Mahler measure of K3 surfaces (Lecture 4)

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Introduction

Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1\}.$$

Remarks

- $n = 1$
  By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1).$$

So it is related to Lehmer's question (1933)
Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$1 < M(P) < M(P_0) = 1.1762\cdots ?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.
A partial answer by Smyth (1971)

$$M(P) \geq 1.32\cdots$$

if P is non reciprocal.

The story can be explained with polynomials

$$x_0 + x_1 + x_2 + \cdots + x_n.$$

- $m(x_0 + x_1) = 0$ (by Jensen's formula)
- $$m(x_0 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \text{ Smyth (1980)}$$
- $$m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2}\zeta(3) \text{ Smyth (1980)}$$

  These are the first explicit Mahler measures.

- $$m(x_0+x_1+x_2+x_3+x_4) \stackrel{?}{=} \frac{675\sqrt{15}}{16\pi^3} L(f,4)$$ conjectured by Villegas (2004)

  f cusp form of weight 3 and conductor 15
  $L(f,s)$ is also the L-series of the K3 surface defined by

$$x_0+x_1+x_2+x_3+x_4=0$$
$$\frac{1}{x_0}+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=0$$

How such a conjecture possible?

Because of deep insights of two people.

- Deninger (1996) who conjectured

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)$$

  E elliptic curve of conductor 15 defined by the polynomial
  This conjecture was proved recently (May 2011) by Rogers and Zudilin thanks to a previous result due to Lalin. Here the polynomial is reciprocal.
  A new proof is just posted on the arXiv (April 2013) by Zudilin.

- Maillot (2003) using a result of Darboux (1875): the Mahler measure of P which is the integration of a differential form on a variety, when P is non reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety.

Examples

- $n = 2$ The smaller variety is defined by

$$\begin{array}{c} x_0 + x_1 + x_2 = 0 \\ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} = 0 \end{array} \Leftrightarrow x_1^2 + x_2^2 + x_1 x_2 = 0$$

  It is a curve of genus 0. So $m(x_0 + x_1 + x_2)$ is expressed as a Dirichlet L-series.

- $n = 3$ The smaller variety is defined by

$$\begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0 \end{array} \Leftrightarrow (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = 0$$

  It is the intersection of 3 planes. Thus Smyth's result.

- $n = 4$ (Villegas's Conjecture) The smaller variety is defined by

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0$$
$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0$$

  It is the modular $K3$-surface studied by Peters, Top, van der Vlugt defined by a reciprocal polynomial. Its L-series is related to f.

- $n = 5$ (Villegas's Conjecture again)

$$m(x_0 + x_1 + x_2 + x_3 + x_4 + x_5) = **L(g, 5)$$

  g cusp form of weight 4 and conductor 6 related to L-series of the Barth-Nieto quintic.

Barth-Nieto quintic

It the 3-fold compactification of the complete intersection of

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0$$
$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} = 0$$

It has been studied by Hulek, Spandaw, Van Geemen, Van Straten in 2001. They proved that the L-function of the quintic (i.e. of their third etale cohomology group) is modular, a fact predicted by a conjecture of Fontaine and Mazur.
The modular form is the newform of weight 4 for $\Gamma_0(6)$

$$f = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2$$

Motivation

Briefly, to guess the Mahler measure of a non reciprocal polynomial we need results on reciprocal ones.
In particuliar, it is very important to collect many examples of Mahler measures of $K3$-hypersurfaces.
Notice that Maillot's insight predicts only the type of formula expected.
Also Deninger's guess comes from Beilinson's Conjectures.

So replace E by a surface X which is also a Calabi-Yau variety, i.e. a $K3$-surface and try to answer the questions:
What are the analog of Deninger, Boyd, R-Villegas 's results and conjectures?
Which type of Eisenstein-Kronecker series corresponds to $L(X, 3)$?

Basic facts on K3-surfaces

Our results concern polynomials of the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining K3-surfaces Y_k. What's a K3-surface?
It is a smooth surface X satisfying

- $H^1(X, \mathcal{O}_X) = 0$ i.e. X simply connected
- $K_X = 0$ i.e. the canonical bundle is trivial i.e. there exists a unique, up to scalars, holomorphic 2-form ω on X.

