is U in $\mathrm{GL}(n, \mathbb{S})$ such that $U^{-1}AU = B$.

Definition 2.4. A and B are shift equivalent over $\mathbb{S}$ (SE-$\mathbb{S}$) if there are matrices U, V over $\mathbb{S}$ and ℓ in $\mathbb{N}$ such that the following hold:

$$UV = A^\ell \quad VU = B^\ell \quad AU = UB \quad BV = VA\ .$$

Although SE-$\mathbb{S}$ initially looks complicated and SSE-$\mathbb{S}$ initially looks simple, overall it is SE-$\mathbb{S}$ which is much more tractable.

If A and B are SSE over $\mathbb{S}$, then A and B are SE over $\mathbb{S}$. If $\{(U_i, V_i) : 0 \le i < \ell\}$, is a chain of ESSEs from $A_0 = A$ to $A_\ell = B$, then the pair $U = U_0 U_1 \cdots U_{\ell-1}$, $V = V_{\ell-1} \cdots V_1 V_0$ gives an SE of A and B.

If $\mathbb{S}$ is a ring, and A and B are similar over $\mathbb{S}$, then A and B are ESSE over $\mathbb{S}$, via the pair $U, U^{-1}A$; and therefore A and B are also SE over $\mathbb{S}$.

3. Three "eventual" algebraic invariants

Let A be a square matrix over a ring $/S$. Here are three properties of A which I refer to as "eventual" algebraic invariants of A:

1. $\det(I - tA)$
2. The SE-$\mathbb{S}$ class of A.
3. The SSE-$\mathbb{S}$ class of A.

The first property, $\det(I - tA)$, is an encoding of the nonzero spectrum of A, as we've seen. It is an "eventual" version of the spectrum in the sense that by ignoring the multiplicity of zero in the spectrum, it is ignoring information that arises from the nilpotent part of the action of A. The SE-$\mathbb{S}$ and SSE-$\mathbb{S}$ classes can be seen analogously as "eventual" versions of SIM-$\mathbb{S}$. We will see that when $\mathbb{S}$ is a field, the analogy is precise: SE-$\mathbb{S}$ and SSE-$\mathbb{S}$ simply ignore the nilpotent part of the action. For more general rings, the information in these invariants can be considerably more subtle.

The initial motivation (and a continuing motivation) for considering these eventual invariants comes from symbolic dynamics. However, I argue that these are natural algebraic invariants to consider, independent of the symbolic dynamics interest.

The easiest case to argue is the case of $\det(I - tA)$ (the nonzero spectrum). For example, the Spectral Conjecture (Lecture I) and its partial verifications show the value of approaching (even just) the classical problem of understanding the possible spectra of nonnegative real matrices from the eventual viewpoint.

Before we turn to the other invariants, and the relation to nonnegative matrices, we look at some of the motivation from symbolic dynamics.

4. Motivation from symbolic dynamics: $\mathbb{S} = \mathbb{Z}$ and SFTs

The central example involves the ring $\mathbb{S} = \mathbb{Z}$. A square matrix A over $\mathbb{Z}_+$ defines a shift of finite type $\sigma_A : X_A \to X_A$, as follows.

Let $\mathcal{A}$ be the set of edges in the directed graph with adjacency matrix A. Let X_A be the set of doubly infinite sequences $x = ...x_{-1}x_0x_1\dots$ of symbols from $\mathcal{A}$ such that for all i, the terminal vertex of the edge x_i equals the initial vertex of the edge x_{i+1}. (So, x is the itinerary of a biinfinite walk through the directed graph.) $\mathcal{A}$ is given the discrete topology; $\prod \mathcal{A}^{\mathbb{Z}}$ is given the product topology; X_A is given the relative topology. Now X_A is a compact metrizable zero dimensional space. One metric compatible with the topology is given by setting $\mathrm{dist}(x, y) = 1/(n + 1)$ when $x \neq y$ and n is the largest nonnegative number such that $x_i = y_i$ whenever $|i| < n$.

