Explicit formula for the spherical image of the symplectic Hecke series of genus four

Kirill Vankov

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Abstract

We obtain an explicit formula for the spherical image of the polynomial fraction conjectured by Shimura in 1963 [Sh63] for generating Hecke series in particular case of genus 4. As in previous work [PaVa06] we used the Satake spherical map for Sp_4 and formulas by Andrianov [An70].

Keywords: Symplectic group, Hecke's operators, spinor L-function.

Résumé

On obtient une formule explicite pour l'image par l'application sphérique de la fraction conjecturée par Shimura en 1963 pour la série génératrice des opérateurs de Hecke dans le cas de genre 4. Comme dans un de nos travaux antérieurs, on utilise des formules relatives aux applications sphériques de Satake pour les groupes Sp_4 et GL_4, et des formules reliées à celles-ci, étudiées par Andrianov.

Mots-clés : Groupe symplectique, Opérateurs de Hecke, Fonction L spineur.

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1 Introduction

In this article we continue computations of formal power series in Hecke algebra. The previous article [PaVa06] contains all necessary theory and formulas we used in general case of genus n and the explicit result for $n = 3$. Presented below is an explicit result for genus $n = 4$.

Consider the formal generating series of Hecke operators of genus n for symplectic group

$$\mathbf{D}_p(\mathsf{X}) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta)\mathsf{X}^\delta \in \mathcal{L}_{n,\mathbb{Z}}[\![\mathsf{X}]\!]\,, \tag{1}$$

where

p is a prime,

$\mathcal{L}_{n,\mathbb{Z}} = \mathbb{Z}[\mathbf{T}(p),\ \mathbf{T}_1(p^2),\ \ldots\ ,\mathbf{T}_n(p^2)]$, generated by the following Hecke operators:

$$\mathbf{T}(p) = \mathbf{T}(\underbrace{1,\cdots,1}_{n},\underbrace{p,\cdots,p}_{n}),$$

$$\mathbf{T}_i(p^2) = \mathbf{T}(\underbrace{1,\cdots,1}_{n-i},\underbrace{p,\cdots,p}_{i},\underbrace{p^2,\cdots,p^2}_{n-i},\underbrace{p,\cdots,p}_{i}),\ \text{ for }\ i = 1,\cdots,n$$

over $\Gamma = \mathrm{Sp}_n(\mathbb{Z})$ a modular Siegel group of genus n,

and $\mathrm{S} = \mathrm{GSp}_n^+(\mathbb{Q}) = \{\mathbf{M} \in \mathrm{M}_{2n}(\mathbb{Q}) \mid {}^t\mathbf{M}\mathbf{J}_n\mathbf{M} = \mu(\mathbf{M})\mathbf{J}_n\,,\ \mu\mathbf{M} > 0\}$.

Applying the spherical map Ω we can carry out all calculations in polynomial rings instead of the Hecke rings of the symplectic group. This theory is developed in [Sh63], [An87] and [AnZh95]. Our previous article [PaVa06] describes in details the method of Andrianov for finding images of Hecke operators. We adopted the formulas of the article [An70]. Result is presented in form of sym_{i_1,i_2,i_3,i_4} polynomials, which are the same symmetrical polynomials as in [PaVa06], but depend on 4 variables x_1, x_2, x_3 and x_4. Formal algebraic computations were carried out in Maple as for the case $n = 3$. For $n = 4$ the number of $\omega(t)$ images increased from 28 ($n = 3$) to 680. It took hours of processor time to obtain the final result. Intermediate polynomials used megabytes of disk space.

2 The spherical image of $\mathbf{D}_p$

THEOREM 2.1 *For $n = 4$ there is the following explicit polynomial presentation:*

$$\Omega(\mathbf{D}_p(\mathsf{X})) = \frac{P_4(\mathsf{X})}{Q_4(\mathsf{X})}\,, \tag{2}$$

where

$$\begin{aligned} Q_4(\mathsf{X}) =&(1 - x_0\mathsf{X})(1 - x_0x_1\mathsf{X})(1 - x_0x_2\mathsf{X})(1 - x_0x_3\mathsf{X})(1 - x_0x_4\mathsf{X})\times \\ &\times (1 - x_0x_1x_2\mathsf{X})(1 - x_0x_1x_3\mathsf{X})(1 - x_0x_1x_4\mathsf{X})(1 - x_0x_2x_3\mathsf{X})\times \\ &\times (1 - x_0x_2x_4\mathsf{X})(1 - x_0x_3x_4\mathsf{X})(1 - x_0x_1x_2x_3\mathsf{X})(1 - x_0x_1x_2x_4\mathsf{X})\times \\ &\times (1 - x_0x_1x_3x_4\mathsf{X})(1 - x_0x_2x_3x_4\mathsf{X})(1 - x_0x_1x_2x_3x_4\mathsf{X}) \end{aligned} \tag{3}$$

and

$$P_4(\mathsf{X}) = \sum_{k=0}^{14} K_k(p, x_0, x_1, x_2, x_3, x_4)\mathsf{X}^k$$

with coefficients K_k are listed in the following table.

