Compact Manifolds covered by a torus

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dedicated to Gennadi Henkin

Abstract

Let X be a compact complex manifold which is the image of a complex torusa by a holomorphic surjective map $A \to X$. We prove that X is Kähler and that up to a finite étale cover, X is a product of projective spaces by a torus.

Keywords: Complex torus, abelian variety, projective space, Kähler manifold, Albanese morphism, fundamental group, étale cover, ramification divisor, nef divisor, nef tangent bundle, anti-canonical line bundle, numerically flat vector bundle.

Résumé

Soit X une variété analytique complexe compacte qui est l'image d'un tore complexe A par une application holomorphe surjective $A \to X$. Nous montrons que X est kählérienne et que modulo un revêtement étale fini, X est un produit d'espaces projectifs complexes par un tore.

Mots-clés : Tore complexe, variété abélienne, espace projectif, variété kählérienne, morphisme d'Albanese, groupe fondamental, revêtement étale, diviseur de ramification, diviseur nef, fibré tangent nef, fibré anti-canonique, fibré vectoriel numériquement plat.

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Contents

1 Introduction

Consider a compact complex manifold X which is the image of a complex torus A by a surjective map

$$f : A \to X.$$

The purpose of this note is a classification of X up to finite étale cover. When A is a simple abelian variety and f is not an isomorphism, Debarre [Deb89] has shown that $X \simeq \mathbb{P}_n$. In the case of a general abelian variety A, Hwang-Mok [HM01] proved that X is a tower of bundles

$$X = X_0 \to X_1 \to \ldots \to X_k = Y$$

over a base Y which is an unramified quotient of an abelian variety, the fibers of which are projective spaces $\mathbb{P}_{n_j}$. Our result extends this structure theorem to the case of general (non necessarily projective) tori, and clarifies the structure of the tower of projectives bundles by proving that it is in fact a locally trivial bundle whose fibers are products of projective spaces.

1.1. Theorem. *Let X be a connected compact complex manifold which is the image of a complex torus A by a surjective holomorphic map $f : A \to X$. Then*

(1) *X is Kähler.*

(2) *There exists a finite étale cover X' of X such that*

$$X' \simeq \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k} \times B$$

is a product of projective spaces with the Albanese torus $B = A(X')$. Moreover f admits a lifting $f' : A' \to X'$ where A' is an étale cover of A and

$$f' : A' = A_1 \times \ldots \times A_k \times B \to \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k} \times B$$

is given by a product of factors $f_{n_j} : A_j \to \mathbb{P}_{n_j}$ and $\mathrm{id} : B \to B$, for suitable abelian varieties A_j and a (non necessarily algebraic) torus B.

(3) *X is an étale quotient of the bundle $X' = \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k} \times B \to B$, in other words X is a $\mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$-bundle over an étale quotient M of the torus B, such that the trivial fibration $X' \to B$ is the pull-back of $X \to M$ by the quotient map $B \to M$.*

We use in a very essential way results and methods borrowed from the paper [DPS94] – the main goal of which was to study the structure of compact Kähler manifolds with nef tangent bundles – although it is unclear whether X should a priori have a nef tangent bundle in the present setting; certainly T_X is nef on all curves not contained in the ramification locus, but we do not have control over the curves in this locus. On the other hand, one can pull-back divisors to A and obtain, as a simple consequence, that every pseudo-effective divisor on X must be nef. This is also a significant property of manifolds with nef tangent bundles.
The fact that X is Kähler is a direct consequence of the fact that any surjective map $f : A \to X$ as above must be equidimensional – this is true even when X is taken in the category of reduced complex spaces. We then rely on an observation, essentially due to Varouchas [Var84], stating that the Kähler property is preserved by proper equidimensional morphisms. This is discussed in more detail in the appendix, where we present a simple approach of this result.

2 Elementary reductions

First notice that every n-dimensional projective subvariety $Y \subset \mathbb{P}_N$ can be mapped by a finite morphism $Y \to \mathbb{P}_n$ onto projective space by taking a generic linear projection $\mathbb{P}_N \dashrightarrow \mathbb{P}_n$, and in particular $\mathbb{P}_n$ can be obtained as a finite surjective image of an abelian variety. A very elementary construction consists in writing $\mathbb{P}_n$ as the symmetric product $(\mathbb{P}_1)^{(n)}$; then by taking $2:1$ covers $E_j \to \mathbb{P}_1$ by elliptic curves, one realizes $\mathbb{P}^n$ as the image of $E_1 \times \ldots \times E_n$ by a $2^n n! : 1$ finite (ramified) morphism. Therefore any product $A_0 \times \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$ of a torus A_0 by projective spaces can be written as a finite surjective image of an abelian variety A.
Conversely, let $f : A \to X$ be a surjective holomorphic map from a complex torus onto a reduced complex space X (we do not require X to be smooth in this section). Then the generic (smooth) fiber $F_x = f^{-1}(x)$ has a trivial normal bundle, and therefore also a trivial tangent bundle. It follows that the connected components of F_x are translates of subtori. Since subtori form a discrete family, by looking at the Stein factorization $A \to W \to X$ which has precisely the connected components of the F_x as fibers of $A \to W$, we see that there is a subtorus $S \subset A$ and a factorization $g : A/S \to X$ of f such that g is generically finite.

2.1. Lemma. *The quotient map $g : A/S \to X$ is finite.*

Proof. Otherwise, there would exist a positive dimensional component Y in some fiber $g^{-1}(x) \subset A/S$. Then a small translate $Y + \delta$ would map to a compact irreducible subset contained in a small Stein neighborhood V of x, and the image $g(Y + \delta)$ would be a single point $y \in V$. This would imply that the generic fiber of $g : A/S \to X$ of g is positive dimensional, a contradiction. □

2.2. Corollary. *Every surjective holomorphic map $f : A \to X$ from a torus A onto a complex space X factorizes as a finite map $g : A/S \to X$ with respect to some subtorus S of A.*

As a consequence, we may assume from the very beginning that the surjective map $f : A \to X$ is finite.

