MAHLER MEASURE OF MULTIVARIABLE POLYNOMIALS

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1. Definition of Mahler Measure and Lehmer's question

Definition 1. *Given* $P \in \mathbb{C}[x]$, *such that*

$$P(x) = a \prod_i (x - \alpha_i)$$

define the Mahler measure of P *as*

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}.$$

The logarithmic Mahler measure is defined as

$$\mathrm{m}(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|.$$

When does $M(P) = 1$ for $P \in \mathbb{Z}[x]$? We have the following result.

Lemma 2. *(Kronecker,* [Kr57]*) Let* $P = \prod_i (x - \alpha_i) \in \mathbb{Z}[x]$. *If* $|\alpha_i| \leq 1$, *then the* α_i *are zero or roots of unity.*

By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $M(P) = 1$ if and only if P is the product of powers of x and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$\mathrm{m}(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1) = \log(1.176280818\dots) = 0.162357612\dots$$

and asked the following (Lehmer's question, 1933, formulated in a slightly different manner):

Is there a constant $C > 1$ *such that for every polynomial* $P \in \mathbb{Z}[x]$ *with* $M(P) > 1$, *then* $M(P) \geq C$*?*

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

There are several results in the direction of solving Lehmer's question. Some of them consider restricted families of polynomials. The first of such results was found by Breusch [Br51] and (independently) by Smyth [Sm71]. For $P \in \mathbb{Z}[x]$ monic, irreducible, $P \neq \pm P^*$ (nonreciprocal), then

$$M(P) \geq M(x^3 - x - 1) = \theta = 1.324717\dots$$

The algebraic integer θ is nothing else than the smallest Pisot number.

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This result implies in particular that if $P \in \mathbb{Z}[x]$ is monic, irreducible, and of odd degree, then P is nonreciprocal and

$$M(P) \geq \theta.$$

On the other hand, there are results giving lower bounds that depend on the degree. The most fundamental of such results was given by Dobrowolski [Do79]. If $P \in \mathbb{Z}[x]$ is monic, irreducible and noncyclotomic of degree d, then

$$M(P) \geq 1 + c\left(\frac{\log\log d}{\log d}\right)^3, \tag{1}$$

where c is an absolute positive constant.

2. Mahler Measure in several variables

Definition 3. *For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the logarithmic Mahler measure is defined by*

$$\begin{aligned} \mathrm{m}(P) &:= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}, \end{aligned}$$

where $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid |x_1| = \dots = |x_n| = 1\}$.

It is possible to prove that this integral is not singular and that $\mathrm{m}(P)$ always exists. This definition appeared for the first time in the work of Mahler [Ma62].

Because of Jensen's formula [1]:

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log^+ |\alpha|$$

we recover the formula for the one-variable case.

Let us mention some elementary properties.

Proposition 4. *For $P, Q \in \mathbb{C}[x_1, \dots, x_n]$*

$$\mathrm{m}(P \cdot Q) = \mathrm{m}(P) + \mathrm{m}(Q).$$

Because of this formula, we can extend the definition of Mahler measure to rational functions.

Proposition 5. *Let $P \in \mathbb{C}[x_1, \dots, x_n]$ such that $a_{m_1,\dots,m_n}$ is the coefficient of $x_1^{m_1} \dots x_n^{m_n}$ and P has degree d_i in x_i. Then*

$$\begin{aligned} |\, a_{m_1,\dots,m_n}| &\leq \binom{d_1}{m_1} \dots \binom{d_n}{m_n} M(P), \\ M(P) &\leq L(P) \leq 2^{d_1 + \dots + d_n} M(P), \end{aligned}$$

where $L(P)$ is the length of the polynomial, the sum of the absolute values of the coefficients.

[1] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

In fact, the reason why Mahler considered this construction is that he was looking for inequalities of the typical polynomial heights (such as $L(P)$ or the maximum absolute value of the coefficients) between the height of a product of polynomials and the heights of the factors. These kinds of inequalities are useful in transcendence theory. The Mahler measure $M(P)$ is multiplicative and comparable to the typical heights, and that makes it possible to deduce such inequalities.

It is also true that $\mathrm{m}(P) \geq 0$ if P has integral coefficients.

Let us mention the following amazing result.

Theorem 6. *(Boyd* [Bo81]*, Lawton* [Law83]*) For* $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} \mathrm{m}(P(x, x^{k_2}, \dots, x^{k_n})) = \mathrm{m}(P(x_1, \dots, x_n)) \tag{2}$$

It should be noted that the limit has to be taken independently for each variable.

Because of the above theorem, Lehmer's question in the several-variable case reduces to the one-variable case. In addition, this theorem shows us that we are working with the "right" generalization of the original definition for one-variable polynomials.

