p-adic Banach modules of arithmetical modular forms and triple products of Coleman's families

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To dear Jean-Pierre Serre for his eightieth birthday with admiration

Abstract

For a prime number $p \geq 5$, we consider three classical cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} e(nz) \in \mathcal{S}_{k_j}(N_j, \psi_j),\ (j = 1, 2, 3)$$

of weights k_1, k_2, k_3, of conductors N_1, N_2, N_3, and of nebentypus characters ψ_j mod N_j.

According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each f_j ($j = 1, 2, 3$) (under suitable assumptions on p and on f_j) into a p-adic analytic family

$$k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j}) q^n\}$$

of cusp eigenforms f_{j,k_j} of weights k_j in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain p-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.

The purpose of this paper is to describe a four variable p-adic L function attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \left\{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k) q^n \right\}$$

of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha^{(1)}_{p,j}(k_j)) \geq 0$ where $\alpha^{(1)}_{p,j} = \alpha^{(1)}_{p,j}(k_j)$ is an eigenvalue (which depends on k_j) of Atkin's operator $U = U_p$ acting on Fourier expansions by $U(\sum_{n\geq 0}^{\infty} a_n q^n) = \sum_{n\geq 0}^{\infty} a_{np} q^n$.

Let us consider the product of three eigenvalues:

$$\lambda = \lambda(k_1, k_2, k_3) = \alpha^{(1)}_{p,1}(k_1)\alpha^{(1)}_{p,2}(k_2)\alpha^{(1)}_{p,3}(k_3)$$

and assume that the slope of this product

$$\sigma = v_p(\lambda(k_1, k_2, k_3)) = \sigma(k_1, k_2, k_3) = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets (k_1, k_2, k_3) in an appropriate p-adic neighbourhood of the fixed triplet of weights (k_1, k_2, k_3). The each value σ_j is fixed.

We consider the p-adic weight space X containing all (k_j, ψ_j). Our p-adic L-functions are Mellin transforms of certain measures with values in $\mathcal{A}$, where $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denotes an affinoid algebra associated with an affinoid space $\mathcal{B}$ as in [CoPB], where $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integers k_j and fixed Dirichlet characters ψ_j mod N).

We construct such a measure from higher twists of classical Siegel-Eisenstein series, which produce distributions with values in certain Banach $\mathcal{A}$-modules $\mathcal{M} = \mathcal{M}(N; \mathcal{A})$ of triple modular forms with coefficients in the algebra $\mathcal{A}$.

Keywords: Symplectic group, Hecke's operators, Coleman's families, Garrett's triple product, Siegel-Eisenstein series.

Résumé

Soit $p \geq 5$ un nombre premier. On considère trois formes paraboliques classiques $f_j(z)$, fonctions propres d'opérateurs de Hecke, de poids k_1, k_2, k_3, de conducteurs N_1, N_2, N_3, et de caractères ψ_j mod N_j. Selon H.Hida and R.Coleman, on peut inclure chaque f_j dans une famille p-adique analytique $k_j \mapsto \{f_{j,k_j}\}$ de formes paraboliques f_{j,k_j}, fonctions propres d'opérateurs de Hecke, de poids variables k_j de telle façon que $f_{j,k_j} = f_j$, et que tous les coefficients de Fourier $a_n(f_{j,k_j})$ sont donnés par certaines fonctions p-adiques analytiques $k_j \mapsto a_{n,j}(k_j)$. Le but de présent travail est de décrire une fonction L p-adique analytique de quatre variables attachée au produit triple de Garrett de trois familles de Coleman $k_j \mapsto \{f_{j,k_j}\}$ de formes paraboliques, fonctions propres d'opérateurs de Hecke, de trois pentes fixées $\sigma_j = v_p(\alpha^{(1)}_{p,j}(k_j)) \geq 0$ où $\alpha^{(1)}_{p,j} = \alpha^{(1)}_{p,j}(k_j)$ est une valeur propre (une fonction de k_j) d'opérateur d'Atkin $U = U_p$ agissant sur les développements de Fourier par $U(\sum_{n\geq 0}^{\infty} a_n q^n) = \sum_{n\geq 0}^{\infty} a_{np} q^n$.

On considère l'espace des poids p-adique X contenant tous les (k_j, ψ_j). Nos fonctions L p-adiques sont les transformées de Mellin de certaines mesures à valeurs dans $\mathcal{A}$, où $\mathcal{A} = \mathcal{A}(\mathcal{B})$ désigne l'algèbre affinoïde associée à l'espace affinoïde $\mathcal{B}$ introduit par R.Coleman, où $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, est un voisinage affinoïde autour de $(k_1, k_2, k_3) \in X^3$ (avec les entiers donnés k_j et les caractères fixés ψ_j mod N). On construit une telle mesure à partir de twists de séries classiques de Siegel-Eisenstein, produisant des distributions à valeurs dans certain modules de Banach de formes modulaires triples sur l'algèbre $\mathcal{A}$.

Mots-clés : Groupe symplectique, Opérateurs de Hecke, Familles de Coleman, Produits triples de Garrett, Séries de Siegel-Eisenstein.

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Contents

1 Introduction

Why study L-values attached to modular forms?

A popular proceedure in Number Theory is the following:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$, for example $a_n = p(n)$	$\rightsquigarrow$	Compute f via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$	$\rightsquigarrow$	A number (solution)
		$\uparrow$		$\uparrow$
Example 1 [Chand70]: (Hardy-Ramanujan) $p(n) = \frac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} + O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3)$, $\lambda_n = \sqrt{n - 1/24}$,		Good bases, finite dimensions, many relations and identities		Values of L-functions, periods, congruences, ...

Other examples: Birch and Swinnerton-Dyer conjecture, ... L-values attached to modular forms

Our data: three primitive cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} q^n \in \mathcal{S}_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3) \tag{1.1}$$

of weights k_1, k_2, k_3, of conductors N_1, N_2, N_3, and of nebentypus characters $\psi_j \bmod N_j$, $N := \mathrm{LCM}(N_1, N_2, N_3)$.

Let p be a prime, $p \nmid N$.

We view $f_j \in \overline{\mathbb{Q}}[\![q]\!] \overset{i_p}{\hookrightarrow} \mathbb{C}_p[\![q]\!]$ via a fixed embedding $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p$, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ is Tate's field.

Let χ denote a variable Dirichlet character $\bmod Np^v, v \geq 0$.

We view k_j as a variable weight in the weight space $X = X_{Np^v} = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$, $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^* \ni (y_0, y_p)$.

The space X is a p-adic analytic space first used in Serre's [Se73] *Formes modulaires et fonctions zêta p-adiques.* Denote by $(k, \chi) \in X$ the homomorphism $(y_0, y_p) \mapsto \chi(y_0)\chi(y_p \bmod p^v) y_p^k$. We write simply k_j for the couple $(k_j, \psi_j) \in X$.

The purpose of this paper is to describe a four variable p-adic L function

attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \left\{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k_j) q^n \right\}$$

of cusp eigenforms of three constant slopes $\sigma_j = \mathrm{ord}_p(\alpha_{p,j}^{(1)}(k_j)) \geq 0$ where $\alpha_{p,j}^{(1)}(k_j)$, $\alpha_{p,j}^{(2)}(k_j)$ are the Satake parameters given as inverse roots of the Hecke p–polynomial $1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1 - \alpha_{p,j}^{(1)}(p)X)(1 - \alpha_{p,j}^{(2)}(p)X)$.

We assume that $\mathrm{ord}_p(\alpha_{p,j}^{(1)}(k_j)) \leq \mathrm{ord}_p(\alpha_{p,j}^{(2)}(k_j))$.

This extends a previous result: (see [PaTV], Invent. Math. v. 154, N3 (2003)) where a two variable p-adic L-function was constructed interpolating on all k a function $(k, s) \mapsto L^*(f_k, s, \chi)$ $(s = 1, \cdots, k-1)$ for such a family.

We use the theory of p-adic integration with values in spaces of nearly holomorphic modular forms (in the sense of Shimura, see [ShiAr]).

A family of slope $\sigma > 0$ of cusp eigenforms f_k of weight $k \geq 2$:

$k \mapsto f_k = \sum_{n=1}^{\infty} a_n(k) q^n$ $\in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!]$ A model example of a p-adic family (not cusp and $\sigma = 0$): Eisenstein series $a_n(k) = \sum_{d \mid n} d^{k-1}, f_k = E_k$	1) the Fourier coefficients $a_n(k)$ of f_k and one of the Satake p-parameters $\alpha(k) := \alpha_p^{(1)}(k)$ are given by certain p-adic analytic functions $k \mapsto a_n(k)$ for $(n, p) = 1$ 2) the slope is constant and positive: $\mathrm{ord}(\alpha(k)) = \sigma > 0$

The existence of families of slope $\sigma > 0$ was established in [CoPB]

R.Coleman gave an example with $p = 7$, $f = \Delta$, $k = 12$ $a_7 = \tau(7) = -7 \cdot 2392, \sigma = 1$.

A program in PARI for computing such families is contained in [CST98] (see also the Web-page of W.Stein, `http://modular.fas.harvard.edu/`)

Coleman proved that :

- The operator U acts as a completely continuous operator on each $\mathcal{A}$-submodule $\mathcal{M}^\dagger(Np^v;\mathcal{A}) \subset \mathcal{A}[\![q]\!]$ (i.e. U is a limit of finite-dimensional operators)

$\Longrightarrow$ there exists the Fredholm determinant $P_U(T) = \det(Id - T \cdot U) \in \mathcal{A}[\![T]\!]$

- there is a version of the Riesz theory: for any inverse root $\alpha \in \mathcal{A}^*$ of $P_U(T)$ there exists an eigenfunction g, $Ug = \alpha g$

such that $ev_k(g) \in \mathbb{C}_p[\![q]\!]$ are classical cusp eigenforms for all k in a neigbourhood $\mathcal{B} \subset X$ (see in [CoPB])

2 Generalities on triple products

The triple product with a Dirichlet character χ is defined as the following complex L-function (an Euler product of degree eight):

$$L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p)p^{-s}), \tag{2.2}$$

$$\begin{aligned} &\text{where } L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} = \\ &\det\left(1_8 - X \begin{pmatrix} \alpha_{p,1}^{(1)} & 0 \\ 0 & \alpha_{p,1}^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,2}^{(1)} & 0 \\ 0 & \alpha_{p,2}^{(2)} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{p,3}^{(1)} & 0 \\ 0 & \alpha_{p,3}^{(2)} \end{pmatrix}\right) \\ &= \prod_\eta (1 - \alpha_{p,1}^{(\eta(1))} \alpha_{p,2}^{(\eta(2))} \alpha_{p,3}^{(\eta(3))} X) \\ &= (1 - \alpha_{p,1}^{(1)}\alpha_{p,2}^{(1)}\alpha_{p,3}^{(1)}X)(1 - \alpha_{p,1}^{(1)}\alpha_{p,2}^{(1)}\alpha_{p,3}^{(2)}X) \cdot \ldots \cdot (1 - \alpha_{p,1}^{(2)}\alpha_{p,2}^{(2)}\alpha_{p,3}^{(2)}X), \end{aligned} \tag{2.3}$$

product taken over all 8 maps $\eta : \{1, 2, 3\} \to \{1, 2\}$.

