

# Estimates on Monge-Ampère operators derived from a local algebra inequality

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Prépublication de l'Institut Fourier n° 702 (2007)

<http://www-fourier.ujf-grenoble.fr/prepublications.html>

*Dedicated to Professor Christer Kiselman, on the occasion of his retirement*

**Abstract.** The goal of this short note is to relate the integrability property of the exponential  $e^{-2\varphi}$  of a plurisubharmonic function  $\varphi$  with isolated or compactly supported singularities, to a priori bounds for the Monge-Ampère mass of  $(dd^c\varphi)^n$ . The inequality is valid locally or globally on an arbitrary open subset  $\Omega$  in  $\mathbb{C}^n$ . We show that  $\int_{\Omega}(dd\varphi)^n < n^n$  implies  $\int_K e^{-2\varphi} < +\infty$  for every compact subset  $K$  in  $\Omega$ , while functions of the form  $\varphi(z) = n \log |z - z_0|$ ,  $z_0 \in \Omega$ , appear as limit cases. The result is derived from an inequality of pure local algebra, which turns out a posteriori to be equivalent to it, proved by A. Corti in dimension  $n = 2$ , and later extended by L. Ein, T. De Fernex and M. Mustața to arbitrary dimensions.

**Résumé.** Le but de cette note est d'établir une relation entre la propriété d'intégrabilité de l'exponentielle  $e^{-2\varphi}$  d'une fonction plurisubharmonique  $\varphi$  dont les singularités sont isolées ou à support compact, et la donnée de bornes a priori pour la masse de Monge-Ampère de  $(dd^c\varphi)^n$ . L'inégalité obtenue a lieu aussi bien localement que globalement, ceci sur un ouvert arbitraire  $\Omega$  de  $\mathbb{C}^n$ . Nous montrons que l'hypothèse  $\int_{\Omega}(dd\varphi)^n < n^n$  entraîne  $\int_K e^{-2\varphi} < +\infty$  pour tout sous-ensemble compact  $K$  de  $\Omega$ , les fonctions de la forme  $\varphi(z) = n \log |z - z_0|$ ,  $z_0 \in \Omega$ , apparaissant comme des cas limites. Le résultat se déduit d'une pure inégalité d'algèbre locale, qui se trouve a posteriori lui être équivalente, successivement démontrée par A. Corti en dimension  $n = 2$ , puis étendue par L. Ein, T. De Fernex et M. Mustața en dimensions arbitraires.

**Keywords.** Monge-Ampère operator, local algebra, monomial ideal, Hilbert-Samuel multiplicity, log-canonical threshold, plurisubharmonic function, Ohsawa-Takegoshi  $L^2$  extension theorem, approximation of singularities, birational rigidity.

**Mots-clés.** Opérateur de Monge-Ampère, algèbre locale, idéal monomial, multiplicité de Hilbert-Samuel, seuil log-canonique, fonction plurisousharmonique, théorème d'extension  $L^2$  d'Ohsawa-Takegoshi, approximation des singularités, rigidité birationnelle.

**AMS Classification.** 32F07, 14B05, 14C17

## 1. Main result

Here we put  $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$  so that  $dd^c = \frac{i}{\pi}\partial\bar{\partial}$ . The normalization of the  $d^c$  operator is chosen such that we have precisely  $(dd^c \log |z|)^n = \delta_0$  for the Monge-Ampère operator in  $\mathbb{C}^n$ . The Monge-Ampère operator is defined on locally bounded plurisubharmonic functions according to the definition of Bedford-Taylor [BT76, BT82]; it can also be extended to psh functions with isolated or compactly supported poles by [Dem93]. Our main result is the following a priori estimate for the Monge-Ampère operator acting on functions with compactly supported poles.

**(1.1) Main Theorem.** *Let  $\Omega$  be an open subset in  $\mathbb{C}^n$ ,  $K$  a compact subset of  $\Omega$ , and let  $\varphi$  be a plurisubharmonic function on  $\Omega$  such that  $-A \leq \varphi \leq 0$  on  $\Omega \setminus K$  and*

$$\int_{\Omega} (dd^c \varphi)^n \leq M < n^n.$$

*Then there is an a priori upper bound for the Lebesgue integral of  $e^{-2\varphi}$ , namely*

$$\int_K e^{-2\varphi} d\lambda \leq C(\Omega, K, A, M),$$

*where the constant  $C(\Omega, K, A, M)$  depends on the given parameters but not on the function  $\varphi$ .*

We first make a number of elementary remarks.

