Lecture II: Mahler–Sprindzuk Problem.

Let $x \in \mathbb{R} \setminus \overline{\mathbb{Q}}$. How transcendental is x?

For $P(T) = a_0 + a_1 T + \dots + a_d T^d \in \mathbb{Z}[T]$, define $H(P) = \max_i |a_i|$.

One can show that $\forall x \in \mathbb{R}^d$ $\exists$ infinitely many $P \in \mathbb{Z}[T]$, $\deg(P) \leq d$:

$$|P(x)| \leq c(x,d) \cdot H(P)^{-d}.$$

<u>Conj</u> (Mahler '32) For a.e. $x \in \mathbb{R}$, the exponent <u>cannot</u> be improved.

$\uparrow$ proved by Sprindzuk '64.

$$\left|a_0 + \sum_{i=1}^{d} a_i x^i\right| \ll \|a\|^{-d} \longrightarrow$$ Diophantine properties of points on the curve $(x, \dots x^d)$

<u>Def</u> $y \in \mathbb{R}^d$ is <u>well-approximable</u> if $\exists \alpha > d$:

$$|\langle p, y\rangle + q| \leq \|p\|^{-\alpha}$$

has infinitely many solutions $(p,q) \in \mathbb{Z}^d \times \mathbb{Z}$.

By Borel–Cantelli lemma,

a.e. $y \in \mathbb{R}^d$ is not well-approximable.

Conj (Sprindzuk '80) Let $f: U \to \mathbb{R}^d$ be a polynomial map. (U - open subset of $\mathbb{R}^k$)

Assume that $f(U) \not\subset$ proper affine subspace.

Then for a.e. $x \in U$, $f(x)$ is not well-approximable.

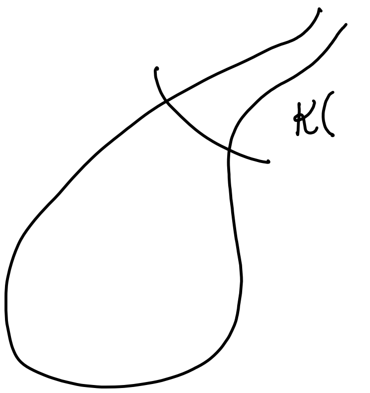
proved by Kleinbock–Margulis '98.

Flows on homogeneous spaces.

$\mathcal{L} = \{$ lattices in $\mathbb{R}^{d+1}$ of covol $= 1 \}$.

For $y \in \mathbb{R}^d$, set $\Lambda_y = \{ (p, \langle p, y \rangle + q) : (p,q) \in \mathbb{Z}^d \times \mathbb{Z} \} \in \mathcal{L}$

$$\Lambda_y = \mathbb{Z}^{d+1} \underbrace{\left(\begin{array}{c|c} I & y \\ \hline 0 & 1 \end{array} \right)}_{u_y}$$



$$a_t = \left(\begin{array}{c|c} e^t I & 0 \\ \hline 0 & e^{-dt} \end{array} \right) \curvearrowright \mathcal{L}.$$

$$K(\varepsilon) = \{ \Lambda \in \mathcal{L} : \min_{v \in \Lambda \setminus \{0\}} \|v\| \le \varepsilon \}.$$

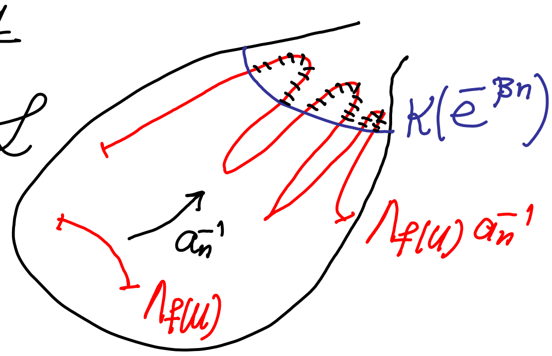
Prop. $y \in \mathbb{R}^d$ is well approximable

$\Updownarrow$

$\exists \beta > 0 : \Lambda_y a_n^{-1} \in K(e^{-\beta n})$ for infinitely many n.