The formula for the one-variable case tells us some information about the nature of the values that Mahler measure can reach. For instance, the Mahler measure of a polynomial in one variable with integer coefficients must be an algebraic number.

It is natural, then, to wonder what happens with the several-variable case. Is there any simple formula, besides the integral from the definition?

Boyd's limit formula (1981) [Bo81]

$$\lim_{N \to +\infty} m(P(x, x^N)) = m(P(x, y))$$

whenever the left hand term contains an infinity of different measures, was a hope for getting small measures in one variable from small measures in two variables. At that time, Boyd computed numerically [Bo81]

$$M((x+1)y^2 + (x^2+x+1)y + x(x+1)) = 1.25542\cdots \tag{3}$$

$$M(y^2 + (x^2+x+1)y + x^2) = 1.28573\cdots \tag{4}$$

and these are the smallest known measures in two variables. Notice that these polynomials define elliptic curves.

The same year, Smyth obtained the first explicit Mahler measures [Bo81]:

$$m(x+y+1) = L'(\chi_{-3}, -1) \tag{5}$$

$$m(x+y+z+1) = \frac{7}{2\pi^2}\zeta(3) \tag{6}$$

where χ_{-3} denotes the odd quadratic character of conductor 3 ($\chi_{-3}(3n+1) = 1$, $\chi_{-3}(3n+2) = -1$, $\chi_{-3}(3n) = 0$) and

$$L'(\chi_{-3}, -1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) =: d_3$$

is derived from the functional equation of the Dirichlet L-series.

Then, we must await Deninger's guess of the formula [De97] (1996)

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) = L'(E, 0), \tag{7}$$

where, the Laurent polynomial defines an elliptic curve E of conductor 15 and $L(E, 2)$ its L-series at $s = 2$ [2]. The last equality comes from the functional equation. (A question-mark over an equals sign means that the relation is verified numerically up to fifty decimals.) Since then, there has been an abundant literature in this area, three Conferences on the Mahler measure and developments in many mathematics domains, see for example [Bo98].

Recently (2011), Rogers and Zudilin proved (7)[RZ]. Also, some weeks ago, Zudilin [Zu13] posted on the arXiv a new proof.

3. Curves of genus 0

Let me take an example. For the polynomial

$$P = y^2(x+1)^2 + 2y(x^2 - 6x + 1) + (x+1)^2,$$

Boyd guessed (1998) [Bo98]

$$m(P) \stackrel{?}{=} 4L'(\chi_{-4}, -1) = \frac{8}{\pi} L(\chi_{-4}, 2) = 4\frac{4\sqrt{4}}{4\pi} L(\chi_{-4}, 2) =: 4d_4$$

where

$$L(\chi_{-4}, 2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} + \cdots = G,$$

G being Catalan's constant.

Here P defines a cubic curve C with $(1,1)$ as double point.

Putting $x = 1 + X$ and $y = 1 + Y$ and completing the square, we find

$$(Y(X+2)^2 + 2X^2)^2 = -16X^2(X+1).$$

Hence we get the parametrization of the curve C

$$x = -t^2 \qquad y = -\left(\frac{1+t}{1-t}\right)^2$$

But,

$$\begin{aligned} m(P) &= \frac{1}{(2\pi i)^2} \int_{|x|=1} \int_{|y|=1} \log |P(x,y)| \frac{dx}{x} \frac{dy}{y} \\ &= \frac{1}{2\pi i} \int_{|x|=1} \log(\max(|y_1|, |y_2|)) \frac{dx}{x} \quad \text{(by Jensen's formula)} \\ &= \frac{1}{2\pi i} \int_{\Gamma} \eta(2)(x,y) \end{aligned}$$

where

$$\eta(x,y) = i \log |y| \mathrm{d} \arg \mathrm{x} - \mathrm{i} \log |\mathrm{x}| \mathrm{d} \arg \mathrm{y}$$

is a differential form on the variety Γ (Maillot's trick [Ma00])

$$\Gamma = \{(x,y) \in \mathbb{C}^2 / \ (x,y) \in C \ |x| = 1, \ \ |y| \geq 1\}.$$

Now η is related to the Bloch-Wigner dilogarithm D

The Bloch-Wigner dilogarithm of a complex number x is defined as

$$D(x) := \Im Li_2(x) + \log |x| \arg(1-x)$$

[2]The L-series of the elliptic curve E is defined by $L(E, s) = \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} \prod_{p|N} \frac{1}{1 - a_p p^{-s}} = \sum_{n \geq 1} \frac{a_n}{n^s}$, with $a_p = p + 1 - N_p$ and $N_p := |E(\mathbb{F}_p)|$.