**(1.2)** The result is optimal as far as the Monge-Ampère bound  $M < n^n$  is concerned, since functions  $\varphi_\varepsilon(z) = (n - \varepsilon) \log |z - z_0|$ ,  $z_0 \in K^\circ \subset \Omega$  satisfy  $\int_{\Omega} (dd^c \varphi_\varepsilon)^n = (n - \varepsilon)^n$ , but  $\int_K e^{-2\varphi_\varepsilon} d\lambda$  tends to  $+\infty$  as  $\varepsilon$  tends to zero.

**(1.3)** The assumption  $-A \leq \varphi \leq 0$  on  $\Omega \setminus K$  is required, as it forces the poles of  $\varphi$  to be compactly supported – a condition needed to define properly the Monge-Ampère measure  $(dd^c \varphi)^n$  (see e.g. [Dem93]). In any case, the functions  $\varphi_\varepsilon(z) = \frac{1}{2} \ln(|z_1|^2 + \varepsilon^2)$  satisfy  $\int_{\Omega} (dd^c \varphi_\varepsilon)^n = 0 < n^n$ , but  $\int_K e^{-2\varphi_\varepsilon} d\lambda$  is unbounded as  $\varepsilon$  tends to 0, whenever  $K$  contains at least one interior point located on the hyperplane  $z_1 = 0$ . The limit  $\varphi(z) = \ln |z_1|$  of course does not have compactly supported poles. In such a circumstance, C.O. Kiselman [Ki84] observed long ago that the Monge-Ampère mass of  $(dd^c \varphi)^n$  need not be finite or well defined.

**(1.4)** The a priori estimate (1.1) can be seen as a non linear analogue of Skoda's criterion for the local integrability of  $e^{-2\varphi}$ . Let us recall Skoda's criterion: *if the Lelong number  $\nu(\varphi, z_0)$  satisfies  $\nu(\varphi, z_0) < 1$ , then  $e^{-2\varphi}$  is locally integrable near  $z_0$ , and if  $\nu(\varphi, z_0) \geq n$ , then  $\int_V e^{-2\varphi} d\lambda = +\infty$  on every neighborhood  $V$  of  $z_0$ .* The gap between 1 and  $n$  is an important feature of potential theory in several complex variables, and it therefore looks like an interesting bonus that there is no similar discrepancy for the estimate given by Theorem 1.1. One of the reasons is that  $(dd^c \varphi)^n$  takes into account all dimensions simultaneously, while the Lelong number only describes the minimal vanishing order with respect to arbitrary lines (or holomorphic curves).  $\square$

The proof consists of several steps, the main of which is a reduction to the following result of local algebra, due to A. Corti [Cor00] in dimension 2 and L. Ein, T. De Fernex and M. Mustața [dFEM04] in general.

**(1.5) Theorem.** *Let  $\mathcal{J}$  be an ideal in the rings of germs  $\mathcal{O}_{\mathbb{C}^n,0}$  of holomorphic functions in  $n$  variables, such that the zero variety  $V(\mathcal{J})$  consists of the single point  $\{0\}$ . Let  $e(\mathcal{J})$  be the Samuel multiplicity of  $\mathcal{J}$ , i.e.*

$$e(\mathcal{J}) = \lim_{k \rightarrow +\infty} \frac{n!}{k^n} \dim \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}^k,$$

*the leading coefficient in the Hilbert polynomial of  $\mathcal{J}$ . Then the log canonical threshold of  $\mathcal{J}$  satisfies*

$$\mathrm{lc}(\mathcal{J}) \geq \frac{n}{e(\mathcal{J})^{1/n}},$$

*and the equality case occurs if and only if the integral closure  $\bar{\mathcal{J}}$  of  $\mathcal{J}$  is a power  $\mathfrak{m}^s$  of the maximal ideal.*

Recall that the *log canonical threshold*  $\mathrm{lc}(\mathcal{J})$  of an ideal  $\mathcal{J}$  is the supremum of all numbers  $c > 0$  such that  $(|g_1|^2 + \dots + |g_N|^2)^{-c}$  is integrable near 0 for any set of generators  $g_1, \dots, g_N$  of  $\mathcal{J}$ . If this supremum is less than 1 [this is always the case after replacing  $\mathcal{J}$  by a sufficiently high power  $\mathcal{J}^m$ , which yields  $e(\mathcal{J}^m) = m^n e(\mathcal{J})$  and  $\mathrm{lc}(\mathcal{J}^m) = \frac{1}{m} \mathrm{lc}(\mathcal{J})$ ], the integrability condition exactly means that the divisor  $cD$  associated with a generic element  $D = \mathrm{Div}_f$ ,  $f \in \mathcal{J}$ , is *Kawamata log terminal* (klt), i.e. that after blowing up and resolving the singularities to get a divisor with normal crossings, the associated divisor  $\mu^*(cD) - E$  has coefficients  $< 1$ , where  $\mu$  is the blow-up map and  $E$  its jacobian divisor (see e.g. [DK00] for details).