Strategy

Study distribution of $\mathcal{L}$ the submanifolds $\Lambda_{f(U)}a_n^{-1}$ in $\mathcal{L}$ as $n\to\infty$



$$A_n=\{x\in U:\Lambda_{f(x)}a_n^{-1}\in K(e^{-\beta n})\}.$$

If $\sum_{n\geq 1}|A_n|<\infty$, then the Sprindzuk Conj follows from the Borel–Cantelli Lemma.

Nondivergence estimates.

For discrete $\Delta\subset\mathbb{R}^{d+1}$, $d(\Delta)=\mathrm{vol}(\mathbb{R}\Delta/\Delta)$.

If $\Delta=\mathbb{Z}v_1\oplus\ldots\oplus\mathbb{Z}v_k$, $d(\Delta)=\|v_1\wedge\ldots\wedge v_k\|$.

__Thm__ (Kleinbock–Margulis) Fix interval I and $\rho\in(0,1)$. Let $h: I\to SL_{d+1}(\mathbb{R})$ be a polynomial map. Assume that $\forall\,\Delta<\mathbb{Z}^{d+1}$:

$$\sup_{x\in I} d(\Delta h(x))\geq\rho \qquad (**)$$

Then $\exists\, c,\alpha>0$ (depending only on $\deg(h)$): $\forall\varepsilon\in(0,\rho)$

$$|\{x\in I:\mathbb{Z}^{d+1}h(x)\in K(\varepsilon)\}|\leq c\cdot\left(\frac{\varepsilon}{\rho}\right)^{\alpha}\cdot|I|.$$

We apply Thm. to $h(x) = u_{f(x)} a_n^{-1}$.

For $v \in \mathbb{R}^{d+1}$, we set $B_v(\varepsilon) = \{x \in I : \|v \cdot h(x)\| \leq \varepsilon\}$.

Then $\{x \in I : \mathbb{Z}^{d+1} h(x) \in K(\varepsilon)\} \subset \bigcup_{v \in \mathbb{Z}^{d+1} \setminus \{0\}} B_v(\varepsilon)$.

~~Def.~~ $f : \mathbb{R} \to \mathbb{R}$ is $\underline{(C, \alpha)\text{-good}}$ if

$\forall$ interval $I \subset \mathbb{R}$: setting $\rho = \sup_I |f|$,

$$\underbrace{|\{x \in I : |f(x)| \leq \varepsilon\}|}_{M_I(f)} \leq C \left(\frac{\varepsilon}{\rho}\right)^{\alpha} \cdot |I|.$$

Prop. Every polynomial f is (C, α)-good.

(C, α depend only on $\deg(f)$).

We can choose $x_1, \ldots x_k \in I$: $|x_i - x_j| \geq \frac{M_I(f)}{2k}$, $i \neq j$,

$|f(x_i)| \leq \varepsilon$.

Take $k = \deg(f) + 1$.

Then $f(x) = \sum_{i=1}^{k} f(x_i) \cdot \dfrac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$, and

$$\rho = \sup_I |f| \leq k \cdot \varepsilon \cdot \frac{|I|^{k-1}}{(M_I(f)/2k)^{k-1}}.$$

This implies the estimate.

Proof of Thm (for 2-dim. lattices)

$v \in \Lambda$ is called primitive if $v \notin k\Lambda$ for $k \geq 2$.

Lem. $\forall$ lattice $\Lambda \in \mathcal{L}$: $\Lambda \cap B(0,1) \supset$ unique prim. vector (up to sign).

Γ Suppose that $\exists v_1, v_2 \in \Lambda \cap B(0,1)$ – linearly independent. Then $\Lambda_0 = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \subset \Lambda$ is a lattice.
$\mathrm{vol}(\mathbb{R}^2/\Lambda_0) \leq \|v_1\| \cdot \|v_2\| < 1$, but $\mathrm{vol}(\mathbb{R}^2/\Lambda_0) \geq \mathrm{vol}(\mathbb{R}^2/\Lambda) = 1$. $\lrcorner$

Since polynomials are (C,α)-good,

$$\sum_{\text{prim. } v \in \mathbb{Z}^{d+1}} |B_v(\varepsilon)| \ll \left(\frac{\varepsilon}{\rho}\right)^{\alpha} \cdot \sum_{\text{prim. } v \in \mathbb{Z}^{d+1}} B_v(\rho) =$$

