Global homotopy formulas on q-concave CR manifolds for small degrees

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Prépublication de l'Institut Fourier n^o 706 (2007)
www-fourier.ujf-grenoble.fr/prepublications.html

Abstract

Using functional analysis, we derive from local homotopy formulas for the tangential Cauchy-Riemann operator a global homotopy formula for compact CR manifolds without loss of regularity.

Keywords: Homotopy formula, Tangential Cauchy-Riemann equation, compact CR mamifolds

Résumé

Par des méthodes d'analyse fonctionnelle, nous construisons, à partir de formules d'homotopies locales pour l'opérateur de Cauchy-Riemann tangentiel, une formule d'homotopie globale pour les variétés CR compactes sans perte de régularité.

Mots-clés : formule d'homotopie, équation de Cauchy-Riemann tangentielle, variétés CR compactes

2000 Mathematics Subject Classification : 32V20

P. Polyakov [P3, P4] proved global homotopy formulas for $\overline{\partial}_b$ on CR manifolds and used them to study the embedding problem for CR manifolds. Gluing together local formulas, which were first introduced by Henkin [H] and Airapetian/Henkin [AH] and then further developed by Polyakov [P1, P2, P3, P4], Polyakov first constructs a global formula which is not yet a homotopy formula, but "almost", up to a compact perturbation. Then the main work is to eliminate this compact perturbation. This is successfully done by Polyakov, but with some loss of smoothness.
In the present paper we use and develop a general functional analytic construction from [L] to eliminate such compact perturbations without any loss of smoothness. May be, this can be used to improve the results about the embedding problem for CR manifolds obtained by Polyakov [P4].
This paper is written in the language of functional analysis. No integral formulas are used, but we use the results obtained elsewhere by integral formulas.

1 Notations used throughout the paper

In this paper, $\widetilde{M}$ is a complex manifold of complex dimension n and E is a holomorphic vector bundle on $\widetilde{M}$. Further, $M \subseteq \widetilde{M}$ is a generic, compact CR submanifold of class $\mathcal{C}^\infty$ of $\widetilde{M}$, k is the real codimension of M in $\widetilde{M}$, and $\mathcal{O}$ is the trivial complex line bundle on $\widetilde{M}$.
If $U \subseteq M$ is an open set, then, for $0 \le r \le n-k$, the following notations are used:

- $\mathcal{C}^\infty_{n,r}(U,E)$ is the Fréchet space of E-valued (n,r)-forms on U which are of class $\mathcal{C}^\infty$, endowed with the $\mathcal{C}^\infty$-topology.
- $\mathcal{Z}^\infty_{n,r}(U,E)$ is the subspace of all closed forms in $\mathcal{C}^\infty_{n,r}(U,E)$, endowed with the same topology.
- $\mathcal{C}^{l+\alpha}_{n,r}(\overline{U},E)$, $l \in \mathbb{N}$, $0 \le \alpha < 1$, is the Banach space of l times differentiable E-valued (n,r)-forms whose derivatives up to order l admit extensions to $\overline{U}$ which are Hölder continuous with exponent α, endowed with the $\mathcal{C}^{l+\alpha}$-topology.
- $\mathcal{Z}^{l+\alpha}_{n,r}(\overline{U},E)$ is the subspace of all closed forms in $\mathcal{C}^{l+\alpha}_{n,r}(\overline{U},E)$, endowed with the same topology.
- If $r \ge 1$, then $\mathcal{B}^{l+\alpha\to l}_{n,r}(M,E)$ is the space of all $f \in \mathcal{C}^l_{n,r}(M,E)$ such that $f = du$ for some $u \in \mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$. Sometimes we write also

$$\mathcal{B}^\infty_{n,r}(M,E) := \mathcal{B}^{\infty\to\infty}_{n,r}(M,E) := d\mathcal{C}^\infty_{n,r-1}(M,E).$$

- $(\mathrm{Dom}\, d)^0_{n,r}(M,E)$ is the space of all $f \in \mathcal{C}^0_{n,r}(M,E)$ such that also df is continuous on M.

2 The main result

If $0 < \alpha < 1$ and q is an integer with $1 \le q \le n-k$, then we shall say that **condition** $H(\alpha,q)$ **is satisfied** if, for each point in M, there exist a neighborhood U and linear operators

$$T_r : \mathcal{C}^0_{n,r}(M,\mathcal{O}) \to \mathcal{C}^0_{n,r-1}\big(U,\mathcal{O}\big)\,, \qquad 1 \le r \le q,$$

with the following two properties:
(i) For all $l \in \mathbb{N}$ and $1 \le r \le q$,

$$T_r\Big(\mathcal{C}^l_{n,r}(M,\mathcal{O})\Big) \subseteq \mathcal{C}^{l+\alpha}_{n,r-1}(\overline{U},\mathcal{O})$$

and T_r is continuous as an operator between $\mathcal{C}^l_{n,r}(M,\mathcal{O})$ and $\mathcal{C}^{l+\alpha}_{n,r}(\overline{U},\mathcal{O})$.
(ii) If $f \in (\operatorname{Dom} d)^0_{n,r}(M,\mathcal{O})$, $0 \le r \le q-1$, has compact support in U, then, on U,

$$f = \begin{cases} T_1 df & \text{if } r = 0\,, \\ dT_r f + T_{r+1} df & \text{if } 1 \le r \le q-1\,. \end{cases} \tag{2.1}$$

If M is q-concave in the sense of Henkin [H], then it is known since 1981 [H, AH] that condition $H(\alpha,q)$ is satisfied for $0 < \alpha < 1/2$. More recently it was proved in [BLT] that then also condition $H(1/2,q)$ is satisfied.

Theorem 2.1. *Suppose, for some $0 < \alpha < 1$ and some integer q with $1 \le q \le n-k$, condition $H(\alpha,q)$ is satisfied. Then there exist finite dimensional subspaces $\mathcal{H}_r$ of $\mathcal{Z}^\infty_{n,r}(M,E)$, $1 \le r \le q-1$, where $\mathcal{H}_0 = \mathcal{Z}^\infty_{n,0}(M,E)$, continuous linear operators*

$$A_r : \mathcal{C}^0_{n,r}(M,E) \longrightarrow \mathcal{C}^0_{n,r-1}(M,E)\,, \qquad 1 \le r \le q\,,$$

and continuous linear projections

$$P_r : \mathcal{C}^0_{n,r}(M,E) \longrightarrow \mathcal{C}^0_{n,r}(M,E)\,, \qquad 0 \le r \le q-1\,,$$

with

$$\operatorname{Im} P_r = \mathcal{H}_r\,, \qquad 0 \le r \le q-1\,, \tag{2.2}$$

and

$$\mathcal{B}^{0\to 0}_{n,r}(M,E) \subseteq \operatorname{Ker} P_r\,, \qquad 1 \le r \le q-1, \tag{2.3}$$

such that:

(i) For all $l \in \mathbb{N} \cup \{\infty\}$ and $1 \le r \le q$,

$$A_r\Big(\mathcal{C}^l_{n,r}(M,E)\Big) \subseteq \mathcal{C}^{l+\alpha}_{n,r-1}(M,E) \tag{2.4}$$

and A_r is continuous as operator from $\mathcal{C}^l_{n,r}(M,E)$ to $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$.

(ii) For all $0 \le r \le q-1$ and $f \in (\operatorname{Dom})^0_{n,r}(M,E)$,

$$f - P_r f = \begin{cases} A_1 df & \textit{if } r = 0\,, \\ dA_r f + A_{r+1} df & \textit{if } 1 \le r \le q-1\,. \end{cases} \tag{2.5}$$

(iii) For all $1 \le r \le q$ and $l \in \mathbb{N} \cup \{\infty\}$, the space $\mathcal{B}^{l+\alpha\to l}_{n,r}(M,E)$ is closed in $\mathcal{C}^l_{n,r}(M,E)$,

$$\mathcal{B}^{l+\alpha\to l}_{n,r}(M,E) = \mathcal{B}^{0\to l}_{n,r}(M,E)\,, \tag{2.6}$$

$$dA_r f = f \qquad \textit{for all } f \in \mathcal{B}^{0\to 0}_{n,r}(M,E)\,, \tag{2.7}$$

and the natural map

$$\frac{\mathcal{Z}^\infty_{n,r}(M,E)}{d\mathbb{C}^\infty_{n,r-1}(M,E)} \longrightarrow \frac{\mathcal{Z}^l_{n,r}(M,E)}{\mathcal{B}^{0\to l}_{n,r}(M,E)} \tag{2.8}$$

is injective. If $1 \le r \le q-1$, then moreover, for all $l \in \mathbb{N} \cup \{\infty\}$,

$$\mathcal{Z}^l_{n,r}(M,E)\mathcal{B}^{l+\alpha\to l}_{n,r}(M,E) \oplus \mathcal{H}_r\,, \tag{2.9}$$

and hence (2.8) *is an isomorphism (as $\mathcal{H}_r \subseteq \mathcal{Z}^\infty_{n,r}(M,E)$).*

Remark 2.2. Since $\operatorname{Im} P_r$ is a finite dimensional subspace of $\mathcal{C}^\infty_{n,r}(M,E)$, $0 \le r \le q-1$, the projections P_r are even continuous as operators from $\mathcal{C}^0_{n,r}(M,E)$ to $\mathcal{C}^\infty_{n,r}(M,E)$.

The rest of this paper is devoted to the proof of theorem 2.1.