In fact, Theorem 1.5 follows from Theorem 1.1 by taking  $\Omega$  equal to a small ball  $B(0,r) \subset \mathbb{C}^n$  and  $\varphi(z) = \frac{c}{2} \log \sum_j |g_j|^2$  where  $g_1, \dots, g_N$  are local generators of  $\mathcal{J}$ . For this, one observes that the Monge-Ampère mass of  $(dd^c \varphi)^n$  carried by  $\{0\}$  is equal to  $c^n e(\mathcal{J})$  (Lemma 2.1 below), hence the integrability of  $(|g_1|^2 + \dots + |g_N|^2)^{-c} = e^{-2\varphi}$  holds true as soon as  $c^n e(\mathcal{J}) < n$ ; notice that the integral  $\int_{B(0,r)} (dd^c \varphi)^n$  converges to the mass carried by 0 as the radius  $r$  tends to zero.

However, the strategy of the proof goes the other way round: Theorem 1.1 will actually be derived from Theorem 1.5 by means of the approximation techniques for plurisubharmonic functions developed in [Dem92] and the result on semi-continuity of singularity exponents (= log canonical thresholds) obtained in [DK00]. It is somewhat strange that one has to make a big detour through local algebra (and approximation of analytic objects by polynomials, as in [DK00]), to prove what finally appears to be a pure analytic estimate on Monge-Ampère operators.

It would be interesting to know whether a direct proof can be obtained by methods which are more familiar to analysts (integration by parts, convexity inequalities, integral kernels for  $\bar{\partial}$  ...). One of the consequences of our use

of “purely qualitative” algebraic detour is that the constants  $C(\Omega, K, A, M)$  appearing in Theorem 1.1 are non effective. On the other hand, we would like to know what kind of dependance these constants have e.g. on  $n^n - M > 0$ , and also what are the extremal functions (for instance in the case when  $\Omega$  and  $K$  are concentric balls). The question is perhaps more difficult than it would first appear, since the most obvious guess is that the extremal functions are singular ones with a logarithmic pole  $\varphi(z) \sim \lambda \log |z - z_0|$ ; the reason for this expectation is that the equality case in Theorem 1.5 is achieved precisely when the integral closure of the ideal  $\mathcal{J}$  is equal to a power of the maximal ideal.

I would like to thank I. Chel'tsov, R. Lazarsfeld, L. Ein, T. de Fernex, M. Mustață for explaining to me the algebraic issues involved in the inequalities just discussed (see e.g. [Che05]). It is worth mentioning that inequality 1.5 is related to deep questions of algebraic geometry such as the birational (super)rigidity of Fano manifolds; for instance, following ideas of Corti and Pukhlikov ([Cor95], [Cor00]), it is proved in [dFEM03] that every smooth hypersurface of degree  $N$  in  $\mathbb{P}^N$  is birationally superrigid at least for  $4 \leq N \leq 12$ , hence that such a hypersurface cannot be rational – this is a far reaching generalization of the classical result by Iskovskikh-Manin ([IM72], [Isk01]) that 3-dimensional quartics are not rational.

I am glad to dedicate this paper to Professor C.O. Kiselman whose work has been a great source of inspiration for my own research in complex analysis, especially on all subjects related to Monge-Ampère operators, Lelong numbers and attenuation of singularities of plurisubharmonic functions ([Kis78, 79, 84, 94a, 94b]). Various incarnations of these concepts and results appear throughout the present paper.

## 2. Proof of the integral inequality

The first step is to relate Monge-Ampère masses to Samuel multiplicities. The relevant result is probably known, but we have not been able to find a precise reference in the literature.