- A double covering branched along a plane sextic for example defines a K3-surface X.
  In our case

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

Main properties

- $H_2(X, \mathbb{Z})$ is a free group of rank 22.

Main properties (continued)

- With the intersection pairing, $H_2(X,\mathbb{Z})$ is a lattice and

$$H_2(X,\mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

  $\mathcal{L}$ is the $K3$-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unimodular lattice of rank 8.

- $$Pic(X) \subset H_2(X,\mathbb{Z}) \simeq Hom(H^2(X,\mathbb{Z}),\mathbb{Z})$$

  where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles (since for $K3$ surfaces linear and algebraic equivalence are the same).

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$$\rho(X) := \text{Picard number of } X$$

$$1 \leq \rho(X) \leq 20$$

- $$T(X) := (Pic(X))^{\perp}$$

  is the transcendental lattice of dimension $22 - \rho(X)$
- If $\{\gamma_1, \cdots, \gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,

  $$\int_{\gamma_i} \omega$$

  is called a period of X and

  $$\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).$$
- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 has a Shioda-Inose structure, that means

$$\begin{array}{ccc} X & & A = E \times E / C_N \\ & \searrow \quad \swarrow & \\ & Y = Kum(A/\pm 1) & \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve is CM.

Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ *and*

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6, \quad \eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{n\geq 1}(1 - e^{2\pi in\tau}), \quad q = \exp 2\pi i\tau$$

$$\begin{aligned} m(P_k) = & \frac{\Im\tau}{8\pi^3}\Big\{\sum_{m,\kappa}{}' \Big(-4\Big(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)} + \frac{1}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2}\Big) \\ & + 16\Big(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)} + \frac{1}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2}\Big) \\ & - 36\Big(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)} + \frac{1}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2}\Big) \\ & + 144\Big(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)} + \frac{1}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}\Big)\Big)\Big\} \end{aligned}$$

Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family (X_k) of $K3$-surfaces.

- For $k \in \mathbb{P}^1$, generically $\rho = 19$.
- The family is $\mathcal{M}_k$-polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2-4)(k^2-36)y''' + 6k(k^2-20)y'' + (7k^2-48)y' + ky = 0$$

- The family is modular in the following sense
  if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and τ as in the theorem

$$t(\frac{a\tau + b}{c\tau + d}) = t(\tau) \ \forall \ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6,2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in Sl_2(\mathbb{Z}) \ / \ a \equiv d \equiv 1 \ (6) \ \ c \equiv 0 \ (6)\}$$

$$\Gamma_1(6,2) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6) \ \ c \equiv 6b \ \ (12)\}$$

and

$$\Gamma_1(6,2)^* = \langle \Gamma_1(6,2), w_6 \rangle$$

.

- The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n\geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation
-
$$\frac{dm(P_k)}{dk} = G(\tau)$$
-
$$dm(P_k) = -G(\tau)\frac{dt}{t}\frac{1-t^2}{t}$$

is a weight 4 modular form for $\Gamma_1(6,2)^*$
- so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re(-\pi i\tau + \sum_{n\geq 1}(\sum_{d|n} d^3)(4\frac{q^n}{n} - 8\frac{q^{2n}}{2n} + 12\frac{q^{3n}}{3n} - 24\frac{q^{6n}}{6n}))$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

For some values of k, the corresponding τ is imaginary quadratic.
For example

k	0	2	3	6	10	18
τ	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-2}}{2}$	$\sqrt{\frac{-5}{6}}$

For these quadratic τ called "singular moduli", the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve E of the Shioda-Inose is CM

So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).