3 Gluing together local homotopy formulas

Lemma 3.1. *Suppose, for some $0 < \alpha < 1$ and some integer q with $1 \le q \le n-k$, condition $H(\alpha, q)$ is satisfied. Then there exist linear operators*

$$T_r : \mathcal{C}^0_{n,r}(M,E) \to \mathcal{C}^0_{n,r-1}\big(M,E\big)\,, \qquad 1 \le r \le q, \tag{3.1}$$

and

$$K_r : \mathcal{C}^0_{n,r}(M,E) \to \mathcal{C}^0_{n,r}\big(M,E\big)\,, \qquad 0 \le r \le q-1, \tag{3.2}$$

with the following two properties:
(i) For all $l \in \mathbb{N}$,

$$T_r\Big(\mathcal{C}^l_{n,r}(M,E)\Big) \subseteq \mathcal{C}^{l+\alpha}_{n,r-1}(M,E)\,, \qquad 1 \le r \le q, \tag{3.3}$$

$$K_r\Big(\mathcal{C}^l_{n,r}(M,E)\Big) \subseteq \mathcal{C}^{l+\alpha}_{n,r}(M,E)\,, \qquad 0 \le r \le q-1, \tag{3.4}$$

the operators T_r, $1 \le r \le q$, are continuous as operators acting between $\mathcal{C}^l_{n,r}(M,E)$ and $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$, and the operators K_r, $0 \le r \le q-1$, are continuous as operators acting between $\mathcal{C}^l_{n,r}(M,E)$ and $\mathcal{C}^{l+\alpha}_{n,r}(M,E)$.
(ii) If $f \in (\mathrm{Dom}\, d)^0_{n,r}(M,E)$, $0 \le r \le q-1$, then on M

$$f + K_r f = \begin{cases} T_1 df & \text{if } r = 0\,, \\ dT_r f + T_{r+1} df & \text{if } 1 \le r \le q-1\,. \end{cases} \tag{3.5}$$

Proof. By hypothesis, we can find a finite open covering $\{U_j\}_{j=1,\dots,N}$ of M and linear operators

$$T_{r,j} : \mathcal{C}^0_{n,r}(M,E) \to \mathcal{C}^0_{n,r-1}\big(U_j,E\big)\,, \qquad 1 \le r \le q\,, \quad j = 1,\dots,N,$$

with the following properties:
(i') For each $l \in \mathbb{N}$ and all $1 \le r \le q$, $1 \le j \le N$,

$$T_{r,j}\Big(\mathcal{C}^l_{n,r}(M,E)\Big) \subseteq \mathcal{C}^{l+\alpha}_{n,r-1}(\overline{U}_j,E)$$

and $T_{r,j}$ is continuous as an operator acting between $\mathcal{C}^l_{n,r}(M,E)$ and $\mathcal{C}^{l+\alpha}_{n,r}(\overline{U}_j,E)$.
(ii') If $f \in \mathcal{C}^0_{n,r}(M,E)$ is a form with compact support in U_j, $0 \le r \le q-1$, $1 \le j \le N$, such that also df is continuous on M, then on U_j

$$f = \begin{cases} dT_{r,j} f + T_{r+1,j} df & \text{if } 1 \le r \le q-1 \\ T_{1,j} df & \text{if } r = 0\,. \end{cases} \tag{3.6}$$

Now we take $\mathcal{C}^\infty$ functions $\chi_1,\dots,\chi_N$ such that $\chi_1^2,\dots,\chi_N^2$ is a partition of unity subordinated to $\{U_j\}$. Setting

$$T_r f = \sum_{j=1}^N \chi_j T_{r,j}\big(\chi_j f\big)\,, \qquad 1 \le r \le q,$$

and

$$K_r f = \begin{cases} \sum_{j=1}^N \Big(d\chi_j \wedge T_{r,j}(\chi_j f) - \chi_j T_{r+1,j}(d\chi_j \wedge f)\Big) & \text{if } 1 \le r \le q-1 \\ -\sum_{j=1}^N \chi_j T_{1,j}(d\chi_j \wedge f) & \text{if } r = 0\,, \end{cases}$$

we define operators (3.1) and (3.2). Then condition (i) is satisfied by (i').

To prove (ii), we consider a form $f \in \mathcal{C}^0_{n,r}(M,E)$, $0 \le r \le q-1$, such that also df is continuous on M. Then, for each j, the form $\chi_j f$ has compact support in U_j and

$$d(\chi_j f) = \chi_j df + d\chi_j \wedge f$$

is also continuous on M. Therefore, by condition (ii'), for each j we have on U_j

$$\chi_j f = \begin{cases} dT_{r,j}(\chi_j f) + T_{r+1,j}(\chi_j df) + T_{r+1,j}(d\chi_j \wedge f) & \text{if } 1 \le r \le q-1 \\ T_{1,j}(\chi_j df) + T_{1,j}(d\chi_j \wedge f) & \text{if } r = 0 \end{cases}$$

and hence

$$\chi_j^2 f = \begin{cases} \chi_j dT_{r,j}(\chi_j f) + \chi_j T_{r+1,j}(\chi_j df) + \chi_j T_{r+1,j}(d\chi_j \wedge f) & \text{if } 1 \le r \le q-1 \\ \chi_j T_{1,j}(\chi_j df) + \chi_j T_{1,j}(d\chi_j \wedge f) & \text{if } r = 0\,. \end{cases}$$

If $r = 0$, summing up this gives

$$f = T_1(df) - K_0(f),$$

i.e. we have (3.5) for $r = 0$. Now let $1 \le r \le q-1$. Since

$$\chi_j dT_{r,j}(\chi_j f) = d\Big(\chi_j T_{r,j}(\chi_j f)\Big) - d\chi_j \wedge T_{r,j}(\chi_j f) \quad \text{if } 1 \le r \le q-1,$$

then we get on each U_j

$$\chi_j^2 f = d\Big(\chi_j T_{r,j}(\chi_j f)\Big) - d\chi_j \wedge T_{r,j}(\chi_j f) + \chi_j T_{r+1,j}(\chi_j df) + \chi_j T_{r+1,j}(d\chi_j \wedge f)\,.$$

Summing up this gives (3.5) for $1 \le r \le q-1$. □

4 A functional analytic lemma

Lemma 4.1. *Let B_l, $l \in \mathbb{N}$, be a sequence of Banach spaces, and let $R : B_0 \to B_0$ be a linear operator such that, for each $l \in \mathbb{N}$:*

- *$B_{l+1} \subseteq B_l$, and the imbedding $B_{l+1} \hookrightarrow B_l$ is continuous.*
- *$\bigcap_{\mu \in \mathbb{N}} B_\mu$ is dense in B_l.*
- *$R(B_l) \subseteq B_l$ and $R\big|_{B_l}$ is compact as an endomorphism of B_l.*

Then $I + R$ is a Fredholm endomorphism with index zero of B_0 (this is clear, because R is compact as an endomorphism of B_0), and

$$\operatorname{Ker}(I+R) \subseteq \bigcap_{l \in \mathbb{N}} B_l\,. \tag{4.1}$$

Proof. Since the operators $R\big|_{B_l}$ are compact, for each $l \in \mathbb{N}$, $(I+R)\big|_{B_l}$ is a Fredholm endomorphism with index zero of B_l. Clearly

$$\operatorname{Ker}(I+R)\big|_{B_{l+1}} \subseteq \operatorname{Ker}(I+R)\big|_{B_l}\,, \qquad l \in \mathbb{N}.$$

Since $\dim \operatorname{Ker}(I+R)\big|_{B_l} < \infty$ for all $l \in \mathbb{N}$, this yields the existence of $l_0 \in \mathbb{N}$ with

$$\operatorname{Ker}(I+R)\big|_{B_l} = \operatorname{Ker}(I+R)\big|_{B_{l_0}} \qquad \text{for all } l \ge l_0\,.$$

Then

$$\operatorname{Ker}(I+R)\big|_{B_{l_0}} \subseteq \bigcap_{l\in\mathbb{N}} B_l \,. \tag{4.2}$$

Since the index of the Fredholm operator $(I+R)\big|_{B_{l_0}}$ is zero,

$$s := \dim \operatorname{Ker}(I+R)\big|_{B_{l_0}} = \dim \operatorname{Coker}(I+R)\big|_{B_{l_0}}.$$

Let $v_1,\dots,v_s$ be a basis of $\operatorname{Ker}(I+R)\big|_{B_{l_0}}$, and let $w_1,\dots,w_s$ be a basis of a complement of $(I+R)\big(B_{l_0}\big)$ in B_{l_0}. Since $\bigcap_{l\in\mathbb{N}} B_l$ is dense in B_{l_0}, we may assume that $w_1,\dots,w_s \in \bigcap_{l\in\mathbb{N}} B_l$. By the Hahn-Banach theorem, we can find continuous linear functionals $\Phi_1,\dots,\Phi_s$ on B_0 with

$$\Phi_\mu(v_\nu) = \delta_{\mu\nu}\,, \qquad 1 \le \mu,\nu \le s\,.$$

Setting

$$\Phi(f) = \sum_{\mu=1}^{s} \Big(\Phi_\mu(f)\Big) w_\mu,$$

then we get a finite dimensional linear operator

$$\Phi : B_0 \to \bigcap_{l\in\mathbb{N}} B_l$$

such that, for each $l\in\mathbb{N}$, Φ is continuous as an operator between B_0 and B_l. As the embeddings $B_l \hookrightarrow B_0$ are continuous, then for each $l\in\mathbb{N}$, $\Phi\big|_{B_l}$ is a finite dimensional continuous linear endomorphism of B_l. Since R is compact in all B_l, it follows that, for each $l\in\mathbb{N}$, $(I+R+\Phi)\big|_{B_l}$ is a Fredholm endomorphism of B_l with index zero.
Moreover, $(I+R+\Phi)\big|_{B_{l_0}}$ is even an isomorphism of B_{l_0}, because Φ maps $\operatorname{Ker}(I+R)\big|_{B_{l_0}}$ isomorphically to a complement of $(I+R)\big(B_{l_0}\big)$ in B_{l_0}. Hence

$$B_{l_0} = (I+R+\Phi)\big(B_{l_0}\big) \subseteq (I+R+\Phi)\big(B_0\big).$$

Since B_{l_0} is dense in B_0 and $I+R+\Phi$ is a Fredholm endomorphism of B_0 with index zero, this implies that $I+R+\Phi$ is an isomorphism of B_0. Since Φ is of rank s, this further implies that

$$\dim \operatorname{Ker}(I+R) \le s = \dim \operatorname{Ker}(I+R)\big|_{B_{l_0}}.$$

As

$$\operatorname{Ker}(I+R)\big|_{B_{l_0}} \subseteq \operatorname{Ker}(I+R)\,,$$

it follows that

$$\operatorname{Ker}(I+R)\big|_{B_{l_0}} \operatorname{Ker}(I+R).$$

By (4.2) this implies (4.1). □

5 Further steps toward the proof of theorem 2.1

In this section we assume that, for some $0<\alpha<1$ and some integer q with $1\le q\le n-k$, condition $H(\alpha,q)$ is satisfied.

Lemma 5.1. *Let K_r, $0 \le r \le q-1$, be the operators from lemma 3.1. Then:*
(i) For all $0 \le r \le q-1$, $I + K_r$ is a Fredholm endomorphism of $\mathcal{C}^0_{n,r}(M,E)$ with index zero and

$$\operatorname{Ker}(I + K_r) \subseteq \mathcal{C}^\infty_{n,r}(M,E)\,. \tag{5.1}$$

(ii) We have

$$Z^0_{n,0}(M,E) \subseteq \operatorname{Ker}(I + K_0) \subseteq \mathcal{C}^\infty_{n,0}(M,E) \tag{5.2}$$

and

$$\dim Z^0_{n,0}(M,E) = \dim Z^\infty_{n,0}(M,E) < \infty\,. \tag{5.3}$$

(iii) If $q \ge 2$ and $1 \le r \le q-1$, then

$$(I + K_r)\Big(\mathcal{Z}^0_{n,r}(M,E)\Big) \subseteq \mathcal{B}^{\alpha\to 0}_{n,r}(M,E) \tag{5.4}$$

and $(I + K_r)\big|_{\mathcal{Z}^0_{n,r}(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{Z}^0_{n,r}(M,E)$.
(iv) If $q \ge 2$ and $1 \le r \le q-1$, then $\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$ is a closed subspace of finite codimension in $Z^0_{n,r}(M,E)$.
(v) If $q \ge 2$ and $1 \le r \le q-1$, then

$$(I + K_r)\Big(\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)\Big) \subseteq \mathcal{B}^{\alpha\to 0}_{n,r}(M,E) \tag{5.5}$$

and $(I + K_r)\big|_{\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$.

