III. Playing in three acts:

1. The Artin-Mazur zeta function, for unlabeled graphs and their $\mathbb{Z}_+$ adjacency matrices

2. The same formalism, for G-labeled graphs and their $\mathbb{Z}_+G$ adjacency matrices

3. Polynomial matrices presenting nonpolynomial matrices

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Act 1. The Artin-Mazur zeta function, for unlabeled graphs and their $\mathbb{Z}_+$ adjacency matrices.

This is a warmup act for the main event:

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!!

who will tell you a lot about a variety of zeta functions.

I will do more slowly some elementary formal aspects of the Artin-Mazur zeta function, and appy it to just one case, in which we use the function to encode the number of length n loops in a directed graph (with their weights, if the graph is a labeled graph).

This kind of "zeta function for a directed graph" will also be a step to connecting to more refined "away from zero" algebra for a matrix (more refined than the nonzero spectrum – eventiually, much more refined). That is one reason why I will be emphasizing this zeta function as a formal power series regardless of convergence. For a matrix over a more exotic ring it is good to begin with something that makes sense independent of ideas of convergence.

But, if you do not share this fetish for algebra, and prefer honest functions of a complex variable, worry not. Help is on the way.

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Some formal power series nonsense

For a formal power series with zero constant term,

$$\alpha(z) = \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots ,$$

the formal power series

$$\exp(\alpha(z)) = \sum_{n=0}^{\infty} \frac{(\alpha(z))^n}{n!} z^n$$

is well defined, because for every k its coefficient for z^k is the coefficient of z^k in the polynomial p determined by truncating the two series $\alpha(z)$ and $\exp(z)$ to degree k:

$$p(z) = \sum_{n=0}^{k} \frac{(\alpha_1 z + \cdots + \alpha_k z^k)^n}{n!} z^n \ .$$

Likewise, if we write $\exp(\alpha(z))$ as $1 + w$, where $w = w(z)$, then using the familar series for $\log(1 + w)$ we have

$$\log(\exp(\alpha(z)) = w - \frac{w^2}{2} + \frac{w^3}{3} - \cdots = \alpha(z) \ .$$

This familiar equation is true for any polynomial w within some radius of convergence. But this implies the combinatorial identities of power series coeffiecents which gives equality for formal power series independent of convergence issues. (Or, if you prefer, you can find and prove those identities directly.)

Consequently we can think of the series exp($\alpha(z)$) as a way of encoding the series $\alpha(z)$ (each series determines the other). The combinatorial identities which are the equations of formal power series then show that this encoding works equally well for series α with coefficients in any commutative ring containing $\mathbb{Q}$, as long as α has zero constant term.

Why might one be interested in this encoding?

The Artin-Mazur zeta function

Suppose T is a map $T : X \to X$. For $n \in \mathbb{N}$, T^n is defined by $T^1 = T$, and recursively $T^n = T \circ T^{n-1}$ if $n > 1$. Suppose for every n that T^n has only a finite number of fixed points. The Artin-Mazur zeta function of T is

$$\zeta_T(z) \ := \ \exp \sum_{n=1}^{\infty} \frac{|\text{Fix}(T^n)|}{n} z^n \ .$$

From what we've said before, this series encodes the whole sequence $(|\text{Fix}(T^n)|)/n$ and therefore the whole sequence $(|\text{Fix}(T^n)|)$ (and without regard to issues of convergence).

At first glance, this encoding might seem an unnecessary complication. Let's prepare to see some rationale.

The zeta function of a fixed point

Suppose $T : X \to X$ is the system consisting of a single fixed point. Then

$$\zeta_T(z) \ := \ \exp \sum_{n=1}^{\infty} \frac{z^n}{n}$$

and

$$\exp \sum_{n=1}^{\infty} \frac{z^n}{n} = \frac{1}{1-z} \ .$$

To check the last equation, take log to get

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z)$$

then differentiate to get

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \ .$$

These steps reverse.

