Magnetic bottles on the Poincaré half-plane: spectral asymptotics

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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^\star (id + A)$ on the Poincaré upper-half plane $\mathbb{H}$, when the magnetic field dA is infinite at infinity and such that $-\Delta_A$ has pure discret spectrum. We obtain the asymptotic behavior of the counting function of the eigenvalues.

Keywords: spectral asymptotics, magnetic bottles, hyperbolic plane, minimax principle

Résumé

On considère le Laplacien avec champ magnétique $-\Delta_A = (id+A)^\star (id+A)$ sur le demi-plan de Poincaré $\mathbb{H}$, dans le cas où le champ magnétique dA tend vers l'infini à l'infini, de sorte que $-\Delta_A$ ait un spectre discret. On donne le comportement asymptotique de la fonction de dénombrement des valeurs propres.

Mots-clés : asymptotique du spectre, bouteilles magnétiques, demi-plan hyperbolique, principe du minimax.

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1 Introduction

In this paper we study the asymptotic distribution of large eigenvalues of magnetic bottles on the hyperbolic plane $\mathbb{H}$. Magnetic bottles on $\mathbb{H}$ are Schrödinger operators of the form

$$-\Delta_A \;=\; y^2(D_x - A_1)^2 \;+\; y^2(D_y - A_2)^2 \;, \tag{1.1}$$

where the magnetic field dA is infinite at the infinity . This property ensures that $-\Delta_A$ has a compact resolvent. The precise formulation is given below.

In the Euclidean case the asymptotic distribution of large eigenvalues of magnetic bottles in $\mathbb{R}^d$ has been given by Yves Colin de Verdière [Col], using partition in cubes and estimations for constant magnetic fields in the cubes. This method can still be used here, but cubes are replaced by rectangles adapted to the hyperbolic geometry and the formula we get is of the same type, taking into account the hyperbolic volume and the hyperbolic definition for magnetic fields.

The hyperbolic framework we recall below has been used mainly for studying the Maass Laplacian, which corresponds to the constant magnetic field case. This case has been studied by many authors (see [Gro], [Els], [Com] [D-I-M]). In [In-Sh1] the authors consider asymptotically constant magnetic fields and in [In-Sh3] they deal with Pauli operators. See also [Ike] for relationship between Maass Laplacian and Schrödinger operators with Morse potentials.

From an other point of view , the asymptotic distribution of large eigenvalues in the hyperbolic context has already been studied for Schrödinger operators (without magnetic field) (see [In-Sh2]). The method is based on Feynman-Kac representation of the heat kernel and the Tauberian theorem. As already mentioned our own method involves only min-max techniques so it does not require to study properties of the evolution semigroup. It is also local, so our result is valid for many surfaces of infinite area with fundamental domain $\mathbb{H}$.

Let us now set up the hyperbolic framework of our problem.

In a connected and oriented Riemannian manifold (M, g) of dimension n, for any real one-form A on M, one can define the magnetic Laplacian

$$\begin{gathered} -\Delta_A \;=\; (i\ d + A)^\star (i\ d + A)\;, \\ (\ (i\ d + A)u = i\ du + uA\;,\ \forall\, u\ \in\ C_0^\infty(M)\ )\ . \end{gathered} \tag{1.2}$$

The magnetic field is the exact two-form $\rho_B \;=\; dA$.
The two-form ρ_B is associated with a linear operator B on the tangent space defined by

$$\rho_B(X, Y) \;=\; g(B.X, Y)\;; \quad \forall\, X\ ,\ Y\ \in TM \times TM\ . \tag{1.3}$$

The magnetic intensity $\mathbf{b}$ is given by

$$\mathbf{b} \;=\; \frac{1}{2} tr \left((B^\star B)^{1/2} \right)\ . \tag{1.4}$$

Let us assume that $dim(M) = 2$, and denote by dv the Riemannian measure on M; then $\rho_B \;=\; \widetilde{\mathbf{b}}\, dv$, with $|\widetilde{\mathbf{b}}| \;=\; \mathbf{b}$.
In this case, we can say that the magnetic field is constant iff $\widetilde{\mathbf{b}}$ is constant.

Now, we consider the case where $M \;=\; \mathbb{H}$ is the hyperbolic plane:

$$\mathbb{H} \;=\; \mathbb{R}\times]0, +\infty[\ , \quad g \;=\; \frac{dx^2 + dy^2}{y^2}\ , \quad A = A_1(x, y)\ dx \;+\; A_2(x, y)\ dy\ .$$

We will assume that

$$A_j(x, y)\ \in\ C^2(\mathbb{H}; \mathbb{R})\ , \quad \forall\, j\ . \tag{1.5}$$

Let us define $D_x = \frac{1}{i}\partial_x$ and $D_y = \frac{1}{i}\partial_y$. Then we have

$$-\Delta_A \;=\; y^2(D_x - A_1)^2 \;+\; y^2(D_y - A_2)^2 \;, \tag{1.6}$$

$$\widetilde{\mathbf{b}} \;=\; y^2\left(\partial_x A_2 - \partial_y A_1\right) \quad \mathbf{b} \;=\; |\widetilde{\mathbf{b}}| \;, \quad \text{and} \quad dv \;=\; y^{-2}dxdy \;.$$

It is well known that $-\Delta_A$ is essentially self-adjoint on $L^2(\mathbb{H})$, see for example [Shu].
As we are only interested on the spectrum of $\mathrm{sp}(-\Delta_A)$, we will use that it is gauge invariant:

$$\mathrm{sp}(-\Delta_A) \;=\; \mathrm{sp}(-\Delta_{A+d\varphi}) \;;\quad \forall\,\varphi \;\in\; C^2(\mathbb{H};\mathbb{R}) \;. \tag{1.7}$$

For an operator H , $\mathrm{sp}(H)$, $\mathrm{sp}_{es}(H)$, $\mathrm{sp}_a(H)$, $\mathrm{sp}_d(H)$ and $\mathrm{sp}_p(H)$ denote its spectrum, its essential part, its absolutely continuous part, its discret part and its ponctual part.

We will denote $-\Delta_A$ by $P(A)$.

2 The constant magnetic Laplacian on the hyperbolic plane

In this section, we explain how to get the well-known properties of the spectrum of a constant magnetic Laplacian on the hyperbolic plane. The original study was done by J. Elstrodt in [Els].