where Li_2 is the ordinary dilogarithm. It is a univalued, real analytic function in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$. It satifies the property

$$D(\frac{1}{z}) = D(1-z) = D(\bar{z}) = -D(z),$$

the distribution relation

$$D(z^n) = n \sum_{k=0}^{n-1} D(\zeta^k z)$$

where ζ denotes a primitive n-th root of unity and the five-term relation

$$D(x) + D(y) + D(1-xy) + D(\frac{1-x}{1-xy}) + D(\frac{1-y}{1-xy}) = 0.$$

Moreover, if the wedge differential η is given by

$$\eta(x,y) = \log(\mid x \mid) di \arg y - \log \mid y \mid di \arg x,$$

the derivative of D verifies

$$diD(x) = \eta(x, 1-x)$$

Notice that the differential η is

- multiplicative in each variable
- antisymetric
- and if $\alpha \neq \beta$ satisfies the Tate's relation

$$\eta(t-\alpha, t-\beta) = \eta(\frac{t-\alpha}{\beta-\alpha}, 1 - \frac{t-\alpha}{\beta-\alpha}) + \eta(t-\alpha, \alpha-\beta) + \eta(\beta-\alpha, t-\beta).$$

Thus the Mahler measure can be expressed as

$$\begin{aligned} m(P) &= -\frac{i}{2\pi i} \int_{\gamma_1} 4dD(-t) - 4dD(t) \\ &= \frac{2}{\pi}[D(t) - D(-t)]_{-i}^{i} \\ &= \frac{8}{\pi} D(i) \\ &= 4d_4 (\text{by definition of } D). \end{aligned}$$

Thus, Boyd's guess is proved.

Remark 7. *Let us recall Bloch's formula, where* $\zeta_f = e^{\frac{2\pi i}{f}}$,

$$d_f = \frac{f}{4\pi} \sum_{m=1}^{f} \chi_{-f}(m) D(\zeta_f^m).$$

and also

$$\begin{aligned} L(\chi_{-f}, 2) &= \sum_{n \geq 1} \frac{\chi_{-f}(n)}{n^2} = \frac{1}{\sqrt{f}} \sum_{k=1}^{f-1} \left(\frac{k}{f}\right) \mathrm{Im}(\sum_{n \geq 1} \frac{\zeta_f^{kn}}{n^2}) \\ &= \frac{1}{\sqrt{f}} \sum_{k=1}^{f-1} \left(\frac{k}{f}\right) D(\zeta_f^k) \end{aligned} \tag{8}$$

So, for some genus 0 curves, the Mahler measure encodes the Bloch-Wigner dilogarithm hence the Bloch groups.

Lalin [La07] has proved that Smyth's first result can be treated in that context. Vandervelde [V08] has given a class of polynomials defining rational curves to which this applies. For this class of polynomials, the Mahler measure can be expressed in terms of dilogarithms of algebraic numbers up to possibly a term in $\zeta_F(2)$. One of the conditions for such polynomials is to be *tempered*.

Definition 8. *A polynomial in two variables is tempered if the polynomials corresponding to the faces of its Newton polygon has roots of unity as the only zeros.*

When drawing the convex hull of points (i,j) in $\mathbb{Z}^2$ corresponding to the monomials $a_{i,j}x^iy^j$, $a_{i,j}\neq 0$, you also draw points of $\mathbb{Z}^2$ located on the faces. The polynomial of the face is a polynomial in one variable t which is a combination of the monomials 1, t, t^2, ... The coefficients of the combination are given when going along the face, that is $a_{i,j}$ if the lattice point of the face belongs to the convex hull and 0 otherwise. For example, the polynomial

$$y^2+y+x^2+x+1$$

is tempered, since its Newton polygon corresponds to

$$\begin{array}{ccc} 1 & & \\ 1 & 0 & \\ 1 & 1 & 1 \end{array}$$

The polynomials of the faces are

$$1+t^2, \qquad 1+t+t^2, \qquad 1+t+t^2.$$

The polynomial

$$P:=(x^2+x-1)y^2+(x^2+5x+1)y-x^2+x+1$$

is not tempered, since its Newton polygon corresponds to

$$\begin{array}{ccc} -1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & -1 \end{array}$$

and the polynomials of the faces are all equal to $\pm(t^2+t-1)$.