3 The Kähler property

In order to conclude that our complex manifold (or complex space) X is Kähler, we rely on the following general statement. Recall that a complex space Z is said to be Kähler if there exists a (smooth) Kähler metric ω on the regular part Z_{reg}, which can be extended locally as a Kähler metric in a smooth ambient space, near every singular point of Z. We will also say that Z *has mild singularities* if Z is normal and every point of Z admits a neighborhood U for which there exists a finite ramified cover $\hat{U}$ which is smooth (for example, we can take Z to have quotient singularities).

3.1. Theorem. *Let $f : Y \to X$ be a proper and surjective holomorphic map between complex spaces of pure dimensions. Assume that Y is Kähler, that f has equidimensional fibers and that X has mild singularities. Then X is Kähler.*

Proof. When X and Y are manifolds, the result is due to J. Varouchas [Var84], and is in fact sufficient for our purposes – we even only need the case when f is finite. For the sake of simplicity, we outline the proof in this special case, and refer the reader to the appendix for a proof of the general statement. Let ω be a Kähler metric on Y and α a (smooth) positive definite hermitian metric on X. Then for every compact set $K \subset X$ we have $f^*\alpha \le C\omega$ on $f^{-1}(K)$ for some constant $C_K > 0$, hence

$$f_*\omega \ge C_K^{-1} f_* f^*\alpha \ge C_K^{-1}(\deg f)\,\alpha \quad \text{on } K.$$

This implies that $T = f_*\omega$ is a Kähler current (see e.g. [DP04, 0.5]), i.e. that it is bounded below by a smooth positive definite form. However, a local potential of T can be written as $\psi(x) = \sum \varphi_j(y_j)$ where $f^{-1}(x) = \{y_j\}$ and φ_j is a local potential of ω near y_j. This implies immediately that ψ is continuous. In fact the problem occurs only in a neighborhood of ramification points, and there the continuity follows immediately from the fact that f is finite. Therefore ψ is a continuous strictly plurisubharmonic function. Then it follows by using Richberg's result [Ric68] (see also [Dem82], [Dem92]) that ψ and T can be regularized so as to produce a Kähler metric on X. □

For people only interested in pure algebraic geometry, we present a simpler statement which suffices in the algebraic context.

3.2. Theorem. *Let $f : A \to X$ be a finite surjective map from an abelian variety to a complex manifold X. Then X is projective.*

Proof. It immediately follows from the assumption that X is Moishezon. Let

$$\pi : \hat{X} \to X$$

a bimeromorphic holomorphic map from a projective manifold $\hat{X}$. Choose an ample line bundle $\hat{L}$ on $\hat{X}$ and set

$$L = (\pi_*(\hat{L}))^{**}.$$

Then L is a big line bundle on X. Hence $f^*(L)$ is a big line bundle on A and therefore ample. Hence L is ample. □

4 Preliminary structure results

Let X be a compact manifold covered by a torus, and let $n = \dim X$ be its dimension. By the above, X is Kähler and possesses a finite ramified covering $f : A \to X$ by a torus. Let us denote by $R = \{\mathrm{Jac}(f) = 0\}$ the ramification divisor of f in A, so that

$$K_A = 0 = f^*K_X + R.$$

We also consider the irregularity $q(X) = h^0(X, \Omega^1_X)$ of X and the Albanese map

$$\alpha : X \to A(X), \qquad \dim A(X) = q(X),$$

and define

$$\tilde{q}(X) = \max q(\tilde{X}),$$

where $\tilde{X} \to X$ runs over all finite étale covers $\tilde{X} \to X$, to be the maximal irregularity of these covers; the following result shows in particular that $\tilde{q}(X)$ is always finite in the present context.

4.1. Proposition.

(1) *Every pseudo-effective divisor on X is nef. In particular $-K_X$ is nef.*

(2) *Every big divisor on X is ample.*

(3) *X is Fano iff $-K_X$ is big iff $(-K_X)^n > 0$ iff R is ample.*

(4) *$T_{X|C}$ is nef for all curves not contained in the image $f(R) \subset X$ of the ramification locus.*

(5) *$\pi_1(X)$ is almost abelian, i.e. abelian up to a subgroup of finite index.*

(6) *The Albanese map* $\alpha : X \to A(X)$ *is surjective.*

(7) *The Albanese map is in fact a submersion with connected fibers.*

Proof. (1) If L is pseudo-effective, so is $f^*(L)$. Hence $f^*(L)$ (which is a line bundle on a torus) is nef, and therefore so is L by [DPS94, 1.8]. This applies in particular to $-K_X$, since $f^*(-K_X) = R$ is nef.

(2) If L is big, then X is Moishezon and therefore projective. As it is well-known, L is then the sum of an effective $\mathbb{Q}$-divisor and of an ample $\mathbb{Q}$-divisor, and the effective part is nef by (1), hence L must be ample.

(3) follows immediately from (1) and (2).

(4) is a direct consequence of the sheaf inclusion $\mathcal{O}(T_A) \subset f^*\mathcal{O}(T_X)$ and of the triviality of T_A. In fact, if C is not contained in $f(R)$, the cokernel of the restriction map $\mathcal{O}(T_{A|f^{-1}(C)}) \subset f^*\mathcal{O}(T_{X|C})$ is a torsion sheaf supported on the finite set $C \cap f(R)$.