We consider the case where $y^2(\partial_x A_2(x,y) - \partial_y A_1(x,y))$ is constant. We choose a gauge such that $A_2 = 0$, so $A_1(x,y) = \pm\mathbf{b}y^{-1}$. We can assume that $A_1(x,y) = \mathbf{b}y^{-1}$, even if we change x into $-x$, which is a unitary operator on $L^2(\mathbb{H})$. The operator we are interested in is

$$-\Delta_{A^{\mathbf{b}}} \;=\; y^2(D_x - \mathbf{b}y^{-1})^2 \;+\; y^2 D_y^2 \;,\quad \text{with} \quad \mathbf{b} \geq 0 \quad \text{constant.} \tag{2.1}$$

Let U be the unitary operator

$$U \;:\; L^2(\mathbb{H}) \;\to\; L^2(\mathbb{R}\times\mathbb{R}_+^*) \;,\quad Uf \;=\; y^{-1}f \;; \tag{2.2}$$

$\mathbb{R}\times\mathbb{R}_+^*$ is endowed with the standard Lebesgue measure $dxdy$. Then

$$P_{\mathbf{b}} \;=\; U(-\Delta_{A^{\mathbf{b}}})U^\star \;=\; (D_x - \mathbf{b}y^{-1})y^2(D_x - \mathbf{b}y^{-1}) \;+\; D_y y^2 D_y \;. \tag{2.3}$$

Using partial Fourier transform we get that $\mathrm{sp}(P_{\mathbf{b}}) \;=\; \bigcup_{\xi\in\mathbb{R}} \mathrm{sp}(P_{\mathbf{b}}(\xi))$, where $P_{\mathbf{b}}(\xi)$ is the self-adjoint operator on $L^2(\mathbb{R}_+)$ defined by

$$P_{\mathbf{b}}(\xi)f \;=\; (y\xi - \mathbf{b})^2 f(y) + D_y(y^2 D_y f)(y) \;;\quad \forall\, f \;\in\; C_0^\infty(\mathbb{R}_+) \;. \tag{2.4}$$

When $\xi > 0$, by scaling, $y \;\to \xi^{-1}y$, we get that

$$\mathrm{sp}(P_{\mathbf{b}}(\xi)) \;=\; \mathrm{sp}(P_{\mathbf{b}}(1)) \;,\quad (\text{if} \quad \xi > 0) \;.$$

In the same way, we get that

$$\mathrm{sp}(P_{\mathbf{b}}(\xi)) \;=\; \mathrm{sp}(P_{\mathbf{b}}(-1)) \;,\quad \text{if} \quad \xi \;<\; 0 \;.$$

It is easy to see that $\mathrm{sp}_{es}(P_{\mathbf{b}}(\pm 1)) \;=\; \mathbf{b}^2 \;+\; \mathrm{sp}_{ac}(P_0(1)) \;=\; \mathrm{sp}_{ac}(P_{\mathbf{b}}(\pm 1))$, and, (see for example the exercise I6 p. 1573 in [Du-Sc]),

$$\mathrm{sp}(P_{\mathbf{b}}(-1)) \;=\; \mathrm{sp}_{ac}(P_{\mathbf{b}}(-1)) \;=\; [\mathbf{b}^2 + \frac{1}{4}, +\infty[\;=\; \mathrm{sp}_{ac}(P_{\mathbf{b}}(1)) \;. \tag{2.5}$$

$P_{\mathbf{b}}(1)$ may have some eigenvalues in $[\mathbf{b}, \mathbf{b}^2 + \frac{1}{4}[$.
For the proof, we use the method of [In-Sh1]. We define

$$K_{\mathbf{b}} \;=\; y - \mathbf{b} - 1 - iyD_y \;; \quad \text{so} \quad K_{\mathbf{b}}^{\star} \;=\; y - \mathbf{b} + iyD_y \;. \tag{2.6}$$

Then

$$K_{\mathbf{b}}^{\star} K_{\mathbf{b}} \;=\; P_{\mathbf{b}}(1) \;+\; \mathbf{b} \quad \text{and} \quad K_{\mathbf{b}} K_{\mathbf{b}}^{\star} \;=\; P_{\mathbf{b}+1}(1) \;-\; \mathbf{b} - 1 \;. \tag{2.7}$$

When $\mathbf{b} > 1/2$, we define

$$\varphi_{\mathbf{b}}(y) \;=\; \frac{2^{\mathbf{b}-1/2}}{\sqrt{\Gamma(2\mathbf{b}-1)}} y^{\mathbf{b}-1} e^{-y} \;, \quad \left(\varphi_{\mathbf{b}} \;\in\; \mathrm{Ker}(K_{\mathbf{b}-1}^{\star})\right) \;, \tag{2.8}$$

$\varphi_{\mathbf{b}}$ is the ground state of $P_{\mathbf{b}}(1)$: $\quad P_{\mathbf{b}}(1)\varphi_{\mathbf{b}} \;=\; \mathbf{b}\varphi_{\mathbf{b}}$.
As $K_{\mathbf{b}}\left(P_{\mathbf{b}}(1) + 2\mathbf{b} + 1\right) \;=\; P_{\mathbf{b}+1}(1) K_{\mathbf{b}}$ and $K_{\mathbf{b}}^{-1} f(y) \;=\; y^{-\mathbf{b}-1} e^{y} \int_y^{+\infty} s^{\mathbf{b}} e^{-s} f(s) ds$; $\forall\, f \;\in\; [\varphi_{\mathbf{b}+1}]^{\perp}$;
we get that, if $\mu + 2\mathbf{b} + 1 < \mathbf{b}^2 + \frac{1}{4}$, then

$$\mu \;\in\; \mathrm{sp}_d(P_{\mathbf{b}}(1)) \;\Rightarrow\; \mu + 2\mathbf{b} + 1 \;\in\; \mathrm{sp}_d(P_{\mathbf{b}+1}(1)) \;,$$

and if $\lambda - 2\mathbf{b} - 1 \geq \mathbf{b}$,

$$\lambda \;\in\; \mathrm{sp}_d(P_{\mathbf{b}+1}(1)) \setminus \{\mathbf{b}+1\} \;\Rightarrow\; \lambda - 2\mathbf{b} - 1 \;\in\; \mathrm{sp}_d(P_{\mathbf{b}}) \;.$$

One gets the well-known following theorem:

Theorem 2.1. *The spectrum of $P_{\mathbf{b}}(\pm 1)$ is formed by its absolutely continuous part and its discret part, and*

$$\begin{aligned}
\mathrm{sp}(P_{\mathbf{b}}(-1)) \;&=\; \mathrm{sp}_{ac}(P_{\mathbf{b}}(-1)) \;=\; \mathrm{sp}_{ac}(P_{\mathbf{b}}(1)) \;=\; [\mathbf{b}^2 + \frac{1}{4}, +\infty[\\
\mathrm{sp}(P_{\mathbf{b}}(1)) \;&=\; \mathrm{sp}_{ac}(P_{\mathbf{b}}(1)) \quad , \;\; if\ \mathbf{b} \;\leq\; \frac{1}{2} \\
\mathrm{sp}_d(P_{\mathbf{b}}(1)) \;&=\; \{(2j+1)\mathbf{b} - j(j+1) \;;\; j \;\in\; \mathbb{N} \;,\; j < \mathbf{b} - \frac{1}{2}\} \quad if\ \mathbf{b} \;>\; \frac{1}{2} \;.
\end{aligned}$$

Corollary 2.2. *The spectrum of $-\Delta_{A^{\mathbf{b}}}$ is essential:* $\mathrm{sp}(-\Delta_{A^{\mathbf{b}}}) = \mathrm{sp}_{es}(-\Delta_{A^{\mathbf{b}}})$.
Its absolutely continuous part is given by $\mathrm{sp}_{ac}(-\Delta_{A^{\mathbf{b}}}) \;=\; [\mathbf{b}^2 + \frac{1}{4}, +\infty[$.
The remaining part of its spectrum is empty if $0 \leq \mathbf{b} \leq 1/2$, otherwise it is formed by a finite number of eigenvalues of infinite multiplicity given by

$$\mathrm{sp}_p(-\Delta_{A^{\mathbf{b}}}) \;=\; \{(2j+1)\mathbf{b} - j(j+1) \;;\; j \;\in\; \mathbb{N} \;,\; j < \mathbf{b} - \frac{1}{2}\} \;, \quad (\text{if} \quad \frac{1}{2} < \mathbf{b} \;.)$$

3 The case of a magnetic bottle (with compact resolvent)

The following theorem deals with the case of a magnetic field which fulfills magnetic bottles type assumptions.