The Satake parameters and Hecke p–polynomials of forms f_j:

Here the Satake parameters $\alpha_{p,j}^{(1)}$, $\alpha_{p,j}^{(2)}$ are given as inverse roots of the Hecke p–polynomials

$$1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1 - \alpha_{p,j}^{(1)}(p)X)(1 - \alpha_{p,j}^{(2)}(p)X),$$

We always assume that the weights are “balanced”:

$$k_1 \geq k_2 \geq k_3 \geq 2, \text{ and } k_1 \leq k_2 + k_3 - 2 \tag{2.4}$$

Critical values and functional equation

We use the corresponding normalized L function (see [De79], [Co], [Co-PeRi]), which has the form:

$$\Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \tag{2.5}$$
$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k_3+1)\Gamma_{\mathbb{C}}(s-k_2+1)\Gamma_{\mathbb{C}}(s-k_1+1)L(f_1 \otimes f_2 \otimes f_3, s, \chi),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

The Gamma-factor determines the critical values $s = k_1, \cdots, k_2 + k_3 - 2$ of $\Lambda(s)$, which we explicitely evaluate (like in the classical formula $\zeta(2) = \dfrac{\pi^2}{6}$).

A functional equation of $\Lambda(s)$ has the form:

$$s \mapsto k_1 + k_2 + k_3 - 2 - s.$$

According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each f_j ($j = 1, 2, 3$) (under suitable assumptions on p and on f_j) into a p-adic analytic family

$$\mathbf{f}_j : k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j})q^n\}$$

of cusp eigenforms f_{j,k_j} of weights k_j in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain p-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.

3 Statement of the problem

Given three p-adic analytic families $\mathbf{f}_j$ of slope $\sigma_j \geq 0$, to construct a four-variable p-adic L-function attached to Garrett's triple product of these families.

We show that this function interpolates the special values

$$(s, k_1, k_2, k_2) \longmapsto \Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, s, \chi)$$

at critical points $s = k_1, \cdots, k_2 + k_3 - 2$ for balanced weights $k_1 \leq k_2 + k_3 - 2$; we prove that these values are algebraic numbers after dividing out certain "periods".

However the construction uses directly modular forms, and not the L-values in question, and a comparison of special values of two functions is done after the construction.

Consider the product of the Satake parmeters

$$\lambda_p = \alpha^{(1)}_{p,1}\alpha^{(1)}_{p,2}\alpha^{(1)}_{p,3} = \lambda_p(k_1, k_2, k_3)$$

We assume that $\mathrm{ord}_p\alpha^{(1)}_{p,j} \leq \mathrm{ord}_p\alpha^{(2)}_{p,j}$, and that the slope $\sigma = \mathrm{ord}_p(\lambda_p(k_1, k_2, k_3))$ is constant and positive for all triplets (k_1, k_2, k_3) in a p-adic neighbourhood $\mathcal{B} \subset X^3$ of the fixed triplet of weights (k_1, k_2, k_3).

Our method includes:

• a version of Garrett's integral representation for the triple L-functions of the form: for $r = 0, \cdots, k_2 + k_3 - k_1 - 2$,

$$\Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - r, \chi) = \iiint_{(\Gamma_0(N^2p^{2v})\backslash \mathbb{H})^3} \overline{\tilde{f}_{1,k_1}(z_1)\tilde{f}_{2,k_2}(z_2)\tilde{f}_{3,k_3}(z_3)} \mathcal{E}(z_1, z_2, z_3; -r, \chi) \prod_j (\frac{dx_j dy_j}{y_j^2})$$

where $\tilde{f}_{j,k_j} =: f^0_{j,k_j}$ is an eigenfunction of U_p^* in $\mathcal{M}_{k_j}(Np, \psi_j)$,

$f_{j,k_j,0}$ is the corresponding eigenfunction of U_p,

$$\mathcal{E}(z_1, z_2, z_3; -r, \chi) \in \mathcal{M}_T(N^2p^{2v})$$
$$= \mathcal{M}_{k_1}(N^2p^{2v}, \psi_1) \otimes \mathcal{M}_{k_2}(N^2p^{2v}, \psi_2) \otimes \mathcal{M}_{k_3}(N^2p^{2v}, \psi_3)$$

is the triple modular form of triple weight (k_1, k_2, k_3), and of fixed triple Nebentupus character (ψ_1, ψ_2, ψ_3), obtained from a nearly holomorphic Siegel-Eisenstein series $F_{\chi,r} = G^\star(z, -r; k, (Np^v)^2, \boldsymbol{\psi})$, of degree 3, of weight $k = k_2 + k_3 - k_1$, and the Nebentypus character $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$

We obtain $\mathcal{E}(z_1, z_2, z_3; -r, \chi)$ from a Siegel-Eisenstein series

by applying to $F_{\chi,r}$ Boecherer's higher twist (see (11.22)) and Ibukiyama's differential operator (see (11.23)).

These operations act explicitely on the Fourier expansions.

Then one uses:

• The theory of p-adic integration with values in Serre's type $\mathcal{A}$-modules $\mathcal{M}_T(\mathcal{A})$ of triple arithmetical nearly holomorphic modular forms over p-adic Banach algebras $\mathcal{A}$. Explicit Fourier coefficients $a_{\chi,r}(R, \mathcal{T}) \in \overline{\mathbb{Q}}[R, T]$ of $\mathcal{E}(-r, \chi)$ are given by special polynomials of matricies $\mathcal{T} = (t_{ij})$, $R = (R_{ij})$ and of $\chi(\beta)\beta^r$ (with $\beta \in \mathbb{Z}_p^* \cap \mathbb{Q}$) i.e. the coefficients of $a_{\chi,r}$ by some elementry p-adic measures $\int_Y \chi y^r d\mu_{\mathcal{T}} \in \mathcal{A}$. Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ is a certain p-adic Banach algebra of functions on an open analytic subspace $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X^3$ in the product of three copies of the weight space $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$.

These measures on the group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$ produce the coefficients of $a_{\chi,r}$ of $\mathcal{E}(-r, \chi)$ of $\mathcal{M}_T(\mathcal{A})$ for all p-adic weights $x \in X$, given by $\int_Y x(y) d\mu_{\mathcal{T}} \in \mathcal{A}$ (*an interpolation from $x = \chi y_p^r$ to all $x \in X$*).

• **The** spectral theory of triple Atkin's operator $U = U_{p,T}$

allows to evaluate the integral using at each weight (k_1, k_2, k_3) the equality

$$\langle \underline{f}^0, \mathcal{E}(-r, \chi) \rangle = \langle \underline{f}^0, \pi_\lambda(\mathcal{E}(-r, \chi)) \rangle$$

with the projection π_λ of $\mathcal{M}_T(\mathcal{A})$ to the λ-part $\mathcal{M}_T(\mathcal{A})^\lambda$, defined by :

$$\mathrm{Ker}\, \pi_\lambda := \bigcap_{n \geq 1} \mathrm{Im}\, (U_T - \lambda I)^n, \mathrm{Im}\, \pi_\lambda := \bigcup_{n \geq 1} \mathrm{Ker}\, (U_T - \lambda I)^n.$$

We prove that U is a completely continuos $\mathcal{A}$-linear operator on a certain Coleman's submodule $\mathcal{M}(\mathcal{A})^\dagger$ of Serre's type module $\mathcal{M}(\mathcal{A})$. Then the projection π_λ exists (on this submodule) due to general results of Serre and Coleman, see [CoPB], [SePB].

We show that there exists an element $\tilde{\mathcal{E}}(-r,\chi) \in \mathcal{M}(\mathcal{A})^\dagger$ such that at each weight (k_1,k_2,k_3) the equality holds:
$\langle \underline{f}^0, \mathcal{E}(-r,\chi)\rangle = \left\langle \underline{f}^0, \pi_\lambda(\tilde{\mathcal{E}}(-r,\chi))\right\rangle$, and the product can be expressed through certain coefficients the series $\tilde{\mathcal{E}}(-r,\chi)$ which are the same as those of $\mathcal{E}(-r,\chi)$.

• Key point: modular admissible measures

Let us write for simplicity: $\mathcal{E}(-r,\chi)$ for $\tilde{\mathcal{E}}(-r,\chi)$

$\mathcal{M}_T(\mathcal{A})$ instead of $\mathcal{M}_T(\mathcal{A})^\dagger$ (Coleman's submodule)

One defines admissible p-adic measures $\tilde{\Phi}^\lambda$ with values in Banach $\mathcal{A}$-modules $\mathcal{M}_T^\lambda(\mathcal{A})$ which are locally free of finite rank, using the test functions: $\displaystyle\int_Y \chi y_p^r \tilde{\Phi}^\lambda = \pi_\lambda(\mathcal{E}(-r,\chi))$.

Consider the evaluation maps $ev_{\mathbf{s}} : \mathcal{A} \to \mathbb{C}_p$ for any p-adic triple weights $\mathbf{s} = (s_1,s_2,s_3) \in \mathcal{B}$.

• Passage from values in modular forms to scalar values: apply an algebraic $\mathcal{A}$-linear form $\mathcal{M}_T^\lambda(\mathcal{A}) \stackrel{\ell_T}{\to} \mathcal{A}$ to the constructed measure $\tilde{\Phi}^\lambda$ (in modular forms), and the evaluation maps $\mathcal{A} \stackrel{ev_{\mathbf{s}}}{\to} \mathbb{C}_p$ for any p-adic triple weights $\mathbf{s} \in X^3$.

The linear form ℓ_T is an algebraic version of the Petersson product (a geometric meaning of ℓ_T: the first coordinate in an (orthogonal) $\mathcal{A}$-basis of eigenfunctions of all Hecke operators T_q for $q \nmid Np$, with the first basis element $\mathbf{f}_0 \in \mathcal{M}^\lambda(\mathcal{A})$).

Using the evaluation map and the Mellin transform

We obtain the measure $\mu = \ell_T(\tilde{\Phi}^\lambda)$ with values in $\mathcal{A}$ on the profinite group Y.

• Construct an analytic function $\mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B})$ as the p-adic Mellin transform $\mathcal{L}_\mu(x) = \displaystyle\int_Y x(y)\,d\mu(y) \in \mathcal{A}$, $x \in X$.

• Solution: the function in question $\mathcal{L}_\mu(x,\mathbf{s})$ is given by evaluation of $\mathcal{L}_\mu(x)$ at $\mathbf{s} = (s_1,s_2,s_3) \in \mathcal{B}$: this is a p-adic analytic function in four variables

$$(x,\mathbf{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$$

$\mathcal{L}_\mu(x,\mathbf{s}) := ev_{\mathbf{s}}(\mathcal{L}_\mu(x)) \quad (x \in X,\ \mathbf{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3,\ \mathcal{L}_\mu(x) \in \mathcal{A}).$

Final step: comparison between $\mathbb{C}$ and $\mathbb{C}_p$

• We check an equality relating the values of the constructed analytic function $\mathcal{L}_\mu(x,\mathbf{s})$ at the arithmetical characters
$x = y_p^r\chi \in X$, and at triple weights $\mathbf{s} = (k_1,k_2,k_3) \in \mathcal{B}$, with the normalized critical special values

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2+k_3-2-r,\chi) \quad (r = 0,\cdots,k_2+k_3-k_1-2),$$

for certain Dirichlet characters $\chi \bmod Np^v, v \geq 1$. These are algebraic numbers, embedded into $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ (the Tate field of p-adic numbers). The normalisation of L^* includes at the same time Gauss sums, Petersson scalar products, powers of π, the product $\lambda_p(k_1, k_2, k_3)$, and a certain finite Euler product.

4 Arithmetical nearly holomorphic modular forms

Arithmetical nearly holomorphic modular forms (the elliptic case)

Let $\mathcal{A}$ be a commutative ring (a subring of $\mathbb{C}$ or $\mathbb{C}_p$)

Arithmetical nearly holomorphic modular forms (in the sense of Shimura, [ShiAr] are certain formal series

$$g = \sum_{n=0}^{\infty} a(n; R) q^n \in \mathcal{A}[\![q]\!][R], \text{ with the property}$$

that for $\mathcal{A} = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}$ of a given weight k and Dirichlet character ψ. The coefficients $a(n; R)$ are polynomials in $\mathcal{A}[R]$. If $\deg_R a(n; R) \leq r$ for all n, we call g nearly holomorphic of type r (it is annihilated by $(\frac{\partial}{\partial \bar{z}})^{r+1}$, see [ShiAr]).