Motivated by Smyth's result, Chinburg asked for each real odd Dirichlet character χ_{-f} the existence of a polynomial $P_f(x,y)$ such that

$$m(P_f(x,y))=rL'(\chi_{-f},-1)=r\frac{f^{3/2}}{4\pi}L(\chi_{-f},2)=:rd_f$$

where r denotes a rational number and L the Dirichlet L-series associated to the character χ. His student Ray [Ra87] constructed polynomials for $f=3,4,7,8,20,24$ and Boyd [Bo98] extended the list to $f=11,15,35,39,55,84$. Most of these formulae are experimental, that is, satisfied to a high accuracy. For example Boyd obtained

$$m((x+1)^2y+x^2+x+1)\stackrel{?}{=}\frac{1}{3}d_7 \tag{9}$$

$$m((x^2+x+1)y+x^2+1)\stackrel{?}{=}\frac{1}{12}d_{15} \tag{10}$$

$$m((x+1)^2(x^2+x+1)y+(x^2-x+1)^2) \stackrel{?}{=} \frac{2}{3}d_{11}, \tag{11}$$

The shape of the expression is now well understood thanks to the properties of the Bloch group of a number field. In fact, the quotient of the logarithmic Mahler measure by the corresponding d_f is known to be rational. This rational number can be guessed with a computer. So the point is to prove the formulae. We shall give an example below by proving (9)

First we recall Vandervelde's results on the Mahler measure of parametrisable polynomials, the definition of the Bloch group of a number field and the expression of the zeta function of a number field in terms of Bloch-Wigner dilogarithms, that is Zagier's theorem.

Suppose now the polynomial P of the shape $A(x)y+B(x)$. The logarithmic Mahler measure $m(P)$ using Jensen's formula can be expressed as

$$m(P) = m(A) + \frac{1}{2\pi i}\int_{|x|=1} \log(\max\{|\frac{B(x)}{A(x)}|, 1\})\frac{dx}{x} = m(A) - \frac{1}{2\pi i}\int_\gamma \eta(x,y)$$

where γ is the path defined on the variety intersection of P and P^*, for $P^*(x,y) = P(1/x, 1/y)$,

$$\gamma = \{(x,y)/\ P(x,y)=0,\ P^*(x,y)=0,\ |\ x\ |=1,\ |\ y\ |\geq 1\}.$$

3.1. **The Bloch groups.** Let F be a field and define the abelian groups

$$\mathcal{C}(F) \subset \mathcal{A}(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$$

where $\mathcal{A}(F) = \ker\beta$ if

$$\beta : \mathbb{Z}[\mathbb{P}^1_F] \to \Lambda^2 F^\times$$

is defined by

$$\beta(0) = \beta(1) = \beta(\infty) = 0$$
$$\beta(x) = (x) \wedge (1-x)$$

and

$$\mathcal{C}(F) := \langle [x] + [y] + [1-xy] + [\frac{1-x}{1-xy}] + [\frac{1-y}{1-xy}] \rangle$$

is generated by the five-term relation.

The Bloch group is now defined by the exact sequence

$$0 \to \mathcal{B}_2(F) \to \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{C}(F) \xrightarrow{\beta} \Lambda^2 F^\times$$

The class of x in $\mathcal{B}_2(F)$, $[x]$, behaves like a Bloch-Wigner dilogarithm.

The complex

$$\mathcal{B}_F(2) \bigotimes \mathbb{Q} : \mathcal{B}_2(F)_\mathbb{Q} \xrightarrow{\delta_1^2} (\Lambda^2 F^\times)_\mathbb{Q}$$

has a cohomology related to K-theory by Matsumoto's theorem

$$H^2(\mathcal{B}_F(2)) \simeq K_2(F).$$

These things are related to results by Zagier on $\zeta_F(2)$ [3].

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[3]

$$\zeta_F(2) = \sum_{\mathcal{A}} \frac{1}{N\mathcal{A}^2} = \prod_{\mathcal{P}} \frac{1}{1-N\mathcal{P}^{-2}}$$

3.1.1. *Zagier's theorem.*

Theorem 9. *(Zagier) Let F be a number field, $[F:\mathbb{Q}] = n_+ + n_- = r_1 + 2r_2$, $n_+ = r_1 + r_2$, $n_- = r_2$. Then the Bloch group modulo torsion satisfies*

(1)
$$\mathcal{B}_2(F)/tors \simeq \mathbb{Z}^{n_-}$$

(2) *If $\xi_1, \dots, \xi_{n_-}$ is a basis of $\mathcal{B}(F)/tors$ and $\sigma_1, \dots, \sigma_{n_-}$ the complex embeddings of F into $\mathbb{C}$, then there exists $r \in \mathbb{Q}^\times$ such that*
$$\det(D(\sigma_i(\xi_j)))_{i,j=1,\dots,n_-} \frac{\pi^{2n_+}}{\sqrt{\mid D_F \mid}} = r\zeta_F(2).$$

It is also useful to recall Vandervelde's theorem.