Theorem 3.1. *Under the assumptions (1.5) and (1.6), if*

$$\mathbf{b}(x,y) \;\to\; +\infty \quad \text{as} \quad d(x,y) \;\to\; +\infty \;, \tag{3.1}$$

and if $\exists\, C_0 > 0$ *such that, for any vector field* X *on* $\mathbb{H}$,

$$|X\widetilde{\mathbf{b}}| \leq C_0(|\widetilde{\mathbf{b}}| + 1)\sqrt{g(X,X)}\ ; \tag{3.2}$$

then $P(A) = -\Delta_A$ *has a compact resolvent.*
($d(x,y)$ *denotes the hyperbolic distance of* (x,y) *to* $(0,1)$) .

Proof: The standard proof for elliptic operators on the flat $\mathbb{R}^n$ can be applied using the estimate given by the following Lemma.

Lemma 3.2. *For any* $\epsilon \in]0,1[$, *there exists* $C_\epsilon > 0$ *s.t.*

$$\forall\, f \ \in\ C_0^\infty(\mathbb{H})\ ,\quad \int_{\mathbb{H}} \mathbf{b}|f|^2 dv \ \leq\ (1+\frac{\epsilon}{2})\langle -\Delta_A f | f\rangle_{L^2(\mathbb{H})} + C_\epsilon \|f\|_{L^2(\mathbb{H})}\ .$$

For the proof, one can use the unitary operator defined in 2.2
$U\ :\ L^2(\mathbb{H})\ \to\ L^2(\mathbb{R}\times\mathbb{R}_+^*)\ ,\quad Uf(x,y) = y^{-1}f(x,y).$
We get that $UP(A)U^\star = y^2(D_x - A_1)^2 + y(D_y - A_2)^2 y$
In this form, we can write $UP(A)U^\star \ =\ K^\star K + \widetilde{\mathbf{b}} = \widetilde{K}^\star\widetilde{K} - \widetilde{\mathbf{b}}$
with $K\ =\ y(D_x - A_1)\ -\ i(D_y - A_2)y$ and $\widetilde{K}\ =\ y(D_x - A_1)\ +\ i(D_y - A_2)y$. So

$$\pm\widetilde{\mathbf{b}}\ \leq\ UP(A)U^\star\ .$$

We cover $\mathbb{R}\times\mathbb{R}_+^*$ by two open sets $\mathcal{O}_0,\ \mathcal{O}_1$, such that $\mathcal{O}_0$ is bounded and y and $1/y$ are bounded on $\mathcal{O}_0$, and $1 \leq \mathbf{b}$ on $\mathcal{O}_1$.

Taking an associated partition of unity χ_j , $(j = 0,1)$, and using that $\pm\widetilde{\mathbf{b}} \leq UP(A)U^\star$, we get

$$\int_{\mathbb{R}\times\mathbb{R}_+^*} \mathbf{b}|\chi_1 f|^2 dxdy\ \leq\ \int_{\mathbb{R}\times\mathbb{R}_+^*} UP(A)U^\star(\chi_1 f)\overline{\chi_1 f}\,dxdy\ .$$

The Lemma comes easily from this estimate.

4 Spectral asymptotics for magnetic bottles

4.1 The main theorem

For a self-adjoint operator P, and for any real $\lambda \leq \inf \mathrm{sp}_{es}(P)$, we denote by $N(\lambda; P)$ the number of eigenvalues of P, (counted with their multiplicity), which are in $]-\infty,\lambda[$.

Theorem 4.1. *Under the assumptions of Theorem 3.1,* $P(A) = -\Delta_A$ *has a compact resolvent and for any*
$\delta\ \in]\frac{1}{3},\frac{2}{5}[$, *in (4.15), there exists a constant* $C\ >\ 0$ *such that*

$$\frac{1}{2\pi}\int_{\mathbb{H}}(1 - \frac{C}{(\mathbf{b}(m)+1)^{(2-5\delta)/2}})\mathbf{b}(m)\sum_{k=0}^{+\infty}[\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0\ dv$$
$$\leq\ N(\lambda, -\Delta_A)\ \leq \tag{4.1}$$
$$\frac{1}{2\pi}\int_{\mathbb{H}}(1 + \frac{C}{(\mathbf{b}(m)+1)^{(2-5\delta)/2}})\mathbf{b}(m)\sum_{k=0}^{+\infty}[\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0\ dv$$

$[\rho]_+^0$ *is the Heaviside function:*

$$[\rho]_+^0\ =\ \begin{cases} 1\,, & \text{if}\quad \rho > 0\\ 0\,, & \text{if}\quad \rho \leq 0\,. \end{cases}$$

This result can be compared to the one obtained in [Col]. The difference between the two results is the additional term $-\frac{1}{4}$, which comes from the geometry of the problem. It becomes really significant in the following

Corollary 4.2. *Under the assumptions of Theorem 3.1 and if the function*

$$\omega(\mu) \;=\; \int_{\mathbb{H}} [\mu - \mathbf{b}(m)]_+^0 dv$$

satisfies

$$\exists\, C_1 > 0 \text{ s.t. } \forall\, \mu > C_1 \;,\; \forall\, \tau \;\in\,]0,1[\;, \quad \omega\;((1+\tau)\;\mu) - \omega(\mu) \leq C_1\;\tau\;\omega(\mu)\;, \tag{4.2}$$

then

$$N(\lambda; -\Delta_A) \;\sim\; \frac{1}{2\pi}\int_{\mathbb{H}} \mathbf{b}(m)\sum_{k\in\mathbb{N}}[\lambda - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \; dv\;. \tag{4.3}$$

The assumption (4.2) is satisfied when $\omega(\lambda) \;\sim\; \alpha\lambda^k \ln^j(\lambda)$ when $\lambda \to +\infty$, with $k > 0$, or $k = 0$ and $j > 0$.

For example this allows us to consider magnetic fields of the type $\mathbf{b}(x,y) \;=\; \left(\frac{x}{y}\right)^{2j} + g(y)$, with $j \;\in\; \mathbb{N}^\star$ and $g(y) \;=\; p_1(y) \;+\; p_2(1/y)$, where $p_1(s)$ and $p_2(s)$ are, for large s, polynomial functions of order ≥ 1 . In this case $\omega(\lambda) \;\sim\; \alpha\lambda^{\frac{1}{2j}}\ln(\lambda)$ when $\lambda \to +\infty$.

For the proof of Theorem 4.1, we will establish some transformations, prove some technical lemmas and then use the minimax technique on quadratic forms as in Colin de Verdière's result to get successively a lower bound and an upper bound for $N(\lambda; -\Delta_A)$.