**(2.1) Lemma.** *In a neighborhood of  $0 \in \mathbb{C}^n$ , let  $\varphi(z) = \frac{1}{2} \log \sum_{j=1}^N |g_j|^2$  where  $g_1, \dots, g_N$  are germs of holomorphic functions which have 0 as their only common zero. Then the Monge-Ampère mass of  $(dd^c \varphi)^n$  carried by  $\{0\}$  is equal to the Samuel multiplicity  $e(\mathcal{J})$  of the ideal  $\mathcal{J} = (g_1, \dots, g_N) \subset \mathcal{O}_{\mathbb{C}^n, 0}$ .*

*Proof.* For any point  $a \in G_{N,n}$ , the Grassmannian of  $n$ -dimensional subspaces in  $\mathbb{C}^N$ , we define

$$\varphi_a(z) = \frac{1}{2} \log \sum_{i=1}^n \left| \sum_{j=1}^N \lambda_{ij} g_j(z) \right|^2$$

where  $(\lambda_{jk})$  is the  $n \times N$ -matrix of an orthonormal basis of the subspace  $a$ . It is easily shown that  $\varphi_a$  is defined in a unique way and that we have the Crofton type

formula

$$(2.2) \quad (dd^c \varphi(z))^n = \int_{a \in G_{N,n}} (dd^c \varphi_a(z))^n d\mu(a)$$

where  $\mu$  is the unique  $U(N)$ -invariant probability measure on the Grassmannian. In fact this can be proved from the related equality

$$(dd^c \log |w|^2)^n = \int_{a \in G_{N,n}} (dd^c \log |\pi_a(w)|^2)^n d\mu(a)$$

where  $\mathbb{C}^N \ni w \mapsto \pi_a(w)$  is the orthogonal projection onto  $a \subset \mathbb{C}^n$ , which itself follows by unitary invariance and a degree argument (both sides have degree one as bidegree  $(n, n)$  currents on the projective space  $\mathbb{P}^{N-1}$ ). One then applies the substitution  $w = g(z)$  to get the general case. The right hand side of (2.2) is well defined since the poles of  $\varphi_a$  form a finite set for a generic point  $a$  in the grassmannian; then  $(dd^c \varphi_a(z))^n$  is just a sum of Dirac masses with integral coefficients (the local degree of the corresponding germ of map  $g_a : z \mapsto (\sum_{1 \leq j \leq N} \lambda_{ij} g_j(z))_{1 \leq i \leq N}$  from  $\mathbb{C}^n$  to  $\mathbb{C}^n$  near the given point). By a continuity argument, the coefficient of  $\delta_0$  is constant except on some analytic stratum in the Grassmannian, and by Fubini, the mass carried by  $(dd^c \varphi)^n$  at 0 is thus equal to the degree of  $g_a$  at 0 for generic  $a$ . Now, it is a well-known fact of commutative algebra that the Hilbert-Samuel multiplicity  $e(\mathcal{J})$  is equal to the intersection number of the divisors associated with a generic  $n$ -tuple of elements of  $\mathcal{J}$  (Bourbaki, Algèbre Commutative [BAC83], VIII 7.5, Prop. 7). That intersection number is also equal to the generic value of the Monge-Ampère mass

$$(dd^c \log |\lambda_1 \cdot \varphi|) \wedge \dots \wedge (dd^c \log |\lambda_n \cdot \varphi|)(0).$$

By averaging with respect to the  $\lambda_j$ 's, this appears to be the same as the generic value of  $(dd^c \varphi_a)^n(0)$ .  $\square$

We now briefly recall the ideas involved in the proof of Theorem 1.5, as taken from [dFEM04]. In order to prove the main inequality of 1.5 (which can be rewritten as  $e(\mathcal{J}) \geq n^n / \text{lc}(\mathcal{J})^n$ ), it is sufficient so show that

$$(2.3) \quad \dim \mathcal{O}_{\mathbb{C}^n, 0} / \mathcal{J} \geq n^n / (n! \text{lc}(\mathcal{J})^n).$$

In fact, since by definition  $\text{lc}(\mathcal{J}^k) = \frac{1}{k} \text{lc}(\mathcal{J})$ , a substitution of  $\mathcal{J}$  by  $\mathcal{J}^k$  in (2.3) yields

$$\frac{n!}{k^n} \dim \mathcal{O}_{\mathbb{C}^n, 0} / \mathcal{J}^k \geq n^n / \text{lc}(\mathcal{J})^n$$

and we get the expected inequality 1.5 by letting  $k$  tend to  $+\infty$ . Now, by fixing a multiplicative order on the coordinates  $z_j$ , it is well known that one can construct a flat family  $(\mathcal{J}_s)_{s \in \mathbb{C}, 0}$  depending on a small complex parameter  $s$  such that  $\mathcal{J}_0$

is a monomial ideal and  $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}_s \simeq \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$  for all  $s \neq 0$  (see Eisenbud [Ei95] for a nice discussion in the algebraic case). The semicontinuity property of the log canonical threshold (see for example [DK00]) implies that  $\text{lc}(\mathcal{J}_0) \leq \text{lc}(\mathcal{J}_s)$  for small  $s$ .