Proof. Proof of (i): Let $0 \le r \le q-1$. Then it follows from Ascoli's theorem and statement (i) in lemma 3.1 that the hypotheses of lemma 4.1 is satisfied for $R := K_r$. Therefore part (i) of the lemma under proof follows from lemma 4.1.
Proof of (ii): (5.2) follows from (3.5) and (5.1). Since, by part (i), $I + K_0$ is a Fredholm operator, this implies (5.3).
Proof of (iii): (5.4) follows from (3.5). Since $I + K_r$ is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,r}(M,E)$ and, by (5.4), $\mathcal{Z}^0_{n,r}(M,E)$ is an invariant subspace of $I + K_r$, this implies that $(I + K_r)\big|\mathcal{Z}^0_{n,r}(M,E)$ is a Fredholm endomorphism with index zero of $\mathcal{Z}^0_{n,r}(M,E)$.
Proof of (iv): Since $(I + K_r)\big|_{\mathcal{Z}^0_{n,r}(M,E)}$ is a Fredholm endomorphism of $\mathcal{Z}^0_{n,r}(M,E)$, it follows from (5.4) that also $B^{\alpha\to 0}_{n,r}(M,E)$ is of finite codimension in $\mathcal{Z}^0_{n,r}(M,E)$. Together with the fact that $\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$ is the image of a closed linear operator, it follows that $\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$ is closed.
Proof of (v): (5.5) holds by (3.5). Since $I + K_r$ is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,r}(M,E)$ and, by (5.4), $\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$ is an invariant subspace of $I + K_r$ which is closed by part (iv), this implies that $(I + K_r)\big|\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$ is a Fredholm endomorphism with index zero of $\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$. □

Lemma 5.2. *Let T_1 and K_0 be the operators from lemma 3.1. Then there exist finite dimensional continuous linear operators*

$$\begin{aligned} T_1' &: \mathcal{C}^0_{n,1}(M,E) \to \mathcal{C}^\infty_{n,0}(M,E)\,, \\ K_0' &: \mathcal{C}^0_{n,0}(M,E) \to \mathcal{C}^\infty_{n,0}(M,E) \end{aligned} \tag{5.6}$$

such that $I + K_0 + K_0'$ is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,0}(M,E)$ (this is clear, because $I + K_0$ has this property),

$$(I + K_0 + K_0')f = (T_1 + T_1')df \quad \textit{for all } f \in (\operatorname{Dom} d)^0_{n,0}(M,E)\,, \tag{5.7}$$

$$\operatorname{Ker}(I+K_0+K_0')=\mathcal{Z}^{\infty}_{n,0}(M,E)=\mathcal{Z}^{0}_{n,0}(M,E)\,, \tag{5.8}$$

and, moreover, $\mathcal{C}^0_{n,0}(M,E)$ *splits into the direct sum*

$$\mathcal{C}^0_{n,0}(M,E)=\operatorname{Im}(I+K_0+K_0')\oplus\mathcal{Z}^0_{n,0}(M,E)\,. \tag{5.9}$$

Proof. We proceed in two steps. First we construct finite dimensional continuous linear operators

$$\begin{aligned}T_1^*&:\mathcal{C}^0_{n,1}(M,E)\to\mathcal{C}^\infty_{n,0}(M,E)\,,\\ K_0^*&:\mathcal{C}^0_{n,0}(M,E)\to\mathcal{C}^\infty_{n,0}(M,E)\end{aligned} \tag{5.10}$$

such that

$$K_0^*f=T_1^*df\quad\text{for all } f\in(\operatorname{Dom}d)^0_{n,0}(M,E) \tag{5.11}$$

and

$$\operatorname{Ker}(I+K_0+K_0^*)=\mathcal{Z}^{\infty}_{n,0}(M,E)=\mathcal{Z}^{0}_{n,0}(M,E)\,. \tag{5.12}$$

(The second equality in (5.12) we know already from lemma 5.1 (ii).) Then we construct finite dimensional continuous linear operators

$$\begin{aligned}T_1^{**}&:\mathcal{C}^0_{n,1}(M,E)\to\mathcal{C}^\infty_{n,0}(M,E)\,,\\ K_0^{**}&:\mathcal{C}^0_{n,0}(M,E)\to\mathcal{C}^\infty_{n,0}(M,E)\end{aligned} \tag{5.13}$$

such that

$$K_0^{**}f=T_1^{**}df\quad\text{for all } f\in(\operatorname{Dom}d)^0_{n,0}(M,E)\,, \tag{5.14}$$

$$\operatorname{Ker}(I+K_0+K_0^*+K_0^{**})=\mathcal{Z}^{\infty}_{n,0}(M,E)=\mathcal{Z}^{0}_{n,0}(M,E) \tag{5.15}$$

and, moreover, $\mathcal{C}^0_{n,0}(M,E)$ splits into the direct sum

$$\mathcal{C}^0_{n,0}(M,E)=\operatorname{Im}(I+K_0+K_0^*+K_0^{**})\oplus\mathcal{Z}^0_{n,0}(M,E)\,. \tag{5.16}$$

Then $T_1':=T_1^*+T_1^{**}$ and $K_0':=K_0^*+K_0^{**}$ have the required properties. Indeed (5.8) and (5.9) then follow from (5.15) and (5.16), and (5.7) follows from (3.5), (5.11) and (5.14).

First step: By lemma 5.1 (i) and (ii), $I+K_0$ is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,0}(M,E)$, and there is a finite dimensional subspace Λ of $\mathcal{C}^\infty_{n,0}(M,E)$ such that

$$\operatorname{Ker}(I+K_0)=\mathcal{Z}^{\infty}_{n,0}(M,E)\oplus\Lambda\,. \tag{5.17}$$

Let $m:=\dim\Lambda$. If $m=0$, we set $K_0^*=0$ and $T_1^*=0$.
Now let $m>0$. Then we choose a basis $\lambda_1,\dots,\lambda_m$ of Λ. Since the index of $I+K_0$ is zero and $\mathcal{C}^\infty_{n,0}(M,E)$ is dense in $\mathcal{C}^0_{n,0}(M,E)$, we can find an m-dimensional subspace $\widetilde{\Lambda}$ of $\mathcal{C}^\infty_{n,0}(M,E)$ with

$$\widetilde{\Lambda}\cap\operatorname{Im}(I+K_0)=\{0\}\,. \tag{5.18}$$

Let $\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_m$ be a basis of $\widetilde{\Lambda}$. By (5.17), also the forms $d\lambda_1,\dots,d\lambda_m$ are linearly independent. Therefore we can find[1] forms $\psi_1,\dots,\psi_m\in C^\infty_{n-k-1}(M,E^*)$ (where E^* denotes the dual bundle of E), such that

$$\int_M d\lambda_\nu\wedge\psi_\mu=\delta_{\nu\mu}\,,\quad 1\le\mu,\nu\le m\,.\qquad\text{(Kronecker symbol)}\,. \tag{5.19}$$

[1] Here we use the following fact from linear algebra: *Let L be a complex vector space and let Φ be a linear space of linear forms on L with the property that, for each $x\in L$ with $x\neq 0$, there exists $\varphi\in\Phi$ with $\varphi(x)\neq 0$. Then, for each finite linearly independent system of vectors $h_1,\dots,h_m\in L$, there exist forms $\varphi_1,\dots,\varphi_m\in\Phi$ with $\varphi_\nu(h_\mu)=\delta_{\nu\mu}$. Proof:* Denote by M the linear subspace of $\mathbb{C}^m$ which consists of the vectors of the form $\big(\varphi(h_1),\dots,\varphi(h_m)\big)$ with $\varphi\in\Phi$. We have to prove that $M=\mathbb{C}^m$. Assume $M\neq\mathbb{C}^m$. Then there exists a nonzero vector $\big(\lambda_1,\dots,\lambda_m\big)\in\mathbb{C}^m$ with $\sum\lambda_\nu z_\nu=0$ for all $(z_1,\dots,z_m)\in M$. By definition of M it follows that $\varphi\big(\sum\lambda_\nu h_\nu\big)=\sum\lambda_\nu\varphi(h_\nu)=0$ for all $\varphi\in\Phi$. Hence (by hypothesis on Φ) $\sum\lambda_\nu h_\nu=0$, which contradicts the linear independence of $h_1,\dots,h_m$.

Setting

$$T_1^* f = (-1)^{n+1} \sum_{\nu=1}^{m} \left(\int_M f \wedge \psi_\nu \right) \widetilde{\lambda}_\nu \,, \qquad f \in \mathcal{C}^0_{n,1}(M,E)\,,$$
$$K_0^* f = \sum_{\nu=1}^{m} \left(\int_M f \wedge d\psi_\nu \right) \widetilde{\lambda}_\nu \,, \qquad f \in \mathcal{C}^0_{n,0}(M,E)\,.$$

we define the operators (5.10). From Stokes' theorem we get (5.11). Hence

$$\mathcal{Z}^\infty_{n,0}(M,E) \subseteq \operatorname{Ker} K_0^*. \tag{5.20}$$

Moreover, by (5.19), we get from Stokes' theorem that $K_0^*(\lambda_\mu) = (-1)^{n+1}\widetilde{\lambda}_\mu$, $1 \le \mu \le m$. Since $\lambda_1, \ldots, \lambda_m$ is a basis of Λ and $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_m$ is a basis of $\widetilde{\Lambda}$, this implies that

$$\Lambda \cap \operatorname{Ker} K_0^* = \{0\}\,.$$

Together with (5.20) and (5.17) this implies (5.12).