We use the notation $\mathcal{M}_{k,r}(N, \psi, \mathcal{A})$ or $\widetilde{\mathcal{M}}(N, \psi, \mathcal{A})$ for $\mathcal{A}$-modules of such forms (In our constructions the weight k varies).

A known example (see the introduction to [ShiAr]) is given by the series

$$\begin{aligned} &-12R + E_2 := -12R + 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \\ &= \frac{3}{\pi^2} \lim_{s \to 0} y^s \sum_{m_1, m_2 \in \mathbb{Z}}{}' (m_1 + m_2 z)^{-2} |m_1 + m_2 z|^{-2s}, (R = (4\pi y)^{-1}) \end{aligned}$$

where $\sigma_1(n) = \sum_{d|n} d$.

The action of the Shimura differential operator

$$\delta_k : \mathcal{M}_{k,r}(N, \psi, \mathcal{A}) \to \mathcal{M}_{k+2,r+1}(N, \psi, \mathcal{A}),$$

is given over $\mathbb{C}$ by $\delta_k(f) = (\frac{1}{2\pi i} \frac{\partial}{\partial z} - \frac{k}{4\pi y}) f$.

This operator is a correction of the Ramanujan operator

$$\theta(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} n a_n q^n = \frac{1}{2\pi i} \frac{\partial}{\partial z} (\sum_{n=0}^{\infty} a_n q^n) = q \frac{\partial}{\partial q} (\sum_{n=0}^{\infty} a_n q^n),$$

which does not preserve the modularity. For example $\theta \Delta = E_2 \Delta$, where E_2 is a quasi-modular form (in the sense of Kaneko and Zagier, see [Ka-Za]).

Notice that $\delta_k f = (\theta - kR) f$, and that Serre's operator $f \mapsto \theta f - \frac{k}{12} E_2 f$ takes $\mathcal{M}_k$ to $\mathcal{M}_{k+2}$.

Note that that the arithmetical twist operator

$$\theta_\chi(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} \chi(n) a_n q^n$$

is a natural analog of the Ramanujan operator.

Triple arithmetical modular forms

Let $\mathcal{A}$ be a commutative ring. The tensor product over $\mathcal{A}$

$$\mathcal{M}_{\mathbf{k},r,T}(N,\psi,\mathcal{A}) := \mathcal{M}_{k_1,r}(N,\psi_1,\mathcal{A}) \otimes \mathcal{M}_{k_2,r}(N,\psi_2,\mathcal{A}) \otimes \mathcal{M}_{k_3,r}(N,\psi_3,\mathcal{A})$$

consists of triple arithmetical modular forms as certain formal series of the form

$$\begin{aligned} g = & \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3;R_1,R_2,R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3} \\ & \in \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3], \text{ where } z_j = x_j + iy_j \in \mathbb{H}, \ R_j = (4\pi y_j)^{-1}, \end{aligned}$$

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight (k_1,k_2,k_3) and character (ψ_1,ψ_2,ψ_3), $j = 1,2,3$. The coefficients $a(n_1,n_2,n_3;R_1,R_2,R_3)$ are polynomials in $\mathcal{A}[R_1,R_2,R_3]$. Examples of such modular forms come from the restriction to the diagonal of Siegel modular forms of degree 3.

5 Siegel-Eisenstein series

Siegel modular groups

Let $J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}$. The symplectic group

$$\mathrm{Sp}_m(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2m}(\mathbb{R}) | {}^t g \cdot J_{2m} g = J_{2m} \right\},$$

acts on the Siegel upper half plane

$$\mathbb{H}_m = \left\{ z = {}^t z \in M_m(\mathbb{C}) | \operatorname{Im} z > 0 \right\}$$

by $g(z) = (az+b)(cz+d)^{-1}$, where we use the bloc notation $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2m}(\mathbb{R})$. We use the congruence subgroup $\Gamma_0^m(N) = \{\gamma \in \mathrm{Sp}_m(\mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod N\} \subset \mathrm{Sp}_m(\mathbb{Z})$.

A Siegel modular form

$f \in \mathcal{M}_k(\Gamma_0^m(N),\chi)$ of degree $m > 1$, weight k and a Dirichlet chracter $\chi \bmod N$ is a holomorphic function $f\colon \mathbb{H}_m \to \mathbb{C}$ such that for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^m(N)$ one has

$$f(\gamma(z)) = \chi(\det\, d) \det(cz+d)^k f(z).$$

The Fourier expansion of f uses the symbol $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T}z))$
$= \prod_{i=1}^m q_{ii}^{\mathcal{T}_{ii}} \prod_{i<j} q_{ij}^{2\mathcal{T}_{ij}} \subset \mathbb{C}[\![q_{11}, \dots, q_{mm}]\!][q_{ij},\ q_{ij}^{-1}]_{i,j=1,\cdots,m},$,
$q_{ij} = \exp(2\pi(\sqrt{-1}z_{i,j}))$, and $\mathcal{T}$ in the semi-group $B_m = \{\mathcal{T} = {}^t\mathcal{T} \geq 0 | \mathcal{T} \text{ half-integral}\}$:
$f(z) = \sum_{\mathcal{T}\in B_m} a(\mathcal{T})q^{\mathcal{T}} \in \mathbb{C}[\![q^{B_m}]\!]$(a formal q-expansion $\in \mathbb{C}[\![q^{B_m}]\!]$),

Siegel-Eisenstein series

Example 5.1 ([Nag2], p.408)

$$\begin{aligned}
E_4^{(2)}(z) =& 1 + 240q_{11} + 240q_{22} + 2160q_{11}^2 + (240q_{12}^{-2} + 13440q_{12}^{-1} \\
&+ 30240 + 13440q_{12} + 240q_{12}^2)q_{11}q_{22} + 2160q_{22}^2 + \dots \\
E_6^{(2)}(z) =& 1 - 504q_{11} - 504q_{22} - 16632q_{11}^2 + (-540q_{12}^{-2} + 44352q_{12}^{-1} \\
&+ 166320 + 44352q_{12} - 504q_{12}^2)q_{11}q_{22} - 16632q_{22}^2 + \dots.
\end{aligned}$$

Arithmetical nearly holomorphic Siegel modular forms

Arithmetical Siegel modular forms

Consider a commutative ring $\mathcal{A}$, the formal variables $q = (q_{i,j})_{i,j=1,\dots,m}$, $R = (R_{i,j})_{i,j=1,\dots,m}$, and the ring of *formal Fourier series*

$$\mathcal{A}[\![q^{B_m}]\!][R_{i,j}] = \left\{ f = \sum_{\mathcal{T}\in B_m} a(\mathcal{T}, R)q^{\mathcal{T}} \;\middle|\; a(\mathcal{T}, R) \in \mathcal{A}[R_{i,j}] \right\} \tag{5.6}$$

(over the complex numbers this notation corresponds to $q^{\mathcal{T}} = \exp(2\pi i \mathrm{tr}(\mathcal{T}z))$, $R = (4\pi \mathrm{Im}(z))^{-1}$).

The formal Fourier expansion of a nearly holomorphic Siegel modular form f with coefficients in $\mathcal{A}$ is a certain element of $\mathcal{A}[\![q^{B_m}]\!][R_{i,j}]$. We call f arithmetical in the sense of Shimura [ShiAr], if $\mathcal{A} = \overline{\mathbb{Q}}$.

5.1 Algebraic differential operators of Maass and Shimura

Maass differential operator

Let us consider the Maass differential operator (see [Maa]) Δ_m of degree m, acting on complex $\mathcal{C}^\infty$-functions on $\mathbb{H}_m$ by:

$$\Delta_m = \det(\tilde{\partial}_{ij}), \qquad \tilde{\partial}_{ij} = 2^{-1}(1 + \delta_{ij})\partial/\partial_{ij}, \tag{5.7}$$

its algebraic version is the Ramanujan operator of degree m:

$$\Theta_m := \det(\frac{1}{2\pi i}\tilde{\partial}_{ij}) = \det(\theta_{ij}) = \frac{1}{(2\pi i)^m}\Delta_m, \tag{5.8}$$

where $\Theta_m(q^{\mathcal{T}}) = \det(\mathcal{T})q^{\mathcal{T}}$.

Shimura differential operator

The Shimura differential operator (see [Shi76, ShiAr]):

$$\delta_k f(z) = \det(R)^{k+1-\varkappa}\Theta_m\left[\det(R)^{\varkappa-1-k} f\right], \text{ where } R = (4\pi y)^{-1},$$

acts on arithmetic nearly holomorphic Siegel modular forms, and the composition is defined

$$\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \widetilde{\mathcal{M}}_k^m(N,\psi;\overline{\mathbb{Q}}) \to \widetilde{\mathcal{M}}_{k+2rm}^m(N,\psi;\overline{\mathbb{Q}}), \tag{5.9}$$

where

$$\delta_k f(z) = \left(\frac{-1}{4\pi}\right)^m \det(y)^{-1}\det(z-\bar{z})^{\varkappa-k}\Delta_m\left[\det(z-\bar{z})^{k-\varkappa+1} f(z)\right].$$

Universal polynomials $Q(R,\mathcal{T};k,r)$

Let $f = \sum_{\mathcal{T}\in B_m} c(\mathcal{T})q^{\mathcal{T}} \in \mathcal{M}_k^m(N,\psi)$ be a formal holomorphic Fourier expansion. One shows that $\delta_k^{(r)} f$ is given by

$$\delta_k^{(r)} f = \sum_{\mathcal{T}\in B_m} Q(R,\mathcal{T};k,r)c(\mathcal{T})q^{\mathcal{T}}.$$

Universal polynomials (continued)

Here we use a universal polynomial (5.10) which can be defined for all $k \in \mathbb{C}$, and it expresses the action of the Shimura operator on the exponential (of degree m):

$$\delta_k^{(r)}(q^{\mathcal{T}}) = Q(R,\mathcal{T};k,r)q^{\mathcal{T}}.$$

If $m=1$, r arbitrary (see [Shi76]), $\delta_k^{(r)} = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\frac{\Gamma(k+r)}{\Gamma(k+j)}R^{r-j}\theta^j$, $Q(R,n;k,r) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\frac{\Gamma(k+r)}{\Gamma(k+j)}R^{r-j}n^j$.

Universal polynomials (continued)

If $r=1$, m arbitrary, one has (see [Maa]):

$$\delta_k f(z) = \sum_{\mathcal{T}\in B_m} c(\mathcal{T})\sum_{l=0}^{m}(-1)^{m-l}c_{m-l}(k+1-\varkappa)\mathrm{tr}\left({}^t\rho_{m-l}(R)\cdot\rho_l^\star(\mathcal{T})\right)q^{\mathcal{T}}$$

where $R = (4\pi y)^{-1} = (R_{i,j}) \in \mathrm{M}_m(\mathbb{R})$, $c_m(\alpha) = \frac{\Gamma_m(\alpha+\kappa)}{\Gamma_m(\alpha+\kappa-1)}$, $\Gamma_m(s) = \pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma$ $(s-(j/2))$.

Here we use the natural representation $\rho_r : \mathrm{GL}_m(\mathbb{C}) \longrightarrow \mathrm{GL}(\wedge^r\mathbb{C}^m)$ $(0 \le r \le m)$ of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r\mathbb{C}^m$. Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r} \times \binom{m}{r}$ composed of the subdeterminants of z of degree r. Put $\rho^\star_r(z) = \det(z)\rho_{m-r}({}^tz)^{-1}$.

Then the representations ρ_r and $\rho^\star_r$ turn out to be polynomial representations.