Two shifts of finite type σ_A, σ_B are topologically conjugate (isomorphic) if there exists a homeomorphism $h : X_A \to X_B$ such that $\sigma_A h = h\sigma_B$.

Williams (1973) proved that the following are equivalent for square matrices A, B over $\mathbb{Z}_+$:

- The SFTs σ_A and σ_B are topologically conjugate.
- A and B are SSE over $\mathbb{Z}_+$.

However, after forty years the problem of understanding SSE over $\mathbb{Z}_+$ remains very open. We do not know (for example) for any primitive matrix A other than the trivial example (1) whether there exists an algorithm which given B decides whether A and B are SSE over $\mathbb{Z}_+$.

There is a more tractable invariant. If A and B are SSE over $\mathbb{Z}_+$, then in particular they are SSE over the ring $\mathbb{Z}$. As a start on SSE-$\mathbb{Z}_+$, we should understand SSE-$\mathbb{Z}$.

The invariant $\det(I - zA)$ is motivated here because its reciprocal is the Artin-Mazur zeta function of σ_A.

There are symbolic dynamical systems with a similar classification setup, over another ring. We'll look next at just one more example.

5. Motivation from symbolic dynamics: $\mathbb{S} = \mathbb{Z}G$ and G-extensions

Here the ring is $\mathbb{Z}G$, with G a finite group. A matrix A over $\mathbb{Z}_+G$ defines a (skew product) G-extension S_A of an SFT. For this connection, let A_1 be the matrix over $\mathbb{Z}_+$ which is the image of A under the map defined entrywise by the augmentation map $\sum n_g g \mapsto \sum n_g$. As in lecture III, we can view A as presenting a labeled graph, for which the underlying labeled graph has adjacency matrix A_1. The edge labeling gives a continuous map $\tau : X_{A_1} \to G$ defined simply by setting $\tau(x)$ to be the label of the edge x_0. This lets one define a homeomorphism $S : X_{A_1} \times G \to X_{A_1} \times G$ by the rule $(x, g) \mapsto (\sigma_{A_1} x, \tau(x)g)$. There is then a free G action on $X \times G$, defined by $h : (x, g) \mapsto (x, gh)$, which commutes with the map S (and with the projection $(x, g) \mapsto x$ which collapses the G-obrits). An isomorphisms of two such skew products is a topological conjugacy of the two maps S which also interwines the G-actions.

Bill Parry proved that given matrices A, B over $\mathbb{Z}_+G$, the G-extensions they define are isomorphic and only if A and B are SSE-$\mathbb{Z}_+G$. There is a detailed exposition of this in my Proc. LMS paper with Michael Sullivan. SSE-$\mathbb{Z}_+G$ is not easier to understand than the relation SSE-$\mathbb{Z}_+$ which we do not understand. But as with $\mathbb{Z}$, we can look to see what the easier (but not so easy!) algebraic invariant SSE-$\mathbb{Z}G$ can tell us with regard to understanding isomorphism of the skew products.

6. The meaning of SE-$\mathbb{S}$ for a ring S

Suppose A is $n \times n$ over $\mathbb{S}$.

PROPOSITION: If A is nilpotent, then A is SE over $\mathbb{S}$ to the 1×1 matrix (0). (This works for $\mathbb{S}$ just a semiring.)

PROOF. Suppose $A^\ell = 0$, with $\ell \in \mathbb{N}$. Let $B = (0)$. Let the matrix U be the $n \times 1$ column vector with every entry zero. Let V the transpose of U. Then

$$A^\ell = UV \quad B^\ell = VU \quad AU = UB \quad BV = VB$$

(each side of each equation is a zero matrix of appropriate size). QED

REMARK. This is a quick side remark contrasting SE and SSE. If A and B are ESSE over a field $\mathbb{F}$, and A has a nilpotent Jordan block of size k, then B must have a nilpotent Jordan block of size j with $|j - k| \leq 1$. In particular, if A is nilpotent over $\mathbb{F}$, with k the smallest positive integer such that $A^k = 0$ then a chain of ESSEs from A to the 1×1 matrix (0) must have length at least $k - 1$. (And in fact, working over a field $\mathbb{F}$, such a chain will exist).