Second step: Set

$$V = \mathcal{Z}^\infty_{n,0}(M,E) \cap \operatorname{Im}(I + K_0 + K_0^*) \quad \text{and} \quad p = \dim V\,.$$

Then there is a p-dimensional subspace $\widehat{V}$ of $\mathcal{C}^\infty_{n,0}(M,E)$ with

$$(I + K_0 + K_0^*)(\widehat{V}) = V\,.$$

Then, by (5.12),

$$\widehat{V} \cap Z^0_{n,0}(M,E) = \widehat{V} \cap \operatorname{Ker}(I + K_0 + K_0^*) = \{0\}, \tag{5.21}$$

Let $\widehat{v}_1, \ldots, \widehat{v}_p$ be a basis of $\widehat{V}$ and

$$v_\mu := (I + K_0 + K_0^*)\widehat{v}_\mu\,, \qquad 1 \le \mu \le p\,.$$

By (5.21), then also the forms $d\widehat{v}_1, \ldots, d\widehat{v}_p$ are linearly independent. Therefore, as above, we can find forms $\varphi_1, \ldots, \varphi_p \in C^\infty_{n-k-1}(M,E^*)$ with

$$\int_M d\widehat{v}_\nu \wedge \varphi_\mu = \delta_{\nu\mu}\,, \quad 1 \le \mu, \nu \le p\,. \tag{5.22}$$

Set

$$\Gamma = \left\{ f \in \mathcal{C}^0_{n,0}(M,E) \;\middle|\; \int_M f \wedge d\varphi_\mu = 0 \quad \text{for } 1 \le \mu \le p \right\}.$$

By Stokes' theorem, then

$$\mathcal{Z}^0_{n,0}(M,E) \subseteq \Gamma\,, \tag{5.23}$$

and, by (5.22),

$$\mathcal{C}^0_{n,0}(M,E) = \widehat{V} \oplus \Gamma\,. \tag{5.24}$$

Since $Z^0_{n,0}(M,E)$ is finite dimensional and by (5.23) (and the Hahn-Banach theorem), we can find a closed linear subspace $\widehat{L}$ of Γ with

$$\Gamma = \widehat{L} \oplus \mathcal{Z}^0_{n,0}(M,E)\,.$$

Together with (5.24) and (5.12) this implies

$$\mathcal{C}^0_{n,0}(M,E) = \widehat{V} \oplus \widehat{L} \oplus \mathcal{Z}^0_{n,0}(M,E) = \widehat{V} \oplus \widehat{L} \oplus \operatorname{Ker}(I + K_0 + K_0^*)\,. \tag{5.25}$$

Since $I + K_0 + K_0^*$ maps $\widehat{V}$ isomorphically to V, this yields

$$\operatorname{Im}(I + K_0 + K_0^*) = V \oplus (I + K_0 + K_0^*)(\widehat{L}) = V \oplus L\,, \tag{5.26}$$

where

$$L := (I + K_0 + K_0^*)(\widehat{L})\,.$$

Let W be a subspace of $\operatorname{Ker}(I + K_0 + K_0^*) = Z^0_{n,0}(M,E)$ such that

$$V \oplus W = \operatorname{Ker}(I + K_0 + K_0^*) = \mathcal{Z}^0_{n,0}(M,E)\,. \tag{5.27}$$

Since the index of $I + K_0 + K_0^*$ is zero, then, by (5.26),

$$p + \dim W = \dim \operatorname{Ker}(I + K_0 + K_0^*) = \operatorname{codim} \operatorname{Im}(I + K_0 + K_0^*) = \operatorname{codim}(V \oplus L) \tag{5.28}$$

and moreover by (5.26) and the definition of V,

$$W \cap (V \oplus L) = W \cap \operatorname{Im}(I + K_0 + K_0^*) = \{0\}\,. \tag{5.29}$$

$\mathcal{C}^\infty_{n,0}(M,E)$ is dense in $\mathcal{C}^0_{n,0}(M,E)$, it follows from (5.28) and (5.29), that there exists a p-dimensional subspace $\widetilde{V}$ of $\mathcal{C}^\infty_{n,0}(M,E)$ such that

$$\mathcal{C}^0_{n,0}(M,E) = L \oplus V \oplus W \oplus \widetilde{V}\,,$$

which means by (5.27) that

$$\mathcal{C}^0_{n,0}(M,E) = \operatorname{Ker}(I + K_0 + K_0^*) \oplus L \oplus \widetilde{V}\,. \tag{5.30}$$

Let $\widetilde{v}_1, \ldots, \widetilde{v}_p$ a basis of $\widetilde{V}$. Setting

$$T_1^{**} f = \sum_{\nu=1}^{p} \left(\int_M f \wedge \varphi_\nu \right) (\widetilde{v}_\nu - v_\nu)\,, \qquad f \in \mathcal{C}^0_{n,1}(M,E)\,,$$
$$K_0^{**} f = (-1)^{n+1} \sum_{\nu=1}^{p} \left(\int_M f \wedge d\varphi_\nu \right) (\widetilde{v}_\nu - v_\nu)\,, \qquad f \in \mathcal{C}^0_{n,0}(M,E)\,.$$

we define the operators (5.13). Then (5.14) follows from Stokes' formula. In particular we see that

$$\mathcal{Z}^0_{n,0}(M,E) \subseteq \operatorname{Ker} K_0^{**}.$$

By (5.12), this yields

$$\mathcal{Z}^0_{n,0}(M,E) = \operatorname{Ker}(I + K_0 + K_0^*) \subseteq \operatorname{Ker}(I + K_0 + K_0^* + K_0^{**})\,. \tag{5.31}$$

It remains to prove (5.15) and (5.16). By Stokes's theorem and (5.22),

$$K_0^{**} \widehat{v}_\mu = \sum_{\nu=1}^{p} \left(\int_M \widehat{v}_\mu \wedge d\varphi_\nu \right) (\widetilde{v}_\nu - v_\nu) = \widetilde{v}_\mu - v_\mu = \widetilde{v}_\mu - (I + K_0 + K_0^*)\widehat{v}_\mu,$$

and therefore

$$(I + K_0 + K_0^* + K_0^{**})\widehat{v}_\mu = \widetilde{v}_\mu\,, \qquad 1 \le \mu \le p\,.$$

Hence

$$(I + K_0 + K_0^* + K_0^{**})\big(\widehat{V}\big) = \widetilde{V} \,.$$

By the definitions of Γ and K_0^{**} it is clear that $\Gamma \subseteq \operatorname{Ker} K_0^{**}$. Hence $\widehat{L} \subseteq \operatorname{Ker} K_0^{**}$ and

$$(I + K_0 + K_0^* + K_0^{**})\big(\widehat{L}\big) = (I + K_0 + K_0^*)\big(\widehat{L}\big) = L \,.$$

Together this implies that

$$(I + K_0 + K_0^* + K_0^{**})\big(\widehat{V} \oplus \widehat{L}\big) = L \oplus \widetilde{V} \,. \tag{5.32}$$

By (5.30) this yields

$$\operatorname{codim} \operatorname{Im}(I + K_0 + K_0^* + K_0^{**}) \le \dim \operatorname{Ker}(I + K_0 + K_0^*) \,.$$

As $I + K_0 + K_0^* + K_0^{**}$ is a Fredholm operator with index zero, this further implies that

$$\dim \operatorname{Ker}(I + K_0 + K_0^* + K_0^{**}) \le \dim \operatorname{Ker}(I + K_0 + K_0^*) \,.$$

Taking into account again (5.31), we see that

$$\operatorname{Ker}(I + K_0 + K_0^* + K_0^{**}) = \operatorname{Ker}(I + K_0 + K_0^*) \,. \tag{5.33}$$

By (5.30) and (5.25), this implies that

$$\mathcal{C}^0_{n,0}(M, E) = \operatorname{Ker}(I + K_0 + K_0^* + K_0^{**}) \oplus L \oplus \widetilde{V} \,, \tag{5.34}$$

and

$$\mathcal{C}^0_{n,0}(M, E) = \operatorname{Ker}(I + K_0 + K_0^* + K_0^{**}) \oplus \widehat{L} \oplus \widehat{V} \,. \tag{5.35}$$

From (5.32) and (5.35) it follows that

$$\operatorname{Im}(I + K_0 + K_0^* + K_0^{**}) = L \oplus \widetilde{V}.$$

In view of (5.34) this means (5.16). □

Lemma 5.3. *Let T_r, $1 \le r \le q$, and K_r, $0 \le r \le q-1$, be the operators from lemma 3.1. Then there exist finite dimensional continuous linear operators*

$$\begin{aligned} K_r' &: \mathcal{C}^0_{n,r}(M, E) \to \mathcal{C}^\infty_{n,r}(M, E) \,, \qquad 0 \le r \le q-1, \\ K_r'' &: \mathcal{C}^0_{n,r}(M, E) \to \mathcal{C}^\infty_{n,r}(M, E) \,, \qquad 1 \le r \le q-1, \\ T_r' &: \mathcal{C}^0_{n,r}(M, E) \to \mathcal{C}^\infty_{n,r-1}(M, E) \,, \qquad 1 \le r \le q, \end{aligned}$$

such that with the abbreviations

$$N_0 := I + K_0 + K_0' \quad \textit{and} \quad N_r := I + K_r + K_r' + K_r'' \,, \quad 1 \le r \le q-1,$$

each N_r, $0 \le r \le q-1$, is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,r}(M, E)$ (this is clear, because $I + K_0$ has this property), and:

(i) If $0 \le r \le q-1$ and $f \in (\operatorname{Dom} d)^0_{n,r}(M, E)$, then

$$N_r f = \begin{cases} (T_1 + T_1')df & \textit{if } r = 0 \,, \\ d(T_r + T_r')f + (T_{r+1} + T_{r+1}')df & \textit{if } 1 \le r \le q-1 \,, \end{cases} \tag{5.36}$$

and hence

$$dN_r f = N_{r+1} df \qquad \text{if } 0 \le r \le q-2 . \tag{5.37}$$

(ii) We have

$$\mathcal{C}^0_{n,r}(M,E)\,\mathrm{Im}\,N_r \oplus \mathrm{Ker}\,N_r\,, \qquad 0 \le r \le q-1\,, \tag{5.38}$$

$$\mathrm{Ker}\,N_r \subseteq \mathcal{Z}^\infty_{n,r}(M,E)\,, \qquad 0 \le r \le q-1\,, \tag{5.39}$$

$$\mathcal{Z}^0_{n,r}(M,E) = \begin{cases} \mathrm{Ker}\,N_0 & \text{if } r=0\,, \\ \mathcal{B}^{\alpha\to 0}_{n,r}(M,E) \oplus \mathrm{Ker}\,N_r & \text{if } 1 \le r \le q-1\,, \end{cases} \tag{5.40}$$

and

$$\mathcal{B}^{\alpha\to 0}_{n,r}(M,E) = \mathcal{B}^{0\to 0}_{n,r}(M,E)\,, \qquad 1 \le r \le q-1\,. \tag{5.41}$$