In general (see [CourPa], Theorem 3.14) one has:

$$\begin{aligned} Q(R,\mathfrak{T}) &= Q(R,\mathfrak{T};k,r) \\ &= \sum_{t=0}^{r}\binom{r}{t}\det(\mathfrak{T})^{r-t}\sum_{|L|\le mt-t} R_L(\kappa-k-r)Q_L(R,\mathfrak{T}), \\ Q_L(R,\mathfrak{T}) &= \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\mathfrak{T})\right)\cdot\ldots\cdot\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\mathfrak{T})\right)). \end{aligned} \tag{5.10}$$

In (5.10), L goes over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$, and $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ in (5.10) are polynomials in β of degree $(mt - |L|)$ (used with $\beta = \kappa - k - r$).

Note the differentiation rule of degree m (see [Sh83], p.466):

$$\Delta(fg) = \sum_{r=0}^{m}\mathrm{tr}\left({}^t\rho_r(\tilde\partial/\partial z)f\cdot\rho^\star_{m-r}(\tilde\partial/\partial z)g\right).$$

Example 5.2 (Siegel-Eisenstein series of odd degree and higher level)

$$\begin{aligned} &G^*(z,s;k,\boldsymbol{\psi},N) \\ &= \det(y)^s\sum_{c,d}\boldsymbol{\psi}(\det c)\det(cz+d)^{-k}|\det(cz+d)|^{-2s}\cdot \\ &\cdot\tilde\Gamma(k,s)L_N(k+2s,\boldsymbol{\psi})\left(\prod_{i=1}^{[m/2]}L_N(2k+4s-2i,\boldsymbol{\psi}^2)\right),\ \textit{where} \end{aligned} \tag{5.11}$$

(c,d) runs over all "non-associated coprime symmetric pairs" with $\det(c)$ coprime to N, $\kappa = (m+1)/2$, and for m odd the Γ-factor has the form:
$\tilde\Gamma(k,s) = i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma_m(k+s)$.

We use this series with $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$, $k = k_2 + k_3 - k_1 \ge 2$, $m = 3$, $\kappa = \dfrac{m+1}{2} = 2$, $[m/2] = 1$.

Theorem 5.3 (Siegel, Shimura [Sh83], P. Feit [Fei86]) *Let m be an* odd integer *such that $2k > m$, and $N > 1$ be an integer, then:*

For an integer s such that $s = -r$, $0 \le r \le k - \kappa$, there is the following Fourier expansion

$$G^\star(z,-r) = G^\star(z,-r;k,\boldsymbol{\psi},N) = \sum_{A_m\ni\mathfrak{T}\ge 0} a(\mathfrak{T},R)q^{\mathfrak{T}}, \tag{5.12}$$

where for $s > (m+2-2k)/4$ in (5.12) the only non-zero terms occur for positive definite $\mathfrak{T} > 0$,

Fourier coefficients of Siegel-Eisenstein series (continued)

$$a(\mathcal{T}, R) = M(\mathcal{T}, \boldsymbol{\psi}, k-2r) \cdot \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k-2r, r), \tag{5.13}$$
$$M(\mathcal{T}, k-2r, \boldsymbol{\psi}) = \prod_{\ell | \det(2\mathcal{T})} M_\ell(\mathcal{T}, \boldsymbol{\psi}(\ell)\ell^{-k+2r}) \tag{5.14}$$

polynomials $Q(R, \mathcal{T}; k-2r, r)$ are given by (5.10), and for all $\mathcal{T} > 0$, $\mathcal{T} \in A_m$, is a finite Euler product, in which $M_\ell(\mathcal{T}, x) \in \mathbb{Z}[x]$. □

6 Statement of the Main Result

Main Theorem (on p-adic analytic function in four variables)

1) *The function $\mathcal{L}_{\boldsymbol{f}} : (s, k_1, k_2, k_3) \mapsto \dfrac{\langle \boldsymbol{f}^0, \mathcal{E}(-r,\chi)\rangle}{\langle \boldsymbol{f}^0, \boldsymbol{f}_0\rangle}$ depends p-adic analytically on four variables $(\chi \cdot y_p^r, k_1, k_2, k_3) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$;*

2) *Comparison of complex and p-adic values: for all (k_1, k_2, k_3) in an affinoid neighborhood $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X^3$, satisfying $k_1 \le k_2 + k_3 - 2$: the values at $s = k_2 + k_3 - 2 - r$ coincide with the normalized critical special values*

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \tag{6.15}$$
$$(r = 0, \cdots, k_2 + k_3 - k_1 - 2),$$

for Dirichlet characters χ mod $Np^v, v \ge 1$, such that all three corresponding Dirichlet characters χ_j have Np-complete conductors:

Main Theorem (continued)

$$\chi_1 \bmod Np^v = \chi, \ \ \chi_2 \bmod Np^v = \psi_2\bar{\psi}_3\chi, \tag{6.16}$$
$$\chi_3 \bmod Np^v = \psi_1\bar{\psi}_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3.$$

The normalisation of L^* in (6.15) is the same as in Theorem C below.

3) *Dependence on $x \in X$: let $H = [2\mathrm{ord}_p(\lambda)] + 1$. For any fixed $(k_1, k_2, k_3) \in \mathcal{B}$ and $x = \chi \cdot y_p^r$ the function*

$$x \longmapsto \frac{\langle \boldsymbol{f}^0, \mathcal{E}(-r,\chi)\rangle}{\langle \boldsymbol{f}^0, \boldsymbol{f}_0\rangle}$$

extends to a p-adic analytic function of type $o(\log^H(\cdot))$ of the variable $x \in X$.

Remark. The function $\mathcal{L}_{\mathbf{f}}$ depends on the variables (s, k_1, k_2, k_3) in a different way: it is a mixture of the p-adic Mellin transform (in s), and of a rigid analytic function (in k_1, k_2, k_3).

Outline of the proof

1) • At each classical weight (k_1, k_2, k_3) let us use the equality

$$\left\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\right\rangle = \left\langle \mathbf{f}^0, \pi_\lambda(\mathcal{E}(-r,\chi))\right\rangle$$

which is deduced from the definition of the projector π_λ: $\mathrm{Ker}\,\pi_\lambda := \bigcap_{n\ge 1} \mathrm{Im}\,(U_T - \lambda I)^n$, $\mathrm{Im}\,\pi_\lambda := \bigcup_{n\ge 1} \mathrm{Ker}\,(U_T - \lambda I)^n$.

Notice that the coefficients of $\mathcal{E}(-r,\chi) \in \mathcal{M}(\mathcal{A})$ depend p-adic analytically on $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, where $\mathcal{A} = \mathcal{A}(\mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3)$ is the p-adic Banach algebra of rigid-analytic functions on $\mathcal{B}$.

Interpolation to all p-adic weights:

• At each classical weight (k_1, k_2, k_3) the scalar product $\left\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\right\rangle$ is given by the first coordinate of $\pi_\lambda(\mathcal{E}(-r,\chi))$ with respect to an orthogonal basis of $\mathcal{M}^\lambda(\mathcal{A})$ containing $\mathbf{f}_0$ with respect to Hida's algebraic Petersson product $\langle g, h\rangle_a := \left\langle g^\rho | \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, h \right\rangle$, see [Hi90].

Let us extend the linear form $\ell(h) = \frac{\langle \mathbf{f}^0, h\rangle}{\langle \mathbf{f}^0, \mathbf{f}_0\rangle}$ (defined for classical weights), to Coleman's type submodule of overconvergent families $h \in \mathcal{M}^\lambda(\mathcal{A})^\dagger \subset \mathcal{M}^\lambda(\mathcal{A})$ as the first coordinate of h with respect to some $\mathcal{A}$-basis of eigenfunctions of all (triple) Hecke operators T_q for $q \nmid Np$, having the first basis vector $\mathbf{f}_0 \in \mathcal{M}^\lambda(\mathcal{A})^\dagger$.

The linear form ℓ can be characterized as a normalized eigenfunction of the adjoint Atkin's operator, acting on the dual $\mathcal{A}$-module of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell(\mathbf{f}_0) = 1$.

In order to extend ℓ to $h = \mathcal{E}(-r,\chi)$, we need to choose a certain representative of $\mathcal{E}(-r,\chi)$ in the $\mathcal{A}$-submodule $\mathcal{M}^\lambda(\mathcal{A})^\dagger$, which is locally free of finite rank.

A representative of $\mathcal{E}(-r,\chi)$ in the (locally free of finite rank $\mathcal{A}$-submodule) $\mathcal{M}^\lambda(\mathcal{A})^\dagger$

Choose a (local) basis $\ell^1, \cdots, \ell^n$ given by some *triple Fourier coefficients* of the dual (locally free of finite rank) $\mathcal{A}$-module $\mathcal{M}^\lambda(\mathcal{A})^{\dagger *}$.

Then define

$$\ell = \beta_1 \ell^1 + \cdots + \beta_n \ell^n,$$

where $\beta_i = \ell(\ell_i) \in \mathcal{A}$, and ℓ_i denotes the dual basis of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell^j(\ell_i) = \delta_{ij}$. At each p-adic weight $(k_1, k_2, k_3) \in \mathcal{B}$ let us define

$$\ell(\mathcal{E}(-r,\chi)) := \beta_1 \ell^1(\mathcal{E}(-r,\chi)) + \cdots + \beta_n \ell^n(\mathcal{E}(-r,\chi)) \text{ (belongs to } \mathcal{A}),$$

where $\beta_i = \ell(\ell_i) \in \mathcal{A}$, and $\ell^i(\mathcal{E}(-r,\chi)) \in \mathcal{A}$ are certain Fourier coefficients of the seies $\mathcal{E}(-r,\chi)$.

Conclusion

There exists an element

$$\tilde{\mathcal{E}}(-r,\chi) \in \mathcal{M}^\lambda(\mathcal{A})^\dagger \subset \mathcal{M}(\mathcal{A})^\dagger$$

such that $\boxed{\ell(\mathcal{E}(-r,\chi)) = \ell(\tilde{\mathcal{E}}(-r,\chi))}$ (at each weight (k_1,k_2,k_3)). In fact, let us define

$$\tilde{\mathcal{E}}(-r,\chi) := \ell^1(\mathcal{E}(-r,\chi))\ell_1 + \cdots + \ell^n(\mathcal{E}(-r,\chi))\ell_n$$

$\Rightarrow \ell(\tilde{\mathcal{E}}(-r,\chi)) = \ell(\ell_1)\ell^1(\mathcal{E}(-r,\chi)) + \cdots + \ell(\ell_n)\ell^n(\mathcal{E}(-r,\chi))$
$= \beta_1\ell^1(\mathcal{E}(-r,\chi)) + \cdots + \beta_n\ell^n(\mathcal{E}(-r,\chi))$
$= \ell(\mathcal{E}(-r,\chi))$ (at each weight (k_1,k_2,k_3)).

Thus, the dependence of $\ell(\mathcal{E}(-r,\chi)) \in \mathcal{A}$ on $(k_1,k_2,k_3) \in X^3$ is p-adic analytic.

In order to prove the remaining statements 2), 3), the dependence on $x = \chi \cdot y_p^r$ is studied in the next section.

7 Distributions and admissible measures

Distributions and measures with values in Banach modules

Notation

$\mathcal{A}$	(a p-adic Banach algebra)
V	(an $\mathcal{A}$-module)
$\mathcal{C}(Y,\mathcal{A})$	(the $\mathcal{A}$-Banach algebra
$\cup$	of continuous functions on Y)
$\mathcal{C}^{loc-const}(Y,\mathcal{A})$	(the $\mathcal{A}$-algebra
	of locally constant functions on Y)

Definition 7.1 (Distributions and measures) *a) A* distribution $\mathcal{D}$ *on* Y *with values in* V *is an* $\mathcal{A}$*-linear form*

$$\mathcal{D} : \mathcal{C}^{loc-const}(Y,\mathcal{A}) \to V, \quad \varphi \mapsto \mathcal{D}(\varphi) = \int_Y \varphi d\mathcal{D}.$$

b) A measure μ *on* Y *with values in* V *is a continuous* $\mathcal{A}$*-linear form*

$$\mu : \mathcal{C}(Y,\mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi d\mu.$$

The integral $\int_Y \varphi d\mu$ can be defined for any continuous function φ, and any bounded distribution μ, using the Riemann sums.