Next consider the case that $\mathbb{S}$ is a field and A is not nilpotent. Then A is SIM over $\mathbb{S}$ (hence SE over $\mathbb{S}$) to a matrix with block form $\begin{pmatrix} X & 0 \\ 0 & N \end{pmatrix}$ in which N is nilpotent and X is nonsingular. An argument similar to the last shows that this matrix is SE over $\mathbb{S}$ to the nonsingular matrix X. Thus A and B are SE over $\mathbb{S}$ if and only if their "nonsingular parts"are similar over S.

We can express the last condition in another way. Let V_A be the eventual image of A: the intersection of the (nested) images of $\mathbb{S}^n$ under A^k, $k \in \mathbb{N}$ (which equals the image of $\mathbb{S}^n$ under multiplication by A^n). This is the maximum A-invariant subspace on which the action of A is invertible (V_A is not zero because we are considering A not nilpotent). A is SE over the field $\mathbb{S}$ to B iff the linear transformation A restricted to V_A is isomorphic to the linear transformation B restricted to V_B.

In the case $\mathbb{S} = \mathbb{Z}$, the classification up to SE-$\mathbb{Z}$ is finer than the classification up to SE-$\mathbb{R}$; SIM-$\mathbb{Z}$ refines SE-$\mathbb{Z}$ which refines SE-$\mathbb{R}$. A nice case constrasting SE-$\mathbb{Z}$ and SIM-$\mathbb{Z}$ is described in an appendix.

7. A Pause for Realization Questions

Recall the Spectral Conjecture of Lecture I, of Handelman and myself (recopied in an Appendix). The conjecture states, given a subring $\mathbb{S}$ of $\mathbb{R}$, that a list $\Lambda = (\lambda_1, \dots, \lambda_k)$ of nonzero complex numbers is the nonzero spectrum of a primitive matrix over $\mathbb{S}$ if and only if three necessary conditions hold (the Perron condition; the condition that the polynomial $\prod_i (t - \lambda_i)$ has all coefficients in $\mathbb{S}$; a more complicated condition involving nonnegativity of traces of powers). This Spectral Conjecture has been proved in various cases (e.g. $\mathbb{R}$ and especially $\mathbb{Z}$), and surely must be true. It is then a protypical example in which the constraints of a nonnegativity condition on "eventual algebra"are characterized. Let's consider the following natural analogues.

(Weak) GENERALIZED SPECTRAL CONJECTURE (B-Handelman). Suppose $\mathbb{S}$ is a subring of $\mathbb{R}$, and A is a square matrix over $\mathbb{S}$ whose nonzero spectrum satisfies the necessary conditions of the Spectral Conjecture. Then A is SE over $\mathbb{S}$ to a primitive matrix.

(Strong) GENERALIZED SPECTRAL CONJECTURE (B-Handelman). Suppose $\mathbb{S}$ is a subring of $\mathbb{R}$, and A is a square matrix over $\mathbb{S}$ whose nonzero spectrum satisfies the necessary conditions of the Spectral Conjecture. Then A is SSE over $\mathbb{S}$ to a primitive matrix.

For the case $\mathbb{S} = \mathbb{R}$, each of these is a conjecture that the nonnilpotent part of the Jordan form of a primitive real matrix can be anything compatible with the spectral constraints. This is already a natural problem at the level of linear algebra.

8. A general description of SE-$\mathbb{S}$ for rings.

Suppose $\mathbb{S}$ is a ring. We'll give a "conceptual" description of SE-$\mathbb{S}$, which will also be useful later. Suppose A is $n \times n$ over $\mathbb{S}$.

- Form the direct limit group, which I'll denote $\mathcal{M}_A$:

$$\mathbb{S}^n \longrightarrow \mathbb{S}^n \longrightarrow \mathbb{S}^n \longrightarrow \cdots$$

  where an arrow represents the map $v \mapsto vA$. (You could systematically switch the roles of row and column vector if you prefer.)