(iii) If $1 \le r \le q-1$ (and hence $q \ge 2$), then

$$N_r\big(\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)\big) = \mathcal{B}^{\alpha\to 0}_{n,r}(M,E)\,, \tag{5.42}$$

and $N_r\big|_{\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)}$ is an isomorphism of $\mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$.
(iv) If $0 \le r \le q-2$ (and hence $q \ge 2$), then

$$(\mathrm{Dom}\,d)^0_{n,r}(M,E) = N_r\Big((\mathrm{Dom}\,d)^0_{n,r}(M,E)\Big) \oplus \mathrm{Ker}\,N_r\,, \tag{5.43}$$

and hence $N_r\big|_{\mathrm{Im}\,N_r \cap (\mathrm{Dom}\,d)^0_{n,r}(M,E)}$ is an isomorphism of $\mathrm{Im}\,N_r \cap (\mathrm{Dom}\,d)^0_{n,r}(M,E)$.
(v) Remark: It follows from (5.38), (5.40) and (5.42) that

$$\mathrm{Im}\,N_r \cap \mathcal{Z}^0_{n,r}(M,E) = \mathcal{B}^{\alpha\to 0}_{n,r}(M,E) \qquad \text{if } 1 \le r \le q-1\,. \tag{5.44}$$

Proof. If $q=1$, this is lemma 5.2. We proceed by induction over q. Let $q \ge 2$ and assume that the statement of lemma 5.3 is already proved if we replace in this statement q by $q-1$. Let K'_r, $0 \le r \le q-2$, K''_r, $1 \le r \le q-2$, and T'_r, $1 \le r \le q-1$, be the operators from the statement which we obtain if, in the statement of lemma 5.3, we replace q by $q-1$. Set $N_0 = I + K_0$ and $N_r = I + K'_r + K''_r$ if $q \ge 3$ and $1 \le r \le q-2$.
Set

$$K''_{q-1} := dT'_{q-1}\,. \tag{5.45}$$

Since T'_{q-1} is a finite dimensional continuous linear operator from $\mathcal{C}^0_{n,q-1}(M,E)$ to $\mathcal{C}^\infty_{n,q-2}(M,E)$, then it is clear that also K''_{q-1} is such an operator. Set

$$\widetilde{N}_{q-1} := I + K_{q-1} + K''_{q-1}.$$

Then, by lemma 4.1, $\widetilde{N}_{q-1}$ is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,q-1}(M,E)$, and

$$\mathrm{Ker}\,\widetilde{N}_{q-1} \subseteq \mathcal{C}^\infty_{n,q-1}(M,E)\,. \tag{5.46}$$

Now we first prove that

$$\mathrm{Ker}\,\widetilde{N}_{q-1} \cap \mathcal{B}^{0\to 0}_{n,q-1}(M,E) = \{0\}\,. \tag{5.47}$$

Let $g \in \mathcal{B}^{0\to 0}_{n,q-1}(M,E)$ with $\widetilde{N}_{q-1} g = 0$ be given. Take $f \in \mathcal{C}^0_{n,q-2}(M,E)$ with $g = df$. Then, by definition of $\widetilde{N}_{q-1}$ and K''_{q-1}, we get

$$0 = \widetilde{N}_{q-1} g = \widetilde{N}_{q-1} df = (I + K_{q-1})df + K''_{q-1} df = (I + K_{q-1})df + dT'_{q-1} df\,.$$

By (3.5), this implies

$$0 = (dT_{q-1} + dT'_{q-1})df\,. \tag{5.48}$$

Since, by hypothesis of induction, statement (i) of lemma 5.3 is true if we replace q by $q-1$, we have

$$N_{q-2}f = \begin{cases} (T_1 + T_1')df & \text{if } q = 2\,, \\ d(T_{q-2} + T'_{q-2})f + (T_{q-1} + T'_{q-1})df & \text{if } q > 2\,. \end{cases}$$

Hence $dN_{q-2}f = d(T_{q-1} + T'_{q-1})df$, which implies by (5.48) that

$$N_{q-2}f \in \mathcal{Z}^0_{n,q-2}(M,E). \tag{5.49}$$

By hypothesis of induction, (5.40) and (5.38) are valid for $r = 0$. Therefore

$$\operatorname{Im} N_0 \cap \operatorname{Ker} N_0 = \{0\} \qquad \text{and} \qquad \operatorname{Ker} N_0 = \mathcal{Z}^0_{n,0}(M,E)\,.$$

If $q = 2$, then these relations together with (5.49) imply that $N_0 f = 0$, which means by the second relation that $f \in \mathcal{Z}^0_{n,0}(M,E)$. Hence $g = df = 0$ if $q = 2$.
Now let $q > 2$. Then, by hypothesis of induction, (5.44) is valid for $r = q-2$, and (5.49) implies that

$$N_{q-2}f \in \mathcal{B}^{\alpha\to 0}_{0,q-2}(M,E)\,.$$

Therefore and, since, by hypothesis of induction, (5.42) is valid for $r = q-2$, we can find $\widetilde{f} \in \mathcal{B}^{\alpha\to 0}_{0,q-2}(M,E)$ with

$$N_{q-2}f = N_{q-2}\widetilde{f}\,.$$

Then $f - \widetilde{f} \in \operatorname{Ker} N_{q-2}$. Since, by hypothesis of induction, (5.39) is valid for $r = q-2$, this implies that $f - \widetilde{f} \in \mathcal{Z}^\infty_{n,q-2}(M,E)$. As $\widetilde{f} \in \mathcal{Z}^0_{n,q-2}(M,E)$, this further implies that $f \in \mathcal{Z}^0_{n,q-2}(M,E)$. Hence $g = df = 0$. This completes the proof of (5.47).

Next we prove that

$$\mathcal{Z}^0_{n,q-1}(M,E) = \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E) \oplus \Big(\operatorname{Ker} \widetilde{N}_{q-1} \cap \mathcal{Z}^0_{n,q-1}(M,E)\Big)\,. \tag{5.50}$$

Since $K''_{q-1} = dT'_{q-1}$ and $\operatorname{Im} T'_{q-1} \subseteq \mathcal{C}^\infty_{q-1}(M,E)$, it is clear that

$$\operatorname{Im} K''_{q-1} \subseteq \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)\,. \tag{5.51}$$

By lemma 5.1 (iv), $(I + K_{q-1})\big|_{\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{B}^{\alpha\to 0}_{q-1}(M,E)$. Since K''_{q-1} is finite dimensional and we have (5.51), this implies that $\widetilde{N}_{q-1}\big|_{\mathcal{B}^{\alpha\to 0}_{q-1}(M,E)}$ has the same property. By (5.47), this means that

$$\widetilde{N}_{q-1}\big|_{\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)} \text{ is an isomorphism of } \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E). \tag{5.52}$$

In particular,

$$\operatorname{Im} \widetilde{N}_{q-1}\big|_{\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)} = \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)\,. \tag{5.53}$$

Moreover, by part (iii) of lemma 5.1, $(I + K_{q-1})\big|_{\mathcal{Z}^0_{q-1}(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{Z}^0_{q-1}(M,E)$, where

$$\operatorname{Im}(I + K_{q-1})\big|_{\mathcal{Z}^0_{q-1}(M,E)} \subseteq \mathcal{B}^{\alpha\to 0}_{q-1}(M,E)$$

Since K''_{q-1} is finite dimensional and we have (5.51), this implies that also $\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{q-1}(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{Z}^0_{q-1}(M,E)$, where

$$\operatorname{Im}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)} \subseteq \mathcal{B}^{\alpha\to 0}_{q-1}(M,E)\,.$$

Together with (5.51) this gives

$$\operatorname{Im}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)} = \mathcal{B}^{\alpha\to 0}_{q-1}(M,E)\,. \tag{5.54}$$

Therefore, (5.47) can be written

$$\operatorname{Ker}\widetilde{N}_{q-1}\cap\operatorname{Im}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)} = \{0\}\,.$$

Hence

$$\operatorname{Im}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)}\cap\operatorname{Ker}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)} = \{0\}\,. \tag{5.55}$$

As the index of $\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)}$ is zero, this yields

$$\begin{aligned}\mathcal{Z}^0_{n,q-1}(M,E) &= \operatorname{Im}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)}\oplus\operatorname{Ker}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)}\\ &= \operatorname{Im}\widetilde{N}_{q-1}\big|_{\mathcal{Z}^0_{n,q-1}(M,E)}\oplus\Big(\operatorname{Ker}\widetilde{N}_{q-1}\cap\mathcal{Z}^0_{n,q-1}(M,E)\Big)\,.\end{aligned}$$

Again by (5.53), this proves (5.50).

From (5.47) and (5.50) it follows that

$$\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E) = \mathcal{B}^{0\to 0}_{n,q-1}(M,E)\,. \tag{5.56}$$

Now we construct the operators K'_{q-1} and T'_q. As in the proof of lemma 5.2, we proceed in two steps. (In the first step we construct some operators K^*_{q-1} and T^*_q, and in the second step we construct some operators K^{**}_{q-1} and T^{**}_q and prove that $K'_{q-1} := K^*_{q-1}+K^{**}_{q-1}$ and $T'_q := T^*_q + T^{**}_q$ have the required properties.)

First step: Since $\dim\operatorname{Ker}\widetilde{N}_{q-1}<\infty$ and by (5.46), we can find a finite dimensional subspace Λ of $\mathcal{C}^\infty_{n,q-1}(M,E)$ such that

$$\operatorname{Ker}\widetilde{N}_{q-1}\Big(\mathcal{Z}^0_{n,q-1}(M,E)\cap\operatorname{Ker}\widetilde{N}_{q-1}\Big)\oplus\Lambda\,. \tag{5.57}$$

Let $m := \dim\Lambda$. If $m=0$, we set $K^*_{q-1}=0$ and $T^*_q=0$.