(5) As f is surjective, the induced morphism

$$\pi_1(A) \to \pi_1(X)$$

of fundamental groups has an image of finite index in $\pi_1(X)$, bounded by the degree of f.

(6) If α were not be surjective, then the image $\alpha(X) \subset A(X)$ would have a quotient $Y = \alpha(X)/B$ by a subtorus which is a variety of general type (Ueno [Uen75]). Then there would exist a surjective morphism $A \to Y$ to a variety of general type, which is absurd: the pull-back of a pluricanonical section of Y would yield a section of a certain tensor power $(\Omega_A^p)^{\otimes k}$ possessing zeros, contradicting the triviality of the bundle Ω_A^p.

(7) The composition

$$A \xrightarrow{f} X \xrightarrow{\alpha} A(X)$$

is a surjective map between tori, hence is a linear submersion. Its differential is everywhere surjective, and therefore so is the differential of α. If α has disconnected fibers, we consider its Stein factorization

$$X \to W \to A(X).$$

The finite map $W \to A(X)$ must be unramified, otherwise W would have positive Kodaira dimension, which is impossible since it is an image of a torus. Therefore W itself is a torus, and must be equal to $A(X)$ by the universality of the Albanese map. This actually implies that the fibers of α are connected. □

4.2. Theorem. *Take a finite étale cover* $\tilde{X} \to X$ *such that* $q(\tilde{X}) = \tilde{q}(X)$*. Then the Albanese map*

$$\alpha : \tilde{X} \to A(\tilde{X})$$

is a submersion with connected fibers F *which are covered by tori and have finite fundamental group* $\pi_1(F)$*.*

Proof. There exists a lifting $\tilde{f} : \tilde{A} \to \tilde{X}$ of f to some étale cover of the torus A. Therefore $\tilde{X}$ is also covered by a torus and 4.1 (7) implies that α is a submersion with connected fibers.
Now consider a fiber F of α. Then $\tilde{f}^{-1}(F)$ has a trivial normal bundle in $\tilde{A}$, hence $\tilde{f}^{-1}(F)$ must be a translate of a subtorus. It follows from this that every fiber F is also covered by a torus, as well as any finite étale cover $\tilde{F}$. By what we have just seen, the Albanese map of $\tilde{F}$ is a submersion, and we can thus apply Proposition 4.3 below to obtain

$$\tilde{q}(X) = \tilde{q}(F) + \tilde{q}(A(X)).$$

Since $\tilde{q}(A(X)) = \dim A(X) = q(X) = \tilde{q}(X)$, we conclude that $\tilde{q}(F) = 0$; in other words, $\pi_1(F)$ is finite. □

4.3. Proposition. *Let X and Y be compact Kähler manifolds and $g : X \to Y$ be a surjective submersion with connected fibers. Let F be a fiber of g. Then*

$$\tilde{q}(X) \leq \tilde{q}(F) + \tilde{q}(Y). \qquad (*).$$

Assume moreover that Y is a torus, that $\pi_1(F)$ is almost abelian and that for every finite étale cover $\tilde{F} \to F$, the Albanese map of $\tilde{F} \to A(\tilde{F})$ is a constant rank map. Then equality in $(*)$ holds.

Proof. This is proposition 3.12 in [DPS94]. □

4.4. Theorem. *If $H^0(X, \Omega^p_X) \neq 0$ for some $p \geq 1$, then $\tilde{q}(X) > 0$.*

Proof. We may assume $p \geq 2$ and choose a non-zero holomorphic p-form u. Let

$$\Phi : \bigwedge^{p-1} T_X \to \Omega^1_X$$

be the morphism defined by contraction of $(p-1)$-vectors with u, and let $\mathcal{E} = \mathrm{Im}\Phi \subset \Omega^1_X$ be the image of Φ. A priori $\mathcal{E}$ is a torsion free sheaf of rank r. However, if we pull-back the morphism to A by f^*, we get a composition

$$\bigwedge^{p-1} T_A \to \bigwedge^{p-1} f^*T_X \xrightarrow{f^*\Phi} f^*\Omega^1_X \to \Omega^1_A$$

The bundles on both sides are trivial, hence the composition is given by a constant matrix. This implies that $f^*\Phi$ itself is a bundle morphism of constant rank and that $f^*\mathcal{E}$ is a trivial bundle. As a consequence, Φ is of constant rank and $\mathcal{E}$ is a numerically flat vector bundle on X. By [DPS94, Theorem 1.18], $\mathcal{E}$ admits a filtration by subbundles whose quotients are unitary flat. If we had $\tilde{q}(X) = 0$, then $\pi_1(X)$ would be finite and a suitable étale cover $\tilde{X}$ would be simply connected, thus the pull-back $\tilde{\mathcal{E}} \subset \Omega^1_{\tilde{X}}$ would be trivial. But then we have $H^0(\tilde{X}, \Omega^1_{\tilde{X}}) \neq 0$, contradiction. □

4.5. Corollary. *Assume that $\tilde{q}(X) = 0$ (or equivalently, that $\pi_1(X)$ is finite). Then X is projective.*

Proof. In fact, Theorem 4.4 implies in that case $H^0(X, \Omega_X^2) = 0$. Since X is Kähler, we conclude that X is also projective by the Kodaira embedding theorem. □

4.6. Proposition. *In addition to the initial hypotheses, assume that X admits a smooth fibration $g : X \to Y$. Then :*

(1) *There is a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle g} \\ U & \xrightarrow{\ h\ } & Y \end{array}$$

where $p : A \to U = A/S$ is the quotient map associated with some subtorus S of A and $h : U \to Y$ is finite.