Admissible measures of Amice-Vélu

Admissible measures

Let h be a positive integer. A more delicate notion of an h-admissible measure was introduced in [Am-V] by Y. Amice, J. Vélu (see also [MTT], [V]):

DEFINITION 7.2

a) For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{P}^h(Y, \mathcal{A})$ denote the $\mathcal{A}$-module of locally polynomial functions *of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow \mathcal{A}^\times$; in particular,*

$$\mathcal{P}^1(Y, \mathcal{A}) = \mathcal{C}^{loc-const}(Y, \mathcal{A})$$

(the $\mathcal{A}$-submodule of locally constant functions*). Let also denote $\mathcal{C}^{loc-an}(Y, \mathcal{A})$ the $\mathcal{A}$-module of* locally analytic functions*, so that*

$$\mathcal{P}^1(Y, \mathcal{A}) \subset \mathcal{P}^h(Y, \mathcal{A}) \subset \mathcal{C}^{loc-an}(Y, \mathcal{A}) \subset \mathcal{C}(Y, \mathcal{A}).$$

b) Let V be a normed $\mathcal{A}$-module with the norm $|\cdot|_{p,V}$. For a given positive integer h an h-admissible measure on Y with values in V is an $\mathcal{A}$-module homomorphism

$$\tilde{\Phi} : \mathcal{P}^h(Y, \mathcal{A}) \to V$$

such that for fixed $a \in Y$ and for $v \to \infty$ the following growth condition *is satisfied:*

$$\left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,V} = o(p^{-v(h'-h)}) \tag{7.17}$$
$$\text{for all} \quad h' = 0, 1, \ldots, h-1, a_p := y_p(a)$$

The condition (7.17) allows to integrate the locally-analytic functions on Y along $\tilde{\Phi}$ using Taylor's expansions! This means: there exists a unique extension of $\tilde{\Phi}$ to $\mathcal{C}^{loc-an}(Y, \mathcal{A}) \to V$.

7.1 U_p–Operator and the method of canonical projection

Using the canonical projection π_λ

We construct our H-admissible measure $\widetilde{\Phi}^\lambda : \mathcal{P}^H(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})$ out of a sequence of distributions $\Phi_r : \mathcal{P}^1(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{A})$ defined on local monomials y_p^r of each degree r by the rule

$$\int_Y \chi y_p^r \, d\widetilde{\Phi}^\lambda = \pi_\lambda(\tilde{\mathcal{E}}(-r, \chi)), \text{ where } \tilde{\mathcal{E}}(-r, \chi) \in M = \mathcal{M}(\mathcal{A}).$$

Here $\tilde{\mathcal{E}}(-r, \chi)$ takes values in an $\mathcal{A}$-module

$$M = \mathcal{M}(\mathcal{A}) \subset \mathcal{A}[\![q_1, q_2, q_3]\!][R_1, R_2, R_3]$$

of nearly holomorphic (overconvergent) triple modular forms over $\mathcal{A}$ (for $0 \leq r \leq H-1$, $H = [2\mathrm{ord}_p \lambda_p] + 1$), and the formal series $\tilde{\mathcal{E}}(-r, \chi)$ was constructed in the proof of 1) of Main Theorem.

Definition of the canonical projection π_λ

Here $\mathcal{A}$ is an $\mathbb{C}_p$-algebra, and $\lambda \in \mathcal{A}^\times$ is a fixed non-zero eigenvalue of triple Atkin's operator $U_T = U_{T,p}$, acting on $\mathcal{M}(\mathcal{A})$,

$$\pi_\lambda : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})^\lambda$$

is the canonical projection operator onto the maximal $\mathcal{A}$-submodule $\mathcal{M}(\mathcal{A})^\lambda$ over which the operator $U_T - \lambda I$ is nilpotent (we call $\mathcal{M}(\mathcal{A})^\lambda$ the λ-characteristic submodule of $\mathcal{M}(\mathcal{A})$).
The projector π_λ is defined by its kernel: $\operatorname{Ker}\pi_\lambda := \bigcap_{n\ge 1} \operatorname{Im}(U_T - \lambda I)^n$, $\operatorname{Im}\pi_\lambda := \bigcup_{n\ge 1} \operatorname{Ker}(U_T - \lambda I)^n$.

8 Triple modular forms

Triple modular forms are certain formal series

$$\begin{aligned} g = &\sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3;R_1,R_2,R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3} \\ &\in \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3], \text{ where } z_j = x_j + iy_j \in \mathbb{H},\ R_j = (4\pi y_j)^{-1}, \end{aligned}$$

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight (k_1,k_2,k_3) and character (ψ_1,ψ_2,ψ_3), $j = 1,2,3$.

The coefficients $a(n_1,n_2,n_3;R_1,R_2,R_3)$ are polynomials in $\mathcal{A}[R_1,R_2,R_3]$, and the triple Atkin's operator is given by
$U_T(g) = \sum_{n_1,n_2,n_3=0}^{\infty} a(pn_1,pn_2,pn_3;pR_1,pR_2,pR_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}$.

Eigenfunctions of U_p and of U_p^*.

Functions $f_{j,0}$ and f_j^0

Recall that for any primitive cusp eigenform $f_j = \sum_{n=1}^\infty a_n(f)q^n$, there is an eigenfunction $f_{j,0} = \sum_{n=1}^\infty a_n(f_{j,0})q^n \in \overline{\mathbb{Q}}[\![q]\!]$ of $U = U_p$ with the eigenvalue $\alpha = \alpha_{p,j}^{(1)} \in \overline{\mathbb{Q}}$ $(U(f_0) = \alpha f_0)$ given by

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j|V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \tag{8.18}$$

$$\sum_{n=1}^{\infty} a_n(f_{j,0})n^{-s} = \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} a_n(f_j)n^{-s}(1-\alpha_{p,j}^{(1)}p^{-s})^{-1}.$$

Moreover, there is an eigenfunction f_j^0 of U_p^* given by

$$f_j^0 = f_{j,0}^\rho\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, \text{ where } f_{j,0}^\rho = \sum_{n=1}^{\infty} \overline{a(n,f_0)} q^n. \tag{8.19}$$

Therefore, $U_T(f_{1,0}\otimes f_{2,0}\otimes f_{3,0}) = \lambda(f_{1,0}\otimes f_{2,0}\otimes f_{3,0})$.

9 Critical values of the L function $L(f_1 \otimes f_2 \otimes f_3, s, \chi)$

Choice of Dirichlet characters

For an arbitrary Dirichlet character $\chi \bmod Np^v$ consider the following Dirichlet characters:

$$\begin{aligned} &\chi_1 \bmod Np^v = \chi,\ \ \chi_2 \bmod Np^v = \psi_2\bar\psi_3\chi, \\ &\chi_3 \bmod Np^v = \psi_1\bar\psi_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3; \end{aligned} \tag{9.20}$$

later on we impose the condition that the conductors of the corresponding primitive characters $\chi_{0,1}, \chi_{0,2}, \chi_{0,3}$ are Np-completes (i.e. have the same prime divisors as resp. those of Np).

Theorem A (algebraic properties of the triple product)

Assume that $k_2+k_3-k_1 \ge 2$, then for all pairs (χ, r) such that the corresonding Dirichlet characters χ_j have Np-complete conductors, and $0 \le r \le k_2+k_3-k_1-2$, we have that

$$\frac{\Lambda(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, k_2+k_3-2-r, \psi_1\psi_2\chi)}{\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho\rangle_T} \in \overline{\mathbb{Q}}$$

where

$$\begin{aligned}\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho\rangle_T &:= \langle f_1^\rho, f_1^\rho\rangle_N \langle f_2^\rho, f_2^\rho\rangle_N \langle f_3^\rho, f_3^\rho\rangle_N \\ &= \langle f_1, f_1\rangle_N \langle f_2, f_2\rangle_N \langle f_3, f_3\rangle_N.\end{aligned}$$

10 Theorems B-D

Recall: the p-adic weight space and the Mellin transform

The p-adic weight space is the group $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^\times)$ of (continuouos) p-adic characters of the commutative profinite group $Y = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})^*$

The group X is isomorphic to a finite union of discs $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$.

A p-adic L-function $L_{(p)} : X \to \mathbb{C}_p$ is a certain meromorphic function on X. Such a function usually come from a p-adic measure μ on Y (*bounded* or *admissible* in the sense of Amice-Vélu, see [Am-V]). The p-adic Mellin transform of μ is given for all $x \in X$ by

$$L_{(p)}(x) = \int_{Y_{N,p}} x(y)\mathrm{d}\mu(y), L_{(p)} : X \to \mathbb{C}_p$$

Theorem B (on admissible measures attached to the triple product:fixed balanced weights case)

Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function

$\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p$, given for all $x \in X_{N,p}$ by the integral $\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes$

$f_3) = \int_{Y_{N,p}} x(y)\mathrm{d}\tilde{\mu}^{\lambda}_{f_1\otimes f_2\otimes f_3}(y)$, *and having the following properties:*

(i) *for all pairs* (r,χ) *such that* $\chi \in X^{\mathrm{tors}}_{N,p}$, *and all three corresonding Dirichlet characters* χ_j *have* Np*-complete conductor* $(j = 1,2,3)$, *and* $r \in \mathbb{Z}$ *is an integer with* $0 \le r \le k_2 + k_3 - k_1 - 2$, *the following equality holds:*

$$\mathcal{D}_{(p)}(\chi x_p^r, f_1 \otimes f_2 \otimes f_3) = i_p\Big(\frac{(\psi_1\psi_2)(2)C_\chi^{4(k_2+k_3-2-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1\psi_2\chi_1)\lambda_p^{2v}}$$

$$\frac{\Lambda(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, k_2+k_3-2-r, \psi_1\psi_2\chi)}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0}\otimes f_{2,0}\otimes f_{3,0}\rangle_{T,Np}}\Big)$$

where $v = \mathrm{ord}_p(C_\chi)$, $G(\chi)$ *denotes the Gauß sum of a primitive Dirichlet character* χ_0 *attached to* χ *(modulo the conductor of* χ_0*),*

(ii) *if* $\mathrm{ord}_p\lambda_p = 0$ *then the holomorphic function in (i) is a* bounded $\mathbb{C}_p$-analytic function*;*

(iii) *in the general case (but assuming that* $\lambda_p \ne 0$*) the holomorphic function in (i) belongs to the* type $o(\log(x_p^H))$ with $H = [2\mathrm{ord}_p\lambda_p] + 1$ *and it can be represented as the Mellin transform of the* H*-admissible* $\mathbb{C}_p$*-valued measure* $\tilde{\mu}^{\lambda}_{f_1\otimes f_2\otimes f_3}$ *(in the sense of Amice-Vélu) on* Y

(iv) *Let* $k = k_2+k_3-k_1 \ge 2$. *If* $H \le k-2$ *then the function* $\mathcal{D}_{(p)}$ *is uniquely determined by the above conditions (i).*

Let us describe now p-adic measures attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k)q^n\}(j = 1,2,3). \tag{10.21}$$

Consider the product of three eigenvalues:

$$\lambda = \lambda_p(k_1,k_2,k_3) = \alpha^{(1)}_{p,1}(k_1)\alpha^{(1)}_{p,2}(k_2)\alpha^{(1)}_{p,3}(k_3)$$

and assume that the slope of this product

$$\sigma = \mathrm{ord}_p(\lambda(k_1,k_2,k_3)) = \sigma(k_1,k_2,k_3) = \sigma_1+\sigma_2+\sigma_3$$

is constant and positive for all triplets (k_1,k_2,k_3) in an appropriate p-adic neighbourhood of the fixed triplet of weights (k_1,k_2,k_3).