Now let $m>0$. Choose a basis $\lambda_1,\ldots,\lambda_m$ of Λ. Since the index of $\widetilde{N}_{q-1}$ is zero and and $\mathcal{C}^\infty_{n,q-1}(M,E)$ is dense in $\mathcal{C}^0_{n,q-1}(M,E)$, we can find an m-dimensional subspace $\widetilde{\Lambda}$ of $\mathcal{C}^\infty_{n,q-1}(M,E)$ with

$$\widetilde{\Lambda}\cap\operatorname{Im}\widetilde{N}_{q-1} = \{0\}\,. \tag{5.58}$$

Let $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_m$ be a basis of $\widetilde{\Lambda}$. By (5.57), $\Lambda\subseteq\operatorname{Ker}\widetilde{N}_{q-1}$. Therefore

$$\mathcal{Z}^0_{n,q-1}(M,E)\cap\Lambda\subseteq\mathcal{Z}^0_{n,q-1}(M,E)\cap\operatorname{Ker}\widetilde{N}_{q-1}.$$

Since on the other hand $\mathcal{Z}^0_{n,q-1}(M,E)\cap\Lambda\subseteq\Lambda$, this implies, again by (5.57), that

$$\mathcal{Z}^0_{n,q-1}(M,E)\cap\Lambda = \{0\}\,. \tag{5.59}$$

Since $\lambda_1,\dots,\lambda_m$ is a basis of Λ, it follows that also the forms $d\lambda_1,\dots,d\lambda_m$ are linearly independent. Therefore (as in the proof of lemma 5.2), we can find forms $\psi_1,\dots,\psi_m\in C^\infty_{n-k-q}(M,E^*)$ (where E^* denotes the dual bundle of E) such that

$$\int_M d\lambda_\nu\wedge\psi_\mu=\delta_{\nu\mu}\,,\quad 1\le\mu,\nu\le m\,. \tag{5.60}$$

Setting

$$\begin{aligned} T_q^*f&=\sum_1^m\left(\int_M f\wedge\psi_\nu\right)\widetilde\lambda_\nu\,, && f\in\mathcal C^0_{n,q}(M,E)\,,\\ K_{q-1}^*f&=(-1)^{n+q}\sum_{\nu=1}^m\left(\int_M f\wedge d\psi_\nu\right)\widetilde\lambda_\nu\,, && f\in\mathcal C^0_{n,q-1}(M,E)\,,\end{aligned}$$

now we define finite dimensional continuous linear operators

$$\begin{aligned} T_q^*&:\mathcal C^0_{n,q}(M,E)\longrightarrow\mathcal C^\infty_{n,q-1}(M,E)\,,\\ K_{q-1}^*&:\mathcal C^0_{n,q-1}(M,E)\longrightarrow\mathcal C^\infty_{n,q-1}(M,E)\,.\end{aligned}$$

Then from Stokes' theorem we obtain that

$$K_{q-1}^*f=T_q^*df\quad\text{for all } f\in(\operatorname{Dom}d)^0_{n,q-1}(M,E) \tag{5.61}$$

and

$$\mathcal Z^0_{n,q-1}(M,E)\subseteq\operatorname{Ker}K_{q-1}^*\,. \tag{5.62}$$

Moreover, by (5.60), Stokes' theorem gives

$$K_{q-1}^*\lambda_\mu=(-1)^{n+q}\sum_{\nu=1}^m\left(\int_M\lambda_\mu\wedge d\psi_\nu\right)\widetilde\lambda_\nu=\sum_{\nu=1}^m\left(\int_M d\lambda_\mu\wedge\psi_\nu\right)\widetilde\lambda_\nu=\widetilde\lambda_\mu$$

for all $1\le\mu\le m$. Since $\lambda_1,\dots,\lambda_m$ is a basis of Λ and $\widetilde\lambda_1,\dots,\widetilde\lambda_m$ is a basis of $\widetilde\Lambda$, this implies that

$$\Lambda\cap\operatorname{Ker}K_{q-1}^*=\{0\}\,. \tag{5.63}$$

Together with (5.57) and (5.62) this gives

$$\operatorname{Ker}(\widetilde N_{q-1}+K_{q-1}^*)\subseteq\mathcal Z^0_{n,q-1}(M,E)\subseteq\operatorname{Ker}K_{q-1}^*\,. \tag{5.64}$$

Hence

$$\operatorname{Ker}(\widetilde N_{q-1}+K_{q-1}^*)=\operatorname{Ker}\widetilde N_{q-1}\cap\mathcal Z^0_{n,q-1}(M,E)\,.$$

Together with (5.50) this implies that

$$\mathcal Z^0_{n,q-1}(M,E)\mathcal B^{\alpha\to0}_{n,q-1}(M,E)\oplus\operatorname{Ker}(\widetilde N_{q-1}+K_{q-1}^*)\,. \tag{5.65}$$

Taking into account again (5.53) and (5.62) this further implies that

$$\begin{aligned}(\widetilde N_{q-1}+K_{q-1}^*)\Big(\mathcal Z^0_{n,q-1}(M,E)\Big)&=(\widetilde N_{q-1}+K_{q-1}^*)\Big(\mathcal B^{\alpha\to0}_{n,q-1}(M,E)\Big)\\&=\mathcal B^{\alpha\to0}_{n,q-1}(M,E)\,,\end{aligned} \tag{5.66}$$

and $(\widetilde N_{q-1}+K_{q-1}^*)\big|_{\mathcal B^{\alpha\to0}_{n,q-1}(M,E)}$ is an isomorphism of $\mathcal B^{\alpha\to0}_{n,q-1}(M,E)$.

Second step: Since $\widetilde{N}_{q-1}$ is a Fredholm endomorphism with index zero of $\mathcal{C}^0_{n,q-1}(M,E)$, the operator $\widetilde{N}_{q-1} + K^*_{q-1}$ has the same property. Set

$$V = \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) \cap \operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1}) \quad \text{and} \quad p = \operatorname{Im} V\,.$$

Choose a p-dimensional subspace $\widehat{V}$ of $\mathcal{C}^0_{n,q-1}(M,E)$ with

$$(\widetilde{N}_{q-1} + K^*_{q-1})\big(\widehat{V}\big) = V\,.$$

Then, by (5.65) and (5.66),

$$\widehat{V} \cap \mathcal{Z}^0_{n,q-1}(M,E) = \{0\}\,. \tag{5.67}$$

Let $\widehat{v}_1, \dots, \widehat{v}_p$ be a basis of $\widehat{V}$ and and

$$v_\mu := (\widetilde{N}_{q-1} + K^*_{q-1})\widehat{v}_\mu\,, \qquad 1 \le \mu \le p\,.$$

By (5.67), then also the forms $d\widehat{v}_1, \dots, d\widehat{v}_p$ are linearly independent. Therefore, as above, we can find forms $\varphi_p, \dots, \varphi_p \in C^\infty_{n-k-q}(M,E^*)$ with

$$\int_M d\widehat{v}_\nu \wedge \varphi_\mu = \delta_{\nu\mu}\,, \quad 1 \le \mu,\nu \le p\,. \tag{5.68}$$

Set

$$\Gamma = \left\{ f \in \mathcal{C}^0_{n,q-1}(M,E) \;\middle|\; \int_M f \wedge d\varphi_\mu = 0 \qquad \text{for } 1 \le \mu \le p \right\}.$$

Then, by Stokes' theorem,

$$\mathcal{Z}^0_{n,q-1}(M,E) \subseteq \Gamma\,. \tag{5.69}$$

Since the forms $d\varphi_1, \dots, d\varphi_p$ are linearly independent,

$$\operatorname{codim} \Gamma = p\,. \tag{5.70}$$

(By codim we mean the codimension in $\mathcal{C}^0_{n,q-1}(M,E)$.) Since, by (5.68) and again by Stokes' theorem,

$$\int_M \widehat{v}_\mu \wedge d\varphi_\mu = (-1)^{n+q} \int_M d\widehat{v}_\mu \wedge \varphi_\mu = (-1)^{n+q}\,, \qquad 1 \le \mu \le p\,,$$

we see that $\widehat{V} \cap \Gamma = \{0\}$. Since $\dim \widehat{V} = p = \operatorname{codim} \Gamma$, it follows that

$$\mathcal{C}^0_{n,q-1}(M,E) = \widehat{V} \oplus \Gamma\,. \tag{5.71}$$

It follows from (5.65) and (5.69) that $\operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) \subseteq \Gamma$. Since $\operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1})$ is finite dimensional, therefore we can find a closed linear subspace $\widehat{L}$ of Γ with

$$\Gamma = \widehat{L} \oplus \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) \tag{5.72}$$

and hence, by (5.71),

$$\mathcal{C}^0_{n,q-1}(M,E) = \widehat{V} \oplus \widehat{L} \oplus \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1})\,. \tag{5.73}$$

Since $\widetilde{N}_{q-1} + K^*_{q-1}$ maps $\widehat{V}$ isomorphically to V, this yields

$$\operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1}) = V \oplus (\widetilde{N}_{q-1} + K^*_{q-1})\big(\widehat{L}\big) = V \oplus L\,, \tag{5.74}$$

where

$$L := (\widetilde{N}_{q-1} + K^*_{q-1})(\widehat{L}) .$$

By definition of V, we can find a finite dimensional subspace W of $\operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1})$ with

$$\operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) = V \oplus W. \tag{5.75}$$

Then, by (5.65),

$$\mathcal{Z}^0_{n,q-1}(M,E) = \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E) \oplus V \oplus W . \tag{5.76}$$

Since the index of $\widetilde{N}_{q-1} + K^*_{q-1}$ is zero, it follows form (5.75) and (5.74) that

$$\begin{aligned} p + \dim W &= \dim \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) \\ &= \operatorname{codim} \operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1}) = \operatorname{codim}(V \oplus L) . \end{aligned} \tag{5.77}$$

Moreover, it follows from (5.74) and (5.75) that

$$\begin{aligned} W \cap (V \oplus L) &= W \cap \operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1}) \\ & W \cap \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) \cap \operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1}) = W \cap V = \{0\} . \end{aligned} \tag{5.78}$$

Hence we have a direct sum $V \oplus L \oplus W$, where, by (5.77),

$$\operatorname{codim}(V \oplus L \oplus W) = p.$$

Since $\mathcal{C}^\infty_{n,q-1}(M,E)$ is dense in $\mathcal{C}^0_{n,q-1}(M,E)$, therefore we can find a p-dimensional subspace $\widetilde{V}$ of $\mathcal{C}^\infty_{n,q-1}(M,E)$ such that

$$\mathcal{C}^0_{n,q-1}(M,E) = V \oplus L \oplus W \oplus \widetilde{V} , \tag{5.79}$$

which means, by (5.75),

$$\mathcal{C}^0_{n,q-1}(M,E) = \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) \oplus L \oplus \widetilde{V} . \tag{5.80}$$

Let $\widetilde{v}_1, \dots, \widetilde{v}_p$ be a basis of $\widetilde{V}$. Setting

$$\begin{aligned} T^{**}_q f &= \sum_{\nu=1}^p \left(\int_M f \wedge \varphi_\nu \right) (\widetilde{v}_\nu - v_\nu) , && f \in \mathcal{C}^0_{n,q}(M,E) , \\ K^{**}_{q-1} f &= (-1)^{n+q} \sum_{\nu=1}^p \left(\int_M f \wedge d\varphi_\nu \right) (\widetilde{v}_\nu - v_\nu) , && f \in \mathcal{C}^0_{n,q-1}(M,E) , \end{aligned}$$

we define finite dimensional continuous linear operators

$$\begin{aligned} T^{**}_q &: \mathcal{C}^0_{n,q}(M,E) \longrightarrow \mathcal{C}^\infty_{n,q-1}(M,E) , \\ K^{**}_{q-1} &: \mathcal{C}^0_{n,q-1}(M,E) \longrightarrow \mathcal{C}^\infty_{n,q-1}(M,E) . \end{aligned}$$

Then it follows from Stokes' formula that

$$K^{**}_{q-1} f = T^{**}_q df \qquad \text{for all } f \in (\operatorname{Dom} d)^0_{n,q-1}(M,E) . \tag{5.81}$$