The zeta function of a finite orbit

Next we compute the zeta function when T is a cyclic permutation of k points.
Here $|\mathrm{Fix}(T^n)| = 0$ if k does not divide n, and $|\mathrm{Fix}(T^{kn})| = k$. So,

$$\zeta_T(z) \ := \ \exp \sum_{n=1}^{\infty} \frac{k}{kn} z^{kn}$$

$$= \exp \sum_{n=1}^{\infty} \frac{(z^k)^n}{n}$$

$$= \frac{1}{1 - z^k} \ .$$

The product formula

Suppose $T : X \to X$ with $|\text{Fix}(T^n)| < \infty$ for all n.
The fixed points of powers of T are the periodic points of T. Every periodic point of T has a least period, which is the cardinality of its orbit, $\{T^n(x) : n \in \mathbb{N}\}$.

Let $\pi(k)$ denote the number of T-orbits of cardinality k. A point is a fixed point of T^n if and only if it lies in an orbit of size k such that k divides n. So,

$$|\text{Fix}(T^n)| = \sum_{k:\ k|n} k\pi(k) \ .$$

Conversely, the sequence $(|\text{Fix}(T^n)|)$ determines the sequence $(\pi(n))$, by the Mobius inversion formula:

$$\pi(n) = \sum_{k:k|n} \mu\Big(\frac{n}{k}\Big)|\text{Fix}(T^k)| \ .$$

From the dynamical point of view, it is the sequence $(\pi(n))$ which is more fundamental. But typically it is the sequence $(|\mathsf{Fix}(T^n)|)$ which has a much more pleasant, tractable and even algebraic formulation.

The product formula for the Artin-Mazur zeta function is one facet of the way this zeta function relates the dynamical sequence $(\pi(n))$ and the sequence $(|\mathsf{Fix}(T^n)|)$. Here is the product formula:

$$\exp \sum_{n=1}^{\infty} \frac{|\mathsf{Fix}(T^n)|}{n} z^n = \prod_n \frac{1}{(1-z^n)^{\pi(n)}}$$

Why does the product formula hold? We know it holds if T is $T_{\mathcal{O}}$, the restriction of T to a finite orbit $\mathcal{O}$. If T has just finitely many orbits, then the formula holds because $e^a e^b = e^{a+b}$. Then it holds in general, since for every n the coefficient of z^n on both sides above depends only depends on finitely many orbits (those of size at most n).

The same logic shows that if $S : X \to X$ and $T : Y \to Y$ with X and Y disjoint, and if $R : X \cup Y \to X \cup Y$ is the union of these functions, then

$$\zeta_R(z) = \zeta_S(z)\,\zeta_T(z)\ .$$

Next we prepare to consider an example for which the zeta function can be nicely computed.

Adjacency matrix of a graph.

Unless specified otherwise, by "graph" we always mean finite "directed graph".
$\mathbb{N}$ is the set of positive integers.
$\mathbb{Z}_+$ is the set of nonnegative integers.

Suppose $\mathcal{G}$ is an unlabeled graph, with vertex set $\{1, \ldots, n\}$. Then the adjacency matrix of $\mathcal{G}$ is the $n \times n$ matrix A with entries defined by setting $A(i, j)$ to be the number of edges from vertex i to vertex j.

Choosing another naming of vertices in $\mathcal{G}$ by $\{1, \ldots, n\}$ produces a matrix B such that there is a permutation matrix P such that $B = PAP^{-1}$. The distinction between A and B won't matter to us, so we usually ignore it.

Counting paths in graphs.

Let $\mathcal{G}_A$ be the unlabeled graph with adjacency matrix A.

CLAIM: for any vertices i, j and $n \in \mathbb{N}$, the number of paths of length n from i to j is $A^n(i, j)$.

PROOF BY INDUCTION:
For $n = 1$, trivially true.
For $n > 1$: an n-path from i to j is an $(n-1)$ path from i to some k followed by edge from k to j. Using the induction hypotheses, then, the number of n-paths from i to j is

$$\sum_k A^{n-1}(i, k) A(k, j)$$

which is simply $A^n(i, j)$.

A path in $\mathcal{G}_A$ is a loop if it begins and ends at the same vertex. So, the number of loops of length n is the sum over vertices i of $A^n(i,i)$; i.e., it is $\mathrm{tr}(A^n)$.

A loop is *minimal* if it is not a concatenation of a smaller loop. (A minimal loop is allowed to be a concatenation ab of distinct smaller loops.)