  Here are definitions for this step (one must check welldefinedness). Formally, an element of M_A is a quotient of $\mathbb{S}^n \times \mathbb{N}$ by the relation $(u,k) \sim (v,\ell)$ if there exists $n \in \mathbb{N}$ such that $uA^{\ell+n} = vA^{k+n}$. Addition of quotient classes is induced by e.g. $[(u,k)] + [(v,\ell)] = [(uA^\ell + vA^k, k+\ell]$. The rule $[(u,k)] \mapsto [uA,k]$ defines a group isomorphism $\hat{A} : \mathcal{M}_A \to \mathcal{M}_A$, with $\hat{A}^{-1} : [(u,k)] \mapsto [(u,k+1)]$.
- $\mathcal{M}_A$ is an $\mathbb{S}$-module (where s from $\mathbb{S}$ acts by $[(u,k)] \mapsto (su,k)]$. (If S is not commutative, then S and A must act from opposite sides. Then, if we want st to act as the action of s followed by the action of t, we would use column vectors for $\mathcal{S}^n$.) The maps $\hat{A}$ and $\hat{A}^{-1}$ are $\mathbb{S}$-module isomorphisms.
- Finally, we regard $\mathcal{M}_A$ an $S[t]$-module, extending the action of $\mathbb{S}$ by letting t act by $\hat{A}^{-1}$. (We'll see later the reason for this choice, instead of the choice that would let t act by $\hat{A}$.)

PROPOSITION (Wagoner) A and B are SE-$\mathbb{S}$ if and only if the $\mathbb{S}[t]$-modules $\mathcal{M}_A$ and $\mathcal{M}_B$ are isomorphic.

PROOF. Suppose U, V and ℓ give an SE over $\mathbb{S}$ between A and B. An isomorphism of $\mathbb{S}[t]$ modules from $\mathcal{M}_A$ to $\mathcal{M}_B$ is given by the rule $[(x,m)] \mapsto [xU, m+\ell]$; its inverse is the map $[(y,n)] \mapsto [(yV,n)]$. The converse I leave as an exercise. QED

If $\mathbb{S}$ is a subring of a field $\mathbb{F}$, and A is $n \times n$, then there is a very concrete version ($\widetilde{\mathcal{M}}_A$, say) of the module $\mathcal{M}_A$. Define as before V_A to be the $\mathbb{F}$-vector space which is the eventual image of $\mathbb{F}^n$ under A. As a set,

$$\widetilde{\mathcal{M}}_A = \{x \in V_A \cap \mathbb{F}^n : xA^k \in \mathbb{S}^n \text{ for some } k \in \mathbb{N}\}.$$

As a group and $\mathbb{S}$-module, the operations of $\widetilde{\mathcal{M}}_A$ are just the restrictions of the operations of $\mathbb{S}^n$. The action of $\hat{A}$ on $\widetilde{\mathcal{M}}_A$ is simply multiplication by the matrix A.

For example, suppose $A = (2)$ and $\mathbb{S}$ is $\mathbb{Z}$ and $\mathbb{F}$ is $\mathbb{Q}$. Then $\widetilde{\mathcal{M}}_A$ is $\mathbb{Z}[1/2]$, the dyadic rationals, and $\hat{A}$ is multiplication by 2.

9. The meaning of SSE-$\mathbb{S}$ for a ring $\mathbb{S}$

Let $\mathbb{S}$ be a ring. It is not hard to prove that SSE-$\mathbb{Z}$ is the smallest equivalence relation $\sim$ on square matrices over $\mathbb{S}$ such that

- If A and B are SIM over $\mathbb{S}$, then $A \sim B$.
- If a matrix X over $\mathbb{S}$ has the same number of rows as A, then $A \sim \begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix}$.
- If a matrix X over $\mathbb{S}$ has the same number of columns as A, then $A \sim \begin{pmatrix} A & 0 \\ X & 0 \end{pmatrix}$.

But what does this mean?

A natural starting point is the following question: for a given ring $\mathbb{S}$, does SE over $\mathbb{S}$ imply SSE over $\mathbb{S}$? Some answers:

- YES if $\mathbb{S} = \mathbb{Z}$ (70's Williams; published in 90s)
- YES if $\mathbb{S} =$ a principal ideal domain (80's Effros)
- YES if $\mathbb{S} =$ a Dedekind domain (or more generally, S is Prufer)

That was it, and we had no counterexamples. Very embarassing.