3.1.2. *Vandervelde's theorem.*

Theorem 10. [V08] *Suppose that $P \in \mathbb{C}[x,y]$ can be parametrized by $x = f(t) = \lambda_1 \prod((t-\alpha_r)^{l_r}$ and $y = g(t) = \lambda_2 \prod(t-\beta_s)^{m_s}$. Let S consists of those points in the zero-locus of P for which $\mid x \mid = 1$ and $\mid y \mid \geq 1$ and let $\gamma_1, \dots, \gamma_n$ be the paths which map to S under $t \mapsto (f(t), g(t))$, oriented via $f(\gamma_j) \subset \mathbb{T}^1$. If u_j and v_j denote the initial and terminal points of γ_j and $P(x,y)$ has leading coefficient $\lambda(x)$ as a polynomial in y, then ¡*
$$\begin{aligned}2\pi m(P) =& 2\pi m(\lambda(x)) + \sum_{j=1}^{n}(\sum_{r,s}{}' l_r m_s [D(\frac{u_j-\alpha_r}{\beta_s-\alpha_r}) - D(\frac{v_j-\alpha_r}{\beta_s-\alpha_r})] + \\ & \sum_r l_r \log \mid \tilde{g}(\alpha_r) \mid wind(\gamma_j, \alpha_r) - \sum_s m_s \log \mid \tilde{f}(\beta_s) \mid wind(\gamma_j, \beta_s),\end{aligned}$$
where
$$wind(\gamma, \beta) = \int_\gamma \frac{dz}{z-\beta}.$$
The sum $\sum'$ is taken over r and s such that $\alpha_r \neq \beta_s$. The function $\tilde{g}$ is defined by $\tilde{g}(t) = g(t)$unless $t = \beta_s$ where the corresponding factor in g is omitted before evaluating at t; hence $\tilde{g}(\alpha_r) \neq 0, \infty$.

3.2. **The function $\zeta_F(2)$ for imaginary quadratic fields F.** If F is an imaginary quadratic field or more generally with only one complex embedding, one can associate to F an hyperbolic manifold $\mathbf{M^3}$ of dimension 3 in the hyperbolic plane $\mathbb{H}^3$
$$\mathbf{M^3} = \mathbb{H}^\mathbf{3}/\mathbf{\Gamma_F}$$
where Γ_F is a discrete co-finite subgroup of $Sl_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{H}^3)$. We know since Humbert (1919) that $\zeta_F(2)$ is related to $\mathrm{Vol}(\mathbf{M^3})$. For example, if $F = \mathbb{Q}(\sqrt{-d})$, $d > 0$, then $\Gamma_F = \mathrm{Sl}_2(\mathcal{O}_F) \subset \mathrm{Sl}_2(\mathbb{C})$ and
$$\zeta_F(2) = \frac{4\pi^2}{\mid D_F \mid^{3/2}} \mathrm{Vol}(\mathbb{H}^3/\Gamma_F).$$
By results of Milnor and Thurston, the hyperbolic manifold $\mathbf{M^3}$ can be triangulated by hyperbolic tetrahedra and its volume can be expressed in terms of a sum o f$N \leq 24$ Bloch-Wigner dilogarithms on algebraic numbers. In that case the five-term relation can be interpreted as gluing conditions on the ideal tetrahedra.

Theorem 11. *If the number field F has only one complex embedding, $\zeta_F(2)$ is a sum of Bloch-Wigner dilogarithms on algebraic numbers.*

We can give an hyperbolic interpretation of the five term relation of the Bloch-Wigner dilogarithm.

If $P_1, \ldots, P_5 \in \partial\mathbb{H}^3 = \mathbb{P}^1(\mathbb{C})$, then

$$\sum(-1)^i \mathrm{Vol}(P_1, \ldots, \hat{P}_i, \ldots, P_5) = 0.$$

Besides, if the ideal tetrahedron Δ has vertices A, B, C, D, then

$$\mathrm{Vol}(\Delta) = D((A : B : C : D)) = D\left(\frac{A-B}{A-C}racD - CD - B\right)$$

where D denotes the Bloch-Wigner dilogarithm and $(A : B : C : D)$ the cross-ratio of the corresponding complex numbers. In particular, for the ideal tetrahedron $\Delta_z = (0, 1, \infty, z)$, we get

$$\mathrm{Vol}(\Delta_z) = D(z).$$

For example if $F = \mathbb{Q}(\sqrt{-7})$

$$\zeta_F(2) = \frac{4\pi^2}{21\sqrt{7}}(2D(\frac{1+\sqrt{-7}}{2}) + D(\frac{-1+\sqrt{-7}}{4})). \tag{12}$$