The zeta function of a directed graph

There is an important class of dynamical systems, shifts of finite type, which are presented by unlabeled graphs (or equivalently, by their adjacency matrices).

Let $\mathcal{G}_A$ be the unlabeled graph with adjacency matrix A. Let T be the corresponding shift of finite type. All we need to know at this moment is that the number of fixed points of T^n is equal to the number of loops of length n in $\mathcal{G}_A$. But for more ...

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Meanwhile, we want to compute the Artin-Mazur zeta function ζ_T for the shift of finite type T associated to A.

You can also thing of this ζ_T as simply a zeta function associated to the undirected graph $\mathcal{G}_A$, since $|\text{Fix}(T^n)|$ is simply the number of loops of length n in $\mathcal{G}_A$ (which is tr(A^n)) .
(Caveat: this zeta function is NOT the Ihara zeta function of an undirected graph.)

We compute:

$$\zeta_T(z) = \exp \sum_{n=1}^{\infty} \frac{tr(A^n)}{n} z^n$$

$$= \exp \sum_{n=1}^{\infty} \frac{\sum_\lambda \lambda^n}{n} z^n$$

$$= \prod_\lambda \exp \sum_{n=1}^{\infty} \frac{\lambda^n}{n} z^n = \prod_\lambda \exp \sum_{n=1}^{\infty} \frac{(\lambda z)^n}{n}$$

$$= \prod_\lambda (1-\lambda z)^{-1} = \left[\prod_\lambda (1-\lambda z)\right]^{-1} \ .$$

Suppose A is $N \times N$. Then the numbers λ above are the N roots of χ_A (some might be equal). Then

$$\left[\prod_\lambda (1-\lambda z)\right] = z^N \left[\prod_\lambda (z^{-1}-\lambda)\right]$$

$$= z^N \chi_A(z^{-1}) \ .$$

This is the characteristic polynomial of A (χ_A) "written backwards". If

$$\chi_A = z^N + a_{N-1} z^{N-1} + \cdots + a_j z^j$$

with $a_j \neq 0$, then $z^N \chi_A(z^{-1})$ is

$$z^N (z^{-N} + a_{N-1} z^{-(N-1)} + \cdots + a_j z^{-j})$$

$$= 1 + a_{N-1} z + \cdots + a_j z^{N-j} \ .$$

The roots of $\det(I - zA)$ are precisely the reciprocals of the nonzero roots of $\chi_A(z)$, with appropriate multiplicity.

For example, if $A = (2)$, then $\chi_A(z) = z - 2$ and $\det(I - zA) = 1 - 2z$.

If

$$\chi_A(z) \;=\; z^6 - 2z^5 \;=\; (z-2)\,z^5$$

then $\det(I - zA) = 1 - 2z$.

Summary: each of the following determines the others:
$\det(I - zA)$;
the sequence $(tr(A^n))$;
the sequence $\pi(n)$;
the zeta function of the graph.

Act 2. 2. The same formalism, for G-labeled graphs and their $\mathbb{Z}_+G$ adjacency matrices.

Now we will consider labeled graphs (graph means directed graph).
We allow multiple edges between vertices.
For a labeled graph G, I will let G_1 denote its unlabeled graph. A_1 will denote the adjacency matrix of the unlabeled graph, and A matrix A will present a labeled graph G_A. The adjacency matrix of the supporting unlabeled graph will then be denoted A_1.

The weight of a path of n edges will be the product of the labels along its edges.

EXAMPLE. $A = (1)$ and $A_1 = (1)$.
The is one vertex, with one self loop labeled 1.
For each n, there is one loop of length n. Its

weight is 1.

EXAMPLE. $B = (\frac{1}{2} + \frac{1}{2})$ and $B_1 = (2)$.
There is one vertex, with two self loops, each labeled $\frac{1}{2}$.
There are 2^n loops of length n. Each has weight $(1/2^n)$. The total weight of the length n loops is 1 (as for A above).

Sometimes we only care about the total weights of n-loops (as in lecture II). Then we may as well use the most simple graph, with A_1 a zero one-matrix, and we needn't mention A_1.

But sometimes it is critical to know also the number of n-loops and the distribution of weights among them. Then we need something more.