10. The refinement of SE-$\mathbb{S}$ by SSE-$\mathbb{S}$: an obstruction group from algebraic K theory

Below, let SSE_A denote the set of SSE-$\mathbb{S}$ classes of matrices which are SE over $\mathbb{Z}$ to A. The particular choice of A in a given SE-$\mathbb{S}$ class doesn't matter; this is just a convenient notation to describe the refinement of an SE-$\mathbb{S}$ class into SSE-$\mathbb{S}$ classes.

REFINEMENT THEOREM (B-Schmieding, in progress)
Suppose A is a square matrix over $\mathbb{S}$, B is SE over $\mathbb{S}$ to A and $\mathbb{S}$ is a ring.

(1) There exists a nilpotent matrix N over $\mathbb{S}$ such that $\begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$ is SSE over $\mathbb{S}$ to B.

(2) The map into $\mathrm{GL}(\mathbb{S}[t])$ defined by $N \mapsto \begin{pmatrix} I - tN & 0 \\ 0 & I \end{pmatrix}$ induces a bijection

$$\mathrm{SSE}_{\mathcal{A}} \to \mathrm{NK}_1(\mathbb{S})/H_A$$

where $\mathrm{NK}_1(\mathbb{S})$ is the kernel of the map $K_1(\mathbb{S}[t]) \to K_1(\mathbb{S})$ which is induced by the ring epimorphism $\mathbb{S}[t] \to \mathbb{S}$ induced by $t \mapsto 0$.

A number of definitions are in order to explain this statement.

For a ring R, $\mathrm{GL}(R)$ is the stabilized general linear group of R. An element of $\mathrm{GL}(R)$ is an $\mathbb{N} \times \mathbb{N}$ matrix U such that such that for some n the principal submatrix on indices $\{1, 2, \dots, n\}$ (an upper left corner of the matrix) is a element of $\mathrm{GL}(n, R)$, and entries of U for all other indices agree with the infinite identity matrix.

$\mathrm{El}(R)$ is the subgroup of $\mathrm{Gl}(R)$ generated by the basic elementary matrices. A basic elementary matrix is one of the form $E_{ij}(r)$, with $r \in R$ and $i \neq j$; this matrix by definition has entry r at position (i, j) and otherwise equals the identity. If $M \in \mathrm{GL}(R)$ and $E = E_{ij}(r)$, then EM is the matrix obtained by replacing row i of M with the sum of row i and r times row j.

$\mathrm{El}(R)$ is the commutator subgroup of $\mathrm{Gl}(R)$ (this is the Whitehead Lemma). Then $\mathrm{K}_1(R)$ is defined to be the abelian group $\mathrm{Gl}(R)/\mathrm{El}(R)$. These are groups of fundamental importance in algebraic K-theory (which can be thought of as the linear algebra theory for arbitrary rings). The group $\mathrm{NK}_1(R)$ is also important in algebraic K-theory. Finally, H_A is the subset of elements $[U]$ in $\mathrm{K}_1(\mathbb{S}[t])$ which contain some V for which there exists E in $\mathrm{El}(\mathbb{S}[t])$ such that $E(I - tA)V = I - tA$. H_A is a subgroup of $\mathrm{NK}_1(S)$, and does not depend on the choice of A in a SE-$\mathbb{S}$ class.

11. Some implications

The group H_A is trivial if A is nilpotent or if $\mathbb{S}$ is commutative. In these cases, the refinement of SE-$\mathbb{S}$ by SSE-$\mathbb{S}$ is is given by $\mathrm{NK}_1(\mathbb{S})$ (and therefore is the same for every SE-$\mathbb{S}$ class).

The group $\mathrm{NK}_1(S)$ is known to vanish if $\mathbb{S}$ is left regular Noetherian (the "regular" means that finitely projective modules have finite projective resolutions). This is a huge class and contains for example rings over polynomials or Laurent polynomials with coefficients in $\mathbb{Z}$ or a field. For such rings $\mathbb{S}$, SE-$\mathbb{S}$ implies SSE-$\mathbb{S}$.