Let $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denote an affinoid algebra $\mathcal{A}$ associated with an analytic space $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, a neighbourhood around $(k_1,k_2,k_3) \in X^3$ (with a given k and ψ mod N).

Theorem C (on p-adic measures for families of triple products)

Put $H = [2\mathrm{ord}_p(\lambda)] + 1$. *There exists a sequence of distributions* Φ_r *on* Y *with values in* $\mathcal{M} = \mathcal{M}(\mathcal{A})$ *giving an* H*-admissible measure* $\tilde{\Phi}^\lambda$ *with values in* $\mathcal{M}^\lambda \subset \mathcal{M}$ *with the following properties:*

There exists an $\mathcal{A}$-linear form $\ell = \ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}$ (given by (11.24), such that the composition

$$\tilde\mu = \tilde\mu_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda} := \ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda}(\tilde\Phi^\lambda)$$

is an H-admissible measure with values in $\mathcal{A}$, and for all (k_1,k_2,k_3) in the affinoid neighborhood $\mathcal{B} = \mathcal{B}_1\times\mathcal{B}_2\times\mathcal{B}_3$, as above, satisfying $k_1 \le k_2+k_3-2$

we have that the evaluated integrals

$$ev_{(k_1,k_2,k_3)}\left((\ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda})(\tilde\Phi^\lambda)(y_p^r\chi)\right)$$

on the arithmetical chracters $y_p^r\chi$ coincide with the critical special values

$$\Lambda^*(f_{1,k_1}\otimes f_{2,k_2}\otimes f_{3,k_3}, k_2+k_3-2-r,\chi)$$

for $r = 0,\cdots,k_2+k_3-k_1-2$, and for all Dirichlet characters $\chi \bmod Np^v, v\ge 1$, with all three corresonding Dirichlet characters χ_j given by (6.16), having Np-complete conductors. Here the normalisation of Λ^* includes at the same time certain Gauss sums, Petersson scalar products, powers of π and of $\lambda(k_1,k_2,k_3)$, and a certain finite Euler product.

The p-adic Mellin transform and four variable p-adic analytic functions

Any h-admissible measure $\tilde\mu$ on Y with values in a p-adic Banach algebra $\mathcal{A}$ can be caracterized its *Mellin transform* $\mathcal{L}_{\tilde\mu}(x)$ $\mathcal{L}_{\tilde\mu} : X \to \mathcal{A}$, defined by $\mathcal{L}_{\tilde\mu}(x) = \int_Y x(y)d\tilde\mu(y)$, where $x\in X$, $\mathcal{L}_{\tilde\mu}(x)\in\mathcal{A}$,

Key property of h-admissible measures $\tilde\mu$: its Mellin transform $\mathcal{L}_{\tilde\mu}$ is analytic with values in $\mathcal{A}$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{B}) = \mathcal{A}_1\hat\otimes\mathcal{A}_2\hat\otimes\mathcal{A}_3 = \mathcal{A}(\mathcal{B}_1)\hat\otimes\mathcal{A}(\mathcal{B}_2)\hat\otimes\mathcal{A}(\mathcal{B}_3)$ denote again the Banach algebra $\mathcal{A}$ where $\mathcal{B}$ is an affinoid neighbourhood around $(k_1,k_2,k_3)\in X^3$ (with a given integer k and Dirichlet character $\psi \bmod N$).

Theorem D (on p-adic analytic function in four variables)

Put $H = [2\mathrm{ord}_p(\lambda)]+1$. There exists a p-adic analytic function in four variables $(x,\boldsymbol{s})\in X\times\mathcal{B}_1\times\mathcal{B}_2\times\mathcal{B}_3\subset X\times X\times X\times X$:

$$\mathcal{L}_{\tilde\mu} : (x,\boldsymbol{s}) \longmapsto ev_{\boldsymbol{s}}(\mathcal{L}_{\tilde\mu(x)}) \quad (x\in X,\quad \mathcal{L}_{\tilde\mu}(x)\in\mathcal{A}).$$

with values in $\mathbb{C}_p$, such that for all (k_1,k_2,k_3) in the affinoid neighborhood as above $\mathcal{B} = \mathcal{B}_1\times\mathcal{B}_2\times\mathcal{B}_3$, satisfying $k_1\le k_2+k_3-2$, we have that the special values $\mathcal{L}_{\tilde\mu}(x,\boldsymbol{s})$ at the arithmetical chracters $x = y_p^r\chi$, and $\boldsymbol{s} = (k_1,k_2,k_3)\in\mathcal{B}$ coincide with the normalized critical special values

$$L^*(f_{1,k_1}\otimes f_{2,k_2}\otimes f_{3,k_3}, k_2+k_3-2-r,\chi) \quad (r = 0,\cdots,k_2+k_3-k_1-2),$$

for Dirichlet characters $\chi \bmod Np^v, v\ge 1$, such that all three corresonding Dirichlet characters χ_j given by (6.16), have Np-complete conductors where the same normalisation of L^* as in Theorem C.

Moreover, for any fixed $\boldsymbol{s} = (k_1, k_2, k_3) \in \mathcal{B}$ *the function*

$$x \longmapsto \mathcal{L}_{\tilde{\mu}}(x, \boldsymbol{s})$$

is p-adic analytic of type $o(\log^H(\cdot))$.

Indeed, we obtain the function in question $\mathcal{L}_\mu(x, \mathrm{s})$ by evaluation at

$$\mathrm{s} = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in \mathcal{B} :$$

this is a p-adic analytic function in four variables $(x, \mathrm{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde{\mu}}(x, \mathrm{s}) := ev_{\mathrm{s}}(\mathcal{L}_{\tilde{\mu}})(x) \quad (x \in X,\ \mathrm{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3,\ \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).$$

This is a joint work in progress with S.Boecherer, we use:
1) the higher twists of the Siegel-Eisenstein series, introduced in [Boe-Schm],
2) Ibukiyama's differential operators (see [Ibu], [BSY]).

11 Ideas of the Proof

11.1 Boecherer's higher twist

We define the higher twist of the series $F_{\chi,r} = \sum_{\mathcal{T}} a_{\chi,r}(R, \mathcal{T}) q^{\mathcal{T}}$ by some Dirichlet characters $\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3$ as the following formal nearly holomorphic Fourier expansion:

$$F_{\chi,r} = \sum_{\mathcal{T}} \bar{\chi}_1(t_{12}) \bar{\chi}_2(t_{13}) \bar{\chi}_3(t_{23}) a_{\chi,r}(R, \mathcal{T}) q^{\mathcal{T}}. \tag{11.22}$$

The seies (11.22) is a Siegel modular form of some higher level, but it has additional symmetries with respect to symplectic embedding $\iota_3 : \Gamma_0(Np^{2v}) \times \Gamma_0(Np^{2v}) \times \Gamma_0(Np^{2v}) \to \mathrm{Sp}_3$: its triple Nebentypus character does not depend on $\chi \bmod Np^v$, and is equal to (ψ_1, ψ_2, ψ_3), if we choose Dirichlet characters as in (6.16):

$$\chi_1 \bmod Np^v = \chi,\ \chi_2 \bmod Np^v = \psi_2 \bar{\psi}_3 \chi,$$
$$\chi_3 \bmod Np^v = \psi_1 \bar{\psi}_3 \chi, \boldsymbol{\psi} = \chi^2 \psi_1 \psi_2 \overline{\psi}_3.$$

We use the Siegel-Eisenstein series $F_{\chi,r}$ which depends on the character χ, but its precise nebentypus character is $\boldsymbol{\psi} = \chi^2 \psi_1 \psi_2 \overline{\psi}_3$, and it is defined by

$$F_{\chi,r} = G^\star(z, -r; k, (Np^v)^2, \boldsymbol{\psi}),$$

where z denotes a variable in the Siegel upper half space $\mathbb{H}_3$, and the normalized series $G^\star(z, s; k, (Np^v)^2, \boldsymbol{\psi})$ is given by (5.11).

This series depends on $s = -r$, and for the critical values at integral points $s \in \mathbb{Z}$ such that $2 - k \leq s \leq 0$, it represents a (*nearly*) *holomorphic* Siegel modular form in the sense of Shimura [ShiAr]:

$$F_{\chi,r} = \sum_{\mathcal{T}} \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k - 2r, r) a_{\chi,r}(\mathcal{T}) q^{\mathcal{T}}.$$

11.2 Ibukiyama's differential operator

We use an algebraic version of Ibukiyama's differential operator, which generalizes the algebraic "pull-back": it transforms a nearly holomorphic Siegel modular form of weight k to a nearly holomorphic triple modular form of weight (k_1, k_2, k_3) $(k = k_2 + k_3 - k_1)$.

On a holomorphic Siegel modular form $F = \sum_{\mathcal{T}} a(\mathcal{T}) q^{\mathcal{T}}$, this operator has the form

$$\mathcal{L}_k^{\lambda,\nu}(F) = \sum_{\mathcal{T}} \mathcal{P}(k_1, k_2, k_3, 0, \mathcal{T}) a(\mathcal{T}) q_1^{t_{11}} q_2^{t_{22}} q_3^{t_{33}}, \qquad (11.23)$$

where $\lambda = k_1 - k_3 \geq \mu = k_1 - k_2 \geq 0$, and $\mathcal{P}(k_1, k_2, k_3; r; \mathcal{T})$ is certain Ibukiyama's polynomial, equal to $(t_{11}t_{22}t_{33})^{\lambda}(t_{12}t_{13}t_{23})^{\mu}$, if $r = 0$.

As a result we obtain a sequence of triple modular distributions $\Phi_r(\chi)$ with values in the tensor product $\mathcal{M}_T(\mathcal{A}) = \mathcal{M}(\mathcal{A}) \widehat{\otimes}_{\mathcal{A}} \mathcal{M}(\mathcal{A}) \widehat{\otimes}_{\mathcal{A}} \mathcal{M}(\mathcal{A})$ of three Banach $\mathcal{A}$-modules of arithmetical nearly holomorphic modular forms (the normalizing factor 2^r is neeeded in order to prove certain congruences between Φ_r). Note that $\mathcal{M}_T(\mathcal{A})$ is again a Banach $\mathcal{A}$-module on which U_T acts as a completely continuous operator.

The important property of the triple modular forms $\Phi_r(\chi)$: the nebentypus character is fixed and is equal to (ψ_1, ψ_2, ψ_3) (for all (k_1, k_2, k_3) and χ in question).

Using this property we compute the canonical projection $\pi_\lambda(\Phi_r(\chi))$ of the triple modular form $\Phi_r(\chi)$ onto the λ-characteristic $\mathcal{A}$-submodule ${\mathcal{M}_T}^{\lambda}(\mathcal{A})$ of the triple Atkin's operator $U_{T,p}$:

$$\pi_\lambda : \mathcal{M}_T(\mathcal{A}) \to \mathcal{M}_T^{\lambda}(\mathcal{A}).$$

We prove that the resulting sequence of modular distributions $\pi_\lambda(\Phi_r)$ on the profinite group Y produces a certain p-adic admissible measure $\tilde{\Phi}^{\lambda}$ (in the sense of Amice-Vélu, [Am-V]) with values in a certain locally free $\mathcal{A}$-submodule of finite rank

$$\mathcal{M}_T^{\lambda}(\mathcal{A}) \subset \mathcal{M}_T(\mathcal{A}) \subset \bigcup_{v \geq 0} \mathcal{M}_T(Np^v, \psi_1, \psi_2, \psi_3; \mathcal{A})$$

of formal nearly holomorphic triple modular forms of all levels Np^v and the fixed nebentypus characters (ψ_1, ψ_2, ψ_3). We use congruences between triple modular forms $\Phi_r(\chi) \in \mathcal{M}_T(\mathcal{A})$ (they have explicit formal Fourier coefficients).