The proof is then reduced to the case when  $\mathcal{J}$  is a monomial ideal, i.e. an ideal generated by a family of monomials  $(z^{\beta_j})$ . In the latter situation, the argument proceeds from an explicit formula for  $\text{lc}(\mathcal{J})$  due to J. Howald [Ho01]: let  $P(\mathcal{J})$  be the Newton polytope of  $\mathcal{J}$ , i.e. the convex hull of the points  $\beta \in \mathbb{N}^n$  associated with all monomials  $z^\beta \in \mathcal{J}$ ; then putting  $\mathbf{e} = (1, \dots, 1) \in \mathbb{N}^n$ ,

$$\frac{1}{\text{lc}(\mathcal{J})} = \min \{ \alpha > 0; \alpha \cdot \mathbf{e} \in P(\mathcal{J}) \}$$

(the reader can take this as a clever exercise on the convergence of integrals defined by sums of monomials in the denominator). Let  $F$  be the facet of  $P(\mathcal{J})$  which contains the point  $\frac{1}{\text{lc}(\mathcal{J})}\mathbf{e}$ , and let  $\sum x_j/a_j = 1$ ,  $a_j > 0$  be the equation of this hypersurface in  $\mathbb{R}^n$ . Let us denote also by  $F_+$  and  $F_-$  the open half-spaces delimited by  $F$ , such that  $\mathbb{R}_+^n \cap F_-$  is relatively compact and  $\mathbb{R}_+^n \cap F_+$  is unbounded. Then  $\text{Vol}(\mathbb{R}_+^n \cap F_-) = \frac{1}{n!} \prod a_j$  and therefore, since  $\mathbb{R}_+^n \setminus P(\mathcal{J})$  contains  $\mathbb{R}_+^n \cap F_-$ , we get

$$\text{Vol}(\mathbb{R}_+^n \setminus P(\mathcal{J})) \geq \text{Vol}(\mathbb{R}_+^n \cap F_-) = \frac{1}{n!} \prod a_j.$$

On the other hand,  $\dim \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$  is at least equal to the number of elements of  $\mathbb{N}^n \setminus P(\mathcal{J})$ , which is itself at least equal to  $\text{Vol}(\mathbb{R}_+^n \setminus P(\mathcal{J}))$  since the unit cubes  $\beta + [0, 1]^n$  with  $\beta \in \mathbb{N}^n \setminus P(\mathcal{J})$  cover the complement  $\mathbb{R}_+^n \setminus P(\mathcal{J})$ . This yields

$$\dim \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J} \geq \frac{1}{n!} \prod a_j.$$

As  $\frac{1}{\text{lc}(\mathcal{J})}\mathbf{e}$  belongs to  $F$ , we have  $\sum 1/a_j = \text{lc}(\mathcal{J})$ . The inequality between geometric and arithmetic means implies

$$\left( \prod \frac{1}{a_j} \right)^{1/n} \leq \frac{1}{n} \sum \frac{1}{a_j} = \frac{\text{lc}(\mathcal{J})}{n}$$

and inequality (2.3) follows. We refer to [dFEM04] for the discussion of the equality case.  $\square$

The next ingredient is the following basic approximation theorem for plurisubharmonic functions through the Bergman kernel trick and the Ohsawa-Takegoshi theorem [OT87], the first version of which appeared in [Dem92]. We start with the general concept of complex singularity exponent introduced in [DK00], which extends the concept of log canonical threshold.

**(2.4) Definition.** Let  $X$  be a complex manifold and  $\varphi$  be a plurisubharmonic (psh) function on  $X$ . For any compact set  $K \subset X$ , we introduce the “complex singularity exponent” of  $\varphi$  on  $K$  to be the nonnegative number

$$c_K(\varphi) = \sup \{c \geq 0 : \exp(-2c\varphi) \text{ is } L^1 \text{ on a neighborhood of } K\},$$

and we define the “Arnold multiplicity” to be  $\lambda_K(\varphi) = c_K(\varphi)^{-1}$  :

$$\lambda_K(\varphi) = \inf \{\lambda > 0 : \exp(-2\lambda^{-1}\varphi) \text{ is } L^1 \text{ on a neighborhood of } K\}.$$

In the case where  $\varphi(z) = \frac{1}{2} \log \sum |g_j|^2$ , the exponent  $c_{z_0}(\varphi)$  is the same as the log canonical threshold of the ideal  $\mathcal{J} = (g_j)$  at the point  $z_0$ .