For$G = \mathbb{Z}/n$, $\mathrm{NK}_1(\mathbb{Z}G) = 0$ if and only n is squarefree.

If $\mathrm{NK}_1(\mathbb{S})$ is not trivial, then it is not a finitely generated group. (Farrell 1977)

12. Some applications

(1) A working conjecture of Parry on the classification of G skew products for G a finite abelian group fails when $\mathrm{NK}_1(\mathbb{Z}G)$ is nontrivial.

(2) The Weak Generalized Spectral Conjecture implies the Strong Generalized Spectral Conjecture.
(This requires more argument.)

(3) For a subring $\mathbb{S}$ of $\mathbb{R}$, TFAE.
(i) $\mathrm{NK}_1(S) \neq 0$.
(ii) There are nilpotent matrices over $\mathbb{S}$ which are not SSE over $\mathbb{S}$ to nonnegative matrices.

Ite (3) above shows that a nonzero spectral condition does not always characterize whether a matrix over a ring $\mathbb{S}$ is SSE-$\mathbb{S}$ to a nonnegative matrix. So (even though the GSC only concerns primitive matrices), item (3) is evidence against the Generalized Spectral Conjecture. (There are subrings of $\mathbb{R}$ with $\mathrm{NK}_1(S) \neq 0$.) On the other hand, item (2) is evidence in favor. For now the Generalized Spectral

Conjecture is still standing.

(4) In the long paper "Path methods for strong shift equivalence of positive matrices", a 3-part program was given for understanding when positive matrices over a dense subring $\mathbb{S}$ of $\mathbb{R}$ are SSE over $\mathbb{S}_+$. One part was to understand SSE-$\mathbb{S}$.

(5) A key proof in that "Path methods ... " paper depended on an assumption of SSE-$\mathbb{S}$ (not just SE-$\mathbb{S}$). The Refinement Theorem shows that this assumption is necessary, not just an artifact of the proof. One plus from finding the deeper structure of SSE is to learn that some proof schemes cannot possibly work.

13. The little miracle of Kim-Roush-Wagoner

There is a little miracle discovered by Kim-Roush-Wagoner which leads to the strong connections to algebraic K-theory and more. The little miracle is that a certain kind of multiplication of by elementary matrices over $t\mathbb{Z}_+[t]$ produces a topological conjugacy of SFTs. Let us see how that works by an example.

Define

$$A = \begin{pmatrix} t & t^2 + t^3 \\ t^4 & t^5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} t & t^2 + t^3 \\ t^4 & t^5 \end{pmatrix} .$$

Here we can exhibit how B is obtained from A:

$$B \ = \ \begin{pmatrix} t & t^2 + t^3 \\ t^4 & t^5 \end{pmatrix} - \begin{pmatrix} 0 & t^3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} t^{3+4} & t^{3+5} \\ 0 & 0 \end{pmatrix} .$$

This can be described in terms of a multiplication by an elementary matrix over $\mathbb{Z}[t]$, $E(I-A) = I-B$:

$$\begin{pmatrix} 1 & t^3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-t & -t^2-t^3 \\ -t^4 & 1-t^5 \end{pmatrix} \ = \ \begin{pmatrix} 1-t-t^{3+4} & -t^2-t^{3+5} \\ -t^4 & 1-t^5 \end{pmatrix}$$

Recall from Lecture III how the matrices A and B give rise to directed graphs, let us say $\mathcal{G}_A$ and $\mathcal{G}_B$, with adjacency matrices $A^\sharp$ and $B^\sharp$. Let p_k denote the path in $\mathcal{G}_A$ corresponding to the monomial t^k in A. For example, p_3 is a path from vertex 1 to vertex 2. The passage from A to B can be described in terms of the formation of $\mathcal{G}_B$ from $\mathcal{G}_A$ by two steps:

(1) Take out of $\mathcal{G}_A$ the path p_3 corresponding to t^3.
(2) For each p_i in G_A which can follow p_3, put in a path matching the path p_3p_i in initial vertex (1, here), terminal vertex and length; the new path is a path whose interior vertices are isolated.

