Lecture I: Quasi-independence, mixing and Khinchin Thm.

Given $x \in \mathbb{R}^d$, we would like to find (efficient) Diophantine approximation $x \approx \frac{p}{q} \in \mathbb{Q}^d$.

Dirichlet's Thm. $\forall x \in \mathbb{R}^d\ \forall R > 1:\ \exists p \in \mathbb{Z}^d, q \in \mathbb{N}:$

$$\begin{cases} \|x - \frac{p}{q}\| \leq \frac{R^{-1/d}}{q} \\ q \leq R \end{cases} \Longrightarrow \|x - \frac{p}{q}\| \leq q^{-1-\frac{1}{d}}.$$

Can this be improved?

Fix a continuous nonincreasing $\psi : \mathbb{R}^+ \to (0,1)$.

Def. $x \in \mathbb{R}^d$ is ψ-approximable if

$$\|x - \frac{p}{q}\| \leq \frac{\psi(q)}{q}$$

has infinitely many solutions $(p,q) \in \mathbb{Z}^d \times \mathbb{N}$.

$\mathcal{W}_d(\psi) = \{\psi\text{-approx. vectors in } \mathbb{R}^d\}$.

Khinchin's Thm.

1) $\sum_{q \geq 1} \psi(q)^d < \infty \Rightarrow \mathcal{W}_d(\psi)$ has measure 0.

2) $\sum_{q \geq 1} \psi(q)^d = \infty \Rightarrow \mathcal{W}_d(\psi)$ has full measure.

ex. for a.e. $x \in \mathbb{R}^d$, $\|x - \frac{p}{q}\| \leq q^{-1-\frac{1}{d}} (\log q)^{-1/d}$ has infinitely many solutions $(p,q) \in \mathbb{Z}^d \times \mathbb{N}$.

$W_d(\psi) = \{x : x$ belongs to infinitely many balls $B(\frac{p}{q}, \frac{\psi(q)}{q})\}$

$= \overline{\lim}\, B(\frac{p}{q}, \frac{\psi(q)}{q})$.

Borel-Cantelli Lemma. Let A_n be measurable set in a measure space (X, μ). Then

$$\sum_{n \geq 1} \mu(A_n) < \infty \Rightarrow \overline{\lim}\, A_n \text{ has measure } 0.$$

Hence, (1) follows from Borel-Cantelli Lemma.

Converse of Borel-Cantelli Lemma.

Let A_n be measurable set in a measure space (X, μ).

Assume that:

(i) $\sum_{n \geq 1} \mu(A_n) = \infty$

(ii) $\sum_{n,m \geq 1}^{N} \mu(A_n \cap A_m) \leq \left(\sum_{n=1}^{N} \mu(A_n)\right)^2 + c \cdot \left(\sum_{n=1}^{N} \mu(A_n)\right)$

(quasi-independence property)

Then $\overline{\lim}\, A_n$ has full measure.

Let $S_N(x) = \sum_{n=1}^{N} \chi_{A_n}(x)$ and $E_N = \sum_{n=1}^{N} \mu(A_n) \xrightarrow{(i)} \infty$.

Note that $x \in \overline{\lim} A_n \iff S_N(x) \to \infty$.

We have
$$\|S_N - E_N\|_2^2 = \left\langle \sum_{n=1}^{N} (\chi_{A_n} - \mu(A_n)), \sum_{m=1}^{N} (\chi_{A_m} - \mu(A_m)) \right\rangle$$
$$= \sum_{n,m=1}^{N} (\mu(A_n \cap A_m) - \mu(A_n)\mu(A_m))$$
$$\overset{(ii)}{\leq} c \cdot E_N.$$

Hence, $\left\| \frac{S_N}{E_N} - 1 \right\|_2 \leq \frac{c}{E_N} \to 0$.

Since $\frac{S_N}{E_N} \xrightarrow{L^2} 1$, $\frac{S_{N_i}(x)}{E_{N_i}} \xrightarrow{a.e.} 1$ along a subsequence.

Then $S_{N_i}(x) \xrightarrow{a.e.} \infty$ and $S_N(x) \xrightarrow{a.e.} \infty$ because $S_N(x)$ is monotone.

Space of lattices.

$\mathcal{L} = \{$ lattices in $\mathbb{R}^{d+1}$ with covol $= 1\}$

$\Lambda = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_{d+1}$, (v_i) is a basis of $\mathbb{R}^{d+1}$.

$Sl_{d+1}(\mathbb{R})$ acts transitively on $\mathcal{L}$:
$$\Lambda \longmapsto \Lambda \cdot g.$$

Hence, $\mathcal{L} \simeq Sl_{d+1}(\mathbb{Z}) \backslash Sl_{d+1}(\mathbb{R})$.

We also use that $\mathcal{L}$ supports $SL_{d+1}(\mathbb{R})$-inv. smooth probability measure.

<u>Dani correspondence</u>

For $x \in \mathbb{R}^d$, $\Lambda_x = \{(p+qx, q) : (p,q) \in \mathbb{Z}^{d+1}\} \in \mathcal{L}$

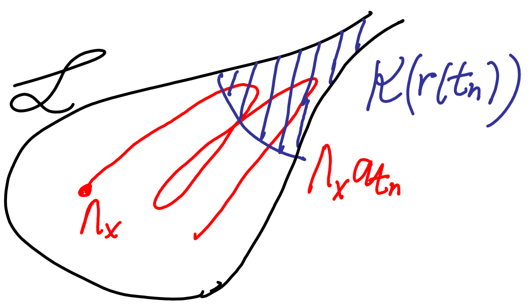
$$= \mathbb{Z}^{d+1} \left(\begin{array}{c|c} I & 0 \\ \hline x & 1 \end{array} \right).$$

Let $a_t = \left(\begin{array}{c|c} e^t I & 0 \\ \hline 0 & e^{-dt} \end{array} \right) \in SL_{d+1}(\mathbb{R})$.

Diophantine properties of $x \in \mathbb{R}^d$ $\Longleftrightarrow$ Distribution of the orbit $\Lambda_x a_t$

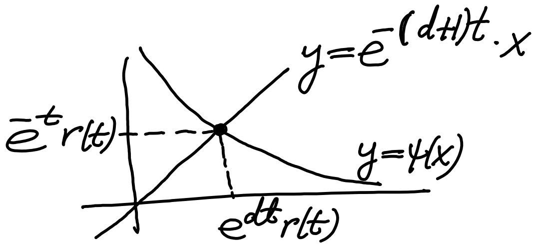
Let $\mathcal{K}(\varepsilon) = \{\Lambda \in \mathcal{L} : \min_{v \in \Lambda \setminus \{0\}} \|v\| \leq \varepsilon\}$.

<u>Prop.</u> (Kleinbock – Margulis) $\exists\, r = r_\psi : \mathbb{R}^+ \to (0,1)$:
$x \in \mathbb{R}^d$ is ψ-approximable $\Leftrightarrow$ $\Lambda_x a_{t_n} \in \mathcal{K}(r(t_n))$ for a sequence $t_n \to \infty$.



Shrinking Target Property

The function r is defined by



$$\psi(e^{dt}r(t))=e^{-t}r(t)$$

$\Longrightarrow$ Suppose that $\|x-\frac{p_n}{q_n}\|\le\frac{\psi(q_n)}{q_n}$ for $q_n\to\infty$.
Take t_n such that $q_n=e^{dt_n}r(t_n)$. Then $t_n\to\infty$,
and $\|x-\frac{p_n}{q_n}\|\le\dfrac{\psi(e^{dt_n}r(t_n))}{q_n}=\dfrac{e^{-t_n}r(t_n)}{q_n}$.
Hence, $\max\{e^{t_n}|q_nx-p_n|,\ e^{-dt_n}|q|\}\le r(t_n)$.
This shows that $\Lambda_x a_{t_n}\in K(r(t_n))$.

<u>Thm.</u> (exponential mixing) $\forall$ smooth $f_1,f_2:\mathcal{L}\to\mathbb{R}$:

$$\int_{\mathcal{L}} f_1(y a_t)f_2(y)\,dy=\int_{\mathcal{L}}f_1\cdot\int_{\mathcal{L}}f_2+O\left(e^{-\delta|t|}\cdot S(f_1)S(f_2)\right)$$

with $\delta>0$. (Here $S(f)$ denotes Sobolev norm).

Proof of Khinchin's Thm (Kleinbock–Margulis)

One can check that:

1) $\sum_{q\geq 1} \psi(q)^d = \infty \iff \sum_{n\geq 1} r(n)^{d+1} = \infty.$

2) $vol(K(\varepsilon)) \asymp \varepsilon^{d+1}.$

We claim that for a.e. x, $\Lambda_x a_n \in K(r(n))$ infinitely often.

We take a smooth function $f_n \approx \chi_{K(r(n))}$:

1) $supp(f_n) \subset K(r(n))$, $f_n \geq 0$.

2) $\int_{\mathcal{L}} f_n \asymp vol(K(r(n))) \asymp r(n)^{d+1}.$

3) $S(f_n) \ll \int_{\mathcal{L}} f_n.$

Then: $- \sum_{n\geq 1} \int_{\mathcal{L}} f_n = \infty,$

$- \sum_{n,m=1}^{N} \langle f_n \cdot a_n, f_m \cdot a_m \rangle \leq \left(\sum_{n=1}^{N} \int_{\mathcal{L}} f_n\right)^2 + c\cdot\left(\sum_{n=1}^{N} \int_{\mathcal{L}} f_n\right).$

Indeed, $\sum_{n,m=1}^{N} \left(\int_{\mathcal{L}} f_n(ya_n) f_m(ya_m)dy - \int_{\mathcal{L}} f_n \cdot \int_{\mathcal{L}} f_m\right)$

$$\ll \sum_{n,m=1}^{N} e^{-\delta|n-m|} S(f_n) S(f_m)$$

$$\ll \sum_{1\leq m\leq n\leq N} e^{-\delta(n-m)} \cdot \int_{\mathcal{L}} f_n \cdot 1 = \sum_{n\geq 1} \underbrace{\left(\sum_{m=1}^{n} e^{-\delta(n-m)}\right)}_{\text{uniformly bounded}} \cdot \int_{\mathcal{L}} f_n$$

As in the converse of Borel-Cantelli Lemma, we deduce that for a.e. $\Lambda \in \mathcal{L}$, $\sum_{n\geq 1} f_n(\Lambda a_n) = \infty$.

Hence, for a.e. $\Lambda \in Y$, $\Lambda a_n \in K(r(n))$ infinitely often.

Write $\Lambda = \mathbb{Z}^{d+1} g$ with $g \in SL_{d+1}(\mathbb{R})$.

For a.e. g, $g = \left(\begin{array}{c|c} I & 0 \\ \hline x & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} A & b \\ \hline 0 & c \end{array}\right)$, and

$$\mathbb{Z}^{d+1} g a_n = \underbrace{\mathbb{Z}^{d+1} \left(\begin{array}{c|c} I & 0 \\ \hline x & 1 \end{array}\right)}_{\Lambda_x} a_n \cdot \underbrace{\left(\begin{array}{c|c} A & e^{-(d+1)n} b \\ \hline 0 & c \end{array}\right)}_{\text{uniformly bounded}}$$

Hence, for a.e. $x \in \mathbb{R}^d$, $\Lambda_x a_n \in K(r(n))$ infinitely often.

By Prop., this implies Khinchin's Thm.