4.2 Technical transformations

4.2.1 Change of variables

Let us consider the diffeomorphism
$\phi \;:\; \mathbb{R}^2 \;\to\; \mathbb{H}\;, \quad (x,y) = \phi(x,t) := (x, e^t)$
which induces a unitary operator
$\widehat{U} \;:\; L^2(\mathbb{H}; dv) \;\to\; L^2(\mathbb{R}^2; dxdt)$
$(\widehat{U}f)(x,t) := e^{-t/2}f(x,e^t)$ for any $f \in L^2(\mathbb{H})$.
$\widehat{U}$ maps $C_0^\infty(\mathbb{H})$ onto $C_0^\infty(\mathbb{R}^2)$ and the inverse $\widehat{U}^{-1}$ is given by
$(\widehat{U}^{-1}g)(x,y) := y^{1/2}g(x,\ln y)$ for each $g \in L^2(\mathbb{R}^2)$.

The quadratic form related to the operator $P(A) = -\Delta_A$ is given, for any $u \in L^2(\mathbb{H})$, by

$$q(u) := \int_{\mathbb{H}^2} \left[|y(D_x - A_1)u|^2 + |y(D_y - A_2)u|^2\right] \frac{dxdy}{y^2}$$

$$= \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)u(\phi)|^2 + |e^t(e^{-t}D_t - \tilde{A}_2)u(\phi)|^2\right] e^{-t}dxdt$$

$$= \int_{\mathbb{R}^2} \left[|e^{t/2}(D_x - \tilde{A}_1)u(\phi)|^2 + |e^{t/2}(e^{-t}D_t - \tilde{A}_2)u(\phi)|^2\right] dxdt$$

with

$$\tilde{A}_i(x,t) := A_i(x,e^t) \quad , \; i = \; 1,2 \;\;.$$

After defining $w := \widehat{U}u$, the preceding form becomes

$$\widehat{q}(w) := \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)w|^2 + |(e^{-t/2}D_t e^{t/2} - e^t \tilde{A}_2)w|^2 \right] dxdt$$

so

$$\widehat{P}(\tilde{A}) := \widehat{U}P(A)\widehat{U}^{-1} = e^{2t}(D_x - \tilde{A}_1)^2 + (D_t - e^t\tilde{A}_2)^2 + 1/4.$$

4.2.2 Gauge

We want to work with a gauge such that $A_2 = 0$. Since

$$\widetilde{\mathbf{b}} \;=\; y^2\,(\partial_x A_2 - \partial_y A_1)$$

we can take

$$A_1(x,y) = -\int_1^y \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds$$

which gives

$$\tilde{A}_1(x,t) \;:= -\int_1^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds \tag{4.4}$$

$$\text{and} \quad \widehat{P}(\tilde{A}) = e^{2t}\left[D_x + \int_1^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds \right]^2 + D_t^2 + 1/4.$$

The associated quadratic form is

$$\widehat{q}^{\tilde{A}}(w) = \int_{\mathbb{R}^2} \left[|e^t(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2 \right] dxdt\ .$$

An application of the assumption (3.2) is the following Lemma.

Lemma 4.3. *For any $a > 0$ and any $\varepsilon_0 > 0$ small enough*

$$(\varepsilon_0 < \min\{\frac{1}{2}, \frac{1}{C_0(a+1)}\})\ ,\textit{there exists}\ \ C_1 > 1\ \ \textit{such that}\ ,$$

if $(x_0, y_0) \in \mathbb{H}$ and $\mathbf{b}(x_0, y_0) > 1$, then

$$\frac{1}{C_1}\ \mathbf{b}(x_0,y_0) \le \mathbf{b}(x,y) \le C_1\ \mathbf{b}(x_0,y_0)\ ; \quad \forall\ (x,y) \in \Omega(x_0,y_0,a,\varepsilon_0)$$

where $\Omega(x_0,y_0,a,\varepsilon_0) := \{(x,y)\ /\ |x-x_0| \le a\varepsilon_0\ y_0,\ |y-y_0| \le \varepsilon_0 y_0\}$.

The proof comes directly from the assumption (3.2). Performing Taylor expansion , we get

$$|\widetilde{\mathbf{b}}(x,y) - \widetilde{\mathbf{b}}(x_0,y_0)| \le (|x-x_0| + |y-y_0|) \sup_{z\in\Omega} (|\partial_x \widetilde{\mathbf{b}}(z)| + |\partial_y \widetilde{\mathbf{b}}(z)|)$$

so $|\widetilde{\mathbf{b}}(x,y) - \widetilde{\mathbf{b}}(x_0,y_0)| \le \varepsilon_0 C_0(a+1) y_0 \sup_{z\in\Omega} \dfrac{\mathbf{b}(z)+1}{y}$
and the proof follows easily.

4.3 Technical lemmas

4.3.1 Localization in a suitable rectangle in $\mathbb{R}^2$

Let $a_0 > 1$ be given.

Any nonnegative constant depending only on a_0, will be denoted invariably C.

Let $X_0 = (x_0, t_0) \in \mathbb{R}^2$ such that $\mathbf{b}(z_0) > 1$; $(z_0 = (x_0, e^{t_0}))$; $|X_0|$ can be very large.

Let us choose $\varepsilon_0 \in]0, 1[$, ε_0 can be very small.

For $a \in]\frac{1}{a_0}, a_0]$, let

$$K := X_0 + K_0 \ , \quad K_0 \ =] - \varepsilon_0 a \frac{e^{t_0}}{2}, \ \varepsilon_0 a \frac{e^{t_0}}{2} [\times] - \frac{\varepsilon_0}{2}, \ \frac{\varepsilon_0}{2} [\ . \tag{4.5}$$

We consider the Dirichlet operator $P_K(\tilde{A})$ on K associated to the quadratic form

$$\widehat{q}_K^{\tilde{A}}(w) = \int_K \left[|e^t(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2 \right] dxdt \quad \forall \ w \ \in \ W_0^1(K) \ .$$

We are interested only by the spectrum of $P_K(\tilde{A})$. It is gauge invariant,

$$sp(P_K(\tilde{A})) \ = \ sp(P_K(\tilde{A} + \nabla\varphi)) \ , \tag{4.6}$$

so by taking $\varphi(x,t) = -\int_0^x \widetilde{A_1}(s, t_0) ds$, we can assume that

$$\tilde{A}_1(x,t) \ := - \int_{e^{t_0}}^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds \quad (\text{and} \quad \tilde{A}_2 = 0 \ ) \ .$$

Let us define the magnetic potential related to a constant magnetic field

$$A^0(x,t) = (A_1^0, 0) \ \text{ with } \quad A_1^0 := -(t - t_0) \ e^{-t_0} \ \widetilde{\mathbf{b}}(x_0, e^{t_0}) \ . \tag{4.7}$$

We want to compare $N(\lambda;\ P_K(\tilde{A}))$ to $N(\lambda;\ P_K^0(A^0))$ for $\lambda >> 1$, where $P_K^0(A^0)$ is the Dirichlet operator on K, associated to the quadratic form

$$\widehat{q}_K^{A^0,0}(w) = \int_K \left[|e^{t_0}(D_x - A_1^0)w|^2 + |D_t w|^2 + 1/4|w|^2 \right] dxdt \quad \forall \ w \ \in \ W_0^1(K) \ .$$

We begin with comparing the associated magnetic potentials.