(2) *The relative anticanonical bundle $-K_{X/Y}$ is nef.*

(3) *If $X = \mathbb{P}(E)$ is the projectivization of some vector bundle E of rank r on Y, then E is numerically projectively flat, i.e. $E \otimes \det(E)^{-1/r}$ is numerically flat (or equivalently, $S^r E \otimes \det(E)^{-1}$ is numerically flat).*

Proof. (1) Since $g \circ f : A \to Y$ is a surjection, Corollary 2.2 shows that it factorizes as $g \circ f = h \circ p$ where $p : A \to A/S$ is a quotient of A and $h : A/S \to Y$ is a finite map.

(2) The diagram induces a generic isomorphism

$$T_{A/U} \to f^*(T_{X/Y}),$$

so that $f^*(T_{X/Y})$ is generically spanned outside the ramification divisor R of f. Hence $T_{X/Y}$ is nef on all curves $C \not\subset f(R)$. In particular

$$-K_{X/Y} \cdot C \geq 0$$

for all $C \not\subset f(R)$. Therefore $-K_{X/Y}$ is pseudo-effective by [BDPP04]. We infer from 4.1 (1) that $-K_{X/Y}$ is actually nef.

(3) Assuming that $X = \mathbb{P}(E) \to Y$, let

$$\zeta = \mathcal{O}_{\mathbb{P}(E)}(1).$$

Since

$$-K_{X/Y} = r\zeta + \phi^*(\det E)^{-1},$$

we conclude that

$$S^r(E) \otimes \det E^{-1} = S^r\big(E \otimes \det E^{-1/r}\big)$$

is nef. Since $c_1(S^r(E) \otimes \det E^{-1}) = 0$, this bundle is numerically flat in the sense of [DPS94], i.e. it admits a filtration whose quotients are hermitian flat (= unitary representations of the fundamental group). □

5 Case of a manifold with finite fundamental group

In this special case, the structure theorem can be stated in a slightly simpler form.

5.1. Proposition. *Assume that X is a compact complex manifold possessing a finite surjective map $f : A \to X$ from a torus. If X has a finite fundamental group (or, equivalently, if $\tilde q(X) = 0$), then*

(1) *A is an abelian variety and $X \simeq \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$ is a product of projective spaces.*

(2) *There is a finite étale quotient $\bar A = A/\Gamma$ such that $\bar A$ splits as a product $\bar A = A_1 \times \ldots \times A_k$, and the map f factorizes through $\bar A$ as a product map $\bar f = f_1 \times \ldots \times f_k$ where $f_j : A_j \to \mathbb{P}_{n_j}$.*

Proof. (1) In fact, we have showed in Corollary 4.5 that X must be projective algebraic, hence the pull-back of an ample line bundle on X is ample on A and A is an abelian variety. By the result of Hwang-Mok [HM01] already cited in the introduction, X is a tower

$$X = X_0 \to X_1 \to \ldots \to X_k = Y$$

of projective bundles, and the base Y has to be a point, otherwise the fundamental group would be infinite. By induction on dimension, we can assume that $\phi : X \to X_1$ is a $\mathbb{P}_{n_1}$-projective bundle over X_1, and since X_1 is also covered by the torus A which maps onto X, that $X_1 = \mathbb{P}_{n_2} \times \ldots \times \mathbb{P}_{n_k}$. Therefore $X = \mathbb{P}(E)$ for a certain vector bundle $E \to X_1$. By proposition 4.6 (3), we conclude that E is projectively numerically flat. However, by Lemma 5.2 below, this implies that $E = L^{\oplus r}$ for some line bundle L on X_1, $r = n_1 + 1$, hence $X = \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$. (Note that, as a consequence, X must in fact be simply connected: this is not surprising, since algebraic automorphisms of $\mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$ always have a fixed point by the Lefschetz fixed point formula, hence $\mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$ cannot possess an étale quotient).

(2) Let L_j be the pull-back of $\mathcal{O}(1)$ by the composition $A \to X \to \mathbb{P}_{n_j}$ with the j-th projection. As L_j is generated by sections, [BL03, 3.3.2] implies that there is a factorization $A \to W_j = A/S_j$ by a subtorus S_j and an ample line bundle G_j over W_j such that L_j is the pull-back of G_j to A. Take $\Gamma = \bigcap S_j$. Then all our bundles L_j descend to line bundles $\bar L_j$ over A/Γ and therefore, since $\bar L_1 + \ldots + \bar L_k$ comes from a very ample line bundle on X, f also factorizes as $\bar f : \bar A = A/\Gamma \to X$ and Γ is finite. We easily see that $\bar A$ is isomorphic to the product of its subtori $A_j = \bigcap_{k \neq j} S_k/\Gamma$, and if $f_j : A_j \to \mathbb{P}_{n_j}$ is the map induced by the composition $A_j \subset \bar A \to X \to \mathbb{P}_{n_j}$, we have $\bar f = f_1 \times \ldots \times f_k$. □

5.2. Lemma. *Let Y be a product of projective spaces, E a vector bundle of rank r on Y. Suppose that E is projectively numerically flat, i.e. that $S^r(E) \otimes \det E^{-1}$ is numerically flat. Then there exists a line bundle L on Y such that*

$$E \simeq L^{\oplus r}.$$

Proof. By a simple argument it suffices to show the lemma for $Y = \mathbb{P}_m$. Let $\ell \subset Y$ be a line. Then $E_{|\ell} = \mathcal{O}(\alpha_1) \oplus \ldots \oplus \mathcal{O}(\alpha_r)$ by Grothendieck's theorem, and it is immediately seen that $E_{|\ell}$ is projectively numerically flat iff all α_j's are equal to the same integer $\alpha \in \mathbb{Z}$, where α is such that $\det E = \mathcal{O}(r\alpha)$. Hence

$$(E \otimes \mathcal{O}(-\alpha))_{|\ell} = \mathcal{O}^{\oplus r}$$

for every line $\ell \subset \mathbb{P}_m$ and we conclude that $E \otimes \mathcal{O}(-\alpha) = \mathcal{O}^{\oplus r}$ (see e.g. [OSS80]). This proves our claim in case the base is $\mathbb{P}_m$. The case of a product of projective spaces follows. □

6 The anti-canonical morphism

Let us consider again a general compact (Kähler) manifold X possessing a finite ramified covering $f : A \to X$ by a torus.