Theorem

Let Y_k the $K3$ hypersurface associated to the polynomial P_k, $L(Y_k, s)$ its L-series, T_Y its transcendental lattice and f_N the unique, up to twist, CM-newform, CM by $\mathbb{Q}(\sqrt{-N})$, of weight 3 and level N with rational coefficients. . Then

Theorem

$$m(P_0) = d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) \quad (B.2005)$$

$$m(P_2) = \frac{|\det T(Y_2)|^{3/2}}{\pi^3} L(Y_2, 3) = \frac{8\sqrt{8}}{\pi^3} L(f_8, 3) \quad (B.\ 2005)$$

$$m(P_{10}) = \frac{|\det T(Y_{10})|^{3/2}}{9\pi^3} L(Y_{10}, 3) + 2d_3 = \frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3 \quad (B.\ 2009)$$

$$m(P_3) = 2\frac{|\det T(Y_3)|^{3/2}}{4\pi^3} L(T(Y_3), 3) = \frac{15\sqrt{15}}{2\pi^3} L(f_{15}, 3) \quad (BFFLM\ 2013)$$

$$m(P_6) = \frac{|\det T(Y_6)|^{3/2}}{2\pi^3} L(Y_6, 3) = \frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3) \quad (BFFLM\ 2013)$$

$$m(P_{18}) = \frac{1}{5}\frac{|\det T(Y_{18})|^{3/2}}{4\pi^3} L(Y_{18}, 3) + \frac{14}{5} d_3 = \frac{120\sqrt{120}}{20\pi^3} L(f_{120}, 3) + \frac{14}{5} d_3$$

$$(BFFLM\ 2013)$$

L-functions

Let Y be a surface. The zeta function is defined by

$$Z(Y,u) = \exp\left(\sum_{n=1}^{\infty} N_n(Y)\frac{u^n}{n}\right), \qquad |u| < \frac{1}{p},$$

where $N_n(Y)$ denotes the number of points on Y in F_{p^n}.
If Y is a $K3$-surface defined over Q, then Y gives a $K3$-surface over F_p for almost all p and

$$Z(Y,u) = \frac{1}{(1-u)(1-p^2u)P_2(u)},$$

where $\deg P_2(u) = 22$. In fact,

$$P_2(u) = Q_p(u)R_p(u),$$

where the polynomial $R_p(u)$ comes from the algebraic cycles and $Q_p(u)$ comes from the transcendental cycles. Hence, for a singular $K3$-surface, $\deg Q_p = 2$ and $\deg R_p = 20$.

L-functions (continued)

Finally, we will work with the part of the L-function of Y coming from the transcendental lattice, which is given by

$$L(T(Y),s) = (*) \prod_{p \,\mathrm{good}} \frac{1}{Q_p(p^{-s})} = \sum_{n=1}^{\infty} \frac{A_n}{n^s},$$

where $(*)$ represents finite factors coming from the primes of bad reduction.

Strategy of the proof

- Understand the transcendental lattice and the group of sections.
- Relate the Mahler measure $m(P_k)$ to the L-function of a modular form.
- Relate the L-function of the surface Y_k to the L-function of that same modular form.

Ingredient: Livné's modularity theorem

Theorem

Let S be a $K3$-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant N. Its transcendental lattice $T(S)$ is a dimension 2 $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-module thus defines a L series, $L(T(S), s)$.
There exists a weight 3 modular form , f, CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S), s) \doteq L(f, s).$$

Moreover, if $NS(S)$ is generated by divisors defined over $\mathbb{Q}$,

$$L(S, s) \doteq \zeta(s-1)^{20} L(f, s).$$

The last ingredient:Schütt's classification of CM-newforms of weight 3

Theorem

Consider the following classifications of singular $K3$ surfaces over $\mathbb{Q}$:

- *by the discriminant d of the transcendental lattice of the surface up to squares,*
- *by the associated newform up to twisting,*
- *by the level of the associated newform up to squares,*
- *by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.*

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

(BFFLM) Marie-José Bertin, Amy Feaver, Jenny Fuselier, Matilde Lalin and Michelle Manes, Mahler measure of some singular K3-surfaces, to appear in Proceedings of WIN2—Women in Numbers 2 CRM Proceedings and Lecture Notes (refereed), arXiv:1208.6240, math.NT

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