Moreover the element

$$\xi = 2[\frac{1+\sqrt{-7}}{2}] + [\frac{-1+\sqrt{-7}}{4}]$$

belongs to the Bloch group of F. To prove this we have to compute $\beta(\xi)$.

$$\beta(\xi) = 2\left(\frac{1+\sqrt{-7}}{2}\right) \wedge \left(\frac{1-\sqrt{-})}{2}\right) + \left(\frac{-1+\sqrt{-})}{4}\right) \wedge \left(\frac{5-\sqrt{-7}}{4}\right).$$

Defining $a := \left(\frac{1-\sqrt{-7}}{2}\right)$ and $b := \left(\frac{-1-\sqrt{-7}}{2}\right)$, it follows

$$\beta(\xi) = 2(-b) \wedge (a) + \left(\frac{1}{b}\right) \wedge \left(\frac{a^2}{b}\right).$$

Now, using the relations $a - b = 1$ and $\frac{1}{b} + \frac{a^2}{b} = 1$, we get $\beta(\xi) = 0$, thus $\xi \in \mathcal{B}_2(F)$.

The element ξ is not unique and can be replaced by another element such the difference satisfies a five-term relation.

3.3. **Proof of formula** (9)**.** The proof will use the following proposition.

Proposition 12. *If $\alpha = \frac{-3+\sqrt{-7}}{4}$, we have the following dilogarithm relation*

$$D(\alpha^3) - 3D(\alpha) + D(\frac{\alpha}{2}) = 0.$$

PROOF. The ingredients are the equation satified by α and the five-term relation

$$D(x) - D(y) + D(\frac{y}{x}) + D(\frac{1-y^{-1}}{1-x^{-1}}) - D(\frac{1-y}{1-x}) = 0 \tag{13}$$

The equation satisfied by α is

$$1 + \alpha + \alpha^2 = -\frac{\alpha}{2} \tag{14}$$

By (13), we get with $x=\alpha^3$ and $y=\alpha$

$$D(\alpha^3)-D(\alpha)=D(\alpha^2)-D(\frac{\alpha^2}{1+\alpha+\alpha^2})+D(\frac{1}{1+\alpha+\alpha^2})$$

and by (14)

$$D(\alpha^3)-D(\alpha)=D(\alpha^2)-D(-2\alpha)+D(-\frac{2}{\alpha})$$

Now using the distribution formula for n=2, the property $D(1/z)=-D(z)$ and the relation

$$D(-2\alpha)=D(-\frac{\alpha}{2}))$$

we get

$$D(\alpha^3)-3D(\alpha)+D(\frac{\alpha}{2})=2D(-\alpha)-2D(-\frac{\alpha}{2})+D(\frac{\alpha}{2}).$$

First we prove that

$$D(-2\alpha)=D(1+\frac{1-\sqrt{-7}}{2})=-D(\frac{-1+\sqrt{-7}}{2})=D(\frac{-1-\sqrt{-7}}{2})$$

and

$$D(-\frac{\alpha}{2})=-D(\frac{8}{3-\sqrt{-7}})=D(\frac{3-\sqrt{-7}}{2})=D(-2\alpha).$$

Also

$$D(-\alpha)=D(1-\frac{1+\sqrt{-7}}{4})=-D(\frac{1+\sqrt{-7}}{4})=-D(\frac{1+\sqrt{-7}}{2}),$$

since

$$D(\frac{1+\sqrt{-7}}{2})=-D(\frac{2}{1+\sqrt{-7}}=-D(\frac{1-\sqrt{-7}}{4})=D(\frac{1+\sqrt{-7}}{4}).$$

Now

$$\begin{aligned}2D(-\alpha)-2D(-2\alpha)&=-2D(\frac{1+\sqrt{-7}}{2})-2D(\frac{-1-\sqrt{-7}}{2})\\&=-D((\frac{1+\sqrt{-7}}{2})^2)=-D(\frac{-3+\sqrt{-7}}{2})\end{aligned}$$

by distribution formula.