6.1. Proposition. *The anti-canonical bundle $-K_X$ is semi-ample. Let $g : X \to Z$ be the associated morphism, which we call the anti-canonical morphism of X. Then g has connected equidimensional fibers and there is a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & X \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle g} \\ V & \xrightarrow{\ h\ } & Z \end{array}$$

where $V = A/S$ is a quotient of A by a subtorus S, and h is finite. Moreover, all fibers of g admit a finite covering by the torus S.

Proof. If $(-K_X)^n > 0$, we know by 4.1 (3) that $-K_X$ is ample, so we have $Z = X$, $g = \mathrm{id}_X$, and we can take $V = A$, $p = \mathrm{id}_A$.
When $(-K_X)^n = 0$, the ramification divisor $R \subset A$ is nef but not ample (let us recall that $\mathcal{O}_A(R) = f^*(-K_X)$). Thus, applying again [BL03, 3.3.2], we see that there exists a map $p : A \to V$ to a quotient torus V and an ample divisor $R_V \subset V$ such that

$$R = p^{-1}(R_V).$$

In particular, a sufficiently large multiple $\mathcal{O}_A(kR)$ is spanned, and the corresponding Kodaira-Iitaka map $\Phi_{|\mathcal{O}_A(kR)|}$ defines p.
Given a nef line bundle L, let $\nu(L)$ denote its numerical dimension, i.e. the maximal number d such that $L^d \not\equiv 0$. In order to prove that $-K_X$ is semi-ample, it suffices to show that

$$\kappa(-K_X) = \nu(-K_X),$$

(cf. [Kaw85, 6.1] and [Nak87, 5.5] ; more generally, the result holds true for nef line bundles L such that $\kappa(L - K_X) = \nu(L - K_X)$, $\nu(aL - K_X) = \nu(L - K_X)$ and $\kappa(aL - K_X) \geq 0$ for some $a > 1$, conditions which are indeed clearly satisfied for $L = -K_X$). The above equality $\kappa(-K_X) = \nu(-K_X)$ is easily verified since

$$\kappa(-K_X) = \kappa(f^*(-K_X)) = \dim V,$$

and

$$\nu(-K_X) = \nu(f^*(-K_X)) = \dim V.$$

Therefore we obtain an associated morphism $g : X \to Z$ such that $-mK_X = g^*(L)$ for a fixed suitable number m and a very ample line bundle L on Z. By a general property of Kodaira-Iitaka maps, the fibers of g are connected if we fix an appropriate multiple m. Consider such a fiber X_z of g; then $f^{-1}(X_z)$ consists of a union of fibers of p, because p is defined by $f^*(-K_X)$ and its sections are constant along the fibers of $p : A \to V$. This implies that $g \circ f$ factors through p, and therefore that there is a map $h : V \to Z$ which makes the diagram commute. We have $\dim Z = \dim V = \kappa(-K_X)$, hence h is a finite map by Corollary 2.2. The same result shows that $g \circ f$ has equidimensional fibers, hence g also has equidimensional fibers, which must then be images of fibers of p by f. □

6.2. Proposition. *In addition to the notation and hypotheses of Proposition* 6.1, *assume that* $q = q(X) = \tilde{q}(X)$ (*possibly after replacing* X *with an étale cover*). *Then*

(1) *The Albanese map* $\alpha : X \to A(X)$ *is a smooth surjective fibration with fibers* $F \simeq \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$. *Moreover,* $-K_X$ *is ample along the fibers* F *and has numerical dimension equal to* $\dim F = \sum n_j = n - q$.

(2) *If* $R \subset A$ *denotes the ramification divisor of* f, *then* $\mathcal{O}_A(R) = f^*(-K_X)$ *is ample along the fibers of the composition* $\alpha \circ f : A \to X \to A(X)$. *Furthermore, the map*

$$\Phi = (p, \alpha \circ f) : A \to V \times A(X)$$

induced by the anti-canonical morphism for the first factor p, *and by* $\alpha \circ f$ *for the second factor, is an isogeny from* A *onto* $V \times A(X)$. *Moreover* $p(f^{-1}f(R))$ *is a proper algebraic subset of* V.

Proof. (1) Theorem 4.2 implies that the fibers F of $\alpha : X \to A(X)$ have finite fundamental group; as they are also covered by tori, Proposition 5.1 shows that $F \simeq \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$. In particular, F is Fano and so $-K_F$ is ample. This implies that $-K_X$ is ample along the fibers F, and therefore the numerical dimension $\nu(-K_X)$ is at least equal to $\dim F = n - q$. If $\nu(-K_X) = \nu(f^*(-K_X))$ was strictly larger that $n - q$, we would get on the torus A

$$H^{n-q}(A, f^*(K_X)) = H^q(A, f^*(-K_X)) = 0$$

(see e.g. [BL04, 3.4.5]), hence

$$H^{n-q}(X, K_X) = H^q(X, \mathcal{O}_X) = 0$$

by taking the direct image. This is absurd.