**(2.5) Theorem** ([Dem92, DK00]). Let  $\varphi$  be a plurisubharmonic function on a bounded pseudoconvex open set  $\Omega \subset \mathbb{C}^n$ . For every real number  $m > 0$ , let  $\mathcal{H}_{m\varphi}(\Omega)$  be the Hilbert space of holomorphic functions  $f$  on  $\Omega$  such that  $\int_{\Omega} |f|^2 e^{-2m\varphi} dV < +\infty$  and let  $\psi_m = \frac{1}{2m} \log \sum_k |g_{m,k}|^2$  where  $(g_{m,k})_k$  is an orthonormal basis of  $\mathcal{H}_{m\varphi}(\Omega)$ . Then:

(a) There are constants  $C_1, C_2 > 0$  independent of  $m$  and  $\varphi$  such that

$$\varphi(z) - \frac{C_1}{m} \leq \psi_m(z) \leq \sup_{|\zeta-z|<r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$$

for every  $z \in \Omega$  and  $r < d(z, \partial\Omega)$ . In particular,  $\psi_m$  converges to  $\varphi$  pointwise and in  $L^1_{\text{loc}}$  topology on  $\Omega$  when  $m \rightarrow +\infty$  and

(b) The Lelong numbers of  $\varphi$  and  $\psi_m$  are related by

$$\nu(\varphi, z) - \frac{n}{m} \leq \nu(\psi_m, z) \leq \nu(\varphi, z) \quad \text{for every } z \in \Omega.$$

(c) For every compact set  $K \subset \Omega$ , the Arnold multiplicity of  $\varphi$ ,  $\psi_m$  and of the multiplier ideal sheaves  $\mathcal{J}(m\varphi)$  are related by

$$\lambda_K(\varphi) - \frac{1}{m} \leq \lambda_K(\psi_m) = \frac{1}{m} \lambda_K(\mathcal{J}(m\varphi)) \leq \lambda_K(\varphi).$$

The final ingredient is the following fundamental semicontinuity result from [DK00].

**(2.6) Theorem** ([DK00]). Let  $X$  be a complex manifold. Let  $\mathcal{Z}_+^{1,1}(X)$  denote the space of closed positive currents of type  $(1,1)$  on  $X$ , equipped with the weak topology, and let  $\mathcal{P}(X)$  be the set of locally  $L^1$  psh functions on  $X$ , equipped with the topology of  $L^1$  convergence on compact subsets (= topology induced by the weak topology). Then

- (a) The map  $\varphi \mapsto c_K(\varphi)$  is lower semi-continuous on  $\mathcal{P}(X)$ , and the map  $T \mapsto c_K(T)$  is lower semi-continuous on  $\mathcal{Z}_+^{1,1}(X)$ .
- (b) (“Effective version”). Let  $\varphi \in \mathcal{P}(X)$  be given. If  $c < c_K(\varphi)$  and  $\psi$  converges to  $\varphi$  in  $\mathcal{P}(X)$ , then  $e^{-2c\psi}$  converges to  $e^{-2c\varphi}$  in  $L^1$  norm over some neighborhood  $U$  of  $K$ .

**(2.7) Proof of Theorem 1.1.** Assume that the conclusion of theorem 1.1 is wrong. Then there exist a compact set  $K \subset \Omega$ , constants  $M < n^n$ ,  $A > 0$  and a sequence  $\varphi_j$  of plurisubharmonic functions such that  $-A \leq \varphi_j \leq 0$  on  $\Omega \setminus K$  and  $\int_{\Omega} (dd^c \varphi_j)^n \leq M$ , while  $\int_K e^{-2\varphi_j} d\lambda$  tends to  $+\infty$  as  $j$  tends to  $+\infty$ . By well-known properties of potential theory, the condition  $-A \leq \varphi_j \leq 0$  on  $\Omega \setminus K$  ensures that the sequence  $(\varphi_j)$  is relatively compact in the  $L_{\text{loc}}^1$  topology on  $\Omega$ : in fact, the Laplacian  $\Delta \varphi_j$  is a uniformly bounded measure on every compact of  $\Omega \setminus K$ , and this property extends to all compact subsets of  $\Omega$  by Stokes’ theorem and the fact that there is a strictly subharmonic function on  $\Omega$ ; we then conclude by an elementary (local) Green kernel argument. Therefore there exists a subsequence of  $(\varphi_j)$  which converges almost everywhere and in  $L_{\text{loc}}^1$  topology to a limit  $\varphi$  such that  $-A \leq \varphi \leq 0$  on  $\Omega \setminus K$  and  $\int_{\Omega} (dd^c \varphi)^n \leq M$ . On the other hand, we must have  $c_K(\varphi) \leq 1$  by (2.6 b) (hence  $\int_K e^{-2(1+\varepsilon)\varphi} d\lambda = +\infty$  for every  $\varepsilon > 0$ ).