Finally

$$2D(-\alpha)-2D(-2\alpha)+D(2\alpha)=-D(\frac{-3+\sqrt{-7}}{2})+D(\frac{-3+\sqrt{-7}}{2})=0$$

since we can prove as previously that

$$D(\frac{\alpha}{2})=D(2\alpha).$$

□

Theorem 13.

$$m((x+1)^2y+x^2+x+1)=\frac{1}{3}d_7=\frac{1}{3}L'(\chi_{-7},-1)=\frac{1}{3}\frac{7^{3/2}}{4\pi}L(\chi_{-7},2)$$

PROOF. We apply Vandervelde 's result. We have only one path γ_1 with initial point $u_1 = \frac{-3+\sqrt{-7}}{4} = \alpha$ and terminal point $\bar{\alpha}$. So

$$2\pi m(P) = 2D(\frac{-3+\sqrt{-7}}{4}j^2) + 2D(\frac{-3+\sqrt{-7}}{4}j) + 4D(\frac{3+\sqrt{-7}}{4}).$$

Applying the distribution formula for $n = 3$ and $n = 2$, it follows

$$\pi m(P) = D(\alpha) - D(\alpha^2) + \frac{1}{3}D(\alpha^3).$$

It is easy to prove that

$$\xi = [\alpha] - [\alpha^2] + \frac{1}{3}[\alpha^3] \in \mathcal{B}(\mathbb{Q}(\sqrt{-7}))$$

since

$$\begin{aligned}\beta(\xi) &= \alpha \wedge (1-\alpha) - \alpha^2 \wedge (1-\alpha^2) + \frac{1}{3}\alpha^3 \wedge (1-\alpha^3)\\ &= \alpha \wedge \frac{(1-\alpha)(1-\alpha^3)}{(1-\alpha)^2(1+\alpha)^2} = \alpha \wedge (-1) = 0\end{aligned}$$

(since $\alpha^2 + \frac{3}{2}\alpha + 1 = 0$ and $\alpha \wedge (-1) = \alpha \wedge (-1)^3 = 3\alpha \wedge (-1)$; so $\alpha \wedge (-1) = 0$)

Thus, by Zagier's theorem

$$\pi^2 D(\xi_1) = r'\sqrt{7}\zeta_F(2)$$

for $r' \in \mathbb{Q}^\times$, that is $m(P) = rd_7$. We must prove now that $r = \frac{1}{3}$ or $r' = 7/2$.

So, using (12), we have to prove that

$$D(\xi_1) = \frac{4}{3}D(\frac{1+\sqrt{-7}}{2}) + \frac{2}{3}D(\frac{-1+\sqrt{-7}}{4}) =: A$$

Using the proposition we can express

$$D(\xi) = -2D(-\alpha) - \frac{1}{3}D(\frac{\alpha}{2}).$$

Since $(\frac{-1+\sqrt{-7}}{4})^2 = \frac{\bar{\alpha}}{2}$ and $D(\frac{1+\sqrt{-7}}{2}) = D((\frac{1+\sqrt{-7}}{4})$, it follows from distribution formula that

$$A = 2D(\frac{1+\sqrt{-7}}{4}) - \frac{1}{3}D(\frac{\alpha}{2}).$$

Thus the equality $D(\xi) = A$, since $D(\frac{1+\sqrt{-7}}{4}) = -D(-\alpha)$.

□

4. The measures of a family of genus-one curves [B-L13]

An elliptic curve (over $\mathbb{C}$) is roughly speaking a curve (zeros of a two-variable polynomial) that is birationally equivalent to an equation of the form

$$E : Y^2 = X^3 + aX + b.$$

For example, the curve given by the equation

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0,$$

where k is a parameter, corresponds to an elliptic curve. We can see this by applying the change of variables

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}, \tag{15}$$

k	s_k	N
1	1	15
2	1	24
3	2	21
5	6	15
6	1/2	120
7	1/2	231
8	4	24
9	1/2	195
10	-1/8	840

TABLE 1. s_k numerically conjectured values from formula (16). N corresponds to the conductor of the elliptic curve. When $k = 4$ the curve has genus zero.

and we get the equation

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

If the elliptic curve is defined over $\mathbb{Q}$ (i.e., $a, b \in \mathbb{Q}$), one can construct the L-function as follows

$$L(E,s) = \prod_{\text{good } p}(1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{\text{bad } p}(1 - a_p p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where for p prime,

$$a_p = 1 + p - \#E(\mathbb{F}_p).$$

The family of two-variable polynomials $P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ was initially studied by Boyd [Bo81], Deninger [De97], and Rodriguez-Villegas [RV97] from different points of view. Boyd found many numerical identities of the form

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \stackrel{?}{=} s_k L'(E_{(k)}, 0) \qquad k \in \mathbb{N} \neq 0, 4 \tag{16}$$

where s_k is a rational number (often integer), and $E_{(k)}$ is the elliptic curve which is the algebraic closure of the zero set of the polynomial (i.e., given by the change of variables (15)). Table 1 shows the first values for s_k conjectured by Boyd. He numerically computed s_k for $k = 1, \ldots 40$.