In our example, let us name these new paths q_{3+4} and q_{3+5}.

Now there is an obvious topological conjugacy h between σ_A and σ_B. A point x in X_A looks like a concatenation of the paths p_i. Define $h(x)$ as follows:

(1) First replace the concatenations p_3p_4 and p_3p_5 occuring in x with q_{3+4} and q_{3+5} respectively.
(2) Leave remaining segments unchanged.

This rule defines h as a shift-commuting homeomorphism.

14. Positive K-theory

The little miracle of Kim-Roush-Wagoner gives rise to the following approach to classification of SFTs (and other symbolic dynamical systems). In the following theorem, matrices are $\mathbb{N} \times \mathbb{N}$. A finite matrix is simply embedded into an infinite matrix which is otherwise zero. I is the infinite identity matrix. An elementary positive equivalence is an equivalence $E(I - C) = I - D$ or $(I - C)E = I - D$ such that C and D have entries in $t\mathbb{Z}_+[t]$ and have all but finitely many entries zero, and E is a basic elementary matrix $E_{ij}(\pm t^k)$, $k \geq 0$.

THEOREM (B-Wagoner) For A, B over $t\mathbb{Z}_+[t]$, the following are equivalent.

(1) σ_A and σ_B are topologically conjugate.
(2) There is a sequence of elementary positive equivalences taking $I - A$ to $I - B$.

Remarks (see B-Wagoner):
(1) The theorem statement is very slightly incorrect, to avoid a technical issue. To be correct, the matrices C, D are allowed to be slightly more general (having entries in $\mathbb{Z}_+[t]$ and with "no zero cycles"), and one has to make sense of how an SFT is defined from the more general matrices.
(2) This scheme generalizes to similarly translate the SSE-$\mathbb{Z}_+G$ classification for other groups G to the infinite polynomial matrix setting.

15. The central result

Here is the central result for connecting the SSE and K-theory. We are identifying a finite matrix A and the $\mathbb{N} \times N$ matrix which has A as upper left corner and is otherwise zero.

THEOREM (B-Schmieding) For matrices A, B over a ring $\mathbb{S}$, the following are equivalent.

(1) A and B are SSE over $\mathbb{S}$.
(2) There are matrice U, V in $\mathrm{El}(\mathbb{S}[t])$ such that $U(I - tA)V = I - tB$.

Strong influences on finding this theorem were the papers B-Sullivan and B-Wagoner.

Note, in the last theorem we are consider $\mathbb{S}$ a ring, and there is not problem with composing elementary positive equivalences. But for example with $= \mathbb{Z}_+$, if we consider have a matrix E which is a composition of basic elementary matrices, we cannot conclude that E is a composition of basic elementary matrices used in a chain of elementary positive equivalences.

16. Appendix: A case of SE-$\mathbb{Z}$ vs. SIM-$\mathbb{Z}$

The material below comes from the AMS Memoir "Resolving Maps and the Dimension Group" by B. Marcus, P. Trow and myself.

Let p be a monic polynomial with coefficients in $\mathbb{Z}$. Also suppose p is irreducible and λ is a nonzero root of p in $\mathbb{C}$. For a matrix A over $\mathbb{Z}$ with characteristic polynomial p, let r be a right eigenvector of A with its entries in $\mathbb{Z}[\lambda]$. (To find r, solve for x in $(A - \lambda I)x = 0$ over the field $\mathbb{Q}(\lambda)$. Then multiply x by a suitable number (to clear denominators in the entries) and let the result be r. Let I_A be the $\mathbb{Z}[\lambda]$ ideal generated by the entries of r. This ideal depends on the choice of r, but its ideal class in $\mathbb{Z}[\lambda]$, denoted here as $(I_A)_{\mathbb{Z}[\lambda]}$, does not.