As  $c_K(\varphi) = \inf_{z \in K} c_{\{z\}}(\varphi)$  and  $z \mapsto c_{\{z\}}(\varphi)$  is lower semicontinuous, there exists a point  $z_0 \in K$  such that  $c_{\{z_0\}}(\varphi) \leq 1$ . By theorem 2.5 applied on a small ball  $B(z_0, r)$ , we can approximate  $\varphi$  by a sequence of psh functions of the form  $\psi_m = \frac{1}{2m} \log \sum |g_{m,k}|^2$  on  $B(z_0, r)$ . Inequality (2.5 c) shows that we have

$$c_{\{z_0\}}(\psi_m) \leq \frac{1}{1/c_{\{z_0\}}(\varphi) - 1/m} \leq \frac{1}{1 - 1/m} < 1 + \varepsilon$$

for  $m$  large, hence  $c_{\{z_0\}}((1 + \varepsilon)\psi_m) \leq 1$ . However, the analytic strata of positive Lelong numbers of  $\varphi$  must be contained in  $K$ , hence they are isolated points in  $\Omega$ , and thus the poles of  $\psi_m$  are isolated. By the weak continuity of the Monge-Ampère operator, we have

$$\int_{\overline{B}(z_0, r')} (dd^c(1 + \varepsilon)\psi_m)^n \leq (1 + \varepsilon)^{n+1} \int_{\overline{B}(z_0, r')} (dd^c \varphi)^n \leq (1 + \varepsilon)^{n+1} M^n$$

for  $m$  large, for any  $r' < r$ . If  $\varepsilon$  is chosen so small that  $(1 + \varepsilon)^{n+1} M^n < n^n$ , then the Monge-Ampère mass of  $(1 + \varepsilon)\psi_m$  at  $z_0$  is strictly less than  $n^n$ , but the log canonical threshold is at most equal to 1. This contradicts inequality 1.5, when using Lemma 2.1 to identify the Monge-Ampère mass with the Samuel multiplicity.  $\square$

**(2.8) Remark.** As the proof shows, the arguments are mostly of a local nature (the main problem is to ensure convergence of the integral of  $e^{-2\psi}$  on a neighborhood of the poles of an approximation  $\psi$  of  $\varphi$  with logarithmic poles). Therefore Theorem 1.1 is also valid for a plurisubharmonic function  $\varphi$  on an arbitrary non

singular complex variety  $X$ , provided that  $X$  does not possess positive dimensional complex analytic subsets (any open subset  $\Omega$  in a Stein manifold will thus do). We leave the reader complete the obvious details.

**(2.9) Remark.** The proof is highly non constructive, so it seems at this point that there is no way of producing an explicit bound  $C(\Omega, K, A, M)$ . It would be interesting to find a method to calculate such a bound, even a suboptimal one.

**(2.10) Remark.** The equality case in Theorem 1.5 suggests that extremal functions with respect to the integral  $\int_K e^{-2\varphi} d\lambda$  might be functions with Monge-Ampère measure  $(dd^c\varphi)^n$  concentrated at one point  $z_0 \in K$ , and a logarithmic pole at  $z_0$ . We are unsure what the correct boundary conditions should be, so as to actually get nice extremal functions of this form. We expect that an adequate condition is to assume that  $\varphi$  has zero boundary values. Further potential theoretic arguments would be needed for this, since prescribing the boundary values is not enough to get the relative compactness of the family in the weak topology (but this might be the case with the granted additional upper bound  $n^n$  on the Monge-Ampère mass).

We end this discussion by stating two generalizations of theorem 1.1 whose algebraic counterparts are useful as well for their applications to algebraic geometry (see [Che05] and [dFEM03]).