Lemma 4.4. *Under the above assumptions there exists a constant C, depending only on a_0 in (4.5), such that for any $(x,t) \in K$:*

$$|\tilde{A}_1(x,t) - A_1^0(x,t)| \ \le \ C \ \varepsilon_0^2 \ e^{-t_0} \ \mathbf{b}(x_0, e^{t_0}) \ .$$

Proof: As

$$\tilde{A}_1(x,t) \ := - \int_{e^{t_0}}^{e^t} \frac{\widetilde{\mathbf{b}}(x,s)}{s^2} ds \ , \tag{4.8}$$

there exists $\tau = \tau(x) \in]t_0, t[$ such that

$$\tilde{A}_1(x,t) = -(e^t - e^{t_0}) \frac{\widetilde{\mathbf{b}}(x, e^\tau)}{e^{2\tau}} = -e^{t_0}(e^{t-t_0} - 1) \frac{\widetilde{\mathbf{b}}(x, e^\tau)}{e^{2\tau}} \tag{4.9}$$

Writing

$$\mathcal{A} := e^{t_0} \frac{\widetilde{\mathbf{b}}(x, e^\tau)}{e^{2\tau}} - \frac{\widetilde{\mathbf{b}}(x_0, e^{t_0})}{e^{t_0}}$$

we get from the definition 4.7

$$|\tilde{A}_1(x,t) - A_1^0(x,t)| \leq C|t-t_0||\mathcal{A}| \ \leq C\ \varepsilon_0\ |\mathcal{A}|\ .$$

But from the lemma 4.3, we get the following estimate for any $(x,\tau) \in K$:

$$|\mathcal{A}| = |e^{t_0}\frac{\widetilde{\mathbf{b}}(x,e^\tau)}{e^{2\tau}} - \frac{\widetilde{\mathbf{b}}(x_0,e^{t_0})}{e^{t_0}}| \leq C\ \varepsilon_0\ e^{-t_0}\ \mathbf{b}(x_0,e^{t_0})\ .$$

To see this we decompose $\mathcal{A}$ in 3 parts

$$\mathcal{A}_1 = e^{t_0-2\tau}\ (\widetilde{\mathbf{b}}(x,e^\tau) - \widetilde{\mathbf{b}}(x_0,e^\tau))$$

$$\mathcal{A}_2 = e^{t_0-2\tau}\ (\widetilde{\mathbf{b}}(x_0,e^\tau) - \widetilde{\mathbf{b}}(x_0,e^{t_0}))$$

$$\mathcal{A}_3 = e^{t_0}\widetilde{\mathbf{b}}(x_0,e^{t_0})\ (\frac{1}{e^{2\tau}} - \frac{1}{e^{2t_0}})$$

According to the assumption (3.2) and to the Lemma 4.3 we have

$$(\widetilde{\mathbf{b}}(x,e^\tau) - \widetilde{\mathbf{b}}(x_0,e^\tau)) \leq e^{t_1-t_0}\ \mathbf{b}(x_0,e^{t_0})\ ,$$

so

$$|\mathcal{A}_1| \leq e^{-t_0}C|x-x_0|e^{-t_0}\ \mathbf{b}(x_0,e^{t_0}) \leq Ca\varepsilon_0 e^{-t_0}\ \mathbf{b}(x_0,e^{t_0})\ ,$$

$$|\mathcal{A}_2| \leq e^{-t_0}C|e^\tau - e^{t_0}|e^{-t_0}\ \mathbf{b}(x_0,e^{t_0})\ \ \leq Ce^{-t_0}|\tau - t_0|\ \mathbf{b}(x_0,e^{t_0}) \leq C\varepsilon_0 e^{-t_0}\mathbf{b}(x_0,e^{t_0})\ ,$$

The third term is also bounded by the same expression

$$|\mathcal{A}_3| \leq C\ \varepsilon_0\ e^{-t_0}\ \mathbf{b}(x_0,e^{t_0})\ \ ,$$

so we finished the proof.

4.3.2 Quadratic forms on K

Let us define

$$\widehat{q}_K^{\tilde{A},0}(w) := \int_K \left[|e^{t_0}(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 + 1/4|w|^2\right] dxdt\ ,\ \forall w \in W_0^1(K)\ .$$

Lemma 4.5. *There exists a constant C depending only on a_0 of (4.5), s.t.*

$$(1-\varepsilon_0 C)\ \widehat{q}_K^{\tilde{A},0}(w) \leq \widehat{q}_K^{\tilde{A}}(w) \leq (1+\varepsilon_0 C)\widehat{q}_K^{\tilde{A},0}(w)\ .$$

Proof : Write

$$\widehat{q}_K^{\tilde{A}}(w) = \int_K \left[e^{2(t-t_0)}|e^{t_0}(D_x - \tilde{A}_1)w|^2 + |D_t w|^2 dxdt| + 1/4|w|^2\right]\ dxdt$$

and use that $|t-t_0| \leq 1$ in K.

Lemma 4.6. *There exists a constant C depending only on a_0 of (4.5), such that, for any* $\tau\ \in]0,1[$, *(with $z_0 = (x_0,e^{t_0})$),*

$$(1-\tau^2)\ \widehat{q}_K^{A^0,0}(w) + (1-\frac{1}{\tau^2})\ C\ \varepsilon_0^4\ \mathbf{b}^2(z_0)\|w\|^2 \leq \widehat{q}_K^{\tilde{A},0}(w)$$

$$\leq\ (1+\tau^2)\widehat{q}_K^{A^0,0}(w) + (1+\frac{1}{\tau^2})\ C\ \varepsilon_0^4\ \mathbf{b}^2(z_0)\|w\|^2\ .$$

Proof : This is a straightforward application of lemma 4.4, when we write

$$e^{t_0}(D_x - \tilde{A}_1)w = e^{t_0}(D_x - A_1^0)w - e^{t_0}(\tilde{A}_1 - A_1^0)w\,.$$

4.3.3 Spectral asymptotics for a rectangle.

An immediate application of Theorem A.2 in the appendix is the following Lemma.

Lemma 4.7. *For any real* λ,

$$N(\lambda, P_K^0(A^0)) \;\leq\; \frac{|K| b(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\lambda - \frac{1}{4} - (2k+1) b(x_0, e^{t_0})]_+^0 \,. \tag{4.10}$$

Moreover, there exists a constant C_0 *depending only on* a_0 *of (4.5), such that, if* $\varepsilon_0^{-2}/C_0 \leq b(x_0, e^{t_0}) \leq \lambda$, *then* $\forall\, \tau \in]0,1[$,

$$(1-\tau)^2 \frac{|K| b(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\lambda - \frac{1}{4} - \frac{C_0}{(\tau \varepsilon_0)^2} - (2k+1) b(x_0, e^{t_0})]_+^0 \;\leq\; N(\lambda, P_K^0(A^0)) \,. \tag{4.11}$$

Proof. Change variables $(x,t) \;\to\; (\xi, \theta) = (e^{-t_0}(x - x_0), t - t_0)$ and apply (A.3) to get (4.10), and (A.4) to get (4.11).

Taking into account (4.6), Lemmas 4.5 - 4.7, we get the following proposition.

Proposition 4.8. *There exists a constant* $C_1 > 1$ *depending only on* a_0 *of (4.5), such that for any* $\varepsilon_0 \in]0, 1/(2C_1)[$, *for any real* $\lambda > 1$ *and for any* $\eta \in]0, 1/2[$,

$$N(\lambda, P_K(\tilde{A})) \;\leq\; \frac{|K| b(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\Lambda_M(\lambda) - \frac{1}{4} - (2k+1) b(x_0, e^{t_0})]_+^0 \,, \tag{4.12}$$

with $\Lambda_M(\lambda) \;=\; (1-\eta^2)^{-1}[\frac{\lambda}{1 - \varepsilon_0 C_1} + \frac{\varepsilon_0^4}{\eta^2} C_1 b^2(x_0, e^{t_0})]$.