$K_0 = 1$
$K_1 = 0$
$K_2 = -\frac{x_0^2}{p^2} \times \begin{pmatrix} p\,(sym_{2211} + sym_{2110} + sym_{1100})+ \\ (p^2 + p + 1)(sym_{2111} + sym_{1110})+ \\ (2p^2 + 4p + 1)\,sym_{1111} \end{pmatrix}$
$K_3 = \frac{x_0^3}{p^3} \times \begin{pmatrix} (p^2 + p)(sym_{3222} + sym_{3221} + sym_{3211} + sym_{3111} + sym_{2220}+ \\ \quad sym_{2210} + sym_{2110} + sym_{1110})+ \\ (p^3 + 5p^2 + 5p + 1)(sym_{2222} + sym_{2221} + sym_{2211} + sym_{2111}+ \\ \quad sym_{1111}) \end{pmatrix}$

$$K_4 = -\frac{x_0^4}{p^4} \times \begin{pmatrix} p^2(sym_{4322} + sym_{4221} + sym_{3220} + sym_{2210}) + \\ p\,(p^2+p+1)(sym_{4222} + sym_{3333} + sym_{3331} + sym_{3311} + \\ \quad sym_{3111} + sym_{2220} + sym_{1111}) + \\ p\,(p^2+4p+1)(sym_{3332} + sym_{3321} + sym_{3211} + sym_{2111}) + \\ p\,(3p^2+6p+4)(sym_{3322} + sym_{3221} + sym_{2211}) + \\ (5p^3+15p^2+6p+1)(sym_{3222} + sym_{2221}) + \\ (12p^3+22p^2+16p+1)sym_{2222} \end{pmatrix}$$

$$K_5 = \frac{x_0^5}{p^4} \times \begin{pmatrix} (p^2+p)(sym_{4433} + sym_{4432} + sym_{4422} + sym_{4331} + sym_{4321} + \\ \quad sym_{4221} + sym_{3311} + sym_{3211} + sym_{2211}) + \\ (4p^2+5p+1)(sym_{4333} + sym_{4332} + sym_{4322} + sym_{4222} + \\ \quad sym_{3331} + sym_{3321} + sym_{3221} + sym_{2221}) + \\ (-p^4+14p^2+18p+5)(sym_{3333} + sym_{3332} + \\ \quad sym_{3322} + sym_{3222} + sym_{2222}) \end{pmatrix}$$

$$K_6 = \frac{x_0^6}{p^6} \times \begin{pmatrix} p^2(p^3-5p-4)(sym_{4432} + sym_{4322} + sym_{3222} + sym_{4443}) + \\ p\,(p^5+5p^4-17p^2-15p-1)(sym_{4333} + sym_{3332}) - \\ p^2(p+1)(sym_{4331} + sym_{3321} + sym_{5332} + sym_{5433}) + \\ p\,(3p^4-12p^2-6p-1)(sym_{4332} + sym_{3322} + sym_{4433}) + \\ p^2(p^3-3p-1)(sym_{4222} + sym_{3331} + sym_{2222} + sym_{4422} + \\ \quad sym_{4442} + sym_{4444} + sym_{5333}) + \\ p^3(sym_{6333} - sym_{5443} - sym_{5432} - sym_{5322} - sym_{4431} - \\ \quad sym_{4321} + sym_{3330} - sym_{3221}) + \\ (2p^6+12p^5-32p^3-22p^2-4p+1)sym_{3333} \end{pmatrix}$$

$$K_7 = -\frac{x_0^7}{p^5} \times \begin{pmatrix} p\,(p^2-1)(sym_{5544} + sym_{5543} + sym_{5533} + sym_{5442} + \\ \quad sym_{5432} + sym_{5332} + sym_{4422} + sym_{4322} + sym_{3322}) + \\ (p^4+4p^3-4p-1)(sym_{5444} + sym_{5443} + sym_{5433} + \\ \quad sym_{5333} + sym_{4442} + sym_{4432} + sym_{4332} + sym_{3332}) + \\ (5p^4+14p^3-14p-5)(sym_{4444} + sym_{4443} + sym_{4433} + \\ \quad sym_{4333} + sym_{3333}) \end{pmatrix}$$