**(2.11) Theorem.** *Let  $\Omega$  be an open subset in  $\mathbb{C}^n$ ,  $K$  a compact subset of  $\Omega$ , and let  $\varphi, \psi$  be plurisubharmonic functions on  $\Omega$  such that  $-A \leq \varphi, \psi \leq 0$  on  $\Omega \setminus K$ , with  $c_K(\psi) \geq \frac{1}{\gamma}$ ,  $\gamma < 1$  and*

$$\int_{\Omega} (dd^c\varphi)^n \leq M < n^n(1 - \gamma)^n.$$

Then

$$\int_K e^{-2\varphi - 2\psi} d\lambda \leq C(\Omega, K, A, \gamma, M),$$

where the constant  $C(\Omega, K, A, \gamma, M)$  depends on the given parameters but not on the functions  $\varphi, \psi$ .

*Proof.* This is an immediate consequence of Hölder's inequality for the conjugate exponents  $p = 1/(1 - \gamma - \varepsilon)$  and  $q = 1/(\gamma + \varepsilon)$ , applied to the functions  $f = \exp(-2\varphi)$  and  $g = \exp(-2\psi)$ : when  $\varepsilon > 0$  is small enough, the Monge-Ampère hypothesis for  $\varphi$  precisely implies that  $f$  is in  $L^p(K)$  thanks to Theorem 1.1, and the assumption  $c_K(\psi) \geq \frac{1}{\gamma}$  implies by definition that  $g$  is in  $L^q(K)$ .  $\square$

In the case where  $D = \sum \gamma_j D_j$  is an effective divisor with normal crossings and  $\psi$  has codimension 1 analytic singularities given by  $D$  (i.e.  $\psi(z) \sim \sum \gamma_j \log |z_j|$  in suitable local analytic coordinates), we see that Theorem 2.11 can be applied

with  $\gamma = \max(\gamma_j)$  and with the Monge-Ampère upper bound  $n^n(1 - \max(\gamma_j))^n$ . In this circumstance, it turns out that the latter bound can be improved.

**(2.12) Theorem.** *Let  $\Omega$  be an open subset in  $\mathbb{C}^n$ ,  $K$  a compact subset of  $\Omega$ , and  $\varphi$  a plurisubharmonic function on  $\Omega$  such that  $-A \leq \varphi \leq 0$  on  $\Omega \setminus K$ . Assume that there are constants  $0 \leq \gamma_1, \dots, \gamma_n < 1$  such that*

$$\int_{\Omega} (dd^c \varphi)^n \leq M < n^n \prod_{1 \leq j \leq n} (1 - \gamma_j).$$

Then

$$\int_K e^{-2\varphi(z)} \prod_{1 \leq j \leq n} |z_j|^{-2\gamma_j} d\lambda \leq C(\Omega, K, A, \gamma_j, M),$$

where the constant  $C(\Omega, K, A, \gamma_j, M)$  depends on the given parameters but not on the function  $\varphi$ .

*Proof.* In the case when  $\gamma_j$  is the form  $\gamma_j = 1 - 1/p_j$  and  $p_j \geq 1$  is an integer, Theorem 2.12 can be derived directly from the arguments of the proof of Theorem 1.1. Since the estimate is essentially local, we only have to check convergence near the poles of  $\varphi$ , in the case when  $\varphi$  has an isolated analytic pole located on the support of the divisor  $D$ . Assume that the pole is the center of a polydisk  $D(0, r) = \prod D(0, r_j)$ , in coordinates chosen so that the components of  $D$  are the coordinate hyperplanes  $z_j = 0$ . We simply apply Theorem 1.1 to the function  $\tilde{\varphi}(z) = \varphi(z_1^{p_1}, \dots, z_n^{p_n})$  (with  $p_j = 1$  if the component  $z_j = 0$  does not occur in  $D$ ). We then get

$$\int_{\prod D(0, r_j^{1/p_j})} (dd^c \tilde{\varphi})^n = p_1 \dots p_n \int_{D(0, r)} (dd^c \varphi)^n = \prod (1 - \gamma_j)^{-1} \int_{D(0, r)} (dd^c \varphi)^n$$

by a covering degree argument, while

$$\int_{\prod \bar{D}(0, \rho_j^{1/p_j})} e^{-2\tilde{\varphi}} d\lambda = \int_{D(0, \rho)} e^{-2\varphi} \left( \prod |z_j|^{2(1-1/p_j)} \right)^{-1} d\lambda$$

by a change of variable  $\zeta = z_j^{p_j}$ . We do not have such a simple argument when the  $\gamma_j$ 's are arbitrary real numbers less than 1. In that case, the proof consists of repeating the steps of Theorem 1.1, with the additional observation that the statement of local algebra corresponding to Theorem 2.12 (i.e. with  $\varphi(z) = c \log \sum |g_j|^2$  possessing one isolated pole) is still valid by [dFEM03], Lemma 2.4.  $\square$

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