(2) Since f is finite, we see by (1) that $f^*(-K_X)$ is ample along the fibers of $\alpha \circ f$. On the other hand, as seen in (6.1), $f^*(-K_X) = \mathcal{O}_A(R)$ is semi-ample on A and defines the morphism $p : A \to V$, hence $\mathcal{O}_A(R)$ is trivial along the fibers of p. Therefore the fibers

of p and those of $\alpha \circ f$ can only have a 0-dimensional intersection, and this implies that the map $(p, \alpha \circ f)$ is finite. Since

$$\dim V = \nu(-f^* K_X) = \nu(-K_X) = n - q = n - \dim A(X),$$

we conclude that $(p, \alpha \circ f)$ must be an isogeny. For the last statement, we notice that the cycle $f_*(R)$ is $\mathbb{Q}$-linearly equivalent to $-K_X$ since $f^*(-K_X) = \mathcal{O}_X(R)$. Thus $f^* f_*(R)$ is $\mathbb{Q}$-linearly equivalent to $f^*(-K_X)$. Hence $f^{-1}f(R)$ cannot meet the general fiber of $p : A \to V$. □

7 Proof of the main theorem

We are now in a position to give all details of the proof of Theorem 1.1.

Proof. (1) The Kähler property follows from Theorem 3.1.

(2) First fix an étale cover $\tilde{X} \to X$ such that $q(\tilde{X}) = \tilde{q}(X)$, and a lifting $\tilde{f} : \tilde{A} \to \tilde{X}$. Then by Proposition 6.2 (2) we get an isogeny $\Phi = (\tilde{p}, \tilde{\alpha} \circ \tilde{f}) : \tilde{A} \to \tilde{V} \times A(\tilde{X})$ such that $\tilde{\alpha} : \tilde{X} \to A(\tilde{X})$ is the Albanese map of $\tilde{X}$, and $\tilde{p} : \tilde{A} \to \tilde{V}$ induces the anti-canonical image $\tilde{Z}$ of $\tilde{X}$ via a commutative diagram

$$\begin{array}{ccccc} \tilde{A} & \xrightarrow{\tilde{f}} & \tilde{X} & \xrightarrow{\tilde{\alpha}} & A(\tilde{X}) \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle \tilde{g}} & & \\ \tilde{V} & \xrightarrow{\tilde{h}} & \tilde{Z}. & & \end{array}$$

On the other hand, we know from 6.2 (1) that $\tilde{\alpha}$ is a smooth (locally trivial) fibration and that the fibers F of $\tilde{\alpha}$ are products of projective spaces $\mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$. The inverse images $(\tilde{f})^{-1}(F)$ are unions of translates of the subtorus $S \subset \tilde{A}$ equal to the connected component of 0 in $\ker(\tilde{\alpha} \circ \tilde{f}) : \tilde{A} \to A(\tilde{X})$, a subtorus which is isogenous to $\tilde{V}$ via $\tilde{p}$. Therefore $\tilde{g}$ maps the fibers F onto $\tilde{Z}$, and the restriction $\tilde{g}_{|F}$ is finite (since $\tilde{h}$ is finite and surjective).
Let $\Sigma \subset \tilde{Z}$ be the union of $\tilde{h}(\tilde{p}(\tilde{f}^{-1}\tilde{f}(R)))$ with the set $\tilde{Z}_{non\ et} \subset \tilde{Z}$ above which $\tilde{h}$ is not étale (this includes of course the set of singular points of $\tilde{Z}$). By Proposition 6.2 (2) and the finiteness of $\tilde{h}$ this is a proper algebraic subset of $\tilde{Z}$. The above argument implies that $\tilde{g}$ is unramified on $F \setminus \tilde{g}^{-1}(\Sigma)$.
We therefore get a "horizontal direction" transverse to the fibers of F by looking at the fibers of $\tilde{g}$, and obtain in this way a monodromy of the fibration $\tilde{\alpha}$ in terms of a morphism of $\mathbb{Z}^{2q} \simeq \pi_1(A(\tilde{X}))$ into the permutation group of the finite set $F \cap \tilde{g}^{-1}(z)$, $z \in \tilde{Z} \setminus \Sigma$. The kernel Λ of this monodromy map is a subgroup of finite index in $\pi_1(A(\tilde{X}))$, and in this way we get a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\alpha'} & A(X') \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ \tilde{X} & \xrightarrow{\tilde{\alpha}} & A(\tilde{X}) \end{array}$$

where the vertical arrows u, v are finite étale covers and v is induced by the inclusion $\Lambda \subset \mathbb{Z}^{2q}$ of the fundamental groups. Our construction shows that α' is a trivial fibration [at least over $X' \setminus u^{-1}(\tilde{g}^{-1}(\Sigma))$, but the finiteness of $\tilde{g}$ implies that the horizontal transport

$$F_1 \cap \big(X' \setminus u^{-1}(\tilde{g}^{-1}(\Sigma))\big) \to F_2 \cap \big(X' \setminus u^{-1}(\tilde{g}^{-1}(\Sigma))\big)$$

between any two fibers F_1, F_2 must extend to isomorphisms of the fibers]. We conclude that

$$X' = \mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k} \times A(X')$$

and that the canonical image Z' of X' is precisely $\mathbb{P}_{n_1} \times \ldots \times \mathbb{P}_{n_k}$. Our arguments also imply that the canonical image $\tilde{Z}$ (as well as the canonical image Z of the original manifold X) is a finite ramified quotient of Z' ; in fact any global section of $-mK_X$ pulls back to a global section of $-mK_{\tilde{X}}$ or $-mK_{X'}$, and in this way we get naturally defined maps $Z' \to \tilde{Z} \to Z$, which are finite by Proposition 6.1 and by obvious commutative diagrams.