In particular,

$$Z^0_{n,q-1}(M,E) \subseteq \operatorname{Ker} K^{**}_{q-1} .$$

By (5.65), this implies that

$$\operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) \subseteq \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})\,. \tag{5.82}$$

By Stokes' theorem and (5.68), we have

$$K^{**}_{q-1}\widehat{v}_\mu = \sum_{\nu=1}^{p}\left(\int_M \widehat{v}_\mu \wedge d\varphi_\nu\right)(\widetilde{v}_\nu - v_\nu) = \widetilde{v}_\mu - v_\mu\,, \quad 1 \le \mu \le p\,.$$

Since $v_\mu = (\widetilde{N}_{q-1} + K^*_{q-1})\widehat{v}_\mu$ (by definition), this implies that

$$(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})\widehat{v}_\mu = \widetilde{v}_\mu\,, \qquad 1 \le \mu \le p.$$

Hence

$$(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})(\widehat{V}) = \widetilde{V}\,.$$

By the definitions of Γ and K^{**}_{q-1} it clear that $\Gamma \subseteq \operatorname{Ker} K^{**}_{q-1}$. Hence $\widehat{L} \subseteq \operatorname{Ker} K^{**}_0$ and

$$(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})(\widehat{L}) = L\,.$$

Together this implies that

$$(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})(\widehat{L} \oplus \widehat{V}) = L \oplus \widetilde{V}\,. \tag{5.83}$$

By (5.80), this yields

$$\begin{aligned}\dim \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})&\\ = \operatorname{codim} \operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) &\le \dim \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1})\,.\end{aligned}$$

Together with (5.82) this gives the equality

$$\operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1}) = \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})\,. \tag{5.84}$$

By (5.65) this yields

$$Z^0_{n,q-1}(M,E) = \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E) \oplus \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})\,. \tag{5.85}$$

By (5.80), it follows from (5.84)

$$\mathcal{C}^0_{n,q-1}(M,E) = \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) \oplus L \oplus \widetilde{V} \tag{5.86}$$

and

$$\mathcal{C}^0_{n,q-1}(M,E) = \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) \oplus \widehat{L} \oplus \widehat{V} \tag{5.87}$$

From (5.83) and (5.87) we get

$$\operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) = L \oplus \widetilde{V}.$$

By (5.86) this means that

$$\mathcal{C}^0_{n,q-1}(M,E) = \operatorname{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) \oplus \operatorname{Im}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1})\,. \tag{5.88}$$

Now we define

$$T'_q = T^*_q + T^{**}_q, \quad \text{and} \quad \mathrm{K}'_{q-1} = K^*_{q-1} + K^{**}_{q-1}\,.$$

Note that then, in the assertion of lemma 5.3, the operator N_{q-1} can be written

$$N_{q-1} = \widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}\,.$$

It remains to prove that the statements of parts (i) - (ii) of lemma 5.3 are valid also for $r = q-1$.

Part (i): Let $f \in (\mathrm{Dom}\, d)^0_{n,q-1}(M,E)$ be given. Then, by (3.5), (5.45), (5.61) and (5.10)

$$\begin{aligned} d(T_{q-1}+T'_{q-1})f + (T_q+T'_q)df &= dT_{q-1}f + T_q df + dT'_{q-1}f + T'_q df \\ &= dT_{q-1}f + T_q df + dT'_{q-1}f + T^*_q df + T^{**}_q df \\ &= (I+K_{q-1})f + K''_{q-1}f + K^*_{q-1}f + K^{**}_{q-1}f \\ &= (I+K_{q-1}+K''_{q-1}+K'_{q-1})f = N_{q-1}f\,. \end{aligned} \tag{5.89}$$

Part (ii): From (5.85) we get

$$\begin{aligned} \mathcal{Z}^0_{n,q-1}(M,E) &= \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E) \oplus \mathrm{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) \\ &= \mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E) \oplus \mathrm{Ker}\, N_{q-1}\,. \end{aligned}$$

As $\mathrm{Ker}\, N_{q-1} \subseteq \mathcal{C}^\infty_{q-1}(M,E)$ (if this is not clear from the construction above, we can say this by lemma 4.1), this implies that $\mathrm{Ker}\, N_{q-1} \subseteq \mathcal{Z}^\infty_{q-1}(M,E)$.
From (5.74) we get

$$\begin{aligned} \mathcal{C}^0_{n,q-1}(M,E) &= \mathrm{Ker}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) \oplus \mathrm{Im}(\widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}) \\ &= \mathrm{Ker}\, N_{q-1} \oplus \mathrm{Im}\, N_{q-1}\,. \end{aligned}$$

That (5.41) is valid for $r = q-1$, we already mentioned above (see (5.56)).

Part (iii): By (5.52), $\widetilde{N}_{q-1}\big|_{\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)}$ is an isomorphism of $\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)$. As $N_{q-1} = \widetilde{N}_{q-1} + K^*_{q-1} + K^{**}_{q-1}$ and, by (5.61) and (5.81), both operators K^*_{q-1} and K^{**}_{q-1} vanish on $\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)$, this implies that

$$N_{q-1}\big|_{\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)} = \widetilde{N}_{q-1}\big|_{\mathcal{B}^{\alpha\to 0}_{n,q-1}(M,E)}$$

is an isomorphism of $\mathcal{B}^{\alpha\to 0}_{q-1}(M,E)$.

Part (iv): Since, by (5.39) and (5.38), $\mathrm{Ker}\, N_r \subseteq (\mathrm{Dom}\, d)^0_{n,r}(M,E)$ and $\mathcal{C}^0(M,E) = \mathrm{Im}\, N_r \oplus \mathrm{Ker}\, N_r$, it is sufficient to prove that

$$\mathrm{Im}\, N_r \cap (\mathrm{Dom}\, d)^0_{n,r}(M,E) = N_r\Big((\mathrm{Dom}\, d)^0_{n,r}(M,E)\Big)\,. \tag{5.90}$$

The relation "$\supseteq$" follows from (5.37). To prove the opposite, we first show that

$$\mathrm{Im}\, N_r \cap \mathcal{Z}^0_{n,r}(M,E) \subseteq N_r\Big((\mathrm{Dom}\, d)^0_{n,r}(M,E)\Big)\,, \qquad 0 \le r \le q-2\,. \tag{5.91}$$

By (5.38) and (5.40),

$$\mathrm{Im}\, N_0 \cap \mathcal{Z}^0_{n,0}(M,E) = \{0\}\,.$$

Therefore (5.91) is trivial for $r = 0$.
Now let $1 \le r \le q-2$ and $f \in \mathrm{Im}\, N_r \cap \mathcal{Z}^0_{n,r}(M,E)$ be given. By (5.40), then f has the form

$$f = g + h \qquad \text{where } g \in \mathcal{B}^{\alpha\to 0}_{n,r}(M,E) \text{ and } h \in \mathrm{Ker}\, N_r\,.$$

Further, from (5.42) we get $\widetilde{g} \in \mathcal{B}^{\alpha\to 0}_{n,r}(M,E)$ with $g = N_r\widetilde{g}$. Then

$$f = g + h = N_r\widetilde{g} + h\,.$$

Since $h \in \operatorname{Ker} N_r$ and, by (5.38), $\operatorname{Im} N_r \cap \operatorname{Ker} N_r = \{0\}$, this implies that $h = 0$ and

$$f = N_r \widetilde{g} \in N_r\Big(\mathcal{B}^{\alpha \to 0}_{n,r}(M,E)\Big) \subseteq N_r\Big((\operatorname{Dom} d)^0_{n,r}(M,E)\Big).$$

This completes the proof of (5.91).
Now we consider an arbitrary $f \in \operatorname{Im} N_r \cap (\operatorname{Dom} d)^0_{n,r}(M,E)$. Then, by (5.41) (recall that $1 \le r \le q-2$),

$$df \in \mathcal{B}^{0 \to 0}_{n,r+1}(M,E) = \mathcal{B}^{\alpha \to 0}_{n,r+1}(M,E)\,.$$

Therefore and by (5.42), we can find $g \in \mathcal{B}^{\alpha \to 0}_{n,r+1}(M,E)$ with $df = N_{r+1}g$. Choose $\widetilde{g} \in (\operatorname{Dom} d)^0_{n,r}(M,E)$ with $g = d\widetilde{g}$. Then, by (5.37),

$$df = N_{r+1}g = N_{r+1}d\widetilde{g} = dN_r\widetilde{g}$$

and therefore

$$(f - N_r\widetilde{g}) \;\in\; \operatorname{Im} N_r \cap \mathcal{Z}^0_{n,r}(M,E)\,.$$

By (5.91) this yields

$$(f - N_r\widetilde{g}) \;\in\; N_r\Big((\operatorname{Dom} d)^0_{n,r}(M,E)\Big)\,.$$

As $\widetilde{g} \in (\operatorname{Dom} d)^0_{n,r}(M,E)$, this implies that

$$f \;\in\; N_r\Big((\operatorname{Dom} d)^0_{n,r}(M,E)\Big)\,.$$

□

6 End of the proof of theorem 2.1

We use the notation of lemma 5.3. Then, by (5.39) and (5.40), $\mathcal{H}_0 = \operatorname{Ker} N_0$. Since N_0 is a Fredholm operator, it follows that $\dim H_0 < \infty$. For $1 \le r \le q-1$ we define

$$\mathcal{H}_r = \operatorname{Ker} N_r$$

Since also the operators N_r, $1 \le r \le q-1$, are Fredholm operators and by (5.39), also each $\mathcal{H}_r$ with $1 \le r \le q-1$ is a finite dimensional subspace of $\mathcal{Z}^\infty_{n,r}(M,E)$.
By (5.38),

$$\mathcal{C}^0_{n,r}(M,E) = \operatorname{Im} N_r \oplus \mathcal{H}_r\,, \qquad 0 \le r \le q-1\,. \tag{6.1}$$

Define P_r, $0 \le r \le q-1$, as the linear projection in $\mathcal{C}^0_{n,r}(M,E)$ with

$$\operatorname{Im} P_r = \mathcal{H}_r \quad \text{and} \quad \operatorname{Ker} P_r = \operatorname{Im} N_r\,, \qquad 0 \le r \le q-1\,. \tag{6.2}$$

Since the spaces $\operatorname{Im} N_r$ and H_r are closed in the $\mathcal{C}^0$-topology, these projections are continuous with respect to the $\mathcal{C}^0$-topology. Since, by (5.41) and (5.42), $\mathcal{B}^{0 \to 0}_{n,r}(M,E) \subseteq \operatorname{Im} N_r$, this implies (2.3).
Set

$$\widehat{N}_r = N_r + P_r\,, \qquad 0 \le r \le q-1\,.$$

Then, by (6.1) and (6.2), for each $0 \le r \le q-1$, $\widehat{N}_r$ is an isomorphism of $\mathcal{C}^0_{n,r}(M,E)$. If $0 \le r \le q-2$, then moreover

$$\widehat{N}_r\Big((\operatorname{Dom} d)^0_{n,r}(M,E)\Big) = (\operatorname{Dom} d)^0_{n,r}(M,E)$$

and therefore $\widehat{N}_r\big|_{(\mathrm{Dom}\, d)^0_{n,r}(M,E)}$ is an isomorphism of $(\mathrm{Dom}\, d)^0_{n,r}(M,E)$. Indeed, since $\mathrm{Ker}\, K_r \subseteq \mathcal{Z}^\infty_{n,r}(M,E) \subseteq (\mathrm{Dom}\, d)^0_{n,r}(M,E)$, this follows from part (iv) of lemma 5.3.
Setting

$$A_r = \widehat{N}^{-1}_{r-1}(T_r + T'_r)\,, \qquad 1 \le r \le q\,,$$

now we define the continuous linear operators

$$A_r : \mathcal{C}^0_{n,r}(M,E) \longrightarrow \mathcal{C}^0_{n,r-1}(M,E)\,, \qquad 1 \le r \le q\,.$$

It remains to prove statements (i) - (iii) of the theorem.