It is a classical result of Olga Taussky Todd that these matrices A, B over $\mathbb{Z}$ with characteristic polynomial p are SIM over $\mathbb{Z}$ if and only if $(I_A)_{\mathbb{Z}[\lambda]} = (I_B)_{\mathbb{Z}[\lambda]}$. The analogous fact is that A and B are SE-$\mathbb{Z}$ if and only if $(I_A)_{\mathbb{Z}[1/\lambda]} = (I_B)_{1/\mathbb{Z}[\lambda]}$.

Any matrix over a principal ideal domain $\mathbb{S}$ (such as a field, or $\mathbb{Z}$) is SE-$\mathbb{S}$ to a nonsingular matrix. (For a Dedekind domain, this is no longer true in general.) So, with p and λ as above, the same classification up to SE-$\mathbb{Z}$ of integer matrices with characteristic polynomial of the form $t^k p(t)$, $k \in \mathbb{Z}_+$, is still given by a correspondence to the ideal classes of $\mathbb{Z}[1/\lambda]$.

The class number (number of ideal classes) of the ring $\mathbb{Z}[\lambda]$ is finite ($\mathbb{Z}[\lambda]$ has finite index in the ring of algebraic integers of $\mathbb{Q}[\lambda]$).

The ring $\mathbb{Z}[1/\lambda]$ contains the ring $\mathbb{Z}[\lambda]$ (because λ is an algebraic integer). The class number of $\mathbb{Z}[\lambda]$ is greater than or equal to that of $\mathbb{Z}[1/\lambda]$, and can be greater. This corresponds to SE-$\mathbb{Z}$ being a coarser relation than SIM-$\mathbb{Z}$.

For an example of a similar flavor involving 2×2 matrices with eigenvalues in $\mathbb{Z}$, see the paper "Algebraic shift equivalence ... " of David Handelman and myself.

The algebra of SE-$\mathbb{Z}$ gets much more complicated in general. However, Kim and Roush proved there is a decision procedure to determine whether two matrices are SE over $\mathbb{Z}$.

17. Appendix: the Spectral Conjecture for primitive matrices

Spectral Conjecture (Boyle-Handelman, Annals of Math. 1991)
Let $\Lambda = (\lambda_1, \dots, \lambda_k)$ be a list of nonzero complex numbers. Let $\mathcal{S}$ be a unital subring of $\mathbb{R}$. Then the following are equivalent.

(1) There exists primitive matrix A of size n whose characteristic polynomial is $t^{n-k}\prod_{i=1}^k (t-\lambda_i)$ (i.e., Λ is the nonzero spectrum of A).
(2) The list Λ satisfies the following conditions:
  (a) (Perron Condition)
  There exists a unique index i such that λ_i is a positive real number and $\lambda_i > |\lambda_j|$ whenever $j \neq i$.
  (b) (Coefficents Condition)
  The polynomial $\prod_{i=1}^k (t-\lambda_i)$ has coefficients in $\mathcal{S}$.
  (c) (Trace Conditions)
    (i) (In the case $\mathcal{S} \neq \mathbb{Z}$.)
    (Let $\mathrm{tr}(\Lambda^n)$ denote $\sum_{i=1}^k (\lambda_i)^n$.)
    For all positive integers n, k the following hold:
      (A) For all n, $\mathrm{tr}(\Lambda^n) \geq 0$.
      (B) If $\mathrm{tr}(\Lambda^n) > 0$, then $\mathrm{tr}(\Lambda^{nk}) > 0$.
    (ii) (In the case $\mathcal{S} = \mathbb{Z}$.)
    (Let $\mathrm{tr}_n(\Lambda)$ denote $\sum_{k|n} \mu(n/k)\mathrm{tr}(\Lambda^n)$.)
    For all positive integers n, $\mathrm{tr}_n(\Lambda) \geq 0$

The three conditions are necessary conditions for existence of the primitive matrix with nonzero spectrum Λ; this is explained below. Also, if a nonzero spectrum can be realized at matrix size $n \times n$, then it can be realized at all larger sizes. So the inverse spectral problem for primitive matrices given the Spectral Conjecture reduces to finding the minimum dimension allowing a given nonzero spectrum.

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