Moreover, there exists a constant C_0 *depending only on* a_0 *of (4.5), such that for any* $\varepsilon_0 \in]0, 1/(2C_1)[$, *for any real* $\lambda > 1$ *and for any* $\eta \in]0, 1/2[$, *if* $\varepsilon_0^{-2}/C_0 \leq b(x_0, e^{t_0})$, *then* $\forall\, \tau \in]0,1[$,

$$(1-\tau)^2 \frac{|K| b(x_0, e^{t_0})}{2\pi e^{t_0}} \sum_{k=0}^{+\infty} [\Lambda_m(\lambda) - \frac{1}{4} - \frac{C_0}{(\tau \varepsilon_0)^2} - (2k+1) b(x_0, e^{t_0})]_+^0 \;\leq\; N(\lambda, P_K(\tilde{A})) \,, \tag{4.13}$$

with $\Lambda_m(\lambda) \;=\; (1+\eta^2)^{-1}[\frac{\lambda}{1 + \varepsilon_0 C_1} - \frac{\varepsilon_0^4}{\eta^2} C_1 b^2(x_0, e^{t_0})]$.

Without the condition that $\epsilon_0^{-2}/C_0 \leq b(x_0, e^{t_0})$, *we have in the same way that*

$$\frac{|K| b(x_0, e^{t_0})}{4\pi e^{t_0}} [\Lambda_m(\lambda) - \frac{1}{4} - C_0(\sqrt{\lambda}(1 + b(x_0, e^{t_0}))^{1/2}] \;\leq\; N(\lambda, P_K(\tilde{A})) \,. \tag{4.14}$$

For the lower bound (4.14), use the same method as for (4.13), by using the lower bound in (A.5) instead of (A.4).

4.4 Lower bound and upper bound for the $N(\lambda; -\Delta_A)$

4.4.1 A partition adapted to b

Let a_0 and δ_0 be given such

$$1 \;<\; a_0 \quad \text{and} \quad \delta_0 \;\in]\frac{1}{3}, \frac{2}{5}[\,. \tag{4.15}$$

For any $\alpha \in \mathbb{Z}^2$, we denote the rectangle

$$K(\alpha) =]-\frac{e^{\alpha_2}}{2} + e^{\alpha_2}\alpha_1 , e^{\alpha_2}\alpha_1 + \frac{e^{\alpha_2}}{2}[\times]-\frac{1}{2} + \alpha_2 , \alpha_2 + \frac{1}{2}[. \qquad (4.16)$$

So $\mathbb{R}^2 = \cup_\alpha \overline{K}(\alpha)$ and $K(\alpha) \cap K(\beta) = \emptyset$ if $\alpha \neq \beta$. Taking into account Lemma 4.3, each $K(\alpha)$ can be parted, (if necessary), into $M(\alpha)$ rectangles:

$$\overline{K}(\alpha) = \cup_{j=1}^{M(\alpha)} \overline{K_{\alpha,j}} , \quad K_{\alpha,j} =]-\frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2} + x_{\alpha,j} , x_{\alpha,j} + \frac{\epsilon_{\alpha,j} e^{t_{\alpha,j}}}{2}[\times]-\frac{\epsilon_{\alpha,j}}{2} + t_{\alpha,j} , t_{\alpha,j} + \frac{\epsilon_{\alpha,j}}{2}[, \qquad (4.17)$$

with

$$\frac{1}{a_0(1 + \mathbf{b}^{\delta_0}(x_{\alpha,j}, e^{t_{\alpha,j}}))} \leq \epsilon_{\alpha,j} \leq \frac{a_0}{(1 + \mathbf{b}^{\delta_0}(x_{\alpha,j}, e^{t_{\alpha,j}}))} , \qquad (4.18)$$

and such that $K_{\alpha,k} \cap K_{\alpha,j} = \emptyset$ if $k \neq j$.

We will denote $\Gamma = \{(\alpha, j) ; \alpha \in \mathbb{Z}^2 , j \in \{1, \ldots, M(\alpha)\}\}$, $X_\gamma = (x_\gamma, t_\gamma)$ the center of K_γ , $(\gamma \in \Gamma)$, and $z_\gamma = (x_\gamma, e^{t_\gamma})$.

4.4.2 The lower bound estimate

Proposition 4.9. *Under the assumptions of Theorem 3.1 and on* δ_0 *in (4.15), there exists a constant* $C_0 > 0$ *such that*

$$\frac{1}{2\pi}\int_{\mathbb{H}} (1 - \frac{C_0}{(\mathbf{b}(m) + 1)^{(2-5\delta_0)/2}})\mathbf{b}(m) \sum_{k=0}^{+\infty} [\lambda(1 - C_0\lambda^{-3\delta_0+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \, dv \leq N(\lambda, -\Delta_A) . \qquad (4.19)$$

Proof. Any constant depending only on the assumptions will be denoted invariably by C.
As $\mathbb{R}^2 = \bigcup_{\gamma\in\Gamma} \overline{K}_\gamma$, and $K_\gamma \cap K_\rho = \emptyset$ if $\gamma \neq \rho$, we get that $\sum_{\gamma\in\Gamma} N(\lambda, P_{K_\gamma}(\tilde{A})) \leq N(\lambda, -\Delta_A)$.
For large $|\gamma|$, we use the lower bound estimate (4.13), with

$$\eta^2 = \epsilon_\gamma = b^{-\delta_0}(z_\gamma) \text{ and } \tau = b^{-(5\delta_0-2)/2}(z_0) ,$$

and also the fact that on K_γ , $|e^{-t}\mathbf{b}(x, e^t) - e^{-t_\gamma}\mathbf{b}(z_\gamma)| \leq \epsilon_\gamma C$.
For small $|\gamma|$, (even the γ such that $b(z_\gamma) \leq \lambda^{1-2(3\delta_0-1)}$), we use (4.14) instead of (4.13), taking into account the Remark A.3. The lower bound (4.19) comes easily.

4.4.3 The upper bound estimate

Proposition 4.10. *Under the assumptions of Theorem 3.1 and on* δ_0 *in (4.15), there exists a constant* $C_0 > 0$ *such that*

$$N(\lambda, -\Delta_A) \leq \qquad (4.20)$$

$$\frac{1}{2\pi}\int_{\mathbb{H}} (1 + \frac{C_0}{(\mathbf{b}(m) + 1)^{(2-5\delta_0)/2}})\mathbf{b}(m) \sum_{k=0}^{+\infty} [\lambda(1 + C_0\lambda^{-3\delta_0+1}) - \frac{1}{4} - (2k+1)\mathbf{b}(m)]_+^0 \, dv .$$

Proof. Any constant depending only on the assumptions will be denoted invariably by C. We keep the partition and the notation used in the lower bound.