For primitive $v_1 \neq \pm v_2$, $B_{v_1}(\rho) \cap B_{v_2}(\rho) = \emptyset$

(Otherwise, $\Lambda = \mathbb{Z}^2 h(x)$ with $x \in B_{v_1}(\rho) \cap B_{v_2}(\rho)$ would contradict Lemma.)

$$= \left(\frac{\varepsilon}{\rho}\right)^{\alpha} \cdot \left| \bigcup_{v} B_v(\rho) \right| \leq \left(\frac{\varepsilon}{\rho}\right)^{\alpha} \cdot |I|. \quad \lrcorner$$

Verifying that $d(\Delta u_{f(x)} a_n^{-1}) \geq ?$

$a_t^{-1} \circlearrowright \wedge^k \mathbb{R}^{d+1}$:

$$\mathbb{R}^{d+1} = \underbrace{\overset{V}{\overset{\parallel}{\langle e_1, \dots e_d \rangle}}}_{\text{contracting}} \oplus \underbrace{\langle e_{d+1} \rangle}_{\text{expanding}}$$

$$a_t = \left(\begin{array}{c|c} e^t I & 0 \\ \hline 0 & e^{-dt} \end{array} \right) \qquad \wedge^k \mathbb{R}^{d+1} = \underbrace{\wedge^k V}_{\text{contracting}} \oplus \underbrace{e_{d+1} \wedge (\wedge^{k-1} V)}_{\text{expanding}}$$

$u_y \circlearrowright \wedge^k \mathbb{R}^{d+1}$:

$$u_y = \left(\begin{array}{c|c} I & y \\ \hline 0 & 1 \end{array} \right)$$

$e_{d+1} \cdot u_y = e_{d+1}$

For $v \in V$, $v \cdot u_y = v + \langle v, y \rangle e_{d+1}$.

$e_{d+1} \wedge (\wedge^{k+1} V)$ is fixed by u_y.

$$(v_1 \wedge \dots \wedge v_k) u_y = (v_1 + \langle v_1, y \rangle e_{d+1}) \wedge \dots \wedge (v_k + \langle v_k, y \rangle e_{d+1})$$

$$= (v_1 \wedge \dots \wedge v_k) + \sum_{i=1}^{k} \pm \langle v_i, y \rangle e_{d+1} \wedge \Big(\bigwedge_{j \neq i} v_i \Big).$$

Given $\Delta = \mathbb{Z}\delta_1 \oplus \dots \oplus \mathbb{Z}\delta_k \leq \mathbb{Z}^{d+1}$, we need to estimate $d(\Delta u_{f(x)} a_n^{-1})$.

Case 1: $e_{d+1} \in \mathbb{R}\Delta$.

Then $\wedge^k \Delta$ is fixed by $u_{f(x)}$ and expanded by a_n^{-1}.

2) $\underbrace{e_{d+1} \notin \mathbb{R}\Delta}$

Take an orthonormal set $\{v_1, \dots v_{k-1}\} \subset \mathbb{R}\Delta \cap V$ and complete it to an orthonormal basis $\{v_1, \dots v_k, e_{d+1}\}$ of $\mathbb{R}\Delta \oplus \mathbb{R}e_{d+1}$.

Then $v_1, \dots v_{k-1}, \alpha v_k + \beta e_{d+1}$ is a basis of $\mathbb{R}\Delta$, so that $\delta_1 \wedge \dots \wedge \delta_k = v_1 \wedge \dots \wedge v_{k-1} \wedge (a v_k + b e_{d+1})$, where $\|\delta_1 \wedge \dots \wedge \delta_k\| = \sqrt{a^2 + b^2} \geq 1$.

$$(\delta_1 \wedge \dots \wedge \delta_k) u_{f(x)} = (b + a\langle v_k, f(x)\rangle)\, v_1 \wedge \dots \wedge v_{k-1} \wedge e_{d+1} + \dots$$

Since the basis is orthonormal,

$$\|(\delta_1 \wedge \dots \wedge \delta_k) u_{f(x)} a_n^{-1}\| \geq |\underbrace{b + a\langle v_k, f(x)\rangle}_{c_0 + \sum_{i=1}^{k} c_i f_i(x)}|$$

with $\sum_{i=0}^{k} c_i^2 \geq 1$.

Since $f(I) \not\subset$ proper affine subspace,

$$\max_{x \in I} \left| c_0 + \sum_{i=1}^{k} c_i f_i(x) \right| > 0.$$

$\square$