There is a simple algebraic device which handles this perfectly. Note, we are assuming we have a way to define weights of paths by multiplying labels. So at the very least, we need the labels to lie in some semigroup. When the semigroup is a multiplicately closed subset of a ring, as in the first two examples, then it's natural to refer to "multiplying" labels for the operation.

Now assume G is a group. Keep in mind the example that G is the group of positive real numbers under multiplication). We will define the integral semigroup ring $\mathbb{Z}G$.

As an additive group, $\mathbb{Z}G$ will be the free abelian group on generator set $\{[g] : g \in G\}$. An element of this group is a formal integral combination

$$\sum_{g \in G} n_g[g]$$

with only finitely many of the integers n_g nonzero. Multiplication is

$$(\sum_g n_g[g])(\sum_h m_h[h]) = \sum_g \sum_h (n_g m_h)[gh] \ .$$

If e is the identity of G, then $[e]$ is the multiplicative identity of $\mathbb{Z}G$, also denoted 1.

With this in mind, for G the positive reals under multiplication we would write the matrix example B more precisely as

$$B = \left([\tfrac{1}{2}] + [\tfrac{1}{2}] \right) \ .$$

The matrix B_1 is the image of B under the map given entrywise by the "augmentation map"

$$\alpha_1 : \mathbb{Z}G \to \mathbb{Z}$$

$$\alpha_1 : \sum n_g[g] \mapsto \sum n_g \ .$$

Now consider the case that G is contained in a ring R and the group operation of G is the ring multiplication. (This fits our examples, with G is the group of positive reals under multiplication.)

Then we have another matrix B_R, the image of B under a map defined entrywise by the ring homomorphism $ZG \to R$ given by the rule

$$\sum n_g[g] = \sum n_g g \ .$$

In general, for a matrix A over $\mathbb{Z}G$:

- $(B_1)^n(i,j)$ is the number of n-paths from i to j
- $(B_R)^n(i,j)$ is the total of the weights of n-paths from i to j
- $(B)^n(i,j)$ tells you the number of n-paths from i to j and the distribution of their weights.

The zeta function for a labeled graph.

From here we assume that the group G labeling a graph is abelian. Then $R = \mathbb{Z}G$ is commutative.

Let A be the adjacency matrix of a G-labeled graph. So, A has entries in $\mathbb{Z}G$. Then

$$\frac{1}{\det(I - zA)} = \exp \sum_{n=1}^{\infty} \frac{tr(A^n)}{n} z^n \ .$$

So, the polynomial $\det(I - zA)$ – the information in the finite list of its coefficients – is encoding the whole sequence $(tr(A^n))$, which determines for every n the number of n-loops and the distribution of their G-weights. The proof is identical to the proof for a matrix A over the integers. We are just working with a different commutative ring.

EXAMPLE. $A = \left([\frac{1}{3}] + [\frac{2}{3}]\right)$.
Then $A_1 = (2)$ and

$$tr(A^n) = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{2^k}{3^n}\right] .$$

Why require G to be abelian? That is the condition that makes the integral group ring $\mathbb{Z}G$ commutative. To use determinant, we need the matrices have entries in a commutative ring.

But, it is even worse than that. We used formal power series identities in developing the zeta function. The underlying combinatorial identities can involve rearrangements of orders of multiplication, and can fail for a noncommutative ring. For A over a noncommutative ring, understanding the sequence $(tr(A^n))$ is a much tougher business.

For example, we relied (rather heavily!) on the identity $e^a e^b = e^{a+b}$. Now, $a + b = b + a$ in any ring, so if e^{a+b} makes sense, then it equals e^{b+a}. But if $ab \neq ba$, then there is no reason for us to expect $e^a e^b = e^b e^a$.

Act 3. Polynomial matrices presenting non-polynomial matrices.

We can see by example how a square matrix A with entries from $t\mathbb{Z}_+[t]$ can be used to define a directed (and unlabeled) graph G_A. (I use the variable t to suggest "time".)

EXAMPLE. $A = \begin{pmatrix} t & t^2 \\ 2t + t^3 & 0 \end{pmatrix}$.
(Draw G_A.)

The graph G_A has an adjacency matrix, which we denote $A^\sharp$. There are some choices for assigning integer names to the new vertices in G_A. As usual, the choices don't matter and we ignore them.