Then we use a general admissibility criterion saying that these congruences imply H-admissibility for their projections in $\mathcal{M}_T^{\lambda}(\mathcal{A})$, where $H = [2\mathrm{ord}_p(\lambda)] + 1$.

11.3 Algebraic linear form

3) From $\mathcal{M}_T^{\lambda}(\mathcal{A})$ to $\mathcal{A}$: we use a $\overline{\mathbb{Q}}$-valued linear forms of type

$$\mathcal{L} \ : h \longmapsto \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, h \rangle}{\langle f_1^0, f_{1,0} \rangle \langle f_2^0, f_{2,0} \rangle \langle f_3^0, f_{3,0} \rangle}$$

where f_j^0 is the eigenfunction (8.18) of the conjugate Atkin's operator U_p^*, and $f_{j,0}$ is the eigenfunction (8.19) of U_p. The linear form $\mathcal{L}$ is defined on the finite dimensional $\overline{\mathbb{Q}}$-vector characteristic subspace

$$h \in \mathcal{M}_{\mathbf{k}}(\overline{\mathbb{Q}})^{\lambda(\mathbf{k})} \subset$$
$$\mathcal{M}_{k_1,r^*}(Np, \psi_1; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_2,r^*}(Np, \psi_2; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_3,r^*}(Np, \psi_3; \overline{\mathbb{Q}}).$$

This map is then extended to an $\mathcal{A}$-linear map

$$\ell = \ell_{\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,\lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}; \tag{11.24}$$

on the locally free $\mathcal{A}$-module of finite rank $\mathcal{M}(\mathcal{A})^\lambda$.

This map produces a sequence of $\mathcal{A}$-valued distributions $\mu_r^\lambda(\chi) \in \mathcal{A}$ in such a way that for all suitable weights $\mathbf{k} \in \mathcal{B}$ one has

$$ev_{\mathbf{k}}(\mu_r^\lambda(\chi)) = \mathcal{L}(ev_{\mathbf{k}}(\pi_\lambda(\Phi_r)(\chi))), \lambda \in \mathcal{A}^\times, \lambda(\mathbf{k}) \in \overline{\mathbb{Q}}^\times,$$

where $\mathbf{k} = (k_1, k_2, k_3) \in \mathcal{B}$, $ev_{\mathbf{k}} : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_{\mathbf{k}} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}_{\mathbf{k}}(\mathbb{C}_p).$$

More precisely, we consider three auxilliary families of modular forms

$$\tilde{f}_{j,k_j}(z) = \sum_{n=1}^{\infty} \tilde{a}_{n,j,k_j} e(nz) \in S_{k_j}(\Gamma_0(N_j p^{\nu_j}), \psi_j),\ (1 \le j \le 3, \nu_j \ge 1), \tag{11.25}$$

with the same eigenvalues as those of (10.21), for all Hecke operators T_q, with q prime to Np. In our construction we use as $\tilde{f}_{j,k_j}$ certain "easy transforms" of primitive cusp forms in (1.1).

In particular, we choose as $\tilde{f}_j$ certain eigenfunctions $\tilde{f}_{j,k_j} = f^0_{j,k_j}$ of the adjoint Atkin's operator U_p^*, in this case we denote by $f_{j,k_j,0}$ the corresponding eigenfunctions of U_p.

The $\overline{\mathbb{Q}}$-linear form $\mathcal{L}$ produces a $\mathbb{C}_p$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ starting from the modular p-adic admissible measure $\tilde{\Phi}^\lambda$ of stage 3), where $\ell : \mathcal{M}_T(\mathbb{C}_p) \to \mathbb{C}_p$ denotes a $\mathbb{C}_p$-linear form, interpolating $\mathcal{L}$.

11.4 Evaluation of p-adic integrals

L-values and p-adic integrals

4) We show that for any appropriate Dirichlet character χ mod Np^v the integral

$$\mu_r^\lambda(\chi) = \mathcal{L}(\pi_\lambda(\Phi_r(\chi))) \in \mathcal{A}$$

evaluated at $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, coincides (up to a normalisation) with the special L-value

$$L^*(f^\rho_{1,k_1} \otimes f^\rho_{2,k_2} \otimes f^\rho_{3,k_3}, k_2 + k_3 - 2 - r, \psi_1\psi_2\chi)$$

under the above assumptions on χ and r).

A general integral representation of Garrett's type

The basic idea how a Dirichlet character χ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out.

Note however that the existence of a $\mathcal{A}$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ established at stage 4), does not depend on this technical computation.

In order to control the denominators of the modular forms

$$\pi_\lambda(\tilde{\mathcal{E}}(-r,\chi)) \in \mathcal{M}^\lambda(\mathcal{A}) =: \Phi_r(\chi),$$

used in the construction (the admissibility condition) we use the following result.

12 Criterion of admissibility

Theorem 12.1 (Criterion of admissibility) *Let $\alpha \in \mathcal{A}^*$, $0 < |\alpha|_p < 1$ and suppose that there exists a positive integer $\varkappa$ such that the following conditions are satisfied:*

1) for all $r = 0, 1, \cdots, h-1$ with $h = [\varkappa \mathrm{ord}_p \alpha] + 1$, and $v \geq 1$,

$$\Phi_r(a + (Np^v)) \in \mathcal{M}(Np^{\varkappa v}) \qquad \text{(the level condition)} \tag{12.26}$$

2) the following congruence for the coefficients holds: for all $t = 0, 1, \cdots, h-1$

$$U^{\varkappa v} \sum_{r=0}^{t} \binom{t}{r} (-a_p)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \bmod p^{vt} \tag{12.27}$$

(the divisibility condition)

Then the linear form given by $\tilde{\Phi}^\alpha(\delta_{a+(Np^v)} y_p^r) := \pi_\alpha(\Phi_r(a + (Np^v))$ on local monomials (for all $r = 0, 1, \cdots, h-1$), is an h-admissible measure: $\tilde{\Phi}^\alpha : \mathcal{P}^h(Y, \overline{\mathbb{Q}}) \to \mathcal{M}^\alpha \subset \mathcal{M}$

Proof uses the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(Np^{v+1}, \psi; \mathcal{A}) & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}) \\ U^v \big\downarrow & & \big\downarrow \wr\, U^v \\ \mathcal{M}(Np, \psi; \mathcal{A}) & \underset{\pi_{\alpha,0}}{\longrightarrow} & \mathcal{M}^\alpha(Np, \psi; \mathcal{A}) \quad = \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}). \end{array}$$

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman's Theorem A.4.3 [CoPB].

On the right: U acts on the locally free $\mathcal{A}$-module $\mathcal{M}^\alpha(Np^{v+1}, \mathcal{A})$ via the matrix:

$$\begin{pmatrix} \alpha & \cdots & \cdots & * \\ 0 & \alpha & \cdots & * \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & \alpha \end{pmatrix} \text{ where } \alpha \in \mathcal{A}^\times$$

$\Longrightarrow U^v$ is an isomorphism over $\mathcal{A}$, and one controls the denominators of the modular forms of all levels v by the relation:

$$\pi_{\alpha,v}(h) = U^{-v}\pi_{\alpha,0}(U^v h) =: \pi_\alpha(h) \tag{12.28}$$

The equality (12.28) can be used as the definition of π_α at any level Np^v.

The growth condition (see (7.17)) for $\pi_\alpha(\Phi_r)$ is deduced from the congruences (12.27) between modular forms, using the relation (12.28).

References

[Am-V] Amice, Y. and Vélu, J., *Distributions p-adiques associées aux séries de Hecke*, Journées Arithmétiques de Bordeaux (Conf. Univ. Bordeaux, 1974), Astérisque no. 24/25, Soc. Math. France, Paris 1975, 119 - 131

[Boe1] Böcherer, S., *Über die Funktionalgleichung automorpher L–Funktionen zur Siegelscher Modulgruppe*, J. reine angew. Math. 362 (1985) 146–168

[Boe2] Böcherer S., *Über die Fourier–Jacobi Entwickelung Siegelscher Eisensteinreihen. I.II.*, Math. Z. 183 (1983) 21-46; 189 (1985) 81–100.

[BHam] Böcherer, S., *Ein Rationalitätssatz für formale Heckereihen zur Siegelschen Modulgruppe.* Abh. Math. Sem. Univ. Hamburg 56, 35–47 (1986)

[Boe-Ha] Boecherer, S., Heim, B. *L-functions for $GSp_2 \times Gl_2$ of mixed weights.* Math. Z. 235, 11-51(2000)

[Boe-Pa6] S. Böcherer and A.A. Panchishkin, *Admissible p-adic measures attached to triple products of elliptic cusp forms*, accepted in Documenta Math. in March 2006 (a special volume dedicated to John Coates).

[BSY] Böcherer, S., Satoh, T., and Yamazaki, T., *On the pullback of a differential operator and its application to vector valued Eisenstem series,* Comm. Math. Univ. S. Pauli, 41 (1992), 1-22.

[Boe-Schm] Böcherer, S., and Schmidt, C.-G., *p-adic measures attached to Siegel modular forms*, Ann. Inst. Fourier 50, N°5, 1375-1443 (2000).

[BoeSP] Böcherer, S. and Schulze-Pillot, R., *On the central critical value of the triple product L-function.* David, Sinnou (ed.), Number theory. Séminaire de Théorie des Nombres de Paris 1993–94. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 235, 1-46 (1996).

[Bue] BUECKER, K., *Congruences between Siegel modular forms on the level of group cohomology.* Ann. Inst. Fourier (Grenoble) 46 (1996), no. 4, 877–897.

[Chand70] CHANDRASEKHARAN, K. Arithmetical functions. Berlin–Heidelberg–New York, Springer–Verlag (1970).

[Co] COATES, J. *On p–adic L–functions.* Sem. Bourbaki, 40eme annee, 1987-88, n° 701, Asterisque (1989) 177–178.

[Co-PeRi] COATES, J. and PERRIN-RIOU, B., *On p-adic L-functions attached to motives over* $\mathbb{Q}$, Advanced Studies in Pure Math. 17, 23–54 (1989)

[CoPB] COLEMAN, ROBERT F., *p-adic Banach spaces and families of modular forms*, Invent. Math. 127, No.3, 417-479 (1997)

[CoM] COLEMAN, R. and MAZUR, B., *The eigencurve*, Scholl, A. J. (ed.) et al., Galois representations in arithmetic algebraic geometry. Proceedings of the symposium, Durham, UK, July 9–18, 1996. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 254, 1-113 (1998).

[CST98] COLEMAN, R., STEVENS, G., TEITELBAUM, J., *Numerical experiments on families of p-adic modular forms*, in: Computational perspectives in Number Theory, ed. by D.A. Buell, J.T. Teitelbaum, Amer. Math. Soc., 143-158 (1998).

[Colm98] COLMEZ, P. *Fonctions L p-adiques*, Séminaire Bourbaki, exposé n° 851, novembre 1998.

[Colm03] COLMEZ, P. La conjecture de Birch et Swinnerton-Dyer p-adique. Séminaire Bourbaki, exposé n° 919, juin 2003.

[Cour] COURTIEU, M., *Familles d'opérateurs sur les formes modulaires de Siegel et fonctions L p-adiques*, `http://www-fourier.ujf-grenoble.fr/THESE/ps/ t101.ps.gz`, Thèse de Doctorat, Institut Fourier, 2000, 1–122.

[CourPa] COURTIEU, M., PANCHISHKIN, A.A., *Non-Archimedean L-Functions and Arithmetical Siegel Modular Forms*, Lecture Notes in Mathematics 1471, Springer-Verlag, 2004 (2nd augmented ed.)