(3) also follows directly from what we have proved. □

7.1. Example. Let us consider $X' = \mathbb{P}_2 \times E_{2\tau}$ where $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is the elliptic curve of periods $(1, \tau)$. We take X to be the finite étale quotient of X' by the involution $(x, t) \mapsto (\sigma(x), t + \tau)$ where σ is the involution of $\mathbb{P}^2$ given (say) by $x \mapsto -x$ on the affine chart $\mathbb{C}^2 \subset \mathbb{P}_2$. In this way we get a fibration $X \to E_\tau$ which is a locally trivial $\mathbb{P}_2$-bundle over E_τ and which is nothing else than the Albanese map $\alpha : X \to A(X)$. In this case, the canonical image Z of X is precisely the (singular) quotient $\mathbb{P}_2/\langle\sigma\rangle$, because sections of $-mK_X$ pull-back to sections of

$$H^0(X', -mK_{X'}) \simeq H^0(\mathbb{P}_2, -mK_{\mathbb{P}_2})$$

which are invariant by the involution σ. This simple example shows that the following two phenomena can occur even under the assumption $q(\tilde{X}) = \tilde{q}(X)$ (here we simply take $\tilde{X} = X$) :

(1) The Albanese map $\tilde{X} \to A(\tilde{X})$ is a non trivial fibration ;

(2) The canonical image of $\tilde{X}$ is singular (and differs from the product of projective spaces obtained by taking the canonical image of a suitable étale cover $X' \to \tilde{X}$).

Of course, it is also easy to produce an example where we additionally have

(3) $q(X') = q(\tilde{X}) = \tilde{q}(X) > 0 = q(X)$.

One can take for instance X to be the quotient of $X' = \mathbb{P}_2 \times (E_{2\tau})^3$ by the finite group $\simeq \mathbb{Z}_2^3$ generated by the involutions

$$\begin{aligned}
g_0 &: (x, t_1, t_2, t_3) \mapsto (\sigma(x), t_1 + \tfrac{1}{2}, t_2 + \tfrac{1}{2}, t_3 + \tfrac{1}{2}),\\
g_1 &: (x, t_1, t_2, t_3) \mapsto (x, +t_1 + \tau, -t_2 + 0, -t_3 + 0),\\
g_2 &: (x, t_1, t_2, t_3) \mapsto (x, -t_1 + 0, +t_2 + \tau, -t_3 + \tau),\\
g_3 &: (x, t_1, t_2, t_3) \mapsto (x, -t_1 + \tau, -t_2 + \tau, +t_3 + \tau),
\end{aligned}$$

(here $g_3 = g_1 \circ g_2 = g_2 \circ g_1$), and let $\tilde{X}$ be the quotient of X' by $\langle g_0 \rangle \simeq \mathbb{Z}_2$. This produces a $\mathbb{P}_2$-fibration $X \to M$ over an étale quotient of $(E_{2\tau})^3$ with $q(M) = 0$, and all phenomena (1), (2), (3) occur simultaneously.

8 Appendix: images of Kähler spaces by flat morphisms

We give here a complete proof of Theorem 3.1. We take the opportunity the present a drastically simplified account of Varouchas' arguments [Var84], which, although not so hard to follow, make up a substantial part of his PhD thesis and do require more advanced results, such as Barlet's theory [Bar75] of cycle spaces for arbitrary analytic spaces (see also [Cam81], [CP94]).

Proof. Let $f : Y \to X$ be a proper and surjective holomorphic between complex spaces and let ω be a Kähler metric on Y. Assuming that the fibers are equidimensional and of pure dimension p, we consider the direct image current

$$T = f_*(\omega^{p+1}). \tag{8.1}$$

Then clearly T is a Kähler current of type $(1,1)$ over X in the sense of [DP04]. In fact, if α is a smooth positive definite form on X, we have $f^*\alpha \wedge \omega^p \le C_K \omega^{p+1}$ on the inverse image $f^{-1}(K)$ of any compact set $K \subset X$, therefore

$$T = f_*(\omega^{p+1}) \ge C_K^{-1} f_*(f^*\alpha \wedge \omega^p) \ge C_K^{-1} (f_*\omega^p)\, \alpha \quad \text{on } K,$$

where $f_*\omega^p$ is a (weakly) d-closed $(0,0)$ positive current, namely a collection of positive constants on each of the irreducible components of X. As explained already in section 3, the main point is to study the continuity of the local potential u of T, since it is then easy to regularize T to obtain a smooth Kähler metric.

Step 1. Assume first that X is non-singular. In this case we claim :

8.2. Lemma. *If X is non-singular, the $(1,1)$ current T defined by (8.1) admits continuous local potentials.*

Proof. By restricting the proper map $f : Y \to X$ over a small neighborhood of a point $x_0 \in X$, we can assume that X is a ball $B(x_0, r)$ in $\mathbb{C}^n$. Modulo smooth terms, the local potential u of T is given by an integral of the form

$$u(z) = \int_{\zeta \in X} \chi(\zeta) T(\zeta) \wedge \log|z - \zeta| \wedge dd^c(\log|z - \zeta|)^{n-1},$$

thanks to the fact that $(dd^c \log|z - \zeta|)^n$ is the current of integration on the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$. Here χ is a cut-off function with compact support in X, equal to 1 on a neighborhood of x_0, and the integral should be viewed as the direct image of a current on $\mathbb{C}^n \times \mathbb{C}^n$ by the first projection $(z, \zeta) \mapsto z$. By the definition of T as a direct image, we can express $u(z)$ as an integral over Y, namely a change of variable $\zeta = f(t)$ yields

$$u(z) = \int_{t \in Y} \chi(f(t)) \omega(t)^{p+1} \wedge \log|z - f(t)| \wedge dd^c(\log|z - f(t)|)^{n-1}$$

where $\dim Y = n + p$. The continuity of u follows from the following more general statement applied to $v_j(t,z) = \log|z - f(t)|$, thanks to the fact that all poles of v_j occur in codimension n on Y by the assumption that f has equidimensional fibers. □