We consider the covering of $\mathbb{R}^2$ by open rectangles:

$$\mathbb{R}^2 \;=\; \bigcup_{\gamma\in\Gamma} \mathcal{K}_\gamma \;, \quad \mathcal{K}_\gamma \;=\; X_\gamma + (1/\tau_\gamma)(K_\gamma - X_\gamma) \;,$$

with $\tau_\gamma \in b^{-(5\delta_0-2)/2}(z_\gamma)[a_0^{-1}, a_0]$. Then there exists a partition of unity $(\chi_\gamma(x,t))$ satisfying

$$\left\{ \begin{array}{c} \sum_\gamma \chi_\gamma^2 = 1 \\ \text{support}(\chi_\gamma) \subset \mathcal{K}_\gamma \\ |D_x\chi_\gamma| \le C/(e^{t_\gamma}\epsilon_\gamma\tau_\gamma) \\ |D_t\chi_\gamma| \le C/(\epsilon_\gamma\tau_\gamma) \end{array} \right\} \tag{4.21}$$

We write $\hat{q}^{\tilde{A}}(w) \;=\; \sum_\gamma \left[\hat{q}^{\tilde{A}}_{\mathcal{K}_\gamma}(\chi_\gamma w) - \int_{\mathcal{K}_\gamma} V|\chi_\gamma w|^2 dxdt \right]$,
with $V(x,t) = \sum_\gamma [|D_x\chi_\gamma(x,t)|^2 + |D_t\chi_\gamma(x,t)|^2]$.
Thus on $\mathcal{K}_\gamma$, $V(x,t) \le C/(\epsilon_\gamma\tau_\gamma)^2$, and it follows easily from the min-max principle that

$$N(\lambda, -\Delta_A) \;\le\; \sum_\gamma N(\lambda + \frac{C}{(\epsilon_\gamma\tau_\gamma)^2},\; P_{\mathcal{K}_\gamma}(\tilde{A})) \;. \tag{4.22}$$

Then we get (4.20) from (4.22), as for (4.19), but using only (4.12), (instead of (4.13) and (4.14)).

APPENDIX

A Constant magnetic laplacian on the flat plane

A.1 The density of states for the euclidian constant magnetic field

Let us consider on $L^2(\mathbb{R}^2)$ the Schrödinger operator with a constant magnetic field $H_0 = (D_x - b\frac{y}{2})^2 + (D_y + b\frac{x}{2})^2$; ($b > 0$ is a constant).
The density of states of H_0, $\mathcal{N}(\lambda, H_0)$, is defined, (see [D-I-M]), by

$$\mathcal{N}(\lambda, H_0) \;=\; \lim_{R\to\infty} \frac{N(\lambda, H_0^{\Omega_R})}{|\Omega_R|} \;; \tag{A.1}$$

Ω_R is any bounded open domain of $\mathbb{R}^2$, with Lipschitz boundary, containing $(]-\frac{R}{2},\ \frac{R}{2}[)^2$,
and $H_0^{\Omega_R}$ is any self-adjoint operator on $L^2(\Omega_R)$ associated to the quadratic form of H_0 , with domain included in the Sobolev space $W^1(\Omega_R)$.

Theorem A.1. *The Colin de Verdière formula holds for any* $\lambda > 0$:

$$\mathcal{N}(\lambda, H_0) \;=\; \frac{b}{2\pi}\sharp\{n \in \mathbb{N}\;;\; (2n+1)b < \lambda\} \;. \tag{A.2}$$

Proof: Its comes easily from THEOREME 1.6 of [Dem].
Let us sketch a proof.
By scaling and dividing λ by b, we need only to establish the formula when $b = 1$.
We take $\Omega_R = (]-\frac{R}{2}, \frac{R}{2}[)^2$, the Dirichlet boundary conditions on $x = \pm\frac{R}{2}$ and the Floquet conditions: $e^{ixy/2}u(x,y)$ is R-periodic in y.

As $u \to ue^{ixy/2}$ is a unitary operator, by performing this gauge transform, $H_0^{\Omega_R}$ becomes $H_0 = D_x^2 + (D_y + x)^2$, for the Dirichlet boundary conditions on $x = \pm\frac{R}{2}$ and the periodic ones on $y = \pm\frac{R}{2}$.

$H_0^{\Omega_R}$ and H_0 have the same spectrum.

Using discret Fourier expansion, we get that $N(\lambda, H_0^{\Omega_R}) = \sum_{k\in\mathbb{Z}} N(\lambda, H_{k,R})$,

where $H_{k,R}$ is the Dirichlet operator on $I_R =]-\frac{R}{2}, \frac{R}{2}[$, associated to the harmonic oscillator $D_x^2 + (\frac{2k\pi}{R} + x)^2$.

As $N(\lambda, H_{k,R}) = 0$ when $|k| > \frac{1}{2\pi}(R\sqrt{\lambda} + \frac{R^2}{2})$, we get

$$\mathcal{N}(\lambda, H_0) \le \sharp\{n \in \mathbb{N};\ 2n+1 < \lambda\} \times \lim_{R\to\infty} \frac{1}{\pi R^2}(R\sqrt{\lambda} + \frac{R^2}{2}) ,$$

or equivalently $\mathcal{N}(\lambda, H_0) \ \le\ \frac{1}{2\pi}\sharp\{n \in \mathbb{N};\ 2n+1 < \lambda\}$.

Now, for any fixed $\epsilon \in]0,1[$, (for example $\epsilon = 1/\sqrt{R}$), and for any k such that $|k| \le (1-\epsilon)\frac{R^2}{4\pi}$, the exponential decreasing of the eigenfunctions of the harmonic oscillator on $\mathbb{R}^2$ leads to $N(\lambda, H_{k,R}) \ \ge \sharp\{n \in \mathbb{N};\ 2n+1 < \lambda - \dfrac{C_\lambda}{\epsilon^2 R^2}\}$, where C_λ depends only on λ.

To see this, with ϵ chosen as previously and $R >> \lambda + 1$, just use the fact that, for any $u \ \in\ \chi(4x/\sqrt{R})E_\lambda(H_{k,\infty})[L^2(\mathbb{R})]$,

$$\int_{-R/2}^{R/2} H_{k,R}\ u\overline{u}\ dx \ \le\ (\lambda + C/R)\int_{-R/2}^{R/2} |u|^2 dx\ .$$

$H_{k,\infty}$ denotes the harmonic oscillator $D_x^2 + (\frac{2k\pi}{R} + x)^2$ on $L^2(\mathbb{R})$, χ is a cut-off function, supported in $[-1,1]$ and equal to 1 in $[-1/2, 1/2]$, and $E_\lambda(H_{k,\infty})$ denotes the spectral projection on $]-\infty, \lambda[$ of the self-adjoint operator $H_{k,\infty}$.

Then, with the same ϵ and using the left-hand side continuity of the function $\lambda \to \sharp\{n \in \mathbb{N};\ 2n+1 < \lambda\}$, we get also that

$$\mathcal{N}(\lambda, H_0) \ \ge\ \frac{1}{2\pi}\sharp\{n \in \mathbb{N};\ 2n+1 < \lambda\}\ .$$

A.2 Eigenvalues estimate in the euclidian rectangle for a constant magnetic field

Let us consider the Dirichlet problem $H_{D,b}^{\Omega_R}$ associated to the Schrödinger operator with a constant magnetic field $H_0 = (D_x - b\dfrac{y}{2})^2 + (D_y + b\dfrac{x}{2})^2$; ($b\ >\ 0$ is a constant), in a rectangle $\Omega_R =]-\frac{R_1}{2}, \frac{R_1}{2}[\times]-\frac{R_2}{2}, \frac{R_2}{2}[$; $R = (R_1, R_2)\ \in\ (\mathbb{R}_+^\star)^2$.