[De69] DELIGNE P., *Formes modulaires et représentations l-adiques*, Sem. Bourb. 1968/69, exp. no 335. Springer-Verlag, Lect. Notes in Math. N°179 (1971) 139 - 172

[De79] DELIGNE P., *Valeurs de fonctions L et périodes d'intégrales*, Proc. Symp. Pure Math AMS 33 (part 2) (1979) 313–342.

[Fei86] FEIT, P., *Poles and Residues of Eisenstein Series for Symplectic and Unitary Groups*, Memoir AMS 61 N°346 (1986), 89 p.

[Ga87] GARRETT, P.B., *Decomposition of Eisenstein series: Rankin triple products*, Ann. of Math. 125 (1987), 209–235

[GaHa] GARRETT, P.B. and HARRIS, M., *Special values of triple product L–functions*, Amer. J. Math 115 (1993) 159–238

[Go02] GORSSE, B., Carrés symétriques de formes modulaires et intégration p-adique. Mémoire de DEA de l'Institut Fourier, June 2002

[Go-Ro] GORSSE B., ROBERT G., Computing the Petersson product $\langle f^0, f_0\rangle$. Prépublication de l'Institut Fourier, N°654, (2004).

[HaH] HA HUY, KHOAI, *p-adic interpolation and the Mellin-Mazur transform*, Acta Math. Viet. 5, 77-99 (1980).

[Ka-Za] KANEKO, M., ZAGIER, DON, *A generalized Jacobi theta function and quasi-modular forms.* The moduli space of curves (Texel Island, 1994), 165–172, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[Ha93] HARRIS, M., and KUDLA, S., *The central critical value of a triple product L-functions*, Ann. Math. 133 (1991), 605–672

[Hasse] HASSE, H., *Vorlesungen über Zahlentheorie.* Zweite neubearbeitete Auflage. Die Grundlehren der Mathematischen Wissenschaften, Band 59 Springer-Verlag, Berlin-New York 1964 xv+504 pp.

[Hi85] HIDA, H., *A p-adic measure attached to the zeta functions associated with two elliptic cusp forms. I*, Invent. Math. 79 (1985) 159–195

[Hi86] HIDA, H., *Galois representations into* $GL_2(\mathbb{Z}_p[[X]])$ *attached to ordinary cusp forms*, Invent. Math. 85 (1986) 545–613

[Hi90] HIDA, H., *Le produit de Petersson et de Rankin p-adique*, Séminaire de Théorie des Nombres, Paris 1988–1989, 87–102, Progr. Math., 91, Birkhäuser Boston, Boston, MA, 1990.

[Hi93] HIDA, H., *Elementary theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts. 26 Cambridge: Cambridge University Press. ix, 386 p. (1993).

[Hi04] H. HIDA, *p-adic automorphic forms on Shimura varieties.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2004. xii+390 pp.

[JoH05] JORY–HUGUE, F., *Unicité des h–mesures admissibles p-adiques données par des valeurs de fonctions L sur les caractères*, Prépublication de l'Institut Fourier (Grenoble), N°676, 1-33, 2005

[Iw] IWASAWA, K., *Lectures on p–adic L–functions*, Ann. of Math. Studies, N°74. Princeton University Press, 1972

[Ibu] IBUKIYAMA, T. *On differential operators on automorphic forms and invariant pluriharmonic polynomials*, Comm. Math. Univ. S. Pauli 48 (1999), 103-118

[Ka76] KATZ, N.M., *p-adic interpolation of real analytic Eisenstein series*, Ann. of Math. 104 (1976) 459–571

[Ka78] KATZ, N.M., *p-adic L-functions for CM–fields*, Invent. Math. 48 (1978) 199–297

[KiK] KITAGAWA,KOJI, *On standard p-adic L-functions of families of elliptic cusp forms*, Mazur, Barry (ed.) et al.: p-adic monodromy and the Birch and Swinnerton-Dyer conjecture. A workshop held August 12-16, 1991 in Boston, MA, USA. Providence, R: American Mathematical Society. Contemp. Math. 165, 81-110, 1994

[Ku-Le] KUBOTA T. and LEOPOLDT H.-W. , *Eine p-adische Theorie der Zetawerte*, J. reine angew. Math. 214/215 (1964) 328–339

[La] LANG S., *Introduction to Modular Forms*, Springer Verlag, 1976

[Maa] MAASS, H., *Siegel's Modular Forms and Dirichlet Series*, Springer-Verlag, Lect. Notes in Math. N°216 (1971).

[Man73] MANIN, YU.I., *Periods of cusp forms and p-adic Hecke series*, Mat. Sbornik 92 (1973) 378–401(in Russian); Math. USSR, Sb. 21(1973), 371-393 (1975) (English translation).

[Man74] MANIN, YU.I., *The values of p–adic Hecke series at integer points of the critical strip*. Mat. Sbornik 93 (1974) 621 - 626 (in Russian)

[Man76] MANIN, YU.I., *Non–Archimedean integration and p-adic L-functions of Jacquet – Langlands*, Uspekhi Mat. Nauk 31 (1976) 5–54 (in Russian); Russ. Math. Surv. 31, No.1, 5-57 (1976) (English translation).

[Ma-Pa77] MANIN, YU.I. and PANCHISHKIN, A.A. , *Convolutions of Hecke series and their values at integral points*, Mat. Sbornik, 104 (1977) 617–651 (in Russian); Math. USSR, Sb. 33, 539-571 (1977) (English translation).

[Ma-Pa05] MANIN, YU.I. and PANCHISHKIN, A.A. , *Introduction to Modern Number Theory*, Encyclopaedia of Mathematical Sciences, vol. 49 (2nd ed.), Springer-Verlag, 2005, 514 p.

[MTT] MAZUR, B., TATE, J. and TEITELBAUM, J., *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. 84, 1-48 (1986).

[Miy] MIYAKE, TOSHITSUNE, *Modular forms*. Transl. from the Japanese by Yoshitaka Maeda. Berlin etc.: Springer-Verlag. viii, 335 p. (1989).

[Or] ORLOFF T., *Special values and mixed weight triple products (with an Appendix by D.Blasius)*, Invent. Math. 90 (1987) 169–180

[Nag1] NAGAOKA, S., *A remark on Serre's example of p-adic Eisenstein series*, Math. Z. 235 (2000) 227-250.

[Nag2] Nagaoka, S., *Note on* $\bmod p$ *Siegel modular forms*, Math. Z. 235 (2000) 405-420.

[Pa94] Panchishkin, A.A., *Admissible Non-Archimedean standard zeta functions of Siegel modular forms*, Proceedings of the Joint AMS Summer Conference on Motives, Seattle, July 20–August 2 1991, Seattle, Providence, R.I., 1994, vol.2, 251 – 292

[PaViet] Panchishkin, A. A. *Non-Archimedean Mellin transform and p-adic L-Functions*, Vietnam Journal of Mathematics, 1997, N3, 179–202.

[PaSE] Panchishkin, A.A., *On the Siegel–Eisenstein measure. Israel Journal of Mathematics*, Vol. 120, Part B(2000), 467–509

[PaIAS] Panchishkin, A.A., *On p-adic integration in spaces of modular forms and its applications*, J. Math. Sci., New York 115, No.3, 2357-2377 (2003).

[PaMMJ] Panchishkin, A.A., *A new method of constructing p-adic L-functions associated with modular forms*, Moscow Mathematical Journal, 2 (2002), Number 2, 1-16

[PaTV] Panchishkin, A.A., *Two variable p-adic L functions attached to eigenfamilies of positive slope*, Invent. Math. v. 154, N3 (2003), pp. 551 - 615

[PaB1] Panchishkin, A.A., *Arithmetical differential operators on nearly holomorphic Siegel modular forms*, Preprint MPI 2002 - 41, 1-52

[PaB2] Panchishkin, A.A., *Admissible measures for standard L-functions and nearly holomorphic Siegel modular forms* , Preprint MPI 2002 - 42, 1-65

[PaJTNB] Panchishkin A.A., *Sur une condition suffisante pour l'existence des mesures p-adiques admissibles*, Journal de Théorie des Nombres de Bordeaux, v. 15 (2003), pp. 1-24

[PaMMJ2] Panchishkin, A.A., *The Maass–Shimura differential operators and congruences between arithmetical Siegel modular forms*, (accepted in Moscow Mathematical Journal in October 2005 (39 p.).

[PSRa] Piatetski–Shapiro, I.I., and Rallis, S., *Rankin triple L–functions*, Compositio Math. 64 (1987) 333–399

[Puy] Puydt, J., *Valeurs spéciales de fonctions L de formes modulaires adéliques*, Thèse de Doctorat, Institut Fourier (Grenoble), 19 décembre 2003

[Ran39] Rankin, R.A., *Contribution to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions. I.II*, Proc. Camb. Phil. Soc 35 (1939) 351–372

[Ran52] Rankin, R.A., *The scalar product of modular forms*, Proc. London math. Soc. 2 (3) (1952) 198-217

[Ran90] RANKIN, R.A., *The adjoint Hecke operator.* Automorphic functions and their applications, Int. Conf., Khabarovsk/USSR 1988, 163-166 (1990)

[Sa] SATOH,T., Some remarks on triple L-functions. Math.Ann.276, 687-698(1987)

[Schm] SCHMIDT, C.–G., *The p-adic L-functions attached to Rankin convolutions of modular forms*, J. reine angew. Math. 368 (1986) 201–220

[Scho] SCHOLL, A., *Motives for modular forms*, Inv. Math. 100 (1990), 419–430

[SchEu] SCHOLL, A.J., *An introduction to Kato's Euler systems*, Scholl, A. J. (ed.) et al., Galois representations in arithmetic algebraic geometry. Proceedings of the symposium, Durham, UK, July 9-18, 1996. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 254, 379-460 (1998).

[Se73] SERRE, J.–P., *Formes modulaires et fonctions zêta p-adiques*, Lect Notes in Math. 350 (1973) 191–268 (Springer Verlag)

[SePB] SERRE, J.-P., *Endomorphisms completement continus des espaces de Banach p-adiques*, Publ. Math. Inst. Hautes Etud. Sci., 12, 69-85 (1962).

[Shi71] SHIMURA G., *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press, 1971

[Shi75] SHIMURA G., *On the holomorphy of certain Dirichlet series*, Proc. Lond. Math. Soc. 31 (1975) 79–98

[Shi76] SHIMURA G., *The special values of the zeta functions associated with cusp forms.* Comm. Pure Appl. Math. 29 (1976), no. 6, 783–804.

[Shi77] SHIMURA G., *On the periods of modular forms*, Math. Annalen 229 (1977) 211–221

[Shi82] SHIMURA, G., *Confluent Hypergeometric Functions on Tube Domains*, Math. Ann. 260 (1982), p. 269-302.

[Sh83] SHIMURA, G., *On Eisentsein Series*, Duke Math. J; 50 (1983), p. 417-476.

[ShiAr] SHIMURA, G., *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs. 82. Providence, RI: American Mathematical Society (AMS) . x, 302 p. (2000)

[Til-U] TILOUINE, J. and URBAN, E. , *Several variable p-adic families of Siegel-Hilbert cusp eigenforms and their Galois representations*, Ann. scient. Éc. Norm. Sup. 4^{e} série, 32 (1999) 499–574.

[V] VISIK, M.M., *Non-Archimedean measures connected with Dirichlet series*, Math. USSR Sb. 28 (1976), 216-228 (1978).

[MV] VISHIK, M.M. and MANIN, YU.I., *p-adic Hecke series of imaginary quadratic fields*, Math. USSR, Sbornik 24 (1974), 345-371 (1976).

[We56] Weil, A., *On a certain type of characters of the idèle-class group of an algebraic number-field*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 1–7, Science Council of Japan, Tokyo, 1956.

[Wi95] Wiles A., *Modular elliptic curves and Fermat's Last Theorem*, Ann. Math., II. Ser. 141, No.3 (1995) 443–55.

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