$$K_8 = \frac{x_0^8}{p^6} \times \begin{pmatrix} (p^5+15p^4+17p^3-5p-1)(sym_{5444} + sym_{4443}) + \\ p^3(-sym_{7444} + sym_{6554} + sym_{6543} + sym_{6433} + sym_{5542} + \\ \quad sym_{5432} - sym_{4441} + sym_{4332}) + \\ p\,(4p^3+5p^2-1)(sym_{5554} + sym_{5543} + sym_{5433} + sym_{4333}) + \\ p^3(p+1)(sym_{6544} + sym_{6443} + sym_{5442} + sym_{4432}) + \\ p\,(p^3+3p^2-1)(sym_{4442} + sym_{5333} + sym_{5533} + sym_{5555} + \\ \quad sym_{6444} + sym_{5553} + sym_{3333}) - \\ (p^6-4p^5-22p^4-32p^3+12p+2)sym_{4444} + \\ p\,(p^4+6p^3+12p^2-3)(sym_{5544} + sym_{5443} + sym_{4433}) \end{pmatrix}$$

$$K_9 = -\frac{x_0^9}{p^6} \times \begin{pmatrix} p^2(p+1)(sym_{6655} + sym_{6654} + sym_{6644} + sym_{6553} + \\ \quad sym_{6543} + sym_{6443} + sym_{5533} + sym_{5433} + sym_{4433}) + \\ p^2(p^2+5p+4)(sym_{6555} + sym_{6554} + sym_{6544} + sym_{6444} + \\ \quad sym_{5553} + sym_{5543} + sym_{5443} + sym_{4443}) + \\ (5p^4+18p^3+14p^2-1)(sym_{5555} + sym_{5554} + \\ \quad sym_{5544} + sym_{5444} + sym_{4444}) \end{pmatrix}$$

$$K_{10} = \frac{x_0^{10}}{p^5} \times \begin{pmatrix} (p^2+p+1)(sym_{4444} + sym_{5553} + sym_{6444} + sym_{6644} + \\ \quad sym_{6664} + sym_{6666} + sym_{7555}) + \\ (p^2+4p+1)(sym_{5444} + sym_{6544} + sym_{6654} + sym_{6665}) + \\ p\,(sym_{5543} + sym_{6553} + sym_{7554} + sym_{7655}) + \\ (4p^2+6p+3)(sym_{5544} + sym_{6554} + sym_{6655}) + \\ (p^3+6p^2+15p+5)(sym_{5554} + sym_{6555}) + \\ (p^3+16p^2+22p+12)sym_{5555} \end{pmatrix}$$

$$K_{11} = -\frac{x_0^{11}}{p^6} \times \begin{pmatrix} (p^2+p)(sym_{7666} + sym_{7665} + sym_{7655} + sym_{7555} + \\ \quad sym_{6664} + sym_{6654} + sym_{6554} + sym_{5554}) + \\ (p^3+5p^2+5p+1)(sym_{6666} + sym_{6665} + sym_{6655} + \\ \quad sym_{6555} + sym_{5555}) \end{pmatrix}$$

$$K_{12} = \frac{x_0^{12}}{p^6} \times \begin{pmatrix} (p^2+p+1)(sym_{6665} + sym_{7666}) + \\ (p^2+4p+2)sym_{6666} + \\ p\,(sym_{7665} + sym_{7766} + sym_{6655}) \end{pmatrix}$$

$$K_{13} = 0$$

$$K_{14} = -\frac{x_0^{14}}{p^6} \times sym_{7777}$$

3 Relations

We noticed a very interesting symmetry property within the coefficients K_k. Knowing this relation in advance would let to limit computation of coefficients just up to degree 7, skipping the most time consuming higher degree coefficients.

PROPOSITION 3.1 *Polynomial* $P_4(\mathsf{X})$ *has the following functional relation between its coefficients* $K_k(p,\, x_0,\, x_1,\, x_2,\, x_3\, x_4)$, $k = 1, \cdots, 14$*:*

$$K_{14-k}(p, x_0, x_1, x_2, x_3, x_4) = \\ - p^6 (x_0 x_1 x_2 x_3 x_4)^{7-k} K_k \left(\frac{1}{p}, x_0 x_1 x_2 x_3 x_4, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4} \right) \tag{4}$$

REMARK 3.2 *Similar and even more simple functional relations could be also established for* $n < 4$*.*

EXERCISE 3.3 *The explicit formula for* $n = 3$ *can be easy obtained from our result in case* $n = 4$ *by simple projection* $x_4 = 0$*.*

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Kirill Vankov
Institut Fourier, UMR 5582 (CNRS-UJF)
100 rue des Mathématiques
Domaine Universitaire
BP 74
38402 Saint Martin d'Hères - France
kvankov@fourier.ujf-grenoble.fr