8.3. Lemma. *Let Y, Z be complex spaces. Consider plurisubharmonic functions on $Y \times Z$ of the form*

$$v_j(t,z) = \log \sum_k |f_{j,k}(t,z)|^2$$

where the $f_{j,k}$ are holomorphic functions such that the poles of the functions $t \mapsto v_j(t,z)$ are at least of codimension n everywhere on Y, for every $z \in Z$. Then the wedge products

$$v_1(\bullet,z)dd^c v_2(\bullet,z) \wedge \ldots \wedge dd^c v_\ell(\bullet,z), \quad dd^c v_1(\bullet,z) \wedge \ldots \wedge dd^c v_\ell(\bullet,z)$$

are well-defined currents of locally finite mass whenever $\ell \leq n$, and they depend continuously on z in the weak topology.

Proof. Such statements have been known for a long time, see e.g. [Dem93, § 3]. The main point is to get uniform bounds on the mass and uniform integrability with respect to the parameter z. Since the result is local on Y, we can assume that Y is a germ of complex space and use a direct image argument to reduce ourselves to the case of a smooth variety Y : just project by a suitable generic projection

$$\mathbb{C}^N = \mathbb{C}^s \times \mathbb{C}^{N-s} \to \mathbb{C}^s, \quad s = \dim Y,$$

from an open set $\Omega \subset \mathbb{C}^N$ which is a smooth ambient space for Y. One can use a slicing argument to reach the situation where the poles are just isolated points in Y, for every $z \in Z$ (this actually amounts to use certain of the coordinates in Y as new parameters). A suitable application of Stokes theorem and of the comparison principle is enough to obtain a uniform bound

$$\|dd^c v_2(\bullet,z) \wedge \ldots \wedge dd^c v_\ell(\bullet,z)\|_{B(t,r)} \leq Cr^2$$

for the mass on small balls of radius $0 < r \leq r_0$ (alternatively, this is a standard estimate on the Lelong projective masses $\nu(\bullet,r)$ of our currents – since they are not of maximum degree, they are of dimension at least 1). The uniform integrability of

$$v_1(\bullet,z)dd^c v_2(\bullet,z) \wedge \ldots \wedge dd^c v_\ell(\bullet,z)$$

is finally obtained from an obvious Lojaziewicz type estimate

$$|v_j(t,z)| \leq C|\log d(t, P_j(z))|,$$

where $P_j(z) \subset Y$ denotes the set of poles of $t \mapsto v_j(t,z)$; all constants C described here can be taken to be locally uniform in z. □

Step 2. X is a general normal complex space of pure dimension with mild singularities. The main point is to prove the existence and the continuity of the potential of T. Since this is a local question on X, we may assume that X admits an irreducible finite ramified covering $\hat{X} \to X$ such that $\hat{X}$ is smooth. By taking the fiber product with $f : Y \to X$, we get a commutative diagram

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\ \hat{f}\ } & \hat{X} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow{\ f\ } & X, \end{array}$$

where the vertical arrows u, v are finite and the horizontal arrows f, $\hat{f}$ are equidimensional. The fact that X is normal implies moreover that $\hat{Y}$ is of pure dimension. Step 1 shows that the current

$$\hat{T} = (\hat{f})_*(u^*\omega^{p+1}) = v^*(f_*\omega^{p+1}) = v^*T$$

has a continuous potential. Therefore, if δ is the ramification degree of v, the finite direct image $T = \frac{1}{\delta}v_*v^*T = \frac{1}{\delta}v_*\hat{T}$ also has a continuous potential by the arguments explained in § 3. We finally conclude that T can be regularized as a Kähler metric by Richberg's theorem [Ric68] (cf. [Dem82, 92]). □

8.4. Remark. Without a suitable assumption on the singularities, it is unclear whether the current $T = f_*(\omega^{p+1})$ admits a local potential, and even if this potential exists, it need not be continuous. We can take for instance $Y = \mathbb{P}^n$, $n \geq 3$, and $f : Y \to X$ equal to the quotient of $\mathbb{P}^n$ obtained by identifying two disjoint isomorphic smooth curves C_1, C_2 of different degrees, e.g. a line and a conic, through a given isomorphism $C_1 \to C_2$. Then clearly X cannot be Kähler since the pull-back of any smooth closed $(1,1)$-form γ on X must have trivial cohomology class on Y (the restrictions of $f^*\gamma$ to C_1 and C_2 are equal but at the same time the degrees $f^*\gamma \cdot C_1$ and $f^*\gamma \cdot C_2$ differ if $f^*\gamma \not\equiv 0$) ; also, in this case, the push forward $f_*\omega$ of the Fubini-Study Kähler form ω by the quotient map $f : Y \to X$ has a potential which is merely defined outside $f(C_1) = f(C_2)$ and does not extend continuously to X. As a consequence, the assumption that X is normal seems hard to avoid.
In general, one of the main points is to be able to show that $T = f_*(\omega^{p+1})$ admits a local potential. If for some reason it does and the induced $\partial\overline{\partial}$-cohomology class $\{T\}$ has a well-defined restriction to the singularity strata of X, we can hope that an induction on the dimension of these strata and the results of Păun [Pau98] will imply that X is Kähler. It is natural to ask whether the normality of X is sufficient for that purpose.

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