The connection with $L'(E,0)$ was predicted by Deninger using Beilinson's conjectures. However, there are some cases in which this identity can be proved. This happens when Beilinson's conjectures are known, i.e., when the elliptic curve has complex multiplication, or when it is given as a modular curve, and then the Mahler measure may be related to the L-function of a modular form.

In [RV97], Rodriguez-Villegas expressed this Mahler measure as an Eisenstein–Kronecker series:

$$\begin{aligned} \mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+k\right) &= \mathrm{Re}\left(\frac{16\,\mathrm{Im}\,\tau}{\pi^2}\sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\tau)^2(m+n4\bar{\tau})}\right) \\ &= \mathrm{Re}\left(-\pi i\tau + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right), \end{aligned} \tag{17}$$

where the q parameter is coming from

$$q = e^{2\pi i\tau} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right).$$

Rodriguez-Villegas' idea to obtain this formula is as follows. One observes that for $\lambda = -1/k$ such that $|\lambda| < 1/4$, the Mahler measure of this polynomial is given by

$$m(k) = \mathrm{Re}(\tilde{m}(\lambda)),$$

where

$$\begin{aligned} \tilde{m}(\lambda) &= -\log\lambda + \frac{1}{2\pi i)^2}\int_{\mathbb{T}^2}\log(1-\lambda(x+x^{-1}+y+y^{-1}))\frac{dx}{x}\frac{dy}{y} \\ &= -\log\lambda - \sum_{n=1}^{\infty}\frac{b_n}{n}\lambda^n, \end{aligned}$$

where b_n is the constant coefficient of the polynomial $(x+x^{-1}+y+y^{-1})^n$. More specifically,

$$b_n = \begin{cases} \binom{n}{n/2}^2 & n \text{ even,} \\ 0 & n \text{ odd.} \end{cases}$$

Now consider

$$\begin{aligned} u(\lambda) &= \int_{\mathbb{T}^2}\frac{1}{1-\lambda(x+x^{-1}+y+y^{-1})}\frac{dx}{x}\frac{dy}{y} \\ &= \sum_{n=0}^{\infty} b_n\lambda^n. \end{aligned}$$

Then

$$\tilde{m}(\lambda) = -\log\lambda - \int_0^{\lambda}(u(\delta)-1)d\delta.$$

By construction, $u(\lambda)$ is a period of a holomorphic differential on the curve defined by $1-\lambda(x+x^{-1}+y+y^{-1}) = 0$ (see [Gr69]) hence a solution to a Picard-Fuchs differential equation. Thus, it is not surprising that $\tilde{m}(\lambda)$ has a hypergeometric series form.

Formula (17) may in turn be related to the elliptic dilogarithm (using the techniques of Bloch [Bl00]). Then one has to relate the values of the elliptic dilogarithm to the L-function, which is done through Beilinson's conjectures.

For example, Rodriguez-Villegas [RV97] proved

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+4\sqrt{2}\right) = L'(E_{32},0). \tag{18}$$

It should be remarked that it suffices that k^2 be an integer for this equation to have an interpretation in terms of Beilinson's conjectures. In this case, the curve has complex multiplication.

Other examples were given by Rogers and Zudilin: in [RZ] they proved

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)=\frac{15}{4\pi^2}L(E_{15},2)=L'(E_{15},0),$$

and in [RZ12] they proved

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right)=\frac{24}{\pi^2}L(E_{24},2)=4L'(E_{24},0).$$

These results come from relating these hypergeometric expressions to lattice sums by means of Ramanujan's identities for q-series. The lattice sums can be related the special values of L-functions of elliptic curves by using the modularity theorem.

It should be noted that there are other families related to elliptic curves that yield similar results that were already numerically studied by Boyd. After Boyd's paper, some identities for these families were proved also using Beilinson's conjectures.

Brunault [Br05, Br06] considered the curve $X_1(11)$ and proved

$$\mathrm{m}\left((1+x)(1+y)(1+x+y)+xy\right)=\frac{77}{4\pi^2}L(E_{11},2)=7L'(E_{11},0),$$

by giving an explicit version of Beilinson's theorem on modular curves.

Similarly, Mellit [Me] considered the modular curve $X_0(14)$ and proved several identities including, for instance,

$$\mathrm{m}\left(x^3+y^3+1+xy\right)=\frac{7}{\pi^2}L(E_{14},2)=2L'(E_{14},0).$$

What do these polynomials have in common? Boyd noticed that they all satisfy that the faces of their Newton polygon are cyclotomic polynomials (i.e., they have Mahler measure zero). This condition was explained by Rodriguez-Villegas [RV97] in terms of K-theory. Roughly speaking, this condition guarantees that there is an element $\{x,y\}$ in K_2 of the elliptic curve. The regulator is then evaluated in this element.

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