You can see immediately that we can present infinitely many graphs with even just 1×1 matrices over $t\mathbb{Z}_+[t]$. A couple of theorems (there are others) will indicate the richness of possibilities.

THEOREM (Handelman)
The following conditions on a positive real number λ are equivalent.
(1) There exists a 1×1 matrix A over $t\mathbb{Z}_+[t]$ such that $A^\sharp$ is primitive and has spectral radius λ.
(2) λ is a Perron number and there is no other root of the minimal polynomial of λ which is a positive real number.

THEOREM (Perrin)
The following conditions on a positive real number λ are equivalent.
(1) λ is a Perron number.
(2) There exists a 2×2 matrix A over $t\mathbb{Z}_+[t]$ such that $A^\sharp$ has spectral radius λ.

(B-Lind) Moreover, in (2) the matrix $A^\sharp$ can be chosen to be primitive.

We will begin to see now and in Lecture IV that the polynomial matrices provide much more than a concise notation.
Here is the first indication.

Let A be square with entries in $t\mathbb{Z}_+[t]$. Let $A^\sharp$ be as above.

THEOREM [BGMY]
$\det(I - tA^\sharp) = \det(I - A)$.

For a proof: Manyana, manyana.

An example:

$$A = \begin{pmatrix} t^3 \end{pmatrix} \qquad A^\sharp = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $A^\sharp$ is $t^3 - 1$. Then

$$\det(I - tA^\sharp) = 1 - t^3 = \det(I - A) \ .$$

Another example:

$$A = \begin{pmatrix} t & t+t^2 \\ t & 0 \end{pmatrix} \qquad A^\sharp = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ .$$

You can check $\det(I - tA^\sharp) = \det(I - A)$.

Other rings.

Suppose now A is a matrix over $t\mathbb{Z}_+G[t]$, with G and abelian group. Just as we can associate to a matrix A over $t\mathbb{Z}_+[t]$ an unlabeled graph, with adjacency matrix $A^\sharp$ over $\mathbb{Z}_+$, we can associate to a matrix A over $t\mathbb{Z}_+G[t]$ a labeled graph, with adjacency matrix $A^\sharp$ over $\mathbb{Z}_+G$. And then

THEOREM
$\det(I - tA^\sharp) = \det(I - A)$.

How to define the labeled graph from A?
Each entry $A(i,j)$ of A is a sum of terms $n_g[g]t^k$, which we regard as a sum of monomials $[g]t^k$ (possibly some of these are repeated). A term $[g]t^k$ will give rise to an isolated path of length k from i to j (as in the unlabelled case, where there were no coefficient $[g]$).
Now label the first edge on this path by $[g]$

and the other edges by 1 (which is $[e]$). The weight of this path is now $[g]$.

The proof the Theorem for $\mathbb{Z}G$ will be as easy as for $\mathbb{Z}$.

Loop graphs

Recall we used a polynomial or power series with coefficients in $\mathbb{Z}_+$,

$$f(z) = (f_1)z + (f_2)z^2 + \cdots$$

to define a *loop graph* G_f, with some adjacency matrix A. When f is a polynomial, this is the construction of $A^\sharp$ from A in the special case that A is 1×1.

But! When f is a polynomial, by [BGMY],

$$\frac{1}{1 - f(z)} = \exp \sum_{n=1}^{\infty} \frac{tr((A^\sharp)^n)}{n} z^n \ .$$

This is the zeta function of the graph $G_{A^\sharp}$, or the Artin-Mazur zeta function of the associated shift of finite type, as you prefer.

The same statement holds if $f(z)$ is a power series, with infinitely many terms, and the graph is infinite, with $A^\sharp$ a matrix which is $\mathbb{N} \times \mathbb{N}$.

This follows immediately from the polynomial case, because the coefficient of z^n in the power series on both sides does not change if we replace f with $\sum_{k=1}^{n} f_k z^k$.

It is also interesting to note in this special case that zeta function equals that series $t(z)$ which plays such a fundamental role in the theory of infinite nonnegative matrices.

In conclusion :

Thank you for your attention.

I hope this has been of interest to some of you.

But if this approach to zeta functions hasn't rocked your world, you can guess why you still have reason to hope ...

Pollicott is coming !