Proof of (i): Let $1 \le r \le q$ and $l \in \mathbb{N}$ be given. By definition, $\widehat{N}_{r-1}$ is of the form $\widehat{N}_{r-1} = I + R$ where $R\big|_{\mathcal{C}^l_{n,r-1}(M,E)}$ is a continuous linear operator from $\mathcal{C}^l_{n,r-1}(M,E)$ to $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$.
Hence, by Ascoli's theorem, $\widehat{N}_{r-1}\big|_{\mathcal{C}^l_{n,r-}(M,E)}$ is a Fredholm endomorphism with index zero of $\mathcal{C}^l_{n,r-1}(M,E)$. Since

$$\mathrm{Ker}\, \widehat{N}_{r-1}\big|_{\mathcal{C}^l_{n,r-1}(M,E)} \subseteq \mathrm{Ker}\, \widehat{N}_{r-1} = \{0\},$$

this further implies that $\widehat{N}_{r-1}\big|_{\mathcal{C}^l_{n,r-1}(M,E)}$ is an isomorphism of $\mathcal{C}^l_{n,r-1}(M,E)$. In particular $\widehat{N}^{-1}_{r-1}\big|\mathcal{C}^l_{n,r-1}(M,E)$ is a continuous endomorphism of $\mathcal{C}^l_{n,r-1}(M,E)$. Hence $R\widehat{N}^{-1}_r\big|\mathcal{C}^l_{n,r-1}(M,E)$ is a continuous operator from $\mathcal{C}^l_{n,r-1}(M,E)$ to $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$. In particular, $R\widehat{N}^{-1}_r\big|\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$ is a continuous endomorphism of $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$. Since

$$\widehat{N}^{-1}_{r-1} = I - R\widehat{N}^{-1}_{r-1}\,,$$

this implies that $\widehat{N}^{-1}_{r-1}\big|\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$ is a continuous endomorphism of $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$. Since $(T_r + T'_r)\big|\mathcal{C}^l_{n,r}(M,E)$ is a continuous from $\mathcal{C}^l_{n,r}(M,E)$ to $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$, it follows that

$$A_r = \widehat{N}^{-1}_r(T_r + T'_r)$$

is continuous from $\mathcal{C}^l_{n,r}(M,E)$ to $\mathcal{C}^{l+\alpha}_{n,r-1}(M,E)$.

Proof of (ii): Let $1 \le r \le q-1$. We first prove that

$$\widehat{N}^{-1}_r d\big|_{(\mathrm{Dom}\, d)^0_{n,r-1}(M,E)} = d\widehat{N}^{-1}_{r-1}\big|_{(\mathrm{Dom}\, d)^0_{n,r-1}(M,E)}\,. \tag{6.3}$$

Since $\widehat{N}_{r-1}\big|_{(\mathrm{Dom}\, d)^0_{n,r-1}(M,E)}$ is an isomorphism of $(\mathrm{Dom}\, d)^0_{n,r-1}(M,E)$, this is equivalent to

$$d\widehat{N}_{r-1}\big|_{(\mathrm{Dom}\, d)^0_{n,r-1}(M,E)} = \widehat{N}_r d\big|_{(\mathrm{Dom}\, d)^0_{n,r-1}(M,E)}\,. \tag{6.4}$$

Let $g \in (\mathrm{Dom}\, d)^0_{n,r-1}(M,E)$ be given. Then, by (2.3), (5.40) and (5.37),

$$dP_{r-1}g = 0\,, \quad P_r dg = 0 \quad \text{and} \quad dN_{r-1}g = N_r dg\,.$$

Hence

$$d\widehat{N}_{r-1}g = dN_{r-1}g + dP_{r-1}g = dN_{r-1}g = N_r dg = N_r dg + P_r dg = \widehat{N}_r dg\,.$$

Now consider $f \in (\mathrm{Dom}\, d)^0_{n,r}(M,E)$. Then, by (5.36) and definition of the operators A_r,

$$\widehat{N}^{-1}_r N_r f = \begin{cases} A_1 df & \text{if } r = 0\,, \\ \widehat{N}^{-1}_r d(T_r + T'_r)f + A_{r+1}df & \text{if } 1 \le r \le q-1\,. \end{cases} \tag{6.5}$$

Since $\operatorname{Im} P_r = \mathcal{H}_r = \operatorname{Ker} N_r$, we have $N_r P_r = 0$. Hence $\widehat{N}_r(I - P_r) = (N_r + P_r)(I - P_r) = N_r$ and therefore $\widehat{N}_r^{-1} N_r = I - P_r$. Therefore, (6.5) takes the form

$$f - P_r f = \begin{cases} A_1 df & \text{if } r = 0\,, \\ \widehat{N}_r^{-1} d(T_r + T_r')f + A_{r+1} df & \text{if } 1 \le r \le q-1\,, \end{cases} \tag{6.6}$$

This completes the proof for $r = 0$. Let $1 \le r \le q - 1$. Then, by (5.36),

$$(T_r + T_r')f \in (\operatorname{Dom} d)^0_{n,r-1}(M, E),$$

and it follows from (6.4) that

$$\widehat{N}_r^{-1} d(T_r + T_r')f = d\widehat{N}_{r-1}^{-1}(T_r + T_r')f = dA_r f\,.$$

Together with (6.6), this gives (2.5).

Proof of (iii): Let $1 \le r \le q$. Then (2.7) follows from (2.5) and (2.3), and (2.6) follows from (2.7) and (2.4).
Now let $l \in \mathbb{N} \cup \{\infty\}$ be given. To prove that $\mathcal{B}^{l+\alpha\to l}_{n,r}(M, E)$ is closed in the $\mathcal{C}^l$-topology, we consider a sequence $f_\nu \in \mathcal{B}^{l+\alpha\to l}_{n,r}(M, E)$ which converges in the $\mathcal{C}^l$-topology to some $f \in \mathcal{C}^l_{n,r}(M, E)$. Since, by part (i) of the theorem, A_r is continuous as operator from $\mathcal{C}^l_{n,r}(M, E)$ to $\mathcal{C}^{l+\alpha}_{n,r-1}(M, E)$, then the sequence $A_r f_\nu$ converges in the $\mathcal{C}^{l+\alpha}$-topology to some $g \in \mathcal{C}^{l+\alpha}_{n,r-1}(M, E)$, where, by (2.7), $dA_r f_\nu = f_\nu$ for all ν. Since the operator

$$d : \mathcal{C}^{l+\alpha}_{n,r-1}(M, E) \longrightarrow \mathcal{B}^{l+\alpha\to l}_{n,r}(M, E)$$

is closed, this implies that $dg = f$, i.e. $f \in \mathcal{B}^{l+\alpha\to l}_{n,r}(M, E)$.
Since, by (2.4),

$$A_r\Big(\mathcal{C}^\infty_{n,r}(M, E)\Big) \subseteq \mathcal{C}^\infty_{n,r-1}(M, E),$$

it follows from (2.7) that

$$d\mathcal{C}^\infty_{n,r-1}(M, E) = \mathcal{C}^\infty_{n,r}(M, E) \cap \mathcal{B}^{0\to l}_{n,r}(M, E)\,,$$

which means that (2.8) is injective.
Now let $1 \le r \le q - 1$. Then, by (5.40), (5.41) and definition of $\mathcal{H}_r$,

$$Z^0_{n,r}(M, E) = \mathcal{B}^{0\to 0}_{n,r}(M, E) \oplus \mathcal{H}_r\,.$$

Since $\mathcal{H}_r \subseteq \mathcal{C}^\infty_{n,r}(M, E)$, this implies that

$$Z^l_{n,r}(M, E) = \mathcal{B}^{0\to l}_{n,r}(M, E) \oplus \mathcal{H}_r \qquad \text{for all } l \in \mathbb{N} \cup \{\infty\}\,.$$

By (2.6) this means (2.9). □

References

[AH] R. Airapetjan and G. Henkin, *Integral representation of differential forms on Cauchy-Riemann maniufolds and the theory of CR-functions*, Russian Math. Survey **39** (1984), 41-118.

[BLT] M. Y. Barkatou and C. Laurent-Thiébaut, *Solutions fondamentales et estimations optimales pour l'oprateur de Cauchy-Riemann tangentiel,* Michigan Math. J. **54** (2006), 545 - 586.

[H] G. Henkin, *Solution des équations de Cauchy-Riemann tangentielles sur des variétés de Cauchy-Riemann q-concaves,* C. R. Acad. Sci., Paris, Sér. I **292** (1981), 27-30.

[L] J. Leiterer, *From local to global homotopy formulas for $\overline{\partial}$ and $\overline{\partial}_b$,* in: J. Noguchi et al. (ed.), Geometric complex analysis, Proceedings of the conference held at the 3rd International Research Institute of the Mathematical Society of Japan, Hayama, March 19-29, 1995, World Scientific, Singapore 1996, 385-391.

[P1] P. Polyakov, *Sharp estimates for operator $\overline{\partial}_M$ on a q-concave CR manifold,* J. Geom. Anal. **6** (1996), 233-276.

[P2] P. Polyakov, *Sharp Lipschitz estimates for operator $\overline{\partial}_M$ on a q-concave CR-manifold,* J. Geom. Anal. **12** (2002), 103-146.

[P3] P. Polyakov, *Global $\overline{\partial}_M$-homotopy with $\mathcal{C}^k$ estimates for a family of compact, regular q-pseuedoconcave CR manifolds* Math. Z. **247** (2004), 813-862.

[P4] P. Polyakov, *Versal embeddings of compact* 3*-pseudoconcave CR-submanifolds,* Math. Z. **248** (2004), 267-312.

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