Theorem A.2. *The Colin de Verdière upper bound holds for any $\lambda > 1$:*

$$N(\lambda, H_{D,b}^{\Omega_R}) \ \le\ \frac{b|\Omega_R|}{2\pi}\sharp\{n \in \mathbb{N}\ ;\ (2n+1)b < \lambda\}\ . \tag{A.3}$$

For the lower bound, we will need to precise Colin de Verdière's one as follows.
There exists a constant $C_0 > 0$ s.t., if $0 < b < \lambda$ and $1 \le \sqrt{b}\ \min R_j$, then $\forall\ \epsilon\ \in]0,1]$,

$$(1-\epsilon)^2\frac{b|\Omega_R|}{2\pi}\sharp\{n \in \mathbb{N}\ ;\ (2n+1)b < \lambda - \frac{C_0}{(\epsilon \min R_i)^2}\}\ \le\ N(\lambda, H_{D,b}^{\Omega_R})\ . \tag{A.4}$$

The classical Weyl estimate is the following. There exists a constant $C_0 > 0$ such that, if $0 \leq b \leq \lambda$ and $(C_0\sqrt{b})^{-1} \leq R_j$ for $j = 1,\ 2$, then

$$\frac{|\Omega_R|}{4\pi}\lambda[1 - C_0\frac{\sqrt{b}}{\sqrt{\lambda}}] \ \leq \ N(\lambda, H_{D,b}^{\Omega_R}) \ \leq \frac{|\Omega_R|}{4\pi}\lambda[1 + C_0\frac{\sqrt{b}}{\sqrt{\lambda}}] \ . \tag{A.5}$$

Proof: The upper bound (A.3) and the lower bound (A.4) come from the density of state using the same proof as in Colin de Verdière paper [Col].

We sketch the proof of the lower bound (A.4).

We set : $R(b) = R\sqrt{b}$. By scaling, we change Ω_R , b and λ into $\Omega_{R(b)}$, 1 and λ/b.

Then we take a large rectangle $\Omega_{R(b),M,\epsilon} \ = \ \bigcup_{j=1}^{M} \overline{\Omega(j,\epsilon)}$ where $\Omega(j,\epsilon) = z_j + (1-\epsilon)\Omega_{R(b)}$ are open rectangles with center z_j such that $\Omega(j,\epsilon)\bigcap\Omega(k,\epsilon) = \emptyset$ if $j \neq k$.

We consider the large rectangle $\Omega_{R(b),M} \ = \ \bigcup_{j=1}^{M} \Omega(j)$ where $\Omega(j) = z_j + \Omega_{R(b)}$.

So there exists a constant C_0 and a partition of unity (χ_j) s.t.

$$\text{support}(\chi_j) \ \subset \ \overline{\Omega(j)} \ , \quad \sum_{j=1}^{M} \chi_j^2(z) = 1 \text{ on } \Omega_{R(b),M,\epsilon} \quad \text{and} \quad |\nabla\chi_j| \leq \frac{C_0}{\epsilon \min R_k(b)} \ .$$

We can write, for any $u \ \in \ W_0^1(\Omega_{R(b),M,\epsilon})$,

$$\int_{\Omega_{R(b),M,\epsilon}} |(D-A_0)u|^2 dxdy \ = \ \sum_{j=1}^{M} \int_{\Omega(j)} [|(A-A_0)\chi_j u|^2 - V|\chi_j u|^2] \ dxdy \ ,$$

where $D - A_0 = (D_x - \frac{y}{2}, D_y + \frac{x}{2})$ and $V(z) = \sum_{\ell=1}^{M} |\nabla\chi_\ell(z)|^2$; so we get that for any real μ,

$$N(\mu, H_{D,1}^{\Omega_{R(b),M,\epsilon}}) \ \leq \ M \times N(\mu + (\frac{C_0}{\epsilon \min R_k(b)})^2, \ H_{D,1}^{\Omega_{R(b)}}) \ .$$

By the density of states formula, we have

$$\lim_{M\to\infty} \frac{N(\mu, H_{D,1}^{\Omega_{R(b),M,\epsilon}})}{M} \ = \ (1-\epsilon)^2\frac{bR_1R_2}{2\pi}\sharp\{n \in \mathbb{N} \ ; \ 2n+1 < \mu\} \ ,$$

and we get the lower bound (A.4) by taking $\mu = \frac{\lambda}{b} - (\frac{C_0}{\epsilon \min R_k(b)})^2$.

For the proof of the classical Weyl estimates (A.5), by scaling, we change Ω_R , b and λ into $\Omega_{\sqrt{b}R}$, 1 and λ/b.

Then we take a partition $\Omega_{\sqrt{b}R} \ = \ \bigcup_{j=1}^{M} \overline{\Omega(j)}$ where $\Omega(j)$ are open rectangles with sides in $[1/2, 1]$ such that $\Omega(j)\bigcap\Omega(k) = \emptyset$ if $j \neq k$.

So $\sum_{j=1}^{M} N(\frac{\lambda}{b}, H_{D,1}^{\Omega(j)}) \ \leq \ N(\lambda, H_{D,b}^{\Omega_R})$.

We change gauge in each $\Omega(j)$ in order to consider $H_{D,1}^{\Omega(j)}$ as the operator $(D - A(j))^2 = (D_x - \frac{y-y_j}{2})^2 + (D_y + \frac{x-x_j}{2})^2$, where (x_j, y_j) is the center of $\Omega(j)$. Now, it is easy to get the uniform Weyl formula:
$\exists C_0 > 0$ s.t. $\forall\, j + 1, \dots, M$,

$$\frac{|\Omega(j)|}{4\pi}\frac{\lambda}{b}[1 - C_0\frac{\sqrt{b}}{\sqrt{\lambda}}] \leq N(\frac{\lambda}{b}, H_{D,1}^{\Omega(j)}) \leq \frac{|\Omega(j)|}{4\pi}\frac{\lambda}{b}[1 + C_0\frac{\sqrt{b}}{\sqrt{\lambda}}] .$$

To be convinced, see that $\exists\, C_0 > 0$ s.t. $\forall\, \tau\ \in]0,1]$, $\forall\, u\ \in W_0^1(\Omega(j)\ )$,

$$(1-\tau^2)\|\nabla u\|^2 - \frac{C_0}{\tau^2}\|u\|^2 \leq \|(D - A(j))u\|^2 \leq (1+\tau^2)\|\nabla u\|^2 + \frac{C_0}{\tau^2}\|u\|^2 ,$$

and take $\tau = \frac{\sqrt{b}}{\sqrt{\lambda}}$.

So we get the lower bound of (A.5). We get in the same way the upper bound by considering the Neumann operators $H_{N,1}^{\Omega(j)}$ instead of the Dirichlet ones $H_{D,1}^{\Omega(j)}$.

Remark A.3. *As in Theorem A.2,*

$$\frac{\lambda - b}{2} \ \leq\ \sharp\,\{n \in \mathbb{N}\,;\ (2n+1)b < \lambda\}\ \leq\ \frac{\lambda + b}{2} ,$$

so the upper bound (A.3) is sharp compared to the one in (A.5). The lower bound (A.4) is sharp, compared to the one in (A.5), when $\epsilon < \sqrt{(b